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# On the hyperbolicity of bipartite graphs and intersection graphs\*

David Coudert<sup>1,2</sup> and Guillaume Ducoffe<sup>2,1</sup>

<sup>1</sup>Inria, France

<sup>2</sup>Univ. Nice Sophia Antipolis, CNRS, I3S, UMR 7271, 06900 Sophia Antipolis, France

## Abstract

Hyperbolicity is a measure of the tree-likeness of a graph from a metric perspective. Recently, it has been used to classify complex networks depending on their underlying geometry. Motivated by a better understanding of the structure of graphs with bounded hyperbolicity, we here investigate on the hyperbolicity of bipartite graphs. More precisely, given a bipartite graph  $B = (V_0 \cup V_1, E)$  we prove it is enough to consider any one side  $V_i$  of the bipartition of  $B$  to obtain a close approximate of its hyperbolicity  $\delta(B)$  — up to an additive constant 2. We obtain from this result the sharp bounds  $\delta(G) - 1 \leq \delta(L(G)) \leq \delta(G) + 1$  and  $\delta(G) - 1 \leq \delta(K(G)) \leq \delta(G) + 1$  for every graph  $G$ , with  $L(G)$  and  $K(G)$  being respectively the line graph and the clique graph of  $G$ . Finally, promising extensions of our techniques to a broader class of intersection graphs are discussed and illustrated with the case of the biclique graph  $BK(G)$ , for which we prove  $(\delta(G) - 3)/2 \leq \delta(BK(G)) \leq (\delta(G) + 3)/2$ .

**Keywords:** Gromov hyperbolicity; bipartite graph; intersection graph; graph power; line graph; clique graph; biclique graph.

## 1 Introduction

The purpose of this paper is to bound the hyperbolicity of some classes of graphs that are defined in terms of graph operators. Roughly, hyperbolicity is a tree-likeness parameter that measures how close the shortest-path metric of a graph is to a tree metric (the smaller the hyperbolicity the closer the graph is to a metric tree). It has thus been proposed to take hyperbolicity into account to better classify complex networks [22]. For instance, it has been experimentally shown in [22] that social networks and protein interaction networks have bounded hyperbolicity while it is not the case for road networks. Another interest for hyperbolicity is that it helps analyzing some graph heuristics on large-scale networks. A good example to this is the *2-sweep* heuristic for computing the diameter, that provides very good results in practice [23]; such good results can be explained assuming a bounded hyperbolicity [11].

Relating the structural properties of graphs with hyperbolicity can be useful in this context, and it has become a growing line of research (*e.g.*, see [12, 15, 24, 31]). Indeed, we argue that one can obtain from such relations a comprehensive overview of the reasons why some complex networks are hyperbolic and some others are not. Following this line, we proved in [15] that most data center interconnection networks are *not* hyperbolic because they are symmetric graphs. As an attempt to go further in this direction, we here investigate on the hyperbolicity of bipartite graphs. In fact, we were motivated at first to bound the hyperbolicity of *line graphs* [30], that

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are intersection graphs of edges in a graph and have already received some attention in the literature of graph hyperbolicity [8, 9]. In this paper, we fully characterize what can be the defect between the hyperbolicity of a given graph and the hyperbolicity of its line graph, using an original connection with bipartite graphs. To better depict our novel approach, let us first recall that intersection graphs over a ground set  $S$  have for vertices a family of subsets of  $S$  with an edge between every two intersecting subsets. Therefore, they can be naturally represented as a bipartite graph — with vertices of the graph on one side, the ground set  $S$  on the other side, and an edge between every element of  $S$  and the subsets that contain it. Our main contribution is to show how we can use this representation so as to bound the hyperbolicity of intersection graphs—. This simple framework does not only apply to line graphs. We can use it to bound the hyperbolicity of *clique graphs* [20] and (with slightly more work) *biclique graphs* [19]. Overall, our main results can be expressed as follows.

- Given a bipartite graph  $B = (V_0 \cup V_1, E)$ , for every  $i \in \{0, 1\}$  let  $G_i$  be the graph with vertex-set the side  $V_i$  and with an edge between every two vertices that share a common neighbor in  $B$ . We prove that  $2\delta(G_i) \leq \delta(B) \leq 2\delta(G_i) + 2$  and the bounds are sharp (Theorem 4).
- We deduce from the above inequalities that  $\delta(G) - 1 \leq \delta(L(G)) \leq \delta(G) + 1$  for every graph  $G$ , with  $L(G)$  being the line graph of  $G$  (Theorem 6). Furthermore we show that all possible cases (between  $\delta(G) - 1$  and  $\delta(G) + 1$ ) can happen. This complements the bounds in [8, 9] that are proved to be sharp only for cycles (but with an alternative definition of hyperbolicity).
- By applying the same technique as for line graphs, we are the first to bound the hyperbolicity of clique graphs, *a.k.a.*, the intersection graphs of maximal cliques. More precisely, we prove that  $\delta(G) - 1 \leq \delta(K(G)) \leq \delta(G) + 1$  for every graph  $G$ , with  $K(G)$  being the clique graph of  $G$ , and all possible cases between  $\delta(G) - 1$  and  $\delta(G) + 1$  can happen (Theorem 8).
- We introduce graph powers [3] in our framework to obtain bounds on the hyperbolicity of other graphs. As example we prove that  $(\delta(G) - 3)/2 \leq \delta(BK(G)) \leq (\delta(G) + 3)/2$  for every graph  $G$ , with  $BK(G)$  being the biclique graph of  $G$  (Theorem 13). Bicliques are maximal induced complete bipartite subgraphs and they have gained recent attention in graph theory and graph algorithms. We refer to [19] and the papers cited therein for details.
- Finally, we bound the hyperbolicity of some other extensions of line graphs using our framework (Section 4.4), namely the *incidence graph*, the *total graph* [5], the *middle graph* [27], and the *k-edge graph* [26] of  $G$ .

Definitions and useful notations are given in Section 2.

## 2 Definitions and notations

We will follow the graph terminology in [6, 17]. Graphs in this study are connected, unweighted and finite (although part of the results extend to infinite weighted graphs). Given a graph  $G = (V, E)$ , the distance between every two vertices  $u, v$  in  $V$  equals the minimum number of edges on an  $uv$ -path. We will denote the distance between  $u$  and  $v$  by  $d_G(u, v)$ , or simply  $d(u, v)$  when  $G$  is clear from the context. Informally, we are interested in this paper in embedding the vertices of  $G$  into a tree  $T$  (possibly, edge-weighted) while minimizing the additive distortion of the distances in  $G$ . Hyperbolicity is both a lower-bound and an  $\mathcal{O}(\log |V|)$ -approximation for the minimum possible distortion [18]. Finer-grained relations between hyperbolicity and the minimum possible distortion for graphs are discussed in [16].

**Definition 1 (4-points Condition, [18]).** Let  $G = (V, E)$  be a connected graph.

For every 4-tuple  $u, v, x, y$  of  $V$ , we define  $\delta(u, v, x, y)$  as half of the difference between the two largest sums amongst:

$$S_1 = d(u, v) + d(x, y), S_2 = d(u, x) + d(v, y), \text{ and } S_3 = d(u, y) + d(v, x).$$

The graph hyperbolicity, denoted by  $\delta(G)$ , is equal to  $\max_{u, v, x, y \in V} \delta(u, v, x, y)$ .

Moreover, we say that  $G$  is  $\delta$ -hyperbolic for every  $\delta \geq \delta(G)$ .

It is well-known that 0-hyperbolic graphs are exactly those that can be embedded into a tree without any distortion, including trees and complete graphs. In fact, 0-hyperbolic graphs coincide with the *block graphs*, that are graphs whose all biconnected components are cliques (see Figure 1 for an illustration) [4, 21]. The class of 1/2-hyperbolic graphs has also been characterized in [2, 14].

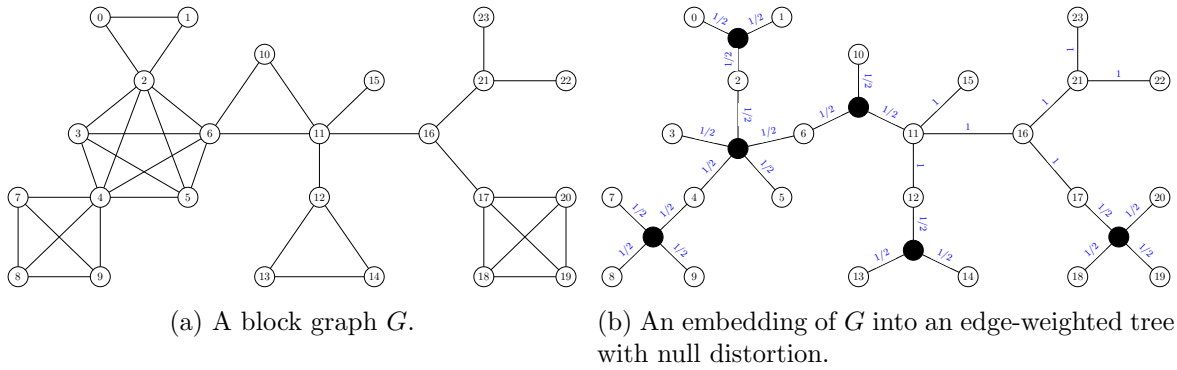


Figure 1: Block graphs are exactly the 0-hyperbolic graphs.

Furthermore, it turns out that not all 4-tuples in the graph need to be considered for the computation of hyperbolicity. This crucial point is the cornerstone of the most efficient algorithms so far to compute this parameter [7, 13]. Here we will use this observation to gain more insights on 4-tuples with maximum hyperbolicity in our proofs. This will require us to introduce the central notion of far-apart pairs.

**Definition 2 (Far-apart pair [25, 28]).** Given  $G = (V, E)$ , the pair  $(u, v)$  is far-apart if for every  $w \in V \setminus \{u, v\}$ , we have  $d(w, u) + d(u, v) > d(w, v)$  and  $d(w, v) + d(u, v) > d(w, u)$ .

Said differently, far-apart pairs are the ends of maximal shortest-paths in the graph. Their key property is that there always exists a 4-tuple with maximum hyperbolicity which contains two far-apart pairs.

**Lemma 3** ([25, 28]). Given  $G = (V, E)$ , there exist two far-apart pairs  $(u, v)$  and  $(x, y)$  satisfying:

- i)  $d_G(u, v) + d_G(x, y) \geq \max\{d_G(u, x) + d_G(v, y), d_G(u, y) + d_G(v, x)\}$ ;
- ii)  $\delta(u, v, x, y) = \delta(G)$ .

### 3 New bounds on the hyperbolicity of bipartite graphs

Let us start proving our main tool for the remaining of the paper, that is Theorem 4. Informally, we will consider a bipartite graph  $B = (V_0 \cup V_1, E)$  as obtained from two smaller intersection

graphs  $G_0$  and  $G_1$ , each having one side of the bipartition as its vertex-set. Our goal is to bound  $\delta(B)$  depending on  $\delta(G_i)$ , for any  $i \in \{0, 1\}$ . In fact, since any side  $V_i$  is a dominating set of  $B$ , then it is not hard to prove that  $\delta(B) \leq 2\delta(G_i) + 4$  (using the the 4-point Condition of Definition 1). The main difficulty is to obtain the sharp upper-bound  $\delta(B) \leq 2\delta(G_i) + 2$ , for which we will need far-apart pairs.

**Theorem 4.** *Let  $B = (V_0 \cup V_1, E)$  be a bipartite graph. We have  $\delta_B(V_i) \leq \delta(B) \leq \delta_B(V_i) + 2$ , where  $\delta_B(V_i) = \max_{u,v,x,y \in V_i} \delta_B(u, v, x, y)$  for every  $i \in \{0, 1\}$ , and these bounds are sharp.*

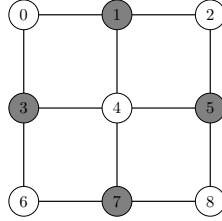


Figure 2: The bipartite graph  $G_{3,3}$  with each side of the bipartition colored differently.

*Proof.* We will only need to consider the upper-bound  $\delta(B) \leq \delta_B(V_i) + 2$ , for the lower-bound  $\delta_B(V_i) \leq \delta(B)$  trivially follows from the 4-points Condition of Definition 1. To prove the upper-bound, let  $(u, v)$  and  $(x, y)$  be two far-apart pairs of  $B$  such that  $S_1 = d(u, v) + d(x, y) \geq \max\{d(u, x) + d(v, y), d(u, y) + d(v, x)\} = S_2$  and  $\delta(u, v, x, y) = \delta(B)$ , that exist by Lemma 3. Note that  $\delta(B) = (S_1 - S_2)/2$ .

We claim that there are  $u', v' \in V_i$  such that  $\delta(u, v, x, y) \leq \delta(u', v', x, y) + 1$ . To prove the claim assume  $\delta(u, v, x, y) > 0$  (or else, it is trivial). The latter implies (by Definition 1) that  $u, v, x, y$  are pairwise different. There are three cases to be considered.

- If  $u, v \in V_i$ , then we are done by setting  $u' = u$  and  $v' = v$ .
- If  $u \in V_i$  and  $v \notin V_i$  (resp.,  $u \notin V_i$  and  $v \in V_i$ ), let us set  $u' = u$  and  $v' \in N(v)$  (resp.,  $u' \in N(u)$  and  $v' = v$ ). In such case let  $S'_1 = d(u', v') + d(x, y)$  and let  $S'_2 = \max\{d(u', x) + d(v', y), d(u', y) + d(v', x)\}$ . By the triangular inequality  $|S_1 - S'_1| \leq 1$  and similarly  $|S_2 - S'_2| \leq 1$ . Therefore, either  $S'_1 < S'_2$  and so,  $\delta(u, v, x, y) = (S_1 - S_2)/2 \leq (S'_1 - S'_2 + 2)/2 < 1 \leq \delta(u', v', x, y) + 1$ , or  $S'_1 \geq S'_2$  and so,  $\delta(u', v', x, y) = (S'_1 - S'_2)/2 \geq (S_1 - S_2 - 2)/2 = \delta(u, v, x, y) - 1$ .
- Else,  $u, v \notin V_i$ . In particular,  $N(u) \subseteq V_i$  and  $N(v) \subseteq V_i$  because  $B$  is bipartite by the hypothesis. We will prove as an intermediate subclaim that for every pair  $(u', v')$  with  $u' \in N(u)$  and  $v' \in N(v)$ , we have either  $d(u', v') = d(u, v)$  or  $d(u', v') = d(u, v) - 2$ . Indeed,  $d(u, v) - 2 \leq d(u', v') \leq d(u, v) + 2$  by the triangular inequality, and so, since the pairs  $(u, v)$  and  $(u', v')$  are in distinct sides of the bipartition of  $B$ , either  $d(u', v') = d(u, v) - 2$  or  $d(u', v') = d(u, v)$  or  $d(u', v') = d(u, v) + 2$ . The latter case,  $d(u', v') = d(u, v) + 2$ , would contradict the fact that  $(u, v)$  is far-apart. Hence either  $d(u', v') = d(u, v)$  or  $d(u', v') = d(u, v) - 2$ , which proves the subclaim. Now there are two subcases to be considered.
  - Suppose there are  $u' \in N(u)$  and  $v' \in N(v)$  such that  $d(u', v') = d(u, v)$ . Let  $S'_1 = d(u', v') + d(x, y)$  and let  $S'_2 = \max\{d(u', x) + d(v', y), d(u', y) + d(v', x)\}$ . By the choice of  $u'$  and  $v'$ , we have  $S'_1 = S_1$  while  $|S_2 - S'_2| \leq 2$  by the triangular inequality. As a result, either  $S'_1 < S'_2$  and so,  $\delta(u, v, x, y) = (S_1 - S_2)/2 \leq (S'_1 - S'_2 + 2)/2 < 1 \leq \delta(u', v', x, y) + 1$ , or  $S'_1 \geq S'_2$  and so,  $\delta(u', v', x, y) = (S'_1 - S'_2)/2 \geq (S_1 - S_2 - 2)/2 = \delta(u, v, x, y) - 1$ .

- Else, for every  $u' \in N(u)$  and  $v' \in N(v)$ , we have  $d(u', v') = d(u, v) - 2$ . Let  $u' \in N(u)$  and  $v' \in N(v)$  satisfy  $d(u, x) = 1 + d(u', x)$  and  $d(v, x) = 1 + d(v', x)$ . We set as before  $S'_1 = d(u', v') + d(x, y)$  and  $S'_2 = \max\{d(u', x) + d(v', y), d(u', y) + d(v', x)\}$ . By the choice of  $u'$  and  $v'$ , we have  $S'_1 = S_1 - 2$ . Furthermore,  $S_2 - 2 \leq S'_2 \leq S_2$  because  $d(u', x) = d(u, x) - 1$ ,  $d(v', x) = d(v, x) - 1$ , and  $|d(u', y) - d(u, y)| \leq 1$  and  $|d(v', y) - d(v, y)| \leq 1$  by the triangular inequality. It follows that either  $S'_1 < S'_2$  and so,  $\delta(u, v, x, y) = (S_1 - S_2)/2 \leq (S'_1 - S'_2 + 2)/2 < 1 \leq \delta(u', v', x, y) + 1$ , or  $S'_1 \geq S'_2$ , and so  $\delta(u', v', x, y) = (S'_1 - S'_2)/2 \geq (S_1 - S_2 - 2)/2 \geq \delta(u, v, x, y) - 1$ , that achieves proving the claim.

Finally, since the pair  $(x, y)$  is also far-apart, there exist  $x', y' \in V_i$  such that  $\delta(u', v', x, y) \leq \delta(u', v', x', y') + 1$ . As a result,  $\delta(B) = \delta(u, v, x, y) \leq \delta(u', v', x, y) + 1 \leq \delta(u', v', x', y') + 2 \leq \delta_B(V_i) + 2$ .

To show that the bounds are sharp, let us consider the square grid  $G_{3,3}$  of side length two as drawn in Figure 2. This bipartite graph has vertex-set  $V = V_0 \cup V_1$ , with  $V_0 = \{0, 2, 4, 6, 8\}$  and  $V_1 = \{1, 3, 5, 7\}$ . We have  $\delta(G_{3,3}) = 2$ , that is reached with the four corners 0, 2, 6 and 8 (i.e.,  $\delta(0, 2, 6, 8) = 2$ ). On the one hand, side  $V_0$  contains the four corners and so  $\delta_{G_{3,3}}(V_0) = \delta(G_{3,3}) = 2$ . On the other hand, vertices on the other side  $V_1$  are exactly the four neighbors of vertex 4, and so  $\delta_{G_{3,3}}(V_1) = 0 = \delta(G_{3,3}) - 2$ .  $\square$

## 4 Applications to intersection graphs

Our main results in this section are (sharp) lower and upper-bounds on the hyperbolicity of intersection graphs that have been considered in the literature. These comprise the two well-known families of line graphs and clique graphs (we refer to [1, 29] for surveys), along with biclique graphs that have been introduced more recently as extensions of line graphs.

### 4.1 Line graph

**Definition 5.** Given  $G = (V, E)$ , the line-graph of  $G$ , denoted by  $L(G)$ , is the intersection graph of  $E$ . That is, it has vertex-set  $E$  and for every  $e, e' \in E$  there is an edge  $\{e, e'\}$  in  $L(G)$  if and only if  $e$  and  $e'$  share an end in  $G$ .

**Theorem 6.** For every graph  $G$ ,  $\delta(G) - 1 \leq \delta(L(G)) \leq \delta(G) + 1$ , and these bounds are sharp.

*Proof.* Let  $B$  be the incidence graph of  $G$ , that is, it has vertex-set  $V \cup E$  and there is an edge in  $B$  between  $u \in V$  and  $e \in E$  if and only if  $u$  is an end of  $e$  in  $G$ . By Theorem 4,  $\delta_B(V) \leq \delta(B) \leq \delta_B(V) + 2$  and similarly  $\delta_B(E) \leq \delta(B) \leq \delta_B(E) + 2$ . Furthermore by construction  $d_B(u, v) = 2d_G(u, v)$  for every  $u, v \in V$  and in the same way  $d_B(e, e') = 2d_{L(G)}(e, e')$  for every  $e, e' \in E$ . As a result,  $\delta_B(V) = 2\delta(G)$ , similarly  $\delta_B(E) = 2\delta(L(G))$ , and so,

$$2\delta(G) \leq \delta(B) \leq 2\delta(G) + 2,$$

$$2\delta(L(G)) \leq \delta(B) \leq 2\delta(L(G)) + 2.$$

By mixing up the two chains of inequality one obtains  $2\delta(G) \leq 2\delta(L(G)) + 2$  and  $2\delta(L(G)) \leq 2\delta(G) + 2$ , whence  $\delta(G) \leq \delta(L(G)) + 1$  and  $\delta(L(G)) \leq \delta(G) + 1$ , as desired.

To show that the bounds are sharp, consider the graphs  $G_{-1}$  and  $G_1$  as drawn respectively in Figures 3a and 3i. We have  $\delta(L(G_{-1})) = \delta(G_{-1}) - 1$  and  $\delta(L(G_1)) = \delta(G_1) + 1$ .  $\square$

In Figure 3 we show that all possible cases of Theorem 6 (with defect between  $-1$  and  $+1$ ) are realized by some graphs. By taking the incidence graphs of  $G_{-1}$  and  $G_1$ , one obtains a new proof that the bounds of Theorem 4 are sharp.

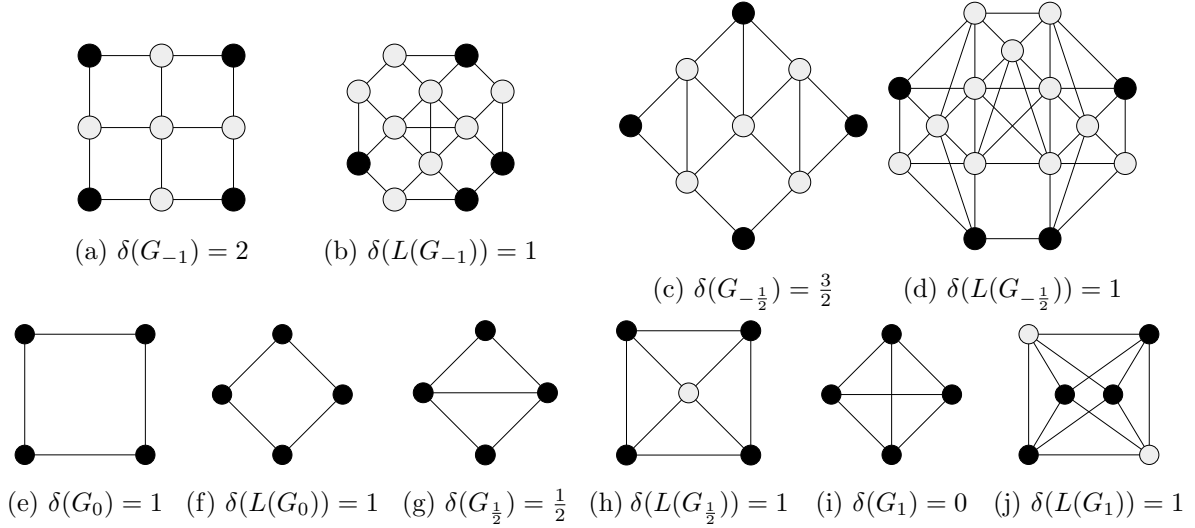


Figure 3: Examples of graphs  $G_i$  with  $\delta(L(G_i)) = \delta(G_i) + i$  for every  $i \in \{-1, -1/2, 0, +1/2, +1\}$ . A 4-tuple with maximum hyperbolicity is drawn in bold on each graph.

## 4.2 Clique graph

**Definition 7.** Given  $G = (V, E)$ , let  $\Omega$  be the set of all maximal cliques of  $G$ . The clique-graph of  $G$ , denoted by  $K(G)$ , is the intersection graph of  $\Omega$ . That is, it has vertex-set  $\Omega$  and for every  $S, S' \in \Omega$  there is an edge  $\{S, S'\}$  in  $K(G)$  if and only if the two cliques  $S$  and  $S'$  intersect.

**Theorem 8.** For every graph  $G$ ,  $\delta(G) - 1 \leq \delta(K(G)) \leq \delta(G) + 1$ , and these bounds are sharp.

*Proof.* Let  $B$  be the bipartite graph defined as follows. It has vertex-set  $V \cup \Omega$  and there is an edge between  $u \in V$  and  $S \in \Omega$  if and only if  $u \in S$ . By Theorem 4,  $\delta_B(V) \leq \delta(B) \leq \delta_B(V) + 2$  and similarly  $\delta_B(\Omega) \leq \delta(B) \leq \delta_B(\Omega) + 2$ . Furthermore,  $d_B(S, S') = 2d_{K(G)}(S, S')$  for every  $S, S' \in \Omega$  by construction, so,  $\delta_B(\Omega) = 2\delta(K(G))$ . We claim in addition that  $d_B(u, v) = 2d_G(u, v)$  for every  $u, v \in V$ . To prove the claim it is enough to prove  $d_B(u, v) = 2$  if and only if  $u$  and  $v$  are adjacent in  $G$ . By construction,  $d_B(u, v) = 2$  if and only if there is  $S \in \Omega$  such that  $u, v \in S$ . If  $u, v \in S$  for some  $S \in \Omega$  then  $u$  and  $v$  are adjacent in  $G$  because  $S$  is a clique of  $G$ , conversely if  $u$  and  $v$  are adjacent in  $G$  then  $u, v \in S$  with  $S$  being any maximal clique containing the edge  $\{u, v\}$ . Therefore, the claim is proved, and so, since  $d_B(u, v) = 2d_G(u, v)$  for every  $u, v \in V$ ,  $\delta_B(V) = 2\delta(G)$ . As a result:

$$2\delta(G) \leq \delta(B) \leq 2\delta(G) + 2,$$

$$2\delta(K(G)) \leq \delta(B) \leq 2\delta(K(G)) + 2.$$

By mixing up the two chains of inequality one obtains  $2\delta(G) \leq 2\delta(K(G)) + 2$  and  $2\delta(K(G)) \leq 2\delta(G) + 2$ , whence  $\delta(G) \leq \delta(K(G)) + 1$  and  $\delta(K(G)) \leq \delta(G) + 1$ , as desired.

To show that the bounds are sharp, consider the graphs  $H_{-1}$  and  $H_1$  as drawn respectively in Figures 4a and 4i. We have  $\delta(K(H_{-1})) = \delta(H_{-1}) - 1$  and  $\delta(K(H_1)) = \delta(H_1) + 1$ .  $\square$

In Figure 4 we show that all possible cases of Theorem 8 (with defect between  $-1$  and  $+1$ ) are realized by some graphs. Note that  $H_{-1} = L(G_{\frac{1}{2}})$ ,  $H_{-\frac{1}{2}} = G_{\frac{1}{2}}$ ,  $H_0 = G_0$  and  $H_1 = L(G_{-1})$ .

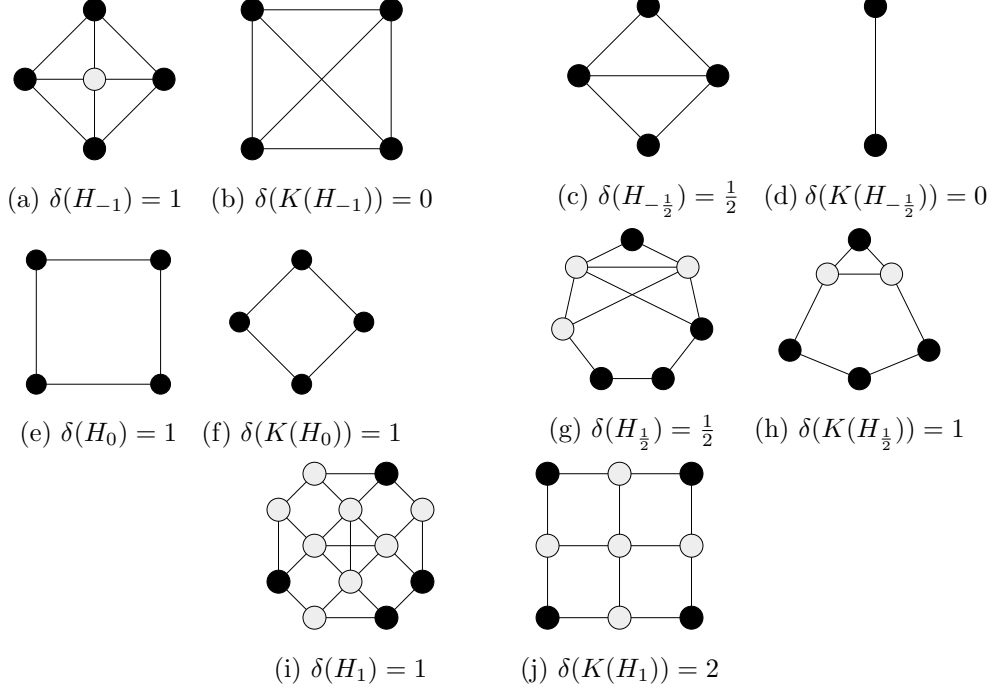


Figure 4: Examples of graphs  $H_i$  with  $\delta(K(H_i)) = \delta(H_i) + i$  for every  $i \in \{-1, -1/2, 0, +1/2, +1\}$ . A 4-tuple with maximum hyperbolicity is drawn in bold on each graph.

### 4.3 Biclique graph

The above two examples of line graphs and clique graphs are intersection graphs of *cliques*. However, there are interesting graph families that are defined as the intersection graphs of some subgraphs of diameter larger than one. As a general method to overcome this difficulty, we now introduce graph powers in our framework.

**Definition 9.** Given  $G = (V, E)$  and  $k \geq 1$ , the  $k^{\text{th}}$ -power of  $G$ , denoted by  $G^k$ , is defined as follows. It has vertex-set  $V$  and for every  $u, v \in V$  there is an edge  $\{u, v\}$  in  $G^k$  if and only if  $d_G(u, v) \leq k$ .

Pushing further a previous result from [14], let us bound the hyperbolicity of graph powers (Proposition 11). We will need the following intermediate lemma.

**Lemma 10** ([3]). Given  $G = (V, E)$  and  $k \geq 1$ ,  $d_{G^k}(u, v) = \left\lceil \frac{d_G(u, v)}{k} \right\rceil$  for every  $u, v \in V$ .

**Proposition 11.** For every graph  $G$  and  $k \geq 2$ ,  $\frac{\delta(G)+1}{k} - 1 \leq \delta(G^k) \leq \frac{\delta(G)-1}{k} + 1$ , and these bounds are sharp.

*Proof.* Let  $u, v, x, y \in V$  be arbitrary. Assume w.l.o.g.  $S_1 = d_G(u, v) + d_G(x, y) \geq S_2 = d_G(u, x) + d_G(v, y) \geq S_3 = d_G(u, y) + d_G(v, x)$ . In order to prove Proposition 11, we will need to prove some relations between the hyperbolicity  $\delta_G(u, v, x, y)$  of the 4-tuple in  $G$  and the hyperbolicity  $\delta_{G^k}(u, v, x, y)$  of the 4-tuple in  $G^k$ . Let  $S'_1 = d_{G^k}(u, v) + d_{G^k}(x, y)$ ,  $S'_2 = d_{G^k}(u, x) + d_{G^k}(v, y)$  and  $S'_3 = d_{G^k}(u, y) + d_{G^k}(v, x)$ . By Lemma 10, we have

$$S'_1 = \left\lceil \frac{d_G(u, v)}{k} \right\rceil + \left\lceil \frac{d_G(x, y)}{k} \right\rceil, S'_2 = \left\lceil \frac{d_G(u, x)}{k} \right\rceil + \left\lceil \frac{d_G(v, y)}{k} \right\rceil \text{ and } S'_3 = \left\lceil \frac{d_G(u, y)}{k} \right\rceil + \left\lceil \frac{d_G(v, x)}{k} \right\rceil.$$



In particular  $S_i/k \leq S'_i \leq S_i/k + 2(1 - 1/k)$  for every  $1 \leq i \leq 3$ . Since  $2(1 - 1/k) < 2$ , there can be no more than two integers between  $S_i/k$  and  $S_i/k + 2(1 - 1/k)$ , that implies either  $S'_i = \lceil S_i/k \rceil$  or  $S'_i = \lceil S_i/k \rceil + 1$ . Now there are two cases to be considered.

- Suppose  $S'_1 < S'_j$  with  $S'_j = \max\{S'_2, S'_3\}$ . Then it must be the case that  $S'_1 = \lceil S_1/k \rceil$  and  $S'_j = \lceil S_j/k \rceil + 1$  because  $\lceil S_1/k \rceil \geq \lceil S_j/k \rceil$ . The latter implies

$$\delta_{G^k}(u, v, x, y) \leq \frac{S'_j - S'_1}{2} \leq \frac{\lceil \frac{S_j}{k} \rceil + 1 - \lceil \frac{S_1}{k} \rceil}{2} \leq \frac{1}{2} \leq 1 - \frac{1}{k} \leq \frac{\delta_G(u, v, x, y) - 1}{k} + 1 \quad (1)$$

$$\delta_G(u, v, x, y) \leq \frac{S_1 - S_j}{2} \leq k \frac{\frac{S_1}{k} - \frac{S_j}{k}}{2} \leq k \left[ \frac{S'_1 - S'_j}{2} + 1 - \frac{1}{k} \right] \leq -\frac{k}{2} + k - 1 \leq \frac{k}{2} - 1 \quad (2)$$

Furthermore if  $\delta_G(u, v, x, y) \leq k/2 - 1$  then  $(\delta_G(u, v, x, y) + 1)/k - 1 < 0 \leq \delta_{G^k}(u, v, x, y)$ .

- Else,  $S'_1 \geq \max\{S'_2, S'_3\}$ . In such case  $\delta_{G^k}(u, v, x, y) = (S'_1 - \max\{S'_2, S'_3\})/2$ . Moreover,  $S_2/k \leq \max\{S'_2, S'_3\} \leq S_2/k + 2(1 - 1/k)$  because  $S_2 \geq S_3$ . Therefore,

$$\delta_{G^k}(u, v, x, y) \geq \frac{\frac{S_1}{k} - \frac{S_2}{k} - 2(1 - \frac{1}{k})}{2} \geq \frac{\delta_G(u, v, x, y)}{k} - 1 + \frac{1}{k} \quad (3)$$

$$\delta_{G^k}(u, v, x, y) \leq \frac{\frac{S_1}{k} + 2(1 - \frac{1}{k}) - \frac{S_2}{k}}{2} \leq \frac{\delta_G(u, v, x, y)}{k} + 1 - \frac{1}{k} \quad (4)$$

It follows that  $(\delta_G(u, v, x, y) + 1)/k - 1 \leq \delta_{G^k}(u, v, x, y) \leq (\delta_G(u, v, x, y) - 1)/k + 1$  in both cases.

- When  $u, v, x, y$  maximizes  $\delta_G$  the first inequality leads to  $(\delta(G) + 1)/k - 1 \leq \delta_{G^k}(u, v, x, y) \leq \delta(G^k)$ .
- When it maximizes  $\delta_{G^k}$  the second inequality leads to  $\delta(G^k) \leq (\delta_G(u, v, x, y) - 1)/k + 1 \leq (\delta(G) - 1)/k + 1$ .

Let us finally show that the bounds of Proposition 11 are sharp. Indeed, on the one hand the cycle  $C_4$  with four vertices satisfies  $\delta(C_4) = 1$  and  $C_4^2 = K_4$ , the clique with four vertices. Therefore,  $\delta(C_4^2) = 0 = (\delta(C_4) + 1)/2 - 1$ . On the other hand the rectangular grid  $G_{2,3}$  (obtained from two  $C_4$ 's sharing exactly one edge) satisfies  $\delta(G_{2,3}) = 1$ , and its four borders induce a  $C_4$  in  $G_{2,3}^2$ . Consequently,  $\delta(G_{2,3}^2) \geq 1 \geq (\delta(G_{2,3}) - 1)/2 + 1$ .  $\square$

We will illustrate the benefit of using graph powers within our framework through the case of *biclique graphs*, that are defined as follows.

**Definition 12.** *Given  $G = (V, E)$ , the set  $S \subseteq V$  is a biclique of  $G$  if it induces a complete bipartite subgraph of  $G$ . Let  $\Sigma$  be all maximal bicliques of  $G$ . The biclique graph of  $G$ , denoted by  $BK(G)$ , is the intersection graph of  $\Sigma$ . That is, it has vertex-set  $\Sigma$  and for every  $S, S' \in \Sigma$  there is an edge  $\{S, S'\}$  in  $BK(G)$  if and only if the two bicliques  $S$  and  $S'$  intersect.*

For instance, the biclique graph of a complete graph is exactly its line graph. In Figure 5 we consider the biclique graph of the grid  $G_{3,3}$ . The maximal bicliques of this grid comprise four cycles of length four, four stars with three branches each and one star with four branches. All of these pairwise intersect at the central vertex of the grid, therefore,  $BK(G_{3,3}) = K_9$ , the complete graph with nine vertices.

**Theorem 13.** *For every graph  $G$ ,  $(\delta(G) - 3)/2 \leq \delta(BK(G)) \leq (\delta(G) + 3)/2$ .*

*Proof.* Let  $B$  be the bipartite graph defined as follows. It has vertex-set  $V \cup \Sigma$  and there is an edge between  $u \in V$  and  $S \in \Sigma$  if and only if  $u \in S$ . By Theorem 4,  $\delta_B(V) \leq \delta(B) \leq \delta_B(V) + 2$

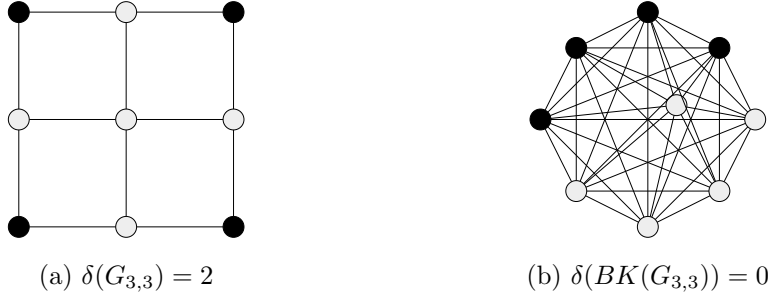


Figure 5: The grid graph  $G_{3,3}$  along with its biclique graph, that is the complete graph  $K_9$  with nine vertices. A 4-tuple with maximum hyperbolicity is drawn in bold on each graph.

and similarly  $\delta_B(\Sigma) \leq \delta(B) \leq \delta_B(\Sigma) + 2$ . Furthermore,  $d_B(S, S') = 2 d_{BK(G)}(S, S')$  for every  $S, S' \in \Sigma$  by construction, so,  $\delta_B(\Sigma) = 2\delta(BK(G))$ . We claim in addition that  $d_B(u, v) = 2 d_{G^2}(u, v)$  for every  $u, v \in V$ , with  $G^2$  be defined as in Definition 9. To prove the claim it is enough to prove  $d_B(u, v) = 2$  if and only if  $d_G(u, v) \leq 2$ . By construction,  $d_B(u, v) = 2$  if and only if there is  $S \in \Sigma$  such that  $u, v \in S$ . If  $u, v \in S$  for some  $S \in \Sigma$  then  $d_G(u, v) \leq 2$  because  $S$  induces a complete bipartite subgraph of  $G$ , conversely if  $d_G(u, v) \leq 2$  then every  $uv$ -shortest-path  $P$  in  $G$  induces a complete bipartite subgraph of  $G$  (with one side containing one vertex and the other side containing one or two vertices) and so,  $u, v \in S$  with  $S$  being any maximal biclique containing  $P$ . Therefore, the claim is proved, and since  $d_B(u, v) = 2 d_{G^2}(u, v)$  for every  $u, v \in V$ , we obtain  $\delta_B(V) = 2\delta(G^2)$ . As a result:

$$2\delta(G^2) \leq \delta(B) \leq 2\delta(G^2) + 2,$$

$$2\delta(BK(G)) \leq \delta(B) \leq 2\delta(BK(G)) + 2.$$

By mixing up the two chains of inequality one obtains  $2\delta(G^2) \leq 2\delta(BK(G)) + 2$  and  $2\delta(BK(G)) \leq 2\delta(G^2) + 2$ , whence  $\delta(G^2) \leq \delta(BK(G)) + 1$  and  $\delta(BK(G)) \leq \delta(G^2) + 1$ . Since by Proposition 11  $(\delta(G) - 1)/2 \leq \delta(G^2) \leq (\delta(G) + 1)/2$ , one obtains  $\delta(BK(G)) \geq \delta(G^2) - 1 \geq (\delta(G) - 3)/2$  and  $\delta(BK(G)) \leq (\delta(G) + 1)/2 + 1 \leq (\delta(G) + 3)/2$ , as desired.  $\square$

In fact, we prove the more precise inequalities  $\delta(G^2) - 1 \leq \delta(BK(G)) \leq \delta(G^2) + 1$  for every graph  $G$ , and we claim these bounds are sharp.

**Corollary 14.** *For every graph  $G$ ,  $\delta(G^2) - 1 \leq \delta(BK(G)) \leq \delta(G^2) + 1$ , and these bounds are sharp.*

*Proof.* The bounds are given by Theorem 13. Also, let us now show that they are sharp.

Consider first the grid graph  $G_{3,3}$ . We have  $\delta(BK(G_{3,3})) = \delta(K_9) = 0$ , while  $\delta(G_{3,3}^2) \geq 1$  because the four corners of the grid induce a cycle of length four in the square graph  $G_{3,3}^2$ . As a result,  $\delta(BK(G_{3,3})) = \delta(G_{3,3}^2) - 1$ , and so the lower-bound is reached.

Now, recall that the biclique graph of a complete graph  $K_n$  is exactly the line graph  $L(K_n)$ . Therefore, consider the graph  $K_4$  and its line graph ( $G_1$  and  $L(G_1)$  in Figures 3i and 3j). Since  $K_4^2 = K_4$  then it is indeed the case that  $\delta(BK(K_4)) = \delta(L(K_4)) = \delta(K_4) + 1 = \delta(K_4^2) + 1$ , and so the upper-bound is also reached.  $\square$

#### 4.4 Additional bounds

Before we conclude this paper, let us present a few other results that are obtained within our framework. More precisely, given a graph  $G = (V, E)$  we consider the following extensions of line graphs (illustrations for each case are given in Figure 6).

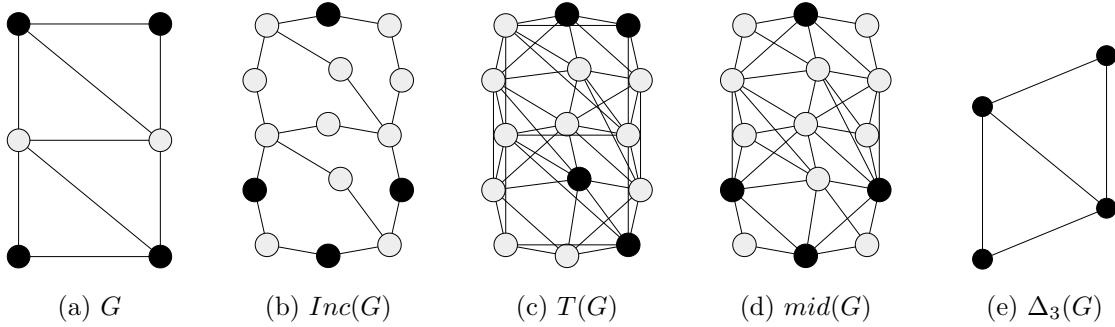


Figure 6: Some intersection graphs obtained from a triangular grid graph  $G$ . We have  $\delta(G) = 1/2$ ,  $\delta(Inc(G)) = 2$ ,  $\delta(T(G)) = 1$ ,  $\delta(mid(G)) = 1$ , and  $\delta(\Delta_3(G)) = 1/2$ .

The *incidence graph* of  $G$ , denoted by  $Inc(G)$ , has vertex set  $V \cup E$  with an edge between every  $u \in V$  and every  $e \in E$  such that  $u$  is an end of  $e$  in  $G$  (see Figure 6b). It follows from the proof of Theorem 6 (for line graphs) that  $2\delta(G) \leq \delta(Inc(G)) \leq 2\delta(G) + 2$  and the bounds are sharp.

The *total graph* of  $G$  [5], denoted by  $T(G)$ , is constructed from  $G$  and  $L(G)$  by adding an edge between every  $u \in V$  and every  $e \in E$  such that  $u$  is an end of  $e$  in  $G$  (see Figure 6c). In fact, this implies  $T(G) = (Inc(G))^2$ , hence by Proposition 11 we have  $(\delta(Inc(G)) - 1)/2 \leq \delta(T(G)) \leq (\delta(Inc(G)) + 1)/2$ , and so,  $\delta(G) - 1/2 \leq \delta(T(G)) \leq \delta(G) + 3/2$ . One can sharpen the lower-bound and write  $\delta(G) \leq \delta(T(G)) \leq \delta(G) + 3/2$  after noticing that  $G$  is an isometric subgraph (*i.e.*, a distance-preserving subgraph) of  $T(G)$ .

The *middle graph* of  $G$  [27], denoted by  $mid(G)$ , is constructed from  $Inc(G)$  by adding an edge between every two “edge-vertices”  $e, e' \in E$  sharing an end in  $G$  (see Figure 6d). Said differently, it is the intersection graph of all cliques of size two or less in  $G$ . Using a bipartite representation that is similar in spirit with those for line graphs (Theorem 6) and clique graphs (Theorem 8), one obtains  $\delta(G) - 1 \leq \delta(mid(G)) \leq \delta(G) + 1$ .

Last, the *k-edge graph* of  $G$  [26], denoted by  $\Delta_k(G)$ , is the intersection graph of all cliques of size  $k$  and maximal cliques of size at most  $k - 1$  in  $G$  (see Figure 6e). Note that if  $k = 2$  then it is exactly the line graph, and if  $k = n$  then it is exactly the clique graph. Again using a bipartite representation with similar properties as those for line graphs and clique graphs, one obtains  $\delta(G) - 1 \leq \delta(\Delta_k(G)) \leq \delta(G) + 1$ .

## 5 Conclusion

We have proved that the hyperbolicity of any bipartite graph can be approximated up to a small additive constant by only considering the smallest side of its bipartition. This means a decrease by half of the number of vertices to be considered, hence a speed-up in the computation of hyperbolicity. On a more theoretical side, we detailed a simple framework so as to bound the hyperbolicity of line graphs and several other intersection graphs. We let open the question whether our techniques could also be applied to more “exotic” generalizations of line graphs – say, edge clique graphs [10].

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