

# Reasoning about Computational Systems using Abella

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# Reasoning about Computational Systems using Abella

<http://abella-prover.org>

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# Overview

## Overview of Abella

Abella is an interactive tactics-based theorem prover for a logic with the following features

- its underlying substrate is an intuitionistic first-order logic over simply typed lambda terms
- it incorporates a mechanism for interpreting atoms through fixed-point definitions
- it allows for inductive and co-inductive forms of reasoning
- it includes logical devices for analyzing binding structure

Abella also builds in a special ability for reasoning about specifications expressed in a separate executable logic

## Abella and Computational Systems

Abella offers intriguing capabilities for reasoning about syntax-directed and rule-based specifications

- such specifications can be formalized succinctly through fixed-point definitions
- formalizations adopt a natural and flexible relational style as opposed to a computational style
- the formalizations allow specifications to be interpreted either inductively or co-inductively in the reasoning process
- binding structure in object systems can be treated via a well-restricted and effective form of higher-order syntax
- a two-level logic approach allows intuitions about the object systems to be reflected into the reasoning process

# Objectives for the Tutorial

We aim to accomplish at least the following goals through the tutorial

- to expose the novel features of the logic underlying Abella
- to provide a feel for Abella so that you will be able to (and interested in) experimenting with it on your own
- to show the applicability of Abella in mechanizing the meta-theory of formal systems
- to indicate the benefits of a special brand of higher-order abstract syntax in treating object-level binding structure

We will assume a basic familiarity with sequent-style logical systems and with intuitionistic logic

# The Structure of the Tutorial

The tutorial will consist of the following conceptual parts

- an exposure to the syntax of formulas in Abella and the basic theorem proving environment
- a presentation of the special logical features of Abella with examples of their use
- an exposition of the two-level logic approach a la Abella to formalization and reasoning
- extensions to reasoning about specifications in a dependently typed lambda calculus

# Outline

- 1 Setup
- 2 The Reasoning Logic  $\mathcal{G}$
- 3 The Two-Level Logic Approach
- 4 Co-Induction
- 5 Extensions



# Setup

# How to Run Abella in your Web-Browser

Go to:

<http://abella-prover.org/try>

- Everything runs inside your browser
- Interface reminiscent of ProofGeneral

# Running Abella Offline

- You will need a working OCaml toolchain + OPAM
- `opam install abella`
- To get ProofGeneral support, read the instructions on:  
<http://abella-prover.org/tutorial/>

## Code for This Tutorial

`http://abella-prover.org/tutorial/try`

Special on-line version just for this tutorial

## Some Concrete Syntax

Types	$A \rightarrow ((B \rightarrow C) \rightarrow D)$	<b>A -&gt; (B -&gt; C) -&gt; D</b>
Application	$(M N) (J K)$	<b>M N (J K)</b>
Abstraction	$\lambda x. M$	<b>x \ M</b>
	$\lambda x:A. M$	<b>(x:A) \ M</b>
Formulas	$\top, \perp$	<b>true, false</b>
	$F \wedge G, F \vee G$	<b>F /\ G, F \/ G</b>
	$F \supset G$	<b>F -&gt; G</b>
	$\forall x, y. F$	<b>forall x y, F</b>
	$\exists x:A, y. F$	<b>exists (x:A) y, F</b>
	$M = N$	<b>M = N</b>
	$\neg F$	<b>F -&gt; false</b>

# Declaring Basic Types and Term Constructors

- New basic types are introduced with **Kind** declarations.

```
Kind nat    type.  
Kind bt     type.  
Kind tm,ty  type.
```

Reserved: **o**, **olist**, and **prop**.

- New term constructors are introduced with **Type** declarations.

```
Type z      nat.  
Type s      nat -> nat.  
  
Type leaf   nat -> bt.  
Type node   bt -> bt -> bt.  
  
Type app    tm -> tm -> tm.  
Type abs    (tm -> tm) -> tm.
```

# Theorems and Proofs

1 - Syntax

# The Reasoning Logic $\mathcal{G}$



# The Reasoning Logic $\mathcal{G}$

## Outline:

- 1 Ordinary Intuitionistic Logic
- 2 Equality
- 3 Fixed Point Definitions
- 4 Induction
  - Inductive data: lists
  - Kinds of induction: simple, mutual, nested
- 5 Higher-Order Abstract Syntax
  - Example: subject reduction for STLC

# Ordinary Intuitionistic Logic

## 2.1 - Basic Logic

# Equality

For closed terms  $M$  and  $N$ , the formula  $M = N$  is true if and only if  $M$  and  $N$  are  $\alpha\beta\eta$ -convertible.

## Consequences

- Two closed **first-order** terms are equal iff they are identical.

```
Kind i      type.  
Type a,b   i.
```

```
Theorem eq1 : a = a /\ b = b.
```

```
Theorem eq2 : a = b -> false.
```

- Different constants are distinct.

# Equality

For closed terms  $M$  and  $N$ , the formula  $M = N$  is true if and only if  $M$  and  $N$  are  $\lambda$ -convertible.

## Consequences

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Theorem eq1 : a = a /\ b = b.
```

```
Theorem eq2 : a = b -> false.
```

- Different constants are distinct.

# The Nature of Variables

Terminology: *variable*, *eigenvariable*, and *universal variable* used interchangeably in Abella.

*Variables are interpreted extensionally in the term model of the underlying logic.*

In other words, a variable stands for **all its possible instances**.

```
Kind nat  type.  
Type z    nat.  
Type s    nat -> nat.
```

The formula  $\forall x:\text{nat}. F$  stands for:

$$[z/x]F \quad \wedge \quad [s \ z/x]F \quad \wedge \quad [s \ (s \ z)/x]F \quad \wedge \quad \dots$$

## Equality and Extensional Variables

**forall** (x:nat) y, x = y -> F x y

We have:

x	y	x = y	x = y -> F x y
z	z	true	F z z
z	anything else	false	true
s z	s z	true	F (s z) (s z)
s z	anything else	false	true
		⋮	

In other words, the formula is equivalent to:

**forall** (x:nat), F x x

## Equality-Left

More generally, given an assumption  $\mathbf{M} = \mathbf{N}$ :

- 1 Find all **unifiers** for  $\mathbf{M}$  and  $\mathbf{N}$ .
  - A unifier of  $\mathbf{M}$  and  $\mathbf{N}$  is a substitution of terms for the free variables of  $\mathbf{M}$  and  $\mathbf{N}$  that makes them  $\lambda$ -convertible.
- 2 For each unifier, apply the unifier to the rest of the subgoal to generate a new subgoal.

Notes:

- There may be **infinitely many** unifiers
- Unification in the general case is **undecidable**
- In practice we work with **complete sets of unifiers** (csu) that cover all possibilities; csus are often finite, even **singletons**.

# Equality Assumptions on Open Terms

Example:

```
Kind i  type
```

```
Type f  i -> i -> i.
```

```
Type g  i -> i.
```

```
Theorem eq3 : forall x y z,  
  f x (g y) = f (g y) z  ->  x = z.
```

- A csu of  $f\ x\ (g\ y)$  and  $f\ (g\ y)\ z$  is the singleton set  $\{(g\ y)/x, (g\ y)/z\}$ .
- This substitution turns  $x = z$  into  $g\ y = g\ y$ , which is **true**.



# Equality Example: Peano's Axioms

2.2 - Peano

# Functions vs. Relations

Say you want to define addition on natural numbers.

- **Functional** approach:

- Declare a new symbol:

```
Type sum    nat -> nat -> nat.
```

- Define a closed set of computational rules:

```
Rule sum z N = N.
```

```
Rule sum (s M) N = s K where sum M N = K.
```

- **Relational** approach:

- Declare a new **predicate**:

```
Type plus  nat -> nat -> nat -> prop.
```

- Declare a closed set of properties of the predicate:

```
forall M, plus z M M.
```

```
forall M N K, plus M N K -> plus (s M) N (s K).
```

## Functions vs. Relations

<b>Functions</b>	<b>Relations</b>
Modifies term language	No change to terms
Modifies equality	No change to equality
Requires confluence	Can be non-deterministic
Fixed inputs and output	Modes can vary
Functional programming	Logic programming

# Relational Definitions

```

                                     type of the relation
Define plus : nat -> nat -> nat -> prop by
clauses [
  plus z N N ;
  plus (s M) N (s K) := plus M N K.
]
      head           body

```

- All defined relations must have target type **prop**.
- Clauses are universally closed over the capitalized identifiers.
- The body implies the head in each clause.
- An omitted body stands for **true**.
- The set of clauses is **closed**.

## Multiple Clauses vs. Single Clause

```
Define plus1 : nat -> nat -> nat -> prop by
  plus1 z N N ;
  plus1 (s M) N (s K) := plus1 M N K.
```

is equivalent to

```
Define plus2 : nat -> nat -> nat -> prop by
  plus2 M N K :=
    (M = z /\ N = K)
  \/ (exists M' K', M = s M' /\ K = s K' /\
      plus2 M' N K').
```

## Proving Defined Atoms

If  $p$  is a defined relation, then to prove  $p\ M1\ \dots\ Mn$ :

- 1 Find a clause whose head **matches** with  $p\ M1\ \dots\ Mn$ ;
- 2 Apply the matching substitution to its body;
- 3 and prove that instance of the body.

Backtracks over clauses and ways to match.

## Proving Defined Atoms: Example

```
Define plus : nat -> nat -> nat -> prop by
  plus z N N ;
  plus (s M) N (s K) := plus M N K.
```

Example: `plus (s z) (s (s z)) (s (s (s z)))`:

- 1 Pick second clause with unifier  $[z/M, s(s z)/N, s(s z)/K]$ .
- 2 Yields goal: `plus z (s (s z)) (s (s z))`.
- 3 Now pick first clause with unifier  $[s(s z)/N]$ .
- 4 Yields goal `true`, and we're done!

## Reasoning About Defined Atoms

To reason about hypothesis  $p \ M1 \ \dots \ Mn$ :

- 1 Find **every** way to unify  $p \ M1 \ \dots \ Mn$  with some head;
- 2 Separately reason about each corresponding instance of the body as a new hypothesis.

Generates one premise (subgoal) per unification solution.

Observe the analogy with equality assumptions!



## Reasoning About Defined Atoms

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Generates one premise (subgoal) per unification solution.

Observe the analogy with equality assumptions!

## Reasoning About Defined Atoms: Example

```
Define plus : nat -> nat -> nat -> prop by  
  plus z N N ;  
  plus (s M) N (s K) := plus M N K.
```

Given hypothesis: `plus M N (s K)`:

- 1 Generate one subgoal for the first clause and unifier  $[z/M, s\ K/N]$ ;
- 2 Another subgoal for the second clause and unifier  $[s\ M' /M]$

```
Theorem plus_s : forall M N K, plus M N (s K) ->  
  (exists J, M = s J) \/\ (exists J, N = s J).
```

# The **case** and **unfold** Tactics

2.3 - case and unfold

# Consistency of Relational Definitions

- Relational definitions are given a **fixed point** interpretation.
- That is, every defined atom is considered to be **equivalent** to the disjunction of its unfolded forms.
- Such an equivalence can introduce **inconsistencies**.

```
Define p : prop by  
  p := p -> false.
```

- Abella's **stratification condition** guarantees consistency.

# Stratification

## 2.4 - Stratification

# The Expressivity of `case` and `unfold`

Consider

```
Define is_nat1 : nat -> prop by
  is_nat1 z ;
  is_nat1 (s N) := is_nat1 N.
```

```
Define is_nat2 : nat -> prop by
  is_nat2 z ;
  is_nat2 (s N) := is_nat2 N.
```

- With `case` and `unfold`, we cannot prove:

```
forall x, is_nat1 x -> is_nat2 x.
```

- Abella actually interprets fixed points as [least fixed points](#).
- This in turn allows us to perform [induction](#) on such definitions.

## The `induction` tactic

Given a goal

```
forall X1 ... Xn, F1 -> ... -> Fk -> ... -> G
```

where `Fk` is a defined atom, the invocation

```
induction on k.
```

- 1 Adds an **inductive hypothesis** (IH):

```
forall X1 ... Xn, F1 -> ... -> Fk * -> ... -> G
```

- 2 Then **changes** the goal to:

```
forall X1 ... Xn, F1 -> ... -> Fk @ -> ... -> G
```

# Inductive Annotations

## Meaning of $F^*$

$F$  has resulted from *at least one* application of **case** to an assumption of the form  $F' @$ .

- These annotations are only maintained on defined atoms.
- Applying **case** to  $F@$  changes the annotation to  $*$  for the resulting bodies in every subgoal.
- The  $*$  annotation **percolates** to:
  - Both operands of  $/\wedge$  and  $\backslash/$ ;
  - Only the right operand of  $\rightarrow$ ; and
  - The bodies of **forall** and **exists**.



# Natural Number Induction

**2.5 - Natural Numbers**

# Lists of Natural Numbers

**2.6 - Lists**

# Nested and Mutual Induction

## 2.7 - Nested and Mutual Induction

# The Reasoning Logic $\mathcal{G}$

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# Principles of Abstract Syntax

[Miller 2015]

- 1 The *names of bound variables* should be treated as the same kind of *fiction* as we treat white space: they are artifacts of how we write expressions and have no semantic content.
- 2 There is “one binder to ring them all.”
- 3 There is no such thing as a free variable.
  - cf. Alan Perlis’ epigram #47
- 4 Bindings have *mobility* and the equality theory of expressions must support such mobility [...].

# Higher-Order Abstract Syntax

Also known as:  $\lambda$ -Tree Syntax

- Binding constructs in syntax are represented with term constructors of higher-order types.
- The normal forms of the representation are in bijection with the syntactic constructs.
- Syntactic substitution is **for free** – part of the  $\lambda$ -converibility inherent in equality.

# HOAS: Representing the Simply Typed Lambda Calculus

Warmup: simple types.

```
Kind ty      type.  
Type bas    ty.  
Type arrow  ty -> ty -> ty.
```

$$\llbracket b \rrbracket = \mathbf{bas}$$

$$\llbracket A \rightarrow B \rrbracket = \mathbf{arrow} \llbracket A \rrbracket \llbracket B \rrbracket$$



# HOAS: Representing the Simply Typed Lambda Calculus

(Closed)  $\lambda$ -terms

```
Kind tm      type.  
Type app     tm -> tm -> tm.  
Type abs     (tm -> tm) -> tm.
```

$$\begin{aligned} \llbracket M N \rrbracket &= \mathbf{app} \llbracket M \rrbracket \llbracket N \rrbracket \\ \llbracket \lambda x. M \rrbracket &= \mathbf{abs} \ (x \backslash \llbracket [x/x] M \rrbracket) \\ \llbracket x \rrbracket &= x \end{aligned}$$

Examples:

$$\begin{aligned} \llbracket \lambda x. \lambda y. x \rrbracket &= \mathbf{abs} \ x \backslash \mathbf{abs} \ y \backslash x \\ \llbracket \lambda x. \lambda y. \lambda z. xz (y z) \rrbracket &= \mathbf{abs} \ x \backslash \mathbf{abs} \ y \backslash \mathbf{abs} \ z \backslash \mathbf{app} \ (\mathbf{app} \ x \ z) \ (\mathbf{app} \ y \ z) \\ \llbracket (\lambda x. x x) (\lambda x. x x) \rrbracket &= \mathbf{app} \ (\mathbf{abs} \ x \backslash \mathbf{app} \ x \ x) \ (\mathbf{abs} \ x \backslash \mathbf{app} \ x \ x) \end{aligned}$$

## HOAS: Representing the Typing Relation

$$\frac{}{\Gamma, x:A \vdash x : A} \quad \frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B}$$
$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

```
Kind ctx  type.
```

```
Type emp  ctx.
```

```
Type add  ctx -> tm -> ty -> ctx.
```

# HOAS: Representing the Typing Relation

$$\frac{}{\Gamma, x:A \vdash x : A} \quad \frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B}$$
$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash M N : B}$$

```
Kind ctx  type.
```

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Type emp  ctx.
```

```
Type add  ctx -> tm -> ty -> ctx.
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## HOAS: Representing the Typing Relation

$$\frac{}{\Gamma, x:A \vdash x : A} \quad \frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B}$$

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

**Kind** ctx    type.

**Type** emp    ctx.

**Type** add    ctx -> tm -> ty -> ctx.

# HOAS: Representing Typing Contexts

```
Define mem : ctx -> tm -> ty -> prop by
  mem (add G X A) X A ;
  mem (add G Y B) X A := mem G X A.
```

$$\frac{}{\Gamma, x:A \vdash x:A} \quad \frac{\Gamma, x:A \vdash M:B}{\Gamma \vdash (\lambda x.M) : A \rightarrow B} \quad \frac{\Gamma \vdash M:A \rightarrow B \quad \Gamma \vdash N:A}{\Gamma \vdash MN:B}$$

```
Define of : ctx -> tm -> ty -> prop by
  of G X A := mem G X A ;

  of G (app M N) B :=
    exists A, of M (arrow A B) /\ of N A ;

  of G (abs x\ M x) (arrow A B) :=
    of (add G ?? A) (M ??) B
```

# HOAS: Representing Typing Contexts

```
Define mem : ctx -> tm -> ty -> prop by
  mem (add G X A) X A ;
  mem (add G Y B) X A := mem G X A.
```

$$\frac{}{\Gamma, x:A \vdash x : A} \quad \frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B} \quad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

```
Define of : ctx -> tm -> ty -> prop by
  of G X A := mem G X A ;

  of G (app M N) B :=
    exists A, of M (arrow A B) /\ of N A ;

  of G (abs x\ M x) (arrow A B) :=
    of (add G ?? A) (M ??) B
```

# Contexts

What does  $\Gamma, x:A$  mean?

- $x \notin \text{fv}(\Gamma)$
- $x \notin \text{fv}(A)$
- $(\Gamma, x:A)(y) = \begin{cases} A & \text{if } x = y \\ \Gamma(y) & \text{otherwise} \end{cases}$

## Names and the $\nabla$ (nabla) Quantifier

$\forall x. F$

For every term  $M$ , it is the case that  $[M/x]F$  is true.

$\nabla x. F$

For *any name*  $n$  that is *not free in*  $F$ , it is the case that  $[n/x]F$  is true.

Every type is inhabited by an **infinite set of names**.

Terminology: sometimes we say *nominal constant* instead of *name*.



## Some Properties of $\nabla$ vs. $\forall$

- $\nabla x. \nabla y. x \neq y$ .
  - For any name  $n \notin \{\}$ , it is that  $\nabla y. n \neq y$ .
  - For any name  $n \notin \{\}$ , for any name  $m \notin \{n\}$ , it is that  $n \neq m$ .
- $\forall x. \forall y. x \neq y$  is not provable.
  - Given any term  $M$ , it must be that  $M = M$ .
- $(\forall x. \forall y. p x y) \supset (\forall z. p z z)$ .
- $(\nabla x. \nabla y. p x y) \supset (\nabla z. p z z)$  is not provable.
  - $\nabla x. \nabla y. p x y$  means that  $p$  holds for any **two distinct** names.
  - $\nabla z. p z z$  means that  $p$  holds for any name, **repeated**.

# Mobility of Binding

The equational theory of  $\lambda$ -terms is restated in terms of  $\nabla$ .

$$(\lambda x. M) = (\lambda x. N) \text{ if and only if } \nabla x. (M = N).$$

Why not  $\forall$ ?

- Differentiate between the identity function  $\lambda x. x$  and the constant function  $\lambda x. c$ .
- $\forall x. (x = c)$  is satisfiable.
- $\nabla x. (x = c)$  is false, i.e.,  $\neg \nabla x. (x = c)$  is provable.

# Names and Equivariance

- Formulas are considered **equivalent up to a permutation of their free names**, known as **equivariance**.
- Example: if  $m$  and  $n$  are distinct names, then:
  - $p\ m \equiv p\ n$ .
  - $p\ m\ n \equiv p\ n\ m$ .
  - $p\ m\ m \not\equiv p\ m\ n$ .
- **Note: terms are not equal up to equivariance!**
- In Abella, any identifier matching the regexp  $n[0-9]^+$  is considered to be a name.

# Raising

Let  $\text{supp}(F)$  stand for the free names in  $F$ .

$\forall x. F$ :

*For every term  $M$ , it is the case that  $[M/x]F$  is true.*

# Raising

Let  $\text{supp}(F)$  stand for the free names in  $F$ .

$\forall x. F$ :

*For every term  $M$  with  $\text{supp}(M) = \{\}$ , it is the case that  $[M \text{supp}(F)/x]F$  is true.*

# Raising

$\forall x. F$ :

For every term  $M$  with  $\text{supp}(M) = \{\}$ , it is the case that  $[M \text{ sup}(\mathbf{F})/x]F$  is true.

- $\forall x. \nabla y. p \ x \ y$ 
  - For every term  $M$ , it is that  $\nabla y. p \ M \ y$ .
  - For every  $M$ , for any name  $n \notin \text{fn}(M)$ , it is that  $p \ M \ n$ .
  - Therefore  $M$  **cannot mention**  $n$ .
  
- $\nabla y. \forall x. p \ x \ y$ 
  - For any name  $n \notin \{\}$ , it is that  $\forall x. p \ x \ n$ .
  - For any name  $n$ , for every term  $M$ , it is that  $p \ (M \ n) \ n$ .
  - In other words,  $M$  is of the form  $\lambda x. M'$  where  $M'$  can have  $x$  free.
  - Therefore,  $M$  **can (indirectly) mention**  $n$ .

## Back to HOAS: The Typing Relation

$$\frac{}{\Gamma, x:A \vdash x : A} \quad \frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B} \quad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

```
Define of : ctx -> tm -> ty -> prop by
  of G X A := mem G X A ;

  of G (app M N) B :=
    exists A, of M (arrow A B) /\ of N A ;

  of G (abs x\ M x) (arrow A B) :=
    nabra x, of (add G x A) (M x) B
```

## Back to HOAS: The Typing Relation

$$\frac{}{\Gamma, x:A \vdash x:A} \quad \frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B} \quad \frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

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  of G (abs x\ M x) (arrow A B) :=
    nabra x, of (add G x A) (M x) B
```



## ∇ in the Body of a Clause

```
of G (abs x\ M x) (arrow A B) :=  
  nabla x, of (add G x A) (M x) B
```

means

```
forall G M A B,  
  of G (abs x\ M x) (arrow A B) <-  
    nabla x, of (add G x A) (M x) B.
```

- None of  $G$ ,  $M$ ,  $A$ ,  $B$  can mention  $x$ .
- $M$  can indirectly mention  $x$ .

# HOAS: Typing Relation

## 2.8 - Properties of the Typing Relation

# HOAS: Substitution

The main promise of HOAS: substitution “for free”

```
Define eval : tm -> tm -> prop by
  eval (abs R) (abs R) ;
  eval (app M N) V :=
    exists R, eval M (abs R) /\ eval (R N) V.
```

Notes:

- $(R\ N)$  may be arbitrarily larger than  $(app\ M\ N)$ .
- However, proving  $(eval\ (R\ N)\ V)$  will require strictly fewer unfolding steps than  $(eval\ (app\ M\ N)\ V)$ .

# HOAS: Subject Reducton (Extended Example)

## 2.9 - Subject Reduction

INTERMISSION

# The Two-Level Logic Approach

# Outline

- 1 Focused Minimal Intuitionistic Logic
- 2 Two-Level Logic Approach
- 3 Context Structure
- 4 Examples

# Meta-Theorems

- We have just seen several examples of **meta-theorems**:
  - Cut (for substituting in contexts)
  - Instantiation (for replacing names with terms)
  - Weakening
- Such theorems can be seen as instances of similar meta-theorems for a **proof system**
- If we can isolate this proof system and prove the meta-theorems **once and for all**, we can avoid a lot of boilerplate.



## Small Aside: A Bit of Proof Theory

Let us start with intuitionistic minimal logic.

$$\begin{aligned} F, G & ::= A \mid F \Rightarrow G \mid \Pi x. F \\ \Gamma & ::= \cdot \mid \Gamma, F \end{aligned}$$

We are going to build a **focused proof system** for this logic.

$$\begin{array}{ll} \Gamma \vdash F & \text{Goal decomposition sequent} \\ \Gamma, [F] \vdash A & \text{Backchaining sequent} \end{array}$$

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Let us start with intuitionistic minimal logic.

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We are going to build a **focused proof system** for this logic.

$\Gamma \vdash F$	Goal decomposition sequent
$\Gamma, [F] \vdash A$	Backchaining sequent

# Focused Proof System

Goal decomposition

$$\frac{\Gamma, F \vdash G}{\Gamma \vdash F \Rightarrow G} \quad \frac{(x \# \Gamma) \quad \Gamma \vdash F}{\Gamma \vdash \Pi x. F}$$

Decision

$$\frac{\Gamma, F, [F] \vdash A}{\Gamma, F \vdash A}$$

Backchaining

$$\frac{\Gamma \vdash F \quad \Gamma, [G] \vdash A}{\Gamma, [F \Rightarrow G] \vdash A} \quad \frac{\Gamma, [[t/x]F] \vdash A}{\Gamma, [\Pi x. F] \vdash A} \quad \frac{}{\Gamma, [A] \vdash A}$$

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## Synthetic (Derived) Rules

Imagine  $\Gamma = R_1, R_2$  where:

$R_1$ :  $\Pi m, n, a, b. \text{of } m (\text{arr } a b) \Rightarrow \text{of } n a \Rightarrow \text{of } (\text{app } m n) b.$

$R_2$ :  $\Pi r, a, b. (\Pi x. \text{of } x a \Rightarrow \text{of } (r x) b) \Rightarrow \text{of } (\text{abs } r) (\text{arr } a b).$

Consider the result of deciding on  $R_1$  and  $R_2$ .

$$\frac{\Gamma, [[M/m, N/n, A/a, B/b] \cdots \Rightarrow \cdots \Rightarrow \cdots] \vdash C}{\Gamma, [R_1] \vdash C} \\ \Gamma \vdash C$$

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## Deciding on $R_2$

$$\frac{\boxed{1} \quad \Gamma, [\mathbf{of}(\mathbf{abs} R) (\mathbf{arr} A B)] \vdash \mathbf{of}(\mathbf{abs} R) (\mathbf{arr} A B)}{\Gamma, [[R/r, A/a, B/b](\Pi x. \dots \Rightarrow \dots) \Rightarrow \dots] \vdash \mathbf{of}(\mathbf{abs} R) (\mathbf{arr} A B)}}{\frac{\Gamma, [R_2] \vdash \mathbf{of}(\mathbf{abs} R) (\mathbf{arr} A B)}{\Gamma \vdash \mathbf{of}(\mathbf{abs} R) (\mathbf{arr} A B)}}$$

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So:

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## Synthetic Rules vs. SOS rules

$$\frac{\Gamma \vdash M : A \rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B}$$

$$\frac{\Gamma \vdash \mathbf{of} M (\mathbf{arr} AB) \quad \Gamma \vdash \mathbf{of} NA}{\Gamma \vdash \mathbf{of} (\mathbf{app} MN) B}$$

$$\frac{\Gamma, x:A \vdash M : B}{\Gamma \vdash (\lambda x. M) : A \rightarrow B}$$

$$\frac{(x \# \Gamma) \quad \Gamma, \mathbf{of} xA \vdash \mathbf{of} (Rx) B}{\Gamma \vdash \mathbf{of} (\mathbf{abs} R) (\mathbf{arr} AB)}$$

*Reasoning about SOS derivations is isomorphic to reasoning about focused derivations for its minimal theory.*

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*Reasoning about SOS derivations is isomorphic to reasoning about focused derivations for its minimal theory.*



## Minimal Logic Definable in $\mathcal{G}$

```
Kind o      type.

Type =>     o -> o -> o.
Type pi    (A -> o) -> o.

Kind olist  type

Type nil   olist.
Type ::    o -> olist -> olist.

Define member : o -> olist -> prop by ...
```

Sequent	Encoding
$\Gamma \vdash F$	seq L F
$\Gamma, [F] \vdash A$	bch L F A

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Sequent	Encoding
$\Gamma \vdash F$	seq L F
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## Focused Minimal Sequent Calculus in $\mathcal{G}$

```
Define seq : olist -> o -> prop,  
      bch : olist -> o -> o -> prop by  
  
% goal reduction  
seq L (F => G)      := seq (F :: L) G ;  
seq L (pi F)        := nabla x, seq L (F x) ;  
  
% decision  
seq L A              :=  
  exists F, member F L /\ bch L F A ;  
  
% backchaining  
bch L (F => G) A := seq L F /\ bch L G A ;  
bch L (pi F) A   := exists T, bch L (F T) A  
bch L A A.
```

# Meta-Theory of Minimal Sequent Calculus

```
Theorem cut : forall L C F,  
  seq L C -> seq (C :: L) F -> seq L F.
```

```
Theorem inst : forall L F, nabla x,  
  seq (L x) (F x) ->  
    forall T, seq (L T) (F T).
```

```
Theorem monotone : forall L1 L2 F,  
  %% L1  $\subseteq$  L2  
  (forall G, member G L1 -> member G L2) ->  
    seq L1 F -> seq L2 F.
```

# The Two Level Logic Approach of Abella

- **Specification Logic**

- Focused sequent calculus for minimal intuitionistic logic
- Shares the type system of  $\mathcal{G}$ , but formulas of type  $\circ$
- Concrete syntax the same as  $\lambda$ Prolog

- **Reasoning Logic**

- Inductive definition of the specification logic proof system
- Inductive reasoning about specification logic derivations
- Syntactic sugar:

<b>seq</b> <b>L F</b>	<b>{L  - F}</b>
<b>bch</b> <b>L F A</b>	<b>{L, [F]  - A}</b>

## Example: STLC Specification

### 3.1 - Typing and Subject Reduction

# Uniqueness of Typing

Change to a Church style representation:

```
type abs   ty -> (tm -> tm) -> tm.
-----
of (abs A R) (arr A B) :-
  pi x \ of x A => of (R x) B.
```

Want to show that every term has a unique type.

```
Theorem type_uniq : forall M A B,
  {of M A} -> {of M B} -> A = B.
```

Need to generalize!

```
Theorem type_uniq_open : forall L M A B,
  {L |- of M A} -> {L |- of M B} -> A = B.
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```

## Structure of Contexts

- The typing **dynamic context**  $L$  is a list of **of** assumptions.
- Already seen how to inductively define the structure of lists.
- Therefore:

```
Define ctx : olist -> prop by
  ctx nil ;
  ctx (of X A :: L) := ctx L.
```

- But this does not capture  $x\#L$ !

## “ $\forall$ In The Head”

Meaning of the second clause:

```
forall L A X,  
  ctx L -> ctx (of X A :: L) .
```

Let us change the “flavor” of  $x$ .

```
forall L A, nabla x,  
  ctx L -> ctx (of x A :: L) .
```

Equivalent to:

```
forall L A, ctx L ->  
  nabla x, ctx (of x A :: L) .
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This suggests:

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# Unification with $\nabla$ In Heads

Clause head: **nabla**  $x$ , **ctx** (of  $x$  **A** :: **L**)

Assumption: **H** : **ctx** (of **U** **B** :: **LL**)

- $u$  must be a name ...
- ...that does not occur in **B** or **LL**!
- Therefore, **case H** picks an  $n \notin \text{supp}(\mathbf{B}) \cup \text{supp}(\mathbf{LL})$  for the unifier for  $u$ .

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Clause head: `nabla x, ctx (of x A :: L)`  
Assumption: `H : ctx (of n1 B :: (LL n1))`  
Tactic: `case H.`

Unification prunes `n1` from `LL n1`.

Clause head: `nabla x, ctx (of x A :: L)`  
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Cannot prune `n1`, so unification fails!

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## Some Puzzles

- Define `name : tm -> prop` that holds only for names.

```
Define name : tm -> prop by
  nabla x, name x.
```

- Define `fresh : tm -> tm -> prop` such that `fresh x Y` means `x` is a name that does not occur in `Y`.

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Define fresh : tm -> tm -> prop by
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## Some Puzzles

- Define `name : tm -> prop` that holds only for names.

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# Extended Example: Uniqueness of Typing

## 3.2 - Type Uniqueness

# Context Relations

No reason for **ctx** relations to be unary.

```
Define ctx_len : olist -> nat -> prop by
  ctx_len nil z ;
  nabla x, ctx_len (of x A :: L) (s N) :=
    ctx_len L N.
```

```
Define ctxs : olist -> olist -> prop by
  ctxs nil nil ;
  nabla x, ctxs (term x :: L) (neutral x :: K) :=
    ctxs L K.
```

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## Example: Partitioning of Lambda Terms

**3.3 - Partitioning**

# Extended Example: Relating HOAS and De Bruijn Representations

## 3.4 - HOAS vs. Indexed



# Co-Induction

# Interpretations of Co-Induction

- Non-termination
- Greatest Fixed Point
- Dual of Induction

```
Define p : prop by  
  p := p.
```

```
Theorem pth : p -> false.
```

```
CoDefine q : prop by  
  q := q.
```

```
Theorem qth : q.
```

## The `coinduction` Tactic

Given a goal

```
forall X1 ... Xn, F1 -> ... -> Fn -> G
```

where `G` is a co-inductively defined atom, the invocation

```
coinduction
```

- 1 Adds a `co-inductive hypothesis` (CH):

```
forall X1 ... Xn, F1 -> ... -> Fn -> G +
```

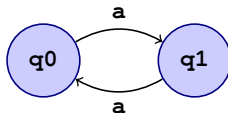
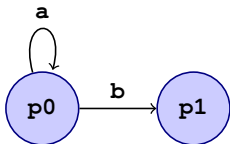
- 2 Then `changes` the goal to:

```
forall X1 ... Xn, F1 -> ... -> Fn -> G #.
```

# Annotations

Annotation	Place	Tactic	Result
@	hypothesis	<b>case</b>	*
@	goal	anything	no change
#	goal	<b>unfold</b>	+
#	hypothesis	anything	no change

## Example: Automata Simulation



**Definition:**  $q$  *simulates*  $p$ , written  $p \lesssim q$ , iff:

- for every  $p'$ ,  $a$  such that  $p \xrightarrow{a} p'$ ,
- there is a  $q'$  such that  $q \xrightarrow{a} q'$ , and
- $p' \lesssim q'$ .

Here,

- $q0 \lesssim p0$ .
- $q1 \lesssim p0$ .
- $p0 \not\lesssim q0$ .

# Example: Automata Simulation

4.1 - Automata

## Example: Diverging $\lambda$ -Terms

4.2 - Divergence

## Summary So Far

You have now seen the *headline features* of Abella.

- Higher-Order Abstract Syntax and  $\nabla$
- Inductive and Co-Inductive Definitions
- Two-Level Logic Approach

Next:

- Re-ification of the type system
- Beyond simple types
- Automation



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# Extensions

## Reasoning about typing

Abella's induction mechanism has two simple principles:

- Every inductive proof is based on an inductive definition
- All inductive definitions are explicit, fixed, and finite

Consequences:

- **Typing** is not itself inductive
- Signatures can always be extended

```
Type z nat.  
Type s nat -> nat.  
  
Theorem nat_str : forall (x:nat),  
  x = z \/  
  exists (y:nat), x = s y.  
% not provable  
skip.  
  
Type p nat -> nat -> nat.
```

Is `nat_str` still true?

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Type p nat -> nat -> nat.
```

Is `nat_str` still true?

## Re-ifying Typing

Sometimes the typing relation can be reified.

```
Define is_nat : nat -> prop by
```

```
  is_nat z ;
```

```
  is_nat (s N) := is_nat N.
```

```
Theorem nat_str : forall x, is_nat x ->
```

```
  x = z /\ exists y, is_nat y /\ x = s y.
```

```
...
```

But not always!

```
Define is_tm : tm -> prop by
```

```
  is_tm (app M N) := is_tm M /\ is_tm N ;
```

```
  is_tm (abs R) := nabla x, is_tm x -> is_tm (R x).
```

This is not stratified.

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This is not stratified.

## Two-Level Reification

```
% typing.sig
type is_nat nat -> o.
type is_tm tm -> o.
----
% typing.mod
is_nat z.
is_nat (s N) :- is_nat N.

is_tm (app M N) :- is_tm M, is_tm N.
is_tm (abs R) :- pi x\ is_tm x => is_tm (R x).
```

Then

```
Theorem nat_str : forall x, {is_nat x} ->
  x = z \/ exists y, {is_nat y} /\ x = s y.

Theorem tm_str : forall T, {is_tm T} ->
  (exists M N, {is_tm M} /\ {is_tm N} /\ T = app M N)
  \/
  (exists R, (forall x, {is_tm x} -> {is_tm R x})
   /\ T = abs R).
```



## Two-Level Reification

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% typing.sig
type is_nat nat -> o.
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is_nat z.
is_nat (s N) :- is_nat N.

is_tm (app M N) :- is_tm M, is_tm N.
is_tm (abs R) :- pi x\ is_tm x => is_tm (R x).
```

Then

```
Theorem nat_str : forall x, {is_nat x} ->
  x = z \/ exists y, {is_nat y} /\ x = s y.

Theorem tm_str : forall T, {is_tm T} ->
  (exists M N, {is_tm M} /\ {is_tm N} /\ T = app M N)
  \/
  (exists R, (forall x, {is_tm x} -> {is_tm R x})
    /\ T = abs R).
```

## Beyond Simple Types: LF (a.k.a. $\lambda\Pi$ )

<http://abella-prover.org/lf>

- All kinds of typing relations can be reified.
- Encoding dependent types (and  $DT\lambda$  terms):

$$\begin{aligned} \llbracket \Pi x:A. U \rrbracket &= \llbracket A \rrbracket \rightarrow \llbracket U \rrbracket & \llbracket M N \rrbracket &= \llbracket M \rrbracket \llbracket N \rrbracket \\ \llbracket a M_1 \cdots M_n \rrbracket &= a M_1 \cdots M_n & \llbracket \lambda x:A. M \rrbracket &= \lambda x:\llbracket A \rrbracket. \llbracket M \rrbracket \\ \llbracket \mathbf{type} \rrbracket &= \mathbf{lf\,type} \end{aligned}$$

- Encoding typing as specification formulas.

$$\begin{aligned} \llbracket M : \Pi x:A. U \rrbracket &= \Pi x. \llbracket x : A \rrbracket \Rightarrow \llbracket M x : U \rrbracket \\ \llbracket M : P \rrbracket &= \mathbf{hastype} \llbracket M \rrbracket \llbracket P \rrbracket \\ \llbracket A : \mathbf{type} \rrbracket &= \mathbf{istype} \llbracket A \rrbracket \end{aligned}$$

- Encoding LF signatures

$$\begin{array}{c} \llbracket c : U \rrbracket = \mathbf{type} \ c \ \llbracket U \rrbracket. \\ \hline \llbracket c : U \rrbracket. \end{array}$$

## Abella/LF Examples

# Automation

- Many theorems about contexts are:
  - Tedious, and
  - Predictable
- This is particularly the case for **regular** contexts.
- We have a proof of concept for some rather sophisticated and certifying automation procedures (LFMTP 2014)
- Look out for it in Abella 2.1!

# More Resources

## Related Material

- See list on:

**<http://abella-prover.org/tutorial/>**

- Extensive tutorial document: [Abella: A System for Reasoning About Relational Specifications](#), J. Formalized Reasoning, 2014.
- Course notes by Gopalan Nadathur for: [Specification and Reasoning About Computational Systems](#)
- Book – Dale Miller and Gopalan Nadathur: [Programming in Higher-Order Logic](#), CUP, 2012

# Some Work in Progress

## That I Know Of

- Compiler verification project in  $\lambda$ Prolog + Abella
  - Using step-indexed logical relations
  - Yuting Wang, Gopalan Nadathur
- ORBI-to-Abella
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- Certified procedures for type checkers
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- Polymorphism and reasoning modules
  - Polymorphic definitions and theorems already part of the upcoming Abella 2.0.4.
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