How to design graphs with low forwarding index and limited number of edges

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Frédéric Giroire[1]**, Stéphane Pérennes[1], and Issam Tahiri[2]

[1] CNRS, Univ. Nice Sophia Antipolis, I3S, UMR 7271, Team Coati, 06900 Sophia Antipolis, France
[2] Université de Bordeaux, Institut de Mathématiques, UMR 5251, CNRS, Inria, 33405 Talence, France

Abstract. The *(edge) forwarding index* of a graph is the minimum, over all possible routings of all the demands, of the maximum load of an edge. This metric is of a great interest since it captures the notion of global congestion in a precise way: the lesser the forwarding-index, the lesser the congestion. In this paper, we study the following design question: Given a number e of edges and a number n of vertices, what is the least congested graph that we can construct? and what forwarding-index can we achieve? Our problem has some distant similarities with the well-known (Δ, D) problem, and we sometimes build upon results obtained on it. The goal of this paper is to study how to build graphs with low forwarding indices and to understand how the number of edges impacts the forwarding index. We answer here these questions for different families of graphs: *general graphs, graphs with bounded degree, sparse graphs with a small number of edges* by providing constructions, most of them asymptotically optimal. For instance, we provide an asymptotically optimal construction for $(n, n+k)$ cubic graphs - its forwarding index is $\sim \frac{n^2}{3k}\log_2(k)$. Our results allow to understand how the forwarding-index drops when edges are added to a graph and also to determine what is the *best (i.e least congested) structure with e edges.* Doing so, we partially answer the practical problem that initially motivated our work: If an operator wants to power only e links of its network, in order to reduce the energy consumption (or wiring cost) of its networks, what should be those links and what performance can be expected?

Keywords: graphs, forwarding index, routing, design problem, energy efficiency, extremal graphs

1 Introduction

Given a graph $G = (V, E)$ with $n = |V|$ vertices, a *routing* R is a collection of paths connecting all the ordered pairs of vertices of G. A routing R induces on every edge e a *load* that is the number of paths going through e. The *edge-forwarding index* (or simply the *forwarding index*) $\pi(G, R)$ of G with respect to R is then the maximum number of paths in R passing through any edge of G. In other words, it corresponds to the maximum load of an edge of G when R is used. So $\pi(G, R)$ measures how congested is the routing R, hence-fore it is important to design routings minimizing this index. The forwarding index $\pi(G)$ of a connected graph G is the minimum $\pi(G, R)$ over all splittable (fractional) routings R's of G (We will also sometimes consider non-splittable (integral) routing and denote the minimum load $\pi_{\mathcal{I}}(G)$ in this case). By definition the forwarding index of a graph measures its intrinsic congestion, so it is a parameter

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** Corresponding author. Email: frederic.giroire@cnrs.fr

as essential, and arguably more important than simpler parameters such as the diameter or the average distance.

Problem. In this paper, our goal is to provide for a given number of vertices n and for a given number of edges k graphs with the minimum forwarding indices, or at least graphs with low forwarding indices. For a given n, we will study how the number of edges of a graph impacts its forwarding index. Formally, we define the following design problem:

MIN CONGESTED (n, e)-GRAPH: *Given $n, e \in \mathbb{N}$, find a graph $(G = V, E)$ with $|V| = n$ vertices and $|E| = e$ edges such that $\pi(G = V, E)$ is minimum. We will denote this number $\pi^*(n, e)$ (when $e < n - 1$, note that $\pi^*(n, e) = \infty$).*

Here is an example. When restricted to the class of cubic graphs, the min congested $(8, 12)$-graph is the cube. Its forwarding index is equal to 8. The routing that achieves this load is the following : for each ordered pair of nodes (u, v), we connect u to v using all shortest paths from u to v. For instance there are 6 paths that connect the node $(0, 0, 0)$ to the node $(1, 1, 1)$. Each of those paths will hold a load of $1/6$ for this ordered pair. Since the cube is edge-transitif, this routing ensures that all edges will get the same load. Every node a is of distance 1 from 3 nodes, distance 2 from 3 nodes, and distance 3 from 1 node. Hence, the total load induced by order pairs that start with a is $1 \cdot 3 + 2 \cdot 3 + 3 \cdot 1 = 12$. Since there are 8 nodes in the cube, The total load on the graph is $12 \cdot 8 = 96$. Therefore the load on each edge is $96/12 = 8$. The optimality of this graph is proven in Section 5. For more examples, check Table 2.

Motivation. Our problem can be viewed as: for a given bound U on the forwarding index, find a spanner F of G with minimum number of edges such that $\pi(F) \leq U$ or reciprocally given a bound on the number of edges minimize $\pi(F)$.

First, to the best of our knowledge the problem of designing a (sub) graph with minimum forwarding index has not been studied when the main other constraint is the number of edges. Indeed, most of the results have been derived either for classical graphs and graphs families or have been considering other constraints, as example the bounded degree one. So even if a constraint such as the number of edges is both natural and of importance it has not been well studied so far. As example, one of our initial goal was to understand how the forwarding index drops from order n^2 for tree like graphs to order $n \log n$ for cubic graphs, and also to understand how adding a single edge can decrease significantly (or not) the forwarding index.

Second, the recent trend of "Energy Saving" has made our problem even more relevant in practice, especially for network operators willing to reduce the energy consumed by their networks. In fact, most of the network links consume a constant energy independently of the amount of traffic they are flowing. Therefore the only way to reduce the energy used by the network links is to turn some of them off, or more conveniently, put them on an idle mode. Outside the rush hours, several studies [1, 2, 4, 5, 7] show that a good choice of the links to turn off can lead to significant energy savings, while keeping the same communication quality. In the case where the throughputs from every node to every other node are of the same order, and where the capacities also lie in same small range, a good choice of those links amount to solve the problem of finding spanners of the network with low forwarding indices.

Related work. The forwarding-index was introduced by Chung & Al in 87 [6], due to its importance this parameter has been studied quite extensively : on one side results have been given for different graph classes (e.g. random graphs [25], transitive and Cayley graphs [10, 22] graphs with small numbers of vertices [3] and well-connected graphs [24]). On the other side deep relations with other expansion-related graph invariants have been established : Laplacian, Cheeger constant (see the survey [18]), Sparsest cut [12] and the "geometry of graphs" [13]. This notion has also been used to prove that some Markov chains mix fast using either canonical paths (routings) or "resistance" [21]. See the recent survey [26] for a global view on the known results. The problem is also known as the *maximum concurrent flow problem* and its dual was probably first introduced in [20] in which the authors also discussed the relation with the network throughput, in [23] a simple oblivous packet routing algorithm achieving network stability for any rate λ with $\lambda\pi < 1$ was provided. Some variants: load on arcs for digraphs ([14]) load on the vertices have also been studied.

The edge forwarding index is strongly related to distance properties of the graph. Indeed a usual naive lower bound on π (Average distance Bound) is:

$$\pi(G = V, E) \geq \frac{\sum_{(u,v)\in V^2} D(u,v)}{|E|} = \frac{\overline{D_G}|V|^2}{|E|} = 2|V|\frac{\overline{D_G}}{\overline{d_G}},$$

where $D(u,v), d(v), \overline{D_G}$ and $\overline{d_G}$ denote respectively the distance function, the degree function, the average distance in G and the average degree in G. This indicates that solving our design problem is strongly related to finding graphs with small average distance. The Degree-Diameter problem or (Δ, D)-Design Problem is about finding the graph with degree Δ and diameter D with the maximum number of vertices (or reciprocally it is about minimizing the diameter of a Δ-regular graph). It is quite a complex problem and it has been studied extensively (see [17] for a recent survey). Even after 30 years of steady efforts, generic constructions are still very far from being optimal. So, since good (n, e)-graphs should resemble (Δ, D) graphs, we may expect our problem to be complex. But we can also hope to be able to use results about the (Δ, D)-problem in our context.

Contributions and plan of the paper

- In Section 2, we consider our design problem for general graphs, that is when the only design constraint is the number of edges. We characterize the graphs with minimum forwarding index. When the number of edges is $k(n-k), k \in \mathbb{N}$, optimum graphs happen to have a simple structure since they are the complete bipartite graphs $K_{k,n-k}$. In between these values, the function $\pi^*(n, e)$ follows, rather surprisingly, a stepwise function (see Propositions 4 and 5).
- In Section 3, motivated by telecommunication networks, we study the case of bounded degree graphs. We provide almost optimal graphs for the different values of maximum degree Δ. We then focus on graphs with a small number of edges ($\Delta = 3$) as they correspond to the range of values for which the forwarding index greatly changes. We determine quite sharply how the minimum forwarding index behaves and evolves from $\Theta(n^2)$ to $\Theta(n \log n)$ when the number edges grows from $n - 1$ to $n + \frac{n}{2}$. We also develop a method that allow us simplify the design problem by considering the *graph skeleton*.
- We then examine the case $e = n + k$ with a fixed small $k \in \{1, 2, 3\}$ in Section 4. We determine the minimum forwarding index exactly for any n. This is possible because the main structure of the graph, that we called skeleton is finite, so we can explore all of them and use weight arguments in order to deal with a finite problem. Some of the results, as example Proposition 11, are strikingly counter intuitive.
- Last, in Section 5, we provide optimal cubic-graphs with *small number of vertices*, that is for $n \in [4, 22]$. Those graphs are not only interesting per se (and some structures again are surprising), but also because, as we shall see, their structure may be used as a *skeleton* to build good graphs with a few edges and arbitrarly size.

Due to the lack of space, all the proofs are omitted and can be found in a research report [9].

2 Minimally congested graphs

In this section, we study the design of minimally congested graphs for given numbers of vertices n and edges e. We first give a trivial lower bound of $\pi^*(n,e)$, the minimum forwarding index of a (n,e)-graph. We then provide families of minimally congested graphs reaching this bound for some couples of values (n,e), e.g. complete bipartite graphs $K_{i,n-i}$, complete k-partite graphs, or Kneser graphs, see Figure 1. These graphs are edge-transitive and of diameter 2. In particular, we show that $K_{i,n-i}$ ($i \in \mathbb{N}, i \leq \lfloor n/2 \rfloor$) are minimally congested $(n, i(n-i))$-graphs with forwarding index $\pi^*(n,e) = 2(\frac{n(n-1)}{e} - 1)$. Last, we study the behavior of $\pi^*(n,e)$ when e varies between two "perfect" cases, from $i(n-i)$ to $(i+1)(n-(i+1))$. Surprisingly, π^* follows a step-wise function in the sense of Propositions 4 and 5 and jumps suddenly from $\pi^*(n, i(n-i))$ to $\pi^*(n, (i+1)(n-(i+1))$.

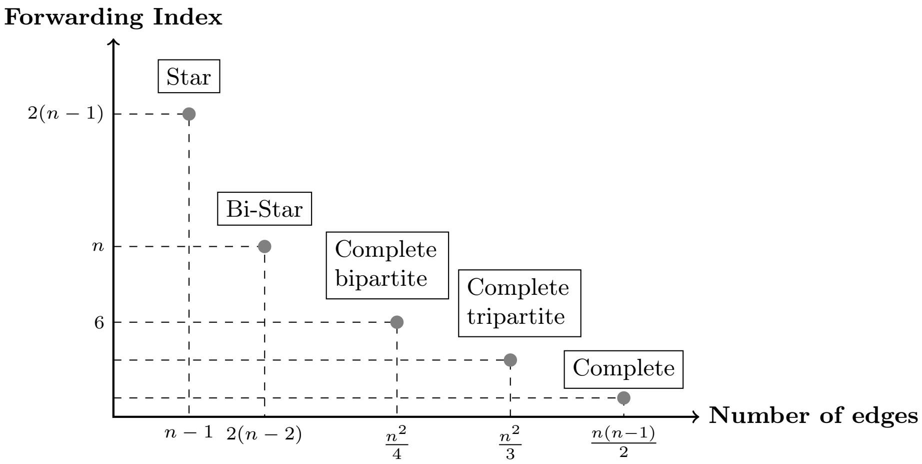


Fig. 1. Forwarding indices of minimaly congested graphs with n vertices as a function of their number of edges.

Proposition 1 (Lower bound on $\pi^*(n,e)$.). *The forwarding index of an (n,e)-graph is lower bounded by:*

$$\pi^*(n,e) \geq \frac{2n(n-1)}{e} - 2.$$

Proposition 2 (Optimal (n,e)-graph). *An (n,e)-graph that is edge-transitive and of diameter 2 is optimal. Its forwarding index is*

$$\frac{2n(n-1)}{e} - 2.$$

Corollary 1 (Families of optimal graphs). *The following families of graphs are optimal:*

– *Complete bipartite graphs, giving:*

$$\pi^*(n, i(n-i)) = \frac{2n(n-1)}{e} - 2, \qquad i \in \mathbb{N}, i \leq \lfloor n/2 \rfloor.$$

– *Turán graphs $T(n,r)$, for which r divides n (that is, complete multipartite regular graphs with r independent subsets of equal sizes), giving:*

$$\pi^*(n, \frac{n}{2}(n - \frac{n}{r})) = \frac{2n(n-1)}{e} - 2, \qquad r \in I\!N, r \leq n.$$

– *Kneser graphs $KN_{\nu,\kappa}$ for which $\kappa \geq \nu/3$ (Kneser graphs of diameter 2), giving:*

$$\pi^*\left(\binom{\nu}{\kappa}, \frac{1}{2}\binom{\nu}{\kappa}\binom{\nu-k}{\kappa}\right) = \frac{2n(n-1)}{e} - 2, \qquad \nu \in I\!N, \nu/3 \leq \kappa \leq \nu.$$

Proposition 3 (Integral Forwarding Index).

– Complete bipartite graphs *are (almost) optimal, in the sense that, for $i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, we have:*

$$\pi_{\mathcal{I}}{}^*(n, i(n-i)) \in \lceil \pi^*(n, i(n-i)) \rceil + \{0, 1, 2, 3, 4\}.$$

– Turán graphs $T(n,r)$, for which r divides n *are (almost) optimal, in the sense that, for $i \in \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, we have:*

$$\pi_{\mathcal{I}}{}^*(n, \frac{n}{2}(n - \frac{n}{r})) = \pi^*(n, \frac{n}{2}(n - \frac{n}{r})) + \{0, 1, 2, 3, 4\}.$$

Since $\pi^*(n,e)$ decreases with e the above results implies that $\pi^*(n,e)$ evolves like $\Theta(\frac{2n^2}{e})$, but we don't know yet the precise behavior of $\pi^*(n,e)$ between two perfect cases (i.e. $e = i(i-k)$). As we shall prove this behavior is not a smooth linear decrease since it indeed proceeds with jumps occurring at values close to those perfect ones. First, we start studying the intermediary cases when e starts at $n-1$ ($\pi^*(n,e) = 2(n-1)$, optimal graph is a star) and grows to $e = 2(n-2)$ ($\pi^*(n,e) = n-2$. optimal graph is $K_{2,n-2}$). The next proposition shows that when e get larger than $n-1$, first π^* does not decrease significantly and stays around $2(n-1)$ then it jumps abruptly down to $n-1$ when e get close to $2(n-2)$.

Proposition 4.

$$\begin{array}{ll} \forall e \in [n-1, 2(n-2) - o(n)] & \pi^*(n,e) = 2(n-1) + o(n) \\ e = 2(n-2) & \pi^*(n,e) = (n-1) + o(n) \end{array}$$

The result can be extended to larger values of e ($e = n+k$ with $k = o(n)$), see Proposition 5.

Proposition 5. *For any fixed $k \in I\!N$:*

$$\forall e \in [kn, (k+1)n - o(n)] \ \pi^*(n,e) = \tfrac{2n}{k} + o(n)$$

3 Bounded degree graphs with low edge forwarding index

In the preceding section, we provided somewhat optimal families of graphs. This solves the question of minimally congested graphs in the general case. We now study graphs with a constraint on the degree (Δ will denote the maximum degree). The motivation comes from telecommunication & real interconnection networks for which the node degree is often small, see for example [19, 8]. In this section, we consider first the general case for $\Delta \geq 3$ ($\Delta = 2$ is trivial) and we suceed in determining how the forwarding index drops from $\pi(n,e) = n^2/4$ to $\frac{2}{3} n \log_2 n$ when the average degree raises from 2 to 3 So, we focus on graphs with a *small number of edges*, namely graphs with average degree $\overline{\Delta} \in [2, 3[$, that is when $e \in [n, \frac{3}{2}n]$, and we study the transition of $\pi(n,e)$ from $\frac{n^2}{4}$ to $\Theta(n \log n)$ when the number of edges e raises from $n-1$ to $\frac{3}{2}n$.

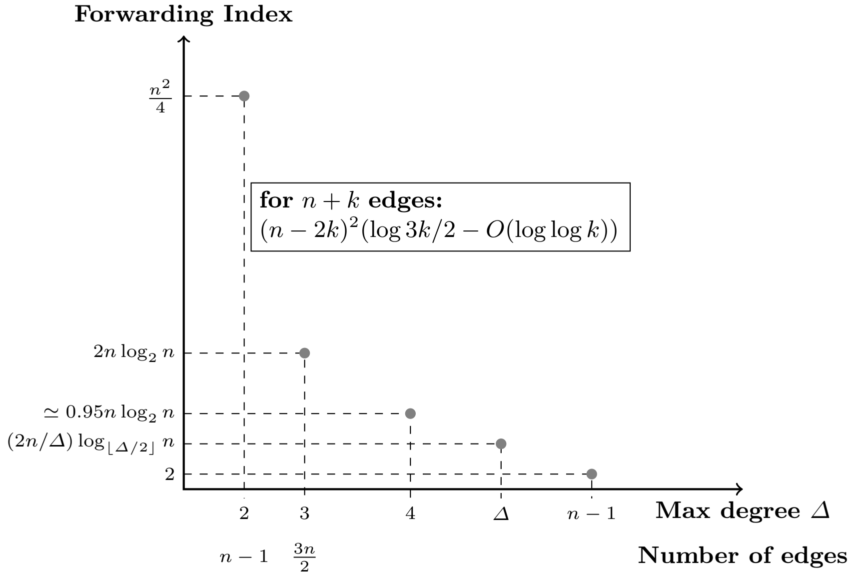


Fig. 2. Forwarding indices of minimaly congested graphs with n vertices as a function of their number of edges.

3.1 Graphs with bounded degree Δ: some remarks.

For $\Delta = 3$, when $e = \frac{3n}{2}$, graphs such like the shuffle exchange provide deterministic generic constructions for which $\pi(G) \leq n\log_2 n$ (this is a folk result for people studying network throughput, one may see [26]). Since using the Moore bound (that bound claims by direct counting that the average distance in a Δ bounded degree graph is of order $\log_{\Delta-1}(n)$, see as example [16]) one can prove that $\pi^*(n, \frac{3n}{2}) \geq \frac{2}{3}n\log_2 n(1+o(1))$ the lower and upper bounds matche up to factor of $\frac{2}{3}$. Moreover we shall prove that random cubic graphs are almost optimal since with high probability they are such that $\pi(G) = \frac{2}{3}n\log_2 n(1+o(1))$. Moreover for larger values of Δ de Bruijn graphs and their variants provide Δ-regular graphs whose forwarding index is of the right order (see Figure 2). So when the degree is bounded by Δ, the value of $\pi(n, \frac{\Delta}{2}n)$ is relatively well understood (see [6, 11]), and structures close to the optimal are obtained using de Bruijn graphs or slight variants of it. Indeed, on the one hand, the Moore bound implies that :

$$\pi^*(n, \frac{\Delta}{2}n) \geq \frac{2}{\Delta}n\log_{\Delta-1} n(1-o(1)).$$

On the other hand, for de Bruijn graphs, one has (see [6, 11])

$$\pi(n, \frac{\Delta}{2}n) \leq \frac{2}{\Delta}n\log_{\lfloor \frac{\Delta}{2}\rfloor} n.$$

The argument that provides the above bound for the de Bruijn graph with degree $\Delta = 2d$ and d^n vertices, is quite simple since it exists in this graph an integral routing that is uniform on the edges and that connects each couple of vertices with a path of length exactly n. This length is only a constant factor larger than the minimum average distance predicted by the Moore bound, hence the ratio between the above upper and lower bound is at most 3 and decreases with Δ.

So our purpose is to understand what is happening between two well understood situations:: $e = n-1, \pi^*(n,e) = \frac{n^2}{4}$ and $e = \frac{3}{2}n, \pi^*(n,e) = \Theta(n \log n)$ that is when e evolves in $[n, \frac{3}{2}n]$, in other words we shall study the evolution of $\pi^*(n,e)$ when the number of edges e raises from $n-1$ to $\frac{3}{2}n$.

3.2 A lower bound for the case $e \in [n, \frac{3}{2}n]$ for $\Delta \leq 3$

In this section, we provide a lower bound on the forwarding indices of graphs with $e \in [n, \frac{3}{2}n]$ and $\Delta \leq 3$.

Proposition 6. *If G is a $(n, n+k)$ graph with $\Delta = 3$ then $\pi(G) \geq \frac{(n-2k)^2}{3k}(\log(3k/2) - O(\log\log(k))$.*

3.3 Construction of minimaly-congested graph with degree $\leq \Delta$

Our construction simply reverts the previous operation and builds graphs with few extra edges from good skeletons.

Definition 1. *Given a graph, we construct $Sub(G, \mathbf{W})$ as follows: we subdivide each edge ab by adding one node x_{ab} and we then attach a binary tree with weight $\mathbf{W}$ on x_{ab}.*

Lemma 1. *Let G be a Δ-regular graph with x vertices, and let $H = sub(G, \mathbf{W})$ then $\pi(H) \leq Max\left\{\pi(G)(\frac{\Delta}{2}\mathbf{W}+1)^2 + \mathbf{W}(\frac{\Delta \mathbf{W}}{2}+1)x, \mathbf{W}((\frac{\Delta \mathbf{W}}{2}+1)x - \mathbf{W})\right\}$*

To our surprise, we could not find the following result in the literature, moreover in the recent survey [11] the best bounds for cubic graphs were provided by shuffle exchange graphs, and more generally, for bounded degree graphs the best bounds known are derived using de Bruijn graphs. Those bounds are rather good since they differ from the lower bound only by a relatively small (always lesser than 2) constant factor. But indeed random regular graph are asymptotically optimal.

Proposition 7. *There exist cubic regular graphs such that $\pi(G) = \frac{2}{3}n\log_2(n)(1+o(1))$, and Δ-regular graphs with $\pi(G) = \frac{2}{\Delta}n\log_{\Delta-1}(n)(1+o(1))$.*

Remark 1. Note that the fair shortest path routing (in which each shortest path carries the same flow) is probably better and for small values of n it may even be significantly better, but we don't have currently a good method to evaluate its load and proving that so doing we get a better load. Probably the forwarding index of random cubic graph is $\frac{2}{3}n\log_2 n + \Theta(n)$, but we proved only a weaker result. Moreover the value of n for which our $(1+o(1))$ becomes smaller than the $\frac{3}{2}$ are relatively high (order of 1000).

Proposition 8. *There exist $(n, e = n+k)$ cubic graphs such that $\pi(G) \leq \frac{n^2}{3k}\log_2(k)(1+o(1))$.*

4 Edge forwarding index of cubic ($\Delta = 3$) graphs with few extra edges: $e = n + k$

When k is large, we provided in Section 3 asymptotically matching upper and lower bounds on the minimum congestion. This implies that $\pi^*(n, n+k)$ behaves like $\Theta(\frac{n^2}{k}\log\frac{n}{k})$ when both k and n are large. So, in order to get a complete picture of the situation, we still need to understand the case of $(n, n+k)$ graphs when k is fixed. In this section, we answer this question, that is we solve the MIN-CONGESTION DESIGN PROBLEM, for graphs with arbitrary n, but small values of k.

4.1 Method: the skeleton approach

From the results of Section 3, we know that $(n, n+k)$ graphs are constructed from a cubic skeleton on which are attached trees with size u. So, when k is small, we may enumerate all the possible skeletons (like we enumerated all the cubic graphs) and determine for each the best way to attach trees. Attaching trees means determining for each edge $e \in E$ the size $\alpha(e)$ of the tree that we attach in the edge. Hence, we want to find the best weight repartition $\alpha : E \to N$ that satisfies $\sum_{e \in E} \alpha(e) = n$ and $\forall e \in E, \alpha(e) \leq w_{max}$, where by best we mean with the smallest forwarding index. So, finding the best way to subdivide edges means solving a problem of the following flavor:

Definition 2 (Best Mass Repartition). *Given a graph G and a maximum weight w_0 find a weight function $w : V \to I\!\!R^+$ with $\forall v \in V, w(V) = 1, w(v) \leq w_0$ such that $\pi(G, w)$ is minimum.*

4.2 Optimal $(n, n-1+k)$ cubic graph for $k = 0, 1, 2, 3$

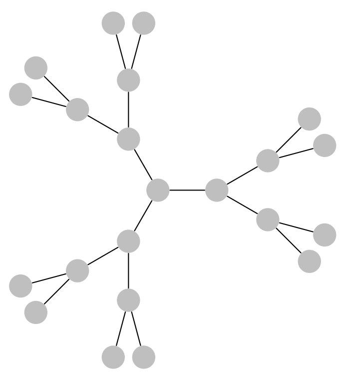
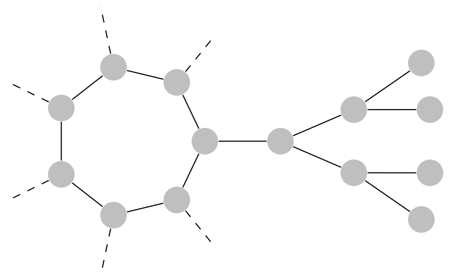
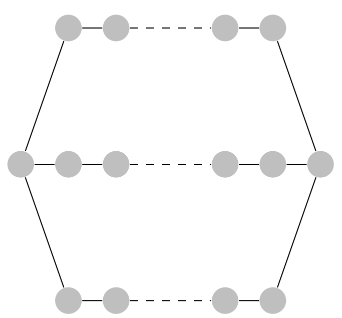
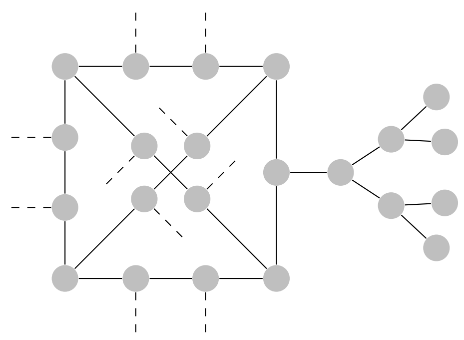
Tree + $k = 0$	Tree + $k = 1$
	
Tree + $k = 2$	Tree + $k = 3$
	

Table 1. Constructions of optimal graphs with n vertices and $n-1+k$ edges for different numbers of extra edges k.

Results are listed below and corresponding constructions are given in Table 1.

When $k = 0$ and $e = n - 1$, the network is a tree with max degree $\Delta = 3$. The case of degree Δ trees is trivial since for such trees, considering the most balanced cut, we get: $\pi(T) \geq 2\Delta(\Delta - 1)\left(\frac{n}{\Delta}\right)^2$ and this value is attained using a balanced Δ-ary tree or a subdivided Δ-star with branches with equal size $\frac{n}{\Delta}$. So, for $\Delta = 3$. we have:

$$\pi^*(n, n-1) = 2\Delta(\Delta - 1)\left(\frac{n}{\Delta}\right)^2 = 2\frac{(\Delta - 1)}{\Delta}n^2 = \frac{4}{3}n^2.$$

In this case, the first intuition is that the cycle C_n should be the optimal structure. Recall that $\pi(C_n) = \frac{n^2}{4}$ when n is even, and $\pi(C_n) = \frac{n-1}{2}\frac{n+1}{2}$ when n is odd (indeed $\pi(C_n) = \lceil \frac{n-1}{2} \rceil \lfloor \frac{n+1}{2} \rfloor$). The cycle is the only 2 connected structure but it is not the min-congested one since some graphs with bridges do have lesser congestion.

Proposition 9. $\pi^*(n, n) = \frac{12}{49}n^2$

We provide a graph G_7 with $\pi(G) = Opt = \frac{12}{49}n^2$: we simply take the cycle C_7 and on each vertex we attach a tree (any tree can will do it) with $\frac{n}{7}$ nodes.

Proposition 10. $\pi^*(n, n+1) = \frac{2}{9}n^2$.

A possible construction is then to use $\forall i \in \{1, 2, 3\}$ a path P_i of length $n/3$ for e_i, then one can cover all the request using 3 cycles of size $\frac{2n}{3}$ $(P_i \cup P_j, i \neq j)$.

The next result result is rather surprising since intuitively a uniform (or at least symmetric) subdivision of the K_4 should provide an optimal solution. But a phenomena similar to the one we already met in the case $k = 1$ (the C_7) happens again in a slightly more complex way.

Proposition 11. $\pi^*(n, n+2) = \frac{20}{11^2}n^2$

A graph reaching this bound is obtained by subdividing 5 edges of K_4 twice and one edge once, thus we add 11 new nodes. Then, we attach a tree with weight $\frac{n}{11}$ on each new node.

5 Graphs with a small number of vertices ($\Delta = 3$)

We have seen in Sections 3 and 4 the importance of having good skeletons to build graphs with low forwarding indices. In Table 2 on page 10, we present graphs with a small number of vertices which have the minimum possible forwarding indices. These graphs can serve as skeletons to build families of graphs with an arbitrary number of vertices. In some cases, optimality is easy to prove using:

- the Moore bound. In a cubic graph, and for a given vertex, the number of vertices that are at distance $0, 1, 2, 3, \ldots$, are respectively, at most $1, 3, 6, 12, \ldots$. When those bounds are reached for all the vertices of a cubic graph, the latter minimizes $\mathcal{L} = 2|V|\frac{\overline{D(G)}}{d(G)}$ among all the graphs with the same size and with degree 3. When the graph is optimal for the Moore bound and is edge-transitive, its forwarding index is minimum. This is the case for $n = 6, 14$;

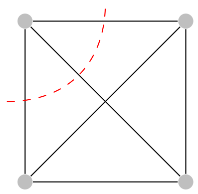
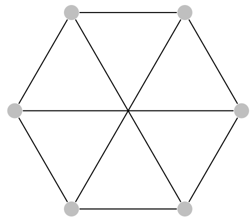
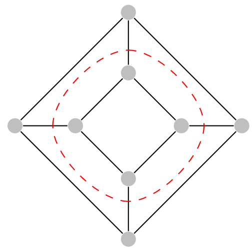
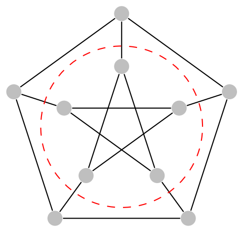
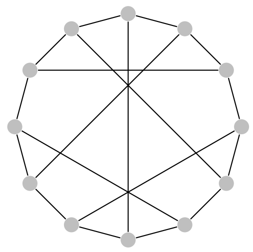
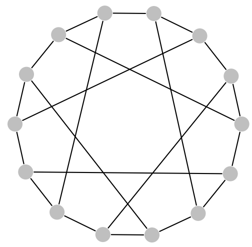
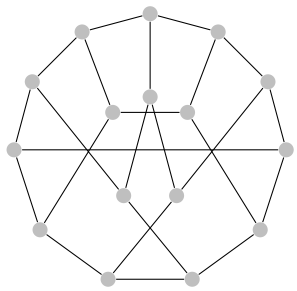
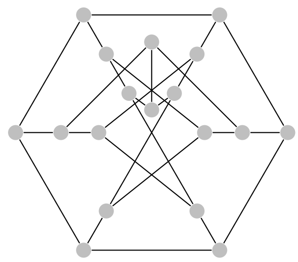
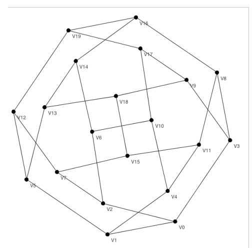
$n=4, \pi=\mathcal{L}=2$ $\mathcal{L}$ given by a cut	$n=6, \pi=\mathcal{L}=4.66...$ $\mathcal{L}$ given by the Moore bound	$n=8, \pi=\mathcal{L}=8$ $\mathcal{L}$ given by a cut
		
$n=10, \pi=\mathcal{L}=10$ $\mathcal{L}$ given by a cut	$n=12, \pi=\mathcal{L}=14.26...$ $\mathcal{L}$ found by brute force	$n=14, \pi=\mathcal{L}=18$ $\mathcal{L}$ given by the Moore bound
		
$n=16, \pi=\mathcal{L}=22$ $\mathcal{L}$ found by brute force	$n=18, \pi=\mathcal{L}=26.66...$ $\mathcal{L}$ found by brute force	$n=20, \pi=30.84..., \mathcal{L}=30$
		

Table 2. Small cubic graphs with minimum edge forwarding index

- cut arguments, for $n = 4, 8, 10$.

In other cases, ($n = 12, 16, 18$), the generic arguments fail to provide matching upper and lower bounds. We had to check all the possible cubic graphs ([15]).

Consequences for unbounded n but a few edges All those graphs can be used as skeletons, as example if one wishes to get a good $(n, 6)$ graph with $e = n+6$ edges one can simply pick the Petersen graph as skeleton and apply lemma 1. We use the uniform weigth function $\mathbf{W} = \frac{n}{15}$ and using the generic routing of the lemma we get : $\pi(n,5) \leq \pi(G) \leq 10\left(\frac{n}{10}\right)^2 + 2\frac{n}{30} \times \frac{14n}{15} = \frac{n^2}{10} + \frac{14n^2}{225}$.

This may be potentially improved by computing the exact forwarding index of the so defined weighted graph (that has only 15 vertices).

Solving the best mass repartition problem would allow us to go quite further, but currently we have no clue about what is the best repartition even for a small structure. It is certainly possible to repeat what we did for $0, 1, 2, 3$ extra edges, but the difficulty shall increase considerably each time we add one edge, finding a method that would scale more than considering cases by "hand" is certainly interesting.

6 Conclusion

In this paper, we provided a basic understanding of the interplay between the forwarding-index of a graph and its number of extra-edges. Our bounds are mostly asymptotically tight and explain as example how the transition happens between highly congested graphs (Trees, Paths, ...) to cubic regular graphs which have much lower congestion.

Some results, like the step-like behavior in Proposition 4 or irregular optimal structures, are also *fun*, since they are unexpected. Last, we believe that our work opens many questions:

- **Small cases**: In the case of a few extra-edges, we stopped at 3 extra edges (and even in those cases the proofs are not immediate). So, it may be interesting to go further and to understand if optimal graphs with k extra-edges are built using an optimal cubic graph with $\frac{k}{2}$ vertices (we determined such graphs till $k = 22$). As example: is the familly of optimal graphs with 5 extra edges built using the Petersen and subdivising it properly? And, if so, how do we find the best subdivision (we saw the the uniform subdivision is not always optimal).
- **Construction from skeletons**: Given a skeleton, we do not know how to affect weights in order to minimize the forwarding-index of the resulting graph. That problem can be expressed as a quadratic non convex problem and we conjecture that it is NP-Complete.

References

1. Araujo, J., Giroire, F., Liu, Y., Modrzejewski, R., Moulierac, J.: Energy efficient content distribution. In: Communications (ICC), 2013 IEEE International Conference on. pp. 4233–4238. IEEE (2013)
2. Baliga, J., Tucker, R., Ayre, R., Hinton, K.W., Sorin, W.: Energy consumption in IP networks. In: Optical Communication, 2008. ECOC 2008. 34th European Conference on. p. 1 (2008)
3. Bouabdallah, A., Sotteau, D.: On the edge forwarding index problem for small graphs. Networks 23(4), 249–255 (1993)
4. Cardona Restrepo, J.C., Gruber, C.G., Mas Machuca, C.: Energy Profile Aware Routing, pp. 1–5 (2009)
5. Chiaraviglio, L., Mellia, M., Neri, F.: Energy-aware umts core network design. In: The 11th International Symposium on Wireless Personal Multimedia Communications (2008)
6. Chung, F.R.K., Jr., E.G.C., Reiman, M.I., Simon, B.: The forwarding index of communication networks. IEEE Transactions on Information Theory 33(2), 224–232 (1987)
7. Giroire, F., Mazauric, D., Moulierac, J., Onfroy, B.: Minimizing routing energy consumption: from theoretical to practical results. In: IEEE/ACM International Conference on Green Computing and Communications (GreenCom'10). p. 8. Hangzhou, China (2010)
8. Giroire, F., Perennes, S., Tahiri, I.: Grid spanners with low forwarding index for energy efficient networks. In: International Network Optimization Conference (INOC). Warsaw, Poland (May 2015)
9. Giroire, F., Pérennes, S., Tahiri, I.: Graphs with optimal forwarding indices: What is the best throughput you can get with a given number of edges? Research Report RR-8752, INRIA Sophia Antipolis ; INRIA (Jun 2015), `https://hal.inria.fr/hal-01172725`

10. Heydemann, M.C., Meyer, J.C., Sotteau, D.: On forwarding indices of networks. Discrete Applied Mathematics 23(2), 103 – 123 (1989)
11. Jun-Ming Xu, M.X.: The forwarding indices of graphs – a survey (2012), xarchiv
12. Leighton, T., Rao, S.: Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. J. ACM 46(6), 787–832 (Nov 1999)
13. Linial, N., London, E., Rabinovich, Y.: The geometry of graphs and some of its algorithmic applications. Combinatorica 15, 577–591 (1994)
14. Manoussakis, Y., Tuza, Z.: The forwarding index of directed networks. Discrete Applied Mathematics 68(3), 279 – 291 (1996), `http://www.sciencedirect.com/science/article/pii/0166218X9500072Y`
15. Meringer, M.: Small cubic graphs, `http://www.flinders.edu.au/science_engineering/csem/research/programs/flinders-hamiltonian-cycle-project/graph-database.cfm`, flinders Univ projet
16. MILLER, Mirka amd ŠIRÁN, J.: Moore graphs and beyond: A survey of the degree/diameter problem. Electronic Journal of Combinatorics 61, 1–63 (2005)
17. Miller, M., Sirán, J.: Moore graphs and beyond: A survey of the degree/diameter problem. The Electronic Journal of Combinatorics (2013), dynamic Survey DS14
18. Mohar, B.: Some applications of laplace eigenvalues of graphs. In: Graph Symmetry, NATO ASI Series, vol. 497, pp. 225–275. Springer Netherlands (1997)
19. Orlowski, S., Wessäly, R., Pióro, M., Tomaszewski, A.: Sndlib 1.0—survivable network design library. Networks 55(3), 276–286 (2010)
20. Shahrokhi, F., Matula, D.W.: The maximum concurrent flow problem. J. ACM 37(2), 318–334 (Apr 1990), `http://doi.acm.org/10.1145/77600.77620`
21. Sinclair, A.: Improved bounds for mixing rates of markov chains and multicommodity flow. Combinatorics, Probability and Computing 1, 351–370 (1992)
22. Solé, P.: Expanding and forwarding. Discrete Appl. Math. 58(1), 67–78 (1995)
23. Tassiulas, L., Ephremides, A.: Stability properties of constrained queueing systems and scheduling policies for maximum throughput in multihop radio networks. Automatic Control, IEEE Transactions on 37(12), 1936–1948 (Dec 1992)
24. de la Vega, W.F., Manoussakis, Y.: The forwarding index of communication networks with given connectivity. Discrete Appl. Math. 37-38, 147–155 (1992)
25. Fernandez de la Vega, W., Gordones, L.M.: The forwarding indices of random graphs. Random Structures and Algorithms 3(1), 107–116 (1992)
26. Xu, J.M., Xu, M.: The forwarding indices of graphs – a survey. CoRR abs/1204.2604 (2012)