



## Jost Bürgi's skillful computation of sines

Denis Roegel

► **To cite this version:**

Denis Roegel. Jost Bürgi's skillful computation of sines. [Research Report] LORIA - Université de Lorraine. 2015. <hal-01220160>

**HAL Id: hal-01220160**

**<https://hal.inria.fr/hal-01220160>**

Submitted on 25 Oct 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Jost Bürgi's skillful computation of sines

Denis Roegel\*

25 october 2015

## 1 Introduction

Folkerts, Launert and Thom have recently announced a very interesting discovery about Jost Bürgi (1552–1632) [3].<sup>1</sup> Bürgi is well known as a (very) skillful mechanic, clockmaker and instrument maker, and also as an inventor of a table of progressions which could be used for the same purpose as logarithms. His table can even be considered a table of logarithms, although we do not consider him as the inventor of logarithms, because there is much more to logarithms than tables [12].

It was also long known that Bürgi had computed a table of sines, but little was known about his methods. His table of sines at intervals of  $2''$  does not seem to have survived, and how accurate the table was remains a mystery. In 1588, Ursus gave a rather cryptic hint about Bürgi's method [17] and it seemed to be related to differences. In an earlier article [12], we had theorized that Bürgi had perhaps anticipated later work where pivot values were computed, then differenced, and then the differences used to compute intermediate values in a manner similar to that used in the *Tables du Cadastre* [13].

Bürgi's newly discovered manuscript on trigonometry gives a number of answers to four centuries of questions. But it also raises new questions.

---

\*Denis Roegel, LORIA, BP 239, 54506 Vandœuvre-lès-Nancy cedex, France, roegel@loria.fr

<sup>1</sup>We thank Fritz Staudacher for having informed us of that recent discovery on October 23, 2015.

## 2 Bürgi's algorithm

### 2.1 Dividing the quadrant in nine parts

Folkerts *et al.* [3] give the following table adapted from Bürgi's manuscript dated from the 1580s. Bürgi used instead sexagesimal values, but we can safely convert them to base 10.

	$c_5$		$c_4$		$c_3$		$c_2$		$c_1$
0	0		0		0		0		0
10	2,235,060	2,235,060	67,912	67,912	2,064	2,064	63	63	2
20	4,402,208	2,167,148	133,760	67,848	4,065	2,001	124	61	4
30	6,435,596	2,033,388	195,543	61,783	5,942	1,877	181	57	6
40	8,273,441	1,837,845	251,384	55,841	7,638	1,696	232	51	7
50	9,859,902	1,586,461	299,587	48,203	9,102	1,464	276	44	8
60	11,146,776	1,286,874	338,688	39,101	10,290	1,188	312	36	9
70	12,094,962	948,186	367,499	28,811	11,166	876	339	27	10
80	12,675,649	580,687	385,144	17,645	11,703	537	356	17	11
90	12,871,192	195,543	391,086	5,942	11,884	181	362	6	12

The purpose of this table is to compute the values of  $\sin 10^\circ$ ,  $\sin 20^\circ$ ,  $\dots$ ,  $\sin 90^\circ$ , to any desired accuracy. Bürgi's algorithm is deceptively simple. There is no bisection, no roots, only two basic operations: many additions, and a few divisions by 2. The computations can be done with integers, or with sexagesimal values, or in any other base.

In order to compute the sines, Bürgi starts with an arbitrary list of values, which can be considered as first approximations of the sines, but need not be. These values are given in column  $c_1$ . In all these columns,  $c_1$  to  $c_5$ , the last value is always the *sinus totus*. That is, the first approximation starts with  $\sin 90^\circ = 12$ . This gives in modern terms for  $\sin 60^\circ$  the value  $\frac{9}{12} = 0.75$ , an approximation of the actual sine which is  $0.866\dots$ . It basically matters very little with what values one starts. Only the first value should be 0. It is even possible to take all other initial values equal to 1, for instance, but they can't all be taken equal to 0.

Column  $c_5$  shows the result of that algorithm, here conducted to four steps, but the table could have been extended at will towards the left. Now, in column  $c_5$ , there is a new value for the *sinus totus*, namely 12871192, and therefore we have as a new approximation of  $\sin 60^\circ$  the fraction  $\frac{11146776}{12871192} = 0.86602515136$ , the exact value being  $\frac{\sqrt{3}}{2} = 0.866025403\dots$

Bürgi's algorithm is an iterative procedure for computing all the values of column  $c_{i+1}$  from those of column  $c_i$ . The computations use an intermediate

column whose last value is half of the previous *sinus totus*. In the above example, the *sinus totus* in column  $c_1$  is 12, and the last value of the column between  $c_2$  and  $c_1$  is 6. The last value of the column between  $c_3$  and  $c_2$  is 181, half of 362. If the *sinus totus* is odd, one might take the exact half, but it actually does not matter, as this algorithm leads to increasingly larger numbers, and ignoring a half integer only has marginal consequences on the convergence.

Once the last value of an intermediate column has been obtained, all other values of that column are obtained by adding the values in the previous column, as if the previous column were differences. So, we have  $6 + 11 = 17$ ,  $17 + 10 = 27$ , and so on.

When the intermediate column has been filled, the new column  $c_{i+1}$  is constructed by starting with 0, and adding the values of the intermediate column.

And that's all!

## 2.2 Other subdivisions

The previous section led to the computation of the sines of  $10^\circ$ ,  $20^\circ$ ,  $\dots$ ,  $90^\circ$ , without any geometric construction. This scheme can be adapted to obtain any set of values  $\sin(\frac{k\pi}{2n})$ , for  $0 \leq k \leq n$ , but it is important to realize that this algorithm provides all the values, and cannot provide only one sine. All the sines are obtained in parallel, with relatively simple operations.

The value of  $n$  corresponds to the number of values in the intermediate columns. In the above example,  $n = 9$ . If we start with  $n = 90$ , we will obtain the sines of all degrees in the quadrant.

We can also conduct simpler examples, such as computing the sine of  $45^\circ$  by taking  $n = 2$ . If we start with the three values 0,1,2 (say), that is, with 0.5 for the approximation of  $\sin 45^\circ$ , we obtain the intermediate values 2,1, from which we get the new approximations 0,2,3 and hence  $\frac{2}{3} = 0.666\dots$  for the sine. Continuing this process, we obtain the intermediate values 3,1 (or 3.5,1.5 if we do not round), leading to the new approximations 0,3,4, and  $\frac{3}{4} = 0.75$  for the sine. Still going on, we find the intermediate values 5,2 and the new approximations 0,5,7, hence  $\frac{5}{7} = 0.71428\dots$  for the sine. This procedure will converge towards the value  $\frac{\sqrt{2}}{2} = 0.707106781\dots$

## 3 Bürgi's intuition

How did Bürgi obtain this algorithm? Folkerts *et al.* [3] do not answer this question. They give a modern proof of the convergence. This proof, by An-

dreas Thom, is quite intricate and is based on the observation that the linear map between one set of approximations and the next one has eigenvalues which are the sought sines.

Bürgi did not have these means, and at first we thought that Bürgi's algorithm must be the expression of a clever geometrical construction. After some search, we gave up this idea.

Then, we found a very simple way to derive Bürgi's algorithm. As mentioned above, Bürgi's procedure alludes to differencing, and we know that Bürgi did use differences to compute other sines between  $\sin 1'$  and  $\sin 1^\circ$ . Now, in the tables of Rheticus and Viète, trigonometrical values were given with first differences, in order to be able to check these values.

Neither Rheticus in 1551 [7], nor Viète in 1579 [18] gave second differences. Rheticus's calculations for the *Opus palatinum* [8] may have contained second differences, but they do not appear in the table published in 1596. Second differences only appeared in Pitiscus's *Thesaurus mathematicus* in 1613 [6].

Bürgi must have been looking with great care at Rheticus' values, and toyed with the computation of differences, including second differences. He must then have noticed that the second differences of the sines mimic the sines themselves. In other words, the second differences are approximations to the sines, and the more one differences, the worse these approximations are. Conversely, if there was a way to start with approximate differences and go backwards towards the actual sines, accuracy would be obtained. But was this feasible?

In order to make it work, or at least to see if one could go backwards, only two things were needed:

- Bürgi had to notice (without proof) that there was almost a constant ratio between the sines and their second differences; in the example for the computation of  $\sin 10^\circ$ ,  $\sin 20^\circ$ , that ratio is about 32.911 (in fact  $\frac{\csc^2(\pi/36)}{4}$  as highlighted by Thom); this ratio depended on the number of subdivisions, but what was important was that the second differences were close to being sines; when second differences are obtained from the sines, there are two values less than at the beginning; the first value could easily be supplied, it had to be 0; the last value could also be supplied, using the known ratio between sines and second differences;
- The second important observation was to notice that the last value of the first differences is very nearly half the value added at the end of the second differences.

Once these two observations were made, and once they were checked to be true whatever the subdivision, Bürgi must have been trying to go

backwards, that is, to apply to algorithm given above. He may have started with approximate values of the sines, and perhaps didn't notice that the algorithm converges even with practically random initial values.

We therefore believe that Bürgi was able to discover some properties of finite differences by a careful observation and manipulation of actual differences, and that he tried to reverse the procedure, and must have been pleased to see that it did in fact make it possible to compute sines with only simple operations, and at any desired accuracy. It seems unlikely to us that Bürgi had a proof, or that he sought one.<sup>2</sup> It is therefore certainly a very skillful and remarkable achievement.

## References

- [1] Jost Bürgi. *Arithmetische und Geometrische Progress Tabulen, sambt gründlichem Unterricht, wie solche nützlich in allerley Rechnungen zugebrauchen, und verstanden werden sol.* Prague, 1620. [These tables were recomputed in 2010 by D. Roegel [12]].
- [2] Menso Folkerts. Eine bisher unbekannte Schrift von Jost Bürgi zur Trigonometrie. In Rainer Gebhardt, editor, *Arithmetik, Geometrie und Algebra in der frühen Neuzeit*, pages 107–114. Annaberg-Buchholz: Adam-Ries-Bund, 2014. [not seen].
- [3] Menso Folkerts, Dieter Launert, and Andreas Thom. Jost Bürgi's method for calculating sines, 2015. [uploaded on arXiv on 12 October 2015, id 1510.03180v1; a preprint dated 19 September 2015 is also available online].
- [4] Dieter Launert, editor. *Astronomischer Grund. Fundamentum Astronomicum (1588) des Nicolaus Reimers Ursus*, volume 47 of *Acta Historica Astronomiae*. Frankfurt am Main: Verlag Harri Deutsch, 2012.
- [5] Dieter Launert. *Bürgis Kunstweg im Fundamentum Astronomiae. Entschlüsselung eines Rätsels*. München: Verlag der Bayerische Akademie der Wissenschaften, 2015. [to appear, not seen].
- [6] Bartholomaeus Pitiscus. *Thesaurus mathematicus sive canon sinuum ad radium 1.00000.00000.00000. et ad dena quæque scrupula secunda*

---

<sup>2</sup>Folkerts *et al.* [3] mention that Peter Ullrich has also proposed a conjecture on how Bürgi may have found his method, but we do not know if Ullrich's conjecture is identical or not with ours.

*quadrantis : una cum sinibus primi et postremi gradus, ad eundem radium, et ad singula scrupula secunda quadrantis : adiunctis ubique differentiis primis et secundis; atque, ubi res tulit, etiam tertijs.*  
 Frankfurt: Nicolaus Hoffmann, 1613. [The tables were reconstructed by D. Roegel in 2010. [9]].

- [7] Georg Joachim Rheticus. *Canon doctrinae triangulorum*. Leipzig: Wolfgang Gunter, 1551. [This table was recomputed in 2010 by D. Roegel [10].].
- [8] Georg Joachim Rheticus and Valentinus Otho. *Opus palatinum de triangulis*. Neustadt: Matthaeus Harnisch, 1596. [This table was recomputed in 2010 by D. Roegel [11].].
- [9] Denis Roegel. A reconstruction of the tables of Pitiscus' *Thesaurus Mathematicus* (1613). Technical report, LORIA, Nancy, 2010. [This is a recalculation of the tables of [6].].
- [10] Denis Roegel. A reconstruction of the tables of Rheticus's *Canon doctrinae triangulorum* (1551). Technical report, LORIA, Nancy, 2010. [This is a recalculation of the tables of [7].].
- [11] Denis Roegel. A reconstruction of the tables of Rheticus's *Opus Palatinum* (1596). Technical report, LORIA, Nancy, 2010. [This is a recalculation of the tables of [8].].
- [12] Denis Roegel. Bürgi's *Progress Tabulen* (1620): logarithmic tables without logarithms. Technical report, LORIA, Nancy, 2010. [This is a recalculation of the tables of [1].].
- [13] Denis Roegel. The great logarithmic and trigonometric tables of the French Cadastre: a preliminary investigation. Technical report, LORIA, Nancy, 2010.
- [14] Denis Roegel. A reconstruction of Viète's *Canon Mathematicus* (1579). Technical report, LORIA, Nancy, 2011. [This is a reconstruction of the main table of [18].].
- [15] Fritz Staudacher. *Jost Bürgi, Kepler und der Keiser: Uhrmacher, Instrumentenbauer, Astronom, Mathematiker, 1552–1632*. Zürich: Verlag Neue Zürcher Zeitung, 2013.
- [16] Fritz Staudacher. Jost Bürgis genialer Artificium-Kunstweg entdeckt, 2015. [6 pages, dated 22 October 2015].

- [17] Nicolaus Reimers Ursus. *Fundamentum Astronomicum; id est nova doctrina sinuum et triangulorum etc.* Strasbourg: Bernhard Jobin, 1588.
- [18] François Viète. *Canon mathematicus seu ad triangula cum appendicibus.* Paris: Jean Mettayer, 1579. [The main table was reconstructed in [14].].