Well Balanced Designs for Data Placement

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Abstract

The problem we consider in this article is motivated by data placement, in particular data replication in distributed storage and retrieval systems. We are given a set V of v servers along with b files (data, documents). Each file is replicated on exactly k servers. A placement consists in finding a family of b subsets of V (representing the files) called blocks, each of size k. Each server has some probability to fail and we want to find a placement which minimizes the variance of the number of available files. It was conjectured that there always exists an optimal placement (with variance better than that of any other placement for any value of the probability of failure). We show that the conjecture is true, if there exists a well balanced design, that is a family of blocks, each of size k, such that each j-element subset of V, $1 \leq j \leq k$, belongs to the same or almost the same number of blocks (difference at most one). The existence of well balanced designs is a difficult problem as it contains as a subproblem the existence of Steiner systems. We completely solve the case $k = 2$ and give bounds and constructions for $k = 3$ and some values of v and b.

1 Introduction

The problem we consider in this article is motivated by data placement in particular data replication in distributed storage and retrieval systems (see [1, 2, 3, 13]). We use here the terminology of design and graph theory (so the notations are somewhat different from the papers mentioned above). We are given a set V of v servers along with b files (data, documents). Each file is replicated (placed) on exactly k servers. The set of servers containing file i is therefore a subset of size k, which will be called a block and denoted B_i. A placement consists of giving a family $\mathcal{F}$ of blocks $B_i, 1 \leq i \leq b$.

A server is available (on-line) with some probability δ and so unavailable (offline, failed) with the probability $1 - \delta$. The file i is said to be available if one of the servers containing it is available or equivalently the file is unavailable if all the servers containing it are unavailable. In [2, 3, 13] the authors studied the random variable Λ, the number of available files and they proved that the mean is $E(\Lambda) = b(1 - (1 - \delta)^k)$; so this mean is independent of the placement. However they proved that the variance of Λ depends on the placement and showed (see [13]) that minimizing the variance corresponds to minimizing the polynomial $P(\mathcal{F}, x) = \sum_{j=0}^{k} v_j x^j$ where $x = \frac{1}{1-\delta}$ (so $x \geq 1$) and v_j denotes the number of ordered pairs of blocks intersecting in exactly j elements. So we can summarize our problem as follows:

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Problem: ***Let v, k, b be given positive integers and x be a real number, $x \geq 1$; find a placement that is a family $\mathcal{F}$ of b blocks, each of size k, on a set of v elements, which minimizes the polynomial $P(\mathcal{F}, x) = \sum_{j=0}^{k} v_j x^j$, where v_j denotes the number of ordered pairs of blocks intersecting in exactly j elements. Such a placement will be called optimal for the value x.***

In [13] the following conjecture is proposed:

Conjecture 1 *For any v, k, b there exists a family $\mathcal{F}^*$ which is optimal for all the values of $x \geq 1$ (that is $P(\mathcal{F}^*, x) \leq P(\mathcal{F}, x)$ for any $\mathcal{F}$ and any $x \geq 1$).*

Note that for $x = 1$, we have $P(\mathcal{F}, 1) = b(b-1)$ as the value is the number of ordered pairs of blocks. So we can restrict to the case $x > 1$. Note also that all the coefficients are even; indeed if B and B' intersect in j elements, then so do B' and B. So, we could have considered only (non ordered) pairs of blocks, in which case the polynomial will have been one half of that for ordered pairs.

Before stating our results let us give some examples. Let $v = 4$, $b = 4$, $k = 2$. We can consider different placements:

- Family $\mathcal{F}_1$: $B_1 = B_2 = B_3 = B_4 = \{1, 2\}$; then $P(\mathcal{F}_1, x) = 12x^2$
- Family $\mathcal{F}_2$: $B_1 = B_2 = \{1, 2\}, B_3 = B_4 = \{3, 4\}$; then $P(\mathcal{F}_2, x) = 4x^2 + 8$
- Family $\mathcal{F}_3$: $B_1 = \{1, 2\}, B_2 = \{1, 3\}, B_3 = \{1, 4\}, B_4 = \{2, 3\}$; then $P(\mathcal{F}_3, x) = 10x + 2$
- Family $\mathcal{F}_4$: $B_1 = \{1, 2\}, B_2 = \{2, 3\}, B_3 = \{3, 4\}, B_4 = \{1, 4\}$; then $P(\mathcal{F}_4, x) = 8x + 4$.

For any $x \geq 1$, $P(\mathcal{F}_4, x) \leq P(\mathcal{F}_i, x)$ and it can be proven that indeed $\mathcal{F}_4$ is an optimal family for any $x \geq 1$. Note that depending on the values of x, $\mathcal{F}_2$ can be better (or worse) than $\mathcal{F}_3$. For $x \leq \frac{3}{2}$, $P(\mathcal{F}_2, x) \leq P(\mathcal{F}_3, x)$ (for example for $x = \frac{5}{4}$, $P(\mathcal{F}_2, \frac{5}{4}) = 14 + \frac{1}{4}$ and $P(\mathcal{F}_3, \frac{5}{4}) = 14 + \frac{1}{2}$). But for $x \geq \frac{3}{2}$, $P(\mathcal{F}_2, x) \geq P(\mathcal{F}_3, x)$ (for example for $x = 2$, $P(\mathcal{F}_2, 2) = 24$ and $P(\mathcal{F}_3, 2) = 22$).

Let now $v = 5$, $b = 3$, and $k = 3$. We claim that the family $\mathcal{F}^*$ consisting of the three blocks $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{3, 4, 5\}$ is optimal for all $x \geq 1$. We have $P(\mathcal{F}^*, x) = 2x^2 + 4x$. Let $\mathcal{F}$ be any other family with a polynomial $P(\mathcal{F}, x) = a_3x^3 + a_2x^2 + a_1x + a_0$. As $v = 5$, there can never be two disjoint blocks; so $a_0 = 0$. Furthermore we always have $a_3 + a_2 + a_1 = b(b-1) = 6$. So $P(\mathcal{F}, x) - P(\mathcal{F}^*, x) = (x-1)(a_3x^2 + (a_3 + a_2 - 2)x)$. If $a_3 \geq 2$ (that is at least one block repeated), then $P(\mathcal{F}, x) - P(\mathcal{F}^*, x) > 0$ for any $x > 1$. If $a_3 = 0$, among 3 blocks necessarily two of them have a pair in common and so $a_2 \geq 2$ and $P(\mathcal{F}, x) - P(\mathcal{F}^*, x) \geq 0$ for all $x \geq 1$.

2 Our results

For a family $\mathcal{F}$ let $\lambda^{\mathcal{F}}_{x_1,\ldots,x_j}$ (or shortly $\lambda_{x_1,\ldots,x_j}$) denote the number of blocks of the family containing the j-element subset $\{x_1, \ldots, x_j\}$. **A family $\mathcal{F}$ is j-balanced** if the $\lambda_{x_1,\ldots,x_j}$ are all equal or almost equal, that is, if for any two j-element subsets $\{x_1, \ldots, x_j\}$ and $\{y_1, \ldots, y_j\}$, $|\lambda_{x_1,\ldots,x_j} - \lambda_{y_1,\ldots,y_j}| \leq 1$. Furthermore, **a family $\mathcal{F}$ is well balanced** if it is j-balanced for $1 \leq j \leq k$, where k is the size of the blocks.

We first show in Section 3 that $P(\mathcal{F}, x) = \Sigma_{j=1}^{k} \Sigma_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j} (x-1)^j - bx^k + b^2$. The form of the above polynomial enables us to prove in Section 3 that a well balanced family is also optimal and therefore Conjecture 1 is proven for the values of b, for which there exists a well balanced family.

The rest of the paper is devoted to the construction of well balanced families and so optimal ones. We consider first the case $k = 2$ (Section 4) where such families are easy to construct for any b. The cases $k > 2$, are much more complicated. Starting with $k = 3$, there are values of v and b for which there do not exist well balanced families (Propositions 6 and 7 of Section 5). In section 6 we develop some tools based on design theory in particular on Steiner Triple Systems (see the handbook [8] for details) to construct some well balanced families.

Note that the problem of constructing well balanced families contains as a subproblem the question of the existence of Steiner systems. Recall that a t-Steiner system (or (v, k, λ) t-design) is a family of blocks such that each t-element subset appears in exactly λ blocks (see [8, 7]). In that case it is well-known that also, for $1 \leq j \leq t$ each j-element subset appears in exactly λ_j blocks, where $\lambda_j = \lambda \frac{\binom{v-j}{t-j}}{\binom{k-j}{t-j}}$. So a t-design is

j-balanced for all j, $1 \leq j \leq t$. In particular, if $t = k-1$ and the blocks are repeated the same or almost the same number of times, then a k-Steiner System is also well balanced. As an example, a Steiner Triple System (STS) consists of a family of triples, such that each pair of elements appears in exactly one triple. In that case each element appears in $\frac{v-1}{2}$ triples and no triple is repeated. Therefore, an STS is a well balanced family. It is well-known that an STS exists if and only if $v \equiv 1$ or $3 \pmod 6$ and then $b = \frac{v(v-1)}{6}$. That gives some sporadic values for which there exist well balanced families.

The results obtained in a preliminary version of this article ([4]) lead us to conjecture that the values excluded by Propositions 6 and 7 are the only ones for which there do not exist well balanced families (Conjecture 2). This conjecture has been recently proved to be true in [19]. We show in Section 6, that well balanced families exist for any b for the values of $v \equiv 3 \pmod 6$ for which there exist a large number of disjoint Kirkman triple systems (see [15, 16]). We also develop various tools and use them to verify Conjecture 2 for small values of v. More detailed constructions for $v = 6t+4$ can be found in [4]. Finally, in Section 7, we present some results for values of $k > 3$.

3 Properties of $P(\mathcal{F}, x)$ and well balanced families

Recall that $\lambda_{x_1,\ldots,x_j}$ denotes the number of blocks of the family containing the j-element subset $\{x_1, \ldots, x_j\}$. By convention $\lambda_\emptyset = b$. In this section, we express the polynomial $P(\mathcal{F}, x)$ in function of $\lambda_{x_1,\ldots,x_j}$ and deduce the optimality of well balanced families.

Proposition 1 $P(\mathcal{F}, x) = \sum_{j=0}^{k} \sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)(x-1)^j$.

Proof. $P(\mathcal{F}, x) = \sum_{h=0}^{k} v_h x^h$. Let us write $P(\mathcal{F}, x) = \sum_{j=0}^{k} \mu_j (x-1)^j$. Using $x^h = (x-1+1)^h = \sum_{j=0}^{h} \binom{h}{j}(x-1)^j$, we get $\mu_j = \sum_{h=j}^{k} \binom{h}{j} v_h$.
We claim that $\mu_j = \sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)$.
Indeed $\lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)$ counts the number of ordered pairs of blocks which contain $x_1, \ldots, x_j$. This number is the sum of the ordered pairs of blocks which intersect in exactly the j elements $x_1, \ldots, x_j$, plus those intersecting in exactly $j+1$ elements containing $x_1, \ldots, x_j$, plus more generally those intersecting in exactly in h elements containing $x_1, \ldots, x_j$, where, $j \leq h \leq k$. When we sum on all the possible j-element subsets to obtain $\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(\lambda_{x_1,\ldots,x_j} - 1)$, we therefore get:

- the number of ordered pairs of blocks intersecting in exactly j elements, that is v_j
- plus the number of ordered pairs of blocks intersecting in exactly $j+1$ elements, which are counted $\binom{j+1}{j}$ times. Indeed, if the intersection of two blocks is $\{x_1, \ldots, x_{j+1}\}$ they are counted for all the j-element subsets included in $\{x_1, \ldots, x_{j+1}\}$ which are in number $\binom{j+1}{j}$. Therefore we have $\binom{j+1}{j} v_{j+1}$ such ordered pairs of blocks.
- plus more generally, for h, $j \leq h \leq k$ we count $\binom{h}{j} v_h$ ordered pairs of blocks intersecting in exactly h elements; indeed if the intersection of two blocks is $\{x_1, \ldots, x_h\}$ they are counted for all the j-element subsets included in $\{x_1, \ldots, x_h\}$, that is $\binom{h}{j}$ times.

  Therefore we get exactly μ_j which is the left-hand side of the equation of the claim.

□

We will use the following equality intensively

$$\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j} = b\binom{k}{j}. \tag{1}$$

It follows from the fact that a given block B is counted once in all the $\lambda_{x_1,\ldots,x_j}$ such that $\{x_1, \ldots, x_j\} \subset B$ and we have $\binom{k}{j}$ such j-element subsets.

Theorem 1 $P(\mathcal{F}, x) = \sum_{j=1}^{k} \sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j}(x-1)^j - bx^k + b^2$.

Proof. Using Equation 1, we get $\sum_{j=0}^{k} \sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}(x-1)^j = \sum_{j=0}^{k} b\binom{k}{j}(x-1)^j = b(x-1+1)^k = bx^k$. Replacing in the expression of $P(\mathcal{F},x)$ given in Proposition 1 and using the fact that $\lambda_\emptyset^2 = b^2$ we obtain the theorem. $\square$

Proposition 2 *$\sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j}$ is minimized when $\mathcal{F}$ is j-balanced.*

Proof. As by Equation 1, $\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j}$ is the constant $b\binom{k}{j}$, then $\sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j}$ is minimized when all the $\lambda_{x_1,\ldots,x_j}$ are equal to $r = b\binom{k}{j}/\binom{v}{j}$ if this value is an integer or are equal either to $\lfloor r \rfloor$ or $\lceil r \rceil$ otherwise. This is equivalent to say that $\mathcal{F}$ is j-balanced. $\square$

So, we can state our main theorem:

Theorem 2 *If $\mathcal{F}^*$ is well balanced, then $\mathcal{F}^*$ is optimal, that is, $P(\mathcal{F}^*,x) \leq P(\mathcal{F},x)$ for any $\mathcal{F}$ and any $x \geq 1$.*

Proof. If $\mathcal{F}^*$ is well balanced, then all the coefficients of the polynomial as expressed in the Theorem 1 are minimized and so $\mathcal{F}^*$ is optimal. $\square$

Note that for a j-balanced family, the coefficient of $(x-1)^j$ in the polynomial $P(\mathcal{F},x)$ is easy to compute. Let $b\binom{k}{j} = q\binom{v}{j} + r$, with $r < \binom{v}{j}$. Then we have r values of the $\lambda_{x_1,\ldots,x_j}$ equal to $q+1$ and $\binom{v}{j} - r$ equal to q. So, $\sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j} = \binom{v}{j}q^2 + 2qr + r$.

When $b = \binom{v}{k}$, the family consisting of all the possible k-element subsets is well balanced and will be called **a complete family**. Furthermore, for any j, the values of the $\lambda_{x_1,\ldots,x_j}$ are all equal to $\lambda_j = \binom{v-j}{k-j}$. By taking h copies we get also a well balanced family for $b = h\binom{v}{k}$.

Proposition 3 *Let v and k be given and let $b' = h\binom{v}{k} + b$ with $b < \binom{v}{k}$. Then, there exists a well balanced family $\mathcal{F}'$ for b' if and only if there exists a well balanced family $\mathcal{F}$ for b.*

Proof. If we have a well balanced family $\mathcal{F}$ for some $b \leq \binom{v}{k}$ we can construct a well balanced family $\mathcal{F}'$ for $b' = h\binom{v}{k} + b$ by adding h complete families to $\mathcal{F}$. Conversely if we have a well balanced family $\mathcal{F}'$ for $b' = h\binom{v}{k} + b$, each k-element subset is repeated h or $h+1$ times and so by deleting h copies of each block, we can deduce a well balanced family for b. $\square$

The next proposition generalizes this idea to optimal families.

Proposition 4 *Let v and k be given and let $b' = h\binom{v}{k} + b$ with $b \leq \binom{v}{k}$. If there exists an optimal family for b', then there exists an optimal family for b and furthermore the optimal family for b' consists of the optimal family for b plus h complete families.*

Proof. Suppose there exists an optimal family $\mathcal{F}'$ for b'. This family is necessarily k-balanced. Indeed suppose it is not the case and let $\mathcal{F}''$ be a k-balanced family (such a family can be easily constructed by taking among the $\binom{v}{k}$ subsets of size k, b of them repeated $h+1$ times and the other $\binom{v}{k} - b$ repeated h times). But, the coefficient of x^k in $P(\mathcal{F}'',x)$ will be strictly less than that of $P(\mathcal{F}',x)$ and so for x large enough $P(\mathcal{F}'',x) < P(\mathcal{F}',x)$ contradicting the optimality of $\mathcal{F}'$. So each k-element subset appears exactly h or $h+1$ times.

Now, deleting h copies of each block we get a family $\mathcal{F}$ with $b = b' - h\binom{v}{k}$ blocks (none of them being repeated). Note that if $\lambda_{x_1,\ldots,x_j}$ (resp. $\lambda'_{x_1,\ldots,x_j}$) denotes the number of blocks of the family $\mathcal{F}$ (resp. $\mathcal{F}'$) containing $\{x_1,\ldots,x_j\}$ we have: $\lambda'_{x_1,\ldots,x_j} = \lambda_{x_1,\ldots,x_j} + h\binom{v-j}{k-j}$. Consider another family $\mathcal{G}$ on b blocks and let $\mathcal{G}'$ be the family on b' blocks obtained by adding h complete families to $\mathcal{G}$. Let $\mu_{x_1,\ldots,x_j}$ (resp. $\mu'_{x_1,\ldots,x_j}$) denote the number of blocks of the family $\mathcal{G}$ (resp. $\mathcal{G}'$) containing $\{x_1,\ldots,x_j\}$. Then we have: $\mu'_{x_1,\ldots,x_j} = \mu_{x_1,\ldots,x_j} + h\binom{v-j}{k-j}$. So, by Equation 1, $\sum_{x_1,\ldots,x_j} \lambda_{x_1,\ldots,x_j} = \sum_{x_1,\ldots,x_j} \mu_{x_1,\ldots,x_j}$ and $\sum_{x_1,\ldots,x_j} \lambda'_{x_1,\ldots,x_j} = \sum_{x_1,\ldots,x_j} \mu'_{x_1,\ldots,x_j}$, then $\sum_{x_1,\ldots,x_j} \lambda^2_{x_1,\ldots,x_j} - \sum_{x_1,\ldots,x_j} \mu^2_{x_1,\ldots,x_j} = \sum_{x_1,\ldots,x_j} \lambda'^2_{x_1,\ldots,x_j} - \sum_{x_1,\ldots,x_j} \mu'^2_{x_1,\ldots,x_j}$ and thus $P(\mathcal{G}',x) - P(\mathcal{F}',x) = P(\mathcal{G},x) - P(\mathcal{F},x)$. Therefore if $\mathcal{F}$ is not optimal there exists a family $\mathcal{G}$ and a value x for which $P(\mathcal{G}, x) < P(\mathcal{F}, x)$ and for this value of x we have $P(\mathcal{G}', x) < P(\mathcal{F}', x)$ and $\mathcal{F}'$ is not optimal, a contradiction. $\square$

We conjecture that the converse is true: that is starting from an optimal family $\mathcal{F}$ for some $b \leq \binom{v}{k}$, the family $\mathcal{F}'$ obtained by adding h complete families is also optimal. This is true, if Conjecture 1 on the existence of an optimal family for any v, b, k is true, as in that case any optimal family is k-balanced.

In what follows we will restrict ourselves to the case $b \leq \binom{v}{k}$. In fact the following proposition shows that we only need to consider the values of $b \leq \frac{1}{2}\binom{v}{k}$.

Proposition 5 *Let v and k be given. An optimal family $\bar{\mathcal{F}}$ for $\bar{b} = \binom{v}{k} - b$ can be obtained from an optimal family $\mathcal{F}$ for $b \leq \binom{v}{k}$ by taking as blocks the k-element subsets which are not blocks of $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be an optimal family with b blocks and let $\bar{\mathcal{F}}$ be the family obtained from $\mathcal{F}$ by taking as blocks the k-element subsets which are not blocks of $\mathcal{F}$. $\bar{\mathcal{F}}$ has $\bar{b} = \binom{v}{k} - b$ blocks. Furthermore, if $\bar{\lambda}_{x_1,\ldots,x_j}$ denotes the number of blocks of the family $\bar{\mathcal{F}}$ containing $\{x_1, \ldots, x_j\}$, we have $\bar{\lambda}_{x_1,\ldots,x_j} = \binom{v-j}{k-j} - \lambda_{x_1,\ldots,x_j}$. Consider another family $\bar{\mathcal{G}}$ with $\bar{b}$ blocks and let $\mathcal{G}$ be the complementary family obtained from $\bar{\mathcal{G}}$ by taking as blocks the k-element subsets which are not blocks of $\bar{\mathcal{G}}$; $\mathcal{G}$ has b blocks. We also have: $\bar{\mu}_{x_1,\ldots,x_j} = \binom{v-j}{k-j} - \mu_{x_1,\ldots,x_j}$ and so we get $P(\bar{\mathcal{G}}, x) - P(\bar{\mathcal{F}}, x) = P(\mathcal{G}, x) - P(\mathcal{F}, x)$. Therefore if $\mathcal{F}$ is an optimal family, then $\bar{\mathcal{F}}$ is also an optimal family. $\square$

4 Case k = 2

Theorem 3 *Let $k = 2$. Then for any v and b there exists a well balanced family.*

Proof. In view of Proposition 4, we only need to consider the case $b \leq \binom{v}{2}$. In the case $k = 2$ the blocks are pairs of elements and so the problem consists of designing a simple graph with v vertices and b edges that is almost regular (the degree of a vertex x being $d(x) = \lfloor \frac{2b}{v} \rfloor$ or $\lceil \frac{2b}{v} \rceil$). We distinguish two cases.

- Case v even. Let $b = q\frac{v}{2} + r$ for $0 \leq r < \frac{v}{2}$. It is well-known that, for v even, the edges of the complete graph K_v can be partitioned into $v-1$ perfect matchings (set of $\frac{v}{2}$ disjoint edges covering the vertices). In that case the family consisting of q perfect matchings plus r edges of the $(q+1)$th perfect matching forms the required family with $b = q\frac{v}{2} + r$ edges, none of them repeated and with the degree of a vertex equal to q or $q + 1$.
- Case v odd. Let $b = qv + r$ for $0 \leq r < v$. It is also well-known that for v odd, the edges of the complete graph K_v can be partitioned into $\frac{v-1}{2}$ hamiltonian cycles (cycles containing each vertex exactly once). In that case consider the family consisting of q hamiltonian cycles plus the following r edges of the $(q+1)$th hamiltonian cycle: if the cycle is $x_0, x_1, \ldots, x_i, \ldots, x_{v-1}$ we take the r edges $\{x_{2j}, x_{2j+1}\}$ for $0 \leq j \leq r - 1$ (indices being taken modulo v). Then it consists of $b = qv + r$ edges none of them being repeated; furthermore the degree of a vertex is $2q$ or $2q + 1$ if $r \leq \frac{v-1}{2}$ and $2q + 1$ or $2q + 2$ otherwise and so in both cases $d(x) = \lfloor \frac{2b}{v} \rfloor$ or $\lceil \frac{2b}{v} \rceil$. $\square$

An algorithm to construct a well balanced family starting from any family.

In some cases related to practical applications, files and servers may be appearing or disappearing over time, leaving the storage system in an unbalanced situation. Instead of starting over, it might be helpful to design an algorithm which, starting from some family, constructs an optimal well balanced family. That is in general a difficult problem; but for $k = 2$, we can easily design such a procedure.

Let v and b be given and $k = 2$ and consider any family $\mathcal{F}$; we will transform it into a well balanced family with the same parameters. First let us construct a 2-balanced family. Suppose, $\mathcal{F}$ is not 2-balanced; so there exist two edges (blocks) $\{x, y\}$ and $\{z, t\}$ with $\lambda_{x,y} \geq \lambda_{z,t} + 2$. Then, delete from $\mathcal{F}$ one edge $\{x, y\}$ and add one edge $\{z, t\}$. Repeating this procedure we end up after a finite number of steps with a family such that for any pair of edges $\{x, y\}$ and $\{z, t\}$ $|\lambda_{x,y} - \lambda_{z,t}| \leq 1$, that is a 2-balanced family. Now let us show how to construct a well balanced family from a 2-balanced one. Let $\mathcal{F}$ be a 2-balanced family with $\lambda_{x,y} = \lambda$ or $\lambda - 1$; suppose it is not 1-balanced; then there exist two vertices x and z with $d(x) \geq d(z) + 2$. So there exists a vertex $y \neq x, z$ with $\lambda_{x,y} \geq \lambda_{z,y}+1$; otherwise $d(x) = \sum_{y\neq x,z} \lambda_{x,y}+\lambda_{x,z} \leq \sum_{y\neq x,z} \lambda_{z,y}+\lambda_{x,z} = d(z)$ a contradiction. Thus, $\lambda_{x,y} = \lambda$ and $\lambda_{z,y} = \lambda - 1$. Deleting from $\mathcal{F}$ one edge $\{x, y\}$ and adding one edge $\{z, y\}$, we still get a 2-balanced family $\mathcal{F}'$ ($\lambda'_{x,y} = \lambda-1$ and $\lambda'_{z,y} = \lambda$); but we have reduced the gap between the degrees of x and z, as $d'(x) = d(x) - 1$ and $d'(z) = d(z) + 1$, while the other degrees remain unchanged. Repeating this procedure we end up after a finite number of steps with a 1-balanced and 2-balanced, so a well balanced family.

5 Case k = 3: Impossible configurations

For $k = 3$, there are values of v and b for which there do not exist well balanced families. In this section, we identify several such sets of parameters. Then, in Section 6, we proceed towards the construction of well balanced families for some other cases.

Consider for instance $v = 4$ and $b = 2$. There are 6 possible different pairs $\{x, y\}$ and 6 pairs in the two blocks, so if there exists a 2-balanced family, then $\lambda_{x,y} = 1$ for all $\{x, y\}$. But this is impossible as $v-1 = 3$ and there cannot exist a partition of the edges of K_4 into triples (non existence of a $(4, 3, 1)$-design). The argument is generalized in the following proposition:

Proposition 6 *Let $k = 3$, v be even and λ be odd. If $\lambda\frac{v(v-1)}{2} - \frac{v}{2} < 3b < \lambda\frac{v(v-1)}{2} + \frac{v}{2}$, then there does not exist a 2-balanced family.*

Proof. Note that the number of possible pairs is $\frac{v(v-1)}{2}$. By Equation 1, $\sum_{x,y} \lambda_{x,y} = 3b$. We distinguish three cases:

- $3b = \lambda\frac{v(v-1)}{2}$. In that case a 2-balanced family will verify $\lambda_{x,y} = \lambda$ for all pairs $\{x, y\}$ and then we should have $\lambda_x = \lambda\frac{v-1}{2}$ which is impossible as λ is odd and v is even (non existence of a $(v, 3, \lambda)$-design for v even and λ odd).

- $3b < \lambda\frac{v(v-1)}{2}$. In that case we cannot have all the $\lambda_{x,y} \geq \lambda$. So we have one of the $\lambda_{x,y} \leq \lambda - 1$ and if the family is 2-balanced all the $\lambda_{x,y} \leq \lambda$. But, then $\lambda_x \leq \lambda\frac{v-1}{2}$ and as $\lambda(v-1)$ is odd, $\lambda_x \leq \lambda\frac{v-1}{2} - \frac{1}{2}$. Using Equation 1, $3b = \sum_x \lambda_x \leq \lambda\frac{v(v-1)}{2} - \frac{v}{2}$. Therefore, there does not exist a 2-balanced family if $\lambda\frac{v(v-1)}{2} - \frac{v}{2} < 3b < \lambda\frac{v(v-1)}{2}$.

- $3b > \lambda\frac{v(v-1)}{2}$. In that case we cannot have all the $\lambda_{x,y} \leq \lambda$. So we have one of the $\lambda_{x,y} \geq \lambda + 1$ and if the family is 2-balanced all the $\lambda_{x,y} \geq \lambda$. But, then $\lambda_x \geq \lambda\frac{v-1}{2}$ and as $\lambda(v-1)$ is odd, $\lambda_x \geq \lambda\frac{v-1}{2} + \frac{1}{2}$. Using Equation 1, $3b = \sum_x \lambda_x \geq \lambda\frac{v(v-1)}{2} + \frac{v}{2}$. Therefore there does not exist a 2-balanced family if $\lambda\frac{v(v-1)}{2} < 3b < \lambda\frac{v(v-1)}{2} + \frac{v}{2}$. $\square$

For example, there do not exist well balanced families for $k = 3$ and $\{v = 6; b \equiv 5 \pmod{10}\}$; $\{v = 8; b \equiv 9, 10, 27, 28, 29, 46, 47 \pmod{56}\}$; $\{v = 10; b \equiv 14, 15, 16 \pmod{30}\}$; $\{v = 12; b \equiv 21, 22, 23 \pmod{44}\}$; $\{v = 16; b \equiv 38, 39, 40, 41, 42 \pmod{80}\}$.

Proposition 7 *Let $k = 3$. If $\lambda\frac{v(v-1)}{6}$ is not an integer, then there does not exist a well balanced family for $b = \lfloor\lambda\frac{v(v-1)}{6}\rfloor$ and $b' = \lceil\lambda\frac{v(v-1)}{6}\rceil$.*

Proof. Let $b = \lfloor\lambda\frac{v(v-1)}{6}\rfloor$. If $\lambda\frac{v(v-1)}{6}$ is not an integer, then $3b = \lambda\frac{v(v-1)}{2} - \epsilon$ where $\epsilon = 1$ or 2. By Equation 1, $3b = \sum_x \lambda_x$ and so if $\mathcal{F}$ is 1-balanced $\lambda_x = \frac{\lambda(v-1)}{2}$ except for ϵ vertices for which the value is one less. Similarly by Equation 1, $3b = \sum_{x,y} \lambda_{x,y}$ and so if $\mathcal{F}$ is 2-balanced $\lambda_{x,y} = \lambda$ except for ϵ pairs appearing $\lambda-1$ times. But for an x_0 with $\lambda_{x_0} = \frac{\lambda(v-1)}{2} - 1$, we have $\lambda(v-1)-2$ pairs containing it (2 pairs per block containing it) and so two pairs appear $\lambda-1$ times. If $\epsilon = 2$ we have another vertex x'_0 with $\lambda_{x'_0} = \frac{\lambda(v-1)}{2} - 1$ and altogether at least 3 pairs appear $\lambda-1$ times (only the pair $\{x_0, x'_0\}$ can be counted twice). So, we have, in all cases, at least $\epsilon+1$ pairs appearing $\lambda-1$ times, contradicting the fact that if $\mathcal{F}$ is 2-balanced only ϵ pairs appear $\lambda-1$ times.

The proof for $b' = \lceil\lambda\frac{v(v-1)}{6}\rceil$ is similar. In that case $3b' = \lambda\frac{v(v-1)}{2} + \epsilon$ where $\epsilon = 1$ or 2. If $\mathcal{F}$ is 1-balanced $\lambda_x = \frac{\lambda(v-1)}{2}$ except for ϵ vertices for which the value is one more. If $\mathcal{F}$ is 2-balanced $\lambda_{x,y} = \lambda$

except for ϵ pairs appearing $\lambda+1$ times. The argument applied for the vertex x_0 (or both x_0 and x'_0), with $\lambda_{x_0} = \frac{\lambda(v-1)}{2} + 1$ gives at least $\epsilon+1$ pairs appearing $\lambda+1$ times, a contradiction. $\square$

Proposition 7 applies for $v \equiv 5 \pmod 6$ and $\lambda \not\equiv 0 \pmod 3$; for example there do not exist well balanced families for $\{v = 5; b \equiv 3, 4, 6, 7 \pmod{10}\}$ or $\{v = 11; b \equiv 18, 19, 36, 37 \pmod{55}\}$. It applies also for $v \equiv 2 \pmod 6$ and $\lambda \not\equiv 0 \pmod 3$; for λ odd it is included in Proposition 6, but for λ even we get new values of non existence of well balanced families for $\{v = 8; b \equiv 18, 19, 37, 38 \pmod{56}\}$.

6 Case k = 3: Construction of well balanced families

6.1 Summary of the results

In a preliminary version of this article ([4]) we developed some tools based on design theory in particular on Steiner Triple Systems (see the handbook [8] for details) to construct some well balanced families. The results obtained lead us to conjecture that the values excluded by Propositions 6 and 7 are the only ones for which there do not exist well balanced families.

Conjecture 2 *Let $k = 3$, there exists a well balanced family for the values of v and b different from those excluded by Propositions 6 and 7.*

In what follows we will construct well balanced families for $b \le \binom{v}{3}$ that have no repeated blocks; indeed due to Proposition 3, it gives all the values of the form $b + h\binom{v}{3}$.

Our results are only partial and rely on the existence of some unknown combinatorial objects. Recently C. Colbourn informed us that a complete solution has been found (see [19]). It relies, among other results, on a lemma (similar to what we did for the case $k = 2$ in the algorithmic part), which shows that given a 3-balanced (no repeated blocks) and 2-balanced family one can construct a well balanced family (that is also 1-balanced). In particular the lemma gives that, if $v \equiv 1$ or $3 \pmod 6$, then there exists a well balanced family for any b (showing the validity of the conjecture for these values). We give in what follows a sketch of our results as the tools used might be interesting and the ideas can motivate some research in this area. In particular, we show how they can be applied to solve the conjecture for small values of v.

6.2 STS and KTS (Steiner Triple Systems and Kirkman Triple Systems)

Recall that a $(v, 3, 1)$ Steiner Triple System (STS(v) shortly) is defined as a family of triples (blocks of size 3), such that every pair of elements belongs to exactly one block ($\lambda_{x,y} = 1$). So it is 2-balanced (and also 3-balanced); it is well-known that every vertex belongs to exactly $\frac{v-1}{2}$ blocks and therefore it is well balanced. Such a design exists if and only if $v \equiv 1$ or $3 \pmod 6$. In that case $b = \frac{v(v-1)}{6}$.

For example, for $v = 7$, the blocks of a $(7, 3, 1)$-design are $B_i = \{i, i+1, i+3\}$, $0 \le i \le 6$, the numbers being taken modulo 7. For $v = 9$, we provide below two STS(9). Those are actually disjoint *Kirkman* triple systems (see the definition below).

Example 1 *Two disjoint Kirkman Triple Systems for $v = 9$:*

$$\begin{array}{lllll} K_A: & \{0,7,8\} & \{0,2,5\} & \{0,3,4\} & \{0,1,6\} \\ & \{1,2,4\} & \{1,3,8\} & \{1,5,7\} & \{2,3,7\} \\ & \{3,5,6\} & \{4,6,7\} & \{2,6,8\} & \{4,5,8\} \end{array} \qquad \begin{array}{lllll} K_B: & \{1,7,8\} & \{1,3,6\} & \{1,4,5\} & \{0,1,2\} \\ & \{2,3,5\} & \{2,4,8\} & \{2,6,7\} & \{3,4,7\} \\ & \{0,4,6\} & \{0,5,7\} & \{0,3,8\} & \{5,6,8\} \end{array}$$

Using directly Steiner Triple Systems provides some sporadic values of v and b for which there exist well balanced families. We can get more values of b by considering more than one STS(v); but we have to ensure that the family is 3-balanced (that is no block is repeated). Fortunately the answer can be obtained due to the existence of families of disjoint STS(v) (see Theorem 4 below). Two STS(v) are said to be disjoint if they have no triple in common. A set of $v-2$ disjoint STS(v) is called a *large set of disjoint STS(v)* and briefly denoted by LSTS(v). An LSTS(v) can be viewed as a partition of the complete family of $\binom{v}{3}$ triples into STS(v). In 1850, Cayley showed that there are only two disjoint STS(7) and so there is no LSTS(7). The same year Kirkman showed that there exists an LSTS(9). Such an LSTS(9) is given by taking as first STS(9) the K_A of Example 1; the 6 other STS(9) are obtained from the first one by developing modulo 7 (that is applying the automorphism fixing 7 and 8 and mapping i to $i+1$). For example, the second STS(9) is obtained by adding 1 to each number (7 and 8 are invariant and $6+1=0 \pmod 7$) and is given in Example 1 as K_B.

Due to the efforts of many authors the following theorem completely settles the existence of LSTS(v).

Theorem 4 *([15, 16, 18] (see [14] for a simple proof)) For $v \equiv 1$ or $3 \pmod 6$, $v > 7$, there exists an LSTS(v).*

Proposition 8 *Let $k=3$, and $v \equiv 1$ or $3 \pmod 6$, $v > 7$, then there exists a well balanced family for any b multiple of $\frac{v(v-1)}{6}$.*

Proof. Let $b = h\frac{v(v-1)}{6}$; $b \leq \binom{v}{3}$ or equivalently, $h \leq v-2$. According to Theorem 4, there exists an LSTS(v), formed of $v-2$ disjoint STS(v). Then, the family consisting of any h disjoint STS(v), extracted from the LSTS(v), is well balanced (with $\lambda_{x,y} = h$ and $\lambda_x = h\frac{v-1}{2}$). For $b \geq \binom{v}{3}$ the result follows by using Proposition 3. □

When $v = 6t+3$, there exist STS(v) which have a stronger property. The triples of the STS(v) can themselves be partitioned into $3t+1$ classes, called *parallel classes*, where a parallel class consists of $2t+1$ blocks forming a partition of the v elements. Such an STS(v) is called *resolvable* or a Kirkman Triple System (briefly KTS(v)). Example 1 gives two KTS(9) where the 4 parallel classes correspond to the 4 columns. It is well-known that a KTS(v) exists for any $v \equiv 3 \pmod 6$ [17].

In our next constructions, we will need families of disjoint STS(v) containing a KTS(v). The existence of mixed STS/KTS structures has not been specifically studied in the literature and we propose some conjectures about them (Conjectures 3, 4, and 5). However we can use results on families of disjoint KTS, which have indeed been studied for a long time. A set of $v-2$ disjoint KTS(v) is called a *large set of disjoint KTS(v)* and briefly denoted by LKTS(v). As mentioned previously, Kirkman showed in 1850 that an LKTS(9) exists and in 1974, Denniston found an LKTS(15). For $v=9$, the LSTS(9) described above, is in fact an LKTS(9) as the resolvabilty is conserved by automorphisms. An example of a KTS(15) denoted K_A is given in Example 2 in appendix. Developing modulo 13, that is, applying the automorphism fixing 13 and 14 and mapping i to $i+1$, we get 13 disjoint KTS(15) and so an LKTS(15). Example 2 shows also $K_B = K_A + 1$.

Since then, many people have done some research on their existence. The most recent paper is [20] where the reader can find other references. The results to date are summarized in the following theorem:

Theorem 5 *[20, Theorems 1.1 and 3.3]*

(a) For any integer $r \in \{1,7,11,13,17,35,53,67,91,123\} \cup \{2^{2p+1}25^q : p,q \geq 1\}$, there exists an LKTS($v$) for $v = 3^a 5^b r \prod_{i=1}^{s}(2\cdot 13^{n_i}+1)\prod_{j=1}^{t}(2\cdot 7^{m_j}+1)$, $a, n_i, m_j \geq 1$ ($1 \leq i \leq s, 1 \leq j \leq t$), $b, s, t \geq 0$ and further $a+s+t \geq 2$ if $b \geq 1$ and $r \neq 1$.

(b) There exists an LKTS($3v$) for $v = \prod_{i=1}^{s}(2q_i^{n_i}+1)\prod_{j=1}^{t}(4^{m_j}-1)$ where $s+t \geq 1$, $n_i, m_j \geq 1$, $q_i \equiv 7 \pmod{12}$ and q_i is a prime power.

6.3 Case $v = 6t+3$

As written above, a full solution for that case has been given in [19]. However, we propose here a simple and explicit construction which gives the answer for $v = 6t+3$, when there exist families of disjoint STS(v), at least one of them being a KTS(v).

Proposition 9 *Let $k=3$ and $v = 6t+3$. If there exists a family of $3t+1$ disjoint STS(v), one of them being a KTS(v), then there exists a well balanced family for any b.*

Proof. By Propositions 3 and 5, we can suppose $b \leq \frac{1}{2}\binom{v}{3} = (2t+1)(3t+1)\frac{6t+1}{2}$. Let the number of blocks be $b = q(2t+1)(3t+1) + r(2t+1) + s$ with $0 \leq q \leq 3t; 0 \leq r < 3t+1; 0 \leq s < 2t+1$. Then a well balanced family for b consists of q disjoint STS(v) taken from the family avoiding the singled-out KTS(v), plus r parallel classes of the KTS(v) and s triples of the $(r+1)$th parallel class of this KTS(v). Indeed, by assumption on the family, all the triples are disjoint and so $\lambda_{x,y,z} = 0$ or 1. In each STS(v) a pair of elements appears exactly once; so $\lambda_{x,y} = q$ or $q+1$ (exactly q if $r = 0, s = 0$). In each parallel class of the KTS(v), each vertex appears exactly once; so $\lambda_x = (3t+1)q + r$ or $(3t+1)q + r + 1$ (exactly $(3t+1)q + r$ if $s = 0$). $\square$

Proposition 9 can be applied when there exists an LKTS(v). There is no need to have a structure as strong as this, but only $3t+1$ disjoint STS, with one of them being a KTS. We conjecture that such a structure always exists for $v = 6t+3$; this conjecture will imply Conjecture 2 for $v \equiv 3 \pmod 6$.

Conjecture 3 *For $v = 6t+3$, there exist $3t+1$ disjoint STS(v) one of them being a KTS(v).*

The following stronger conjecture is also interesting.

Conjecture 4 *For $v = 6t+3$, there exist an LSTS(v) such that one of its STS(v) is a KTS(v).*

6.4 Constructions for $v = 6t + 4$

In this section we present briefly construction techniques for the case $v = 6t+4$ (see [4] for more details and other constructions). We illustrate them for $v = 10$ (Proposition 11) and for $v = 16$ (Proposition 12, proved in the appendix) verifying Conjecture 2 for these values.

6.4.1 Splitting Process: Construction A

The following construction applies to a family containing a KTS($6t+3$) and adds $2(2t+1)$ blocks to it. It consists in "splitting" triples using an extra element.

Construction A. Consider a parallel class of a KTS($6t+3$) and a new element α ($= 6t+4$) and replace each of the $2t+1$ triples $\{x_j, y_j, u_j\}$ of this class ($1 \le j \le 2t+1$) with the 3 triples $\{x_j, y_j, \alpha\}$, $\{x_j, u_j, \alpha\}$, and $\{y_j, u_j, \alpha\}$.

For example take the KTS(9) K_A. We replace the first class consisting of the 3 blocks $\{0,7,8\}$, $\{1,2,4\}$, $\{3,5,6\}$ with the 9 blocks $\{0,7,\alpha\}$, $\{0,8,\alpha\}$, $\{7,8,\alpha\}$, $\{1,2,\alpha\}$, $\{1,4,\alpha\}$, $\{2,4,\alpha\}$, $\{3,5,\alpha\}$, $\{3,6,\alpha\}$, $\{5,6,\alpha\}$. We keep the other blocks of K_A unchanged and add the triples of K_B and so get a well balanced family for $v = 10$ and $b = 30$, where each element appears exactly 9 times and each pair appears exactly twice.

Proposition 10 *Let $k = 3$ and $v = 6t+4$. If there exist, for $p \le 3t+1$, $min(2p, 6t)$ disjoint STS($6t+3$) one of them being a KTS($6t+3$), then there exists a well balanced family for $b_{2p} = 2p(3t+2)(2t+1)$.*

Proof. We apply Construction A for p classes of the KTS K_A by adding a new element α. As the classes are taken in the same KTS, α appears in $3p(2t+1)$ disjoint triples; so $\lambda_\alpha = 3p(2t+1)$. Furthermore each pair $\{\alpha, x\}$ appears exactly $2p$ times; so $\lambda_{\alpha,x} = 2p$. Then, for $1 \le p \le 3t$, we add to this modified K_A, $(2p-1)$ STS($6t+3$), that exist by hypothesis. Any $x \ne \alpha$ appears $3t+1$ times in each of these STS and $3t+1+p$ times in the modified K_A; so, $\lambda_x = 2p(3t+1) + p = 3p(2t+1)$. Each pair $\{x, y\}$ ($x \ne \alpha, y \ne \alpha$) appears exactly once in the modified K_A and in each of the other $2p-1$ STS; so, $\lambda_{x,y} = 2p$. Therefore the family constructed is well balanced. For $p = 3t+1$, the result was already known, as the family obtained is a complete family with $b_{6t+2} = (6t+2)(3t+2)(2t+1) = \binom{6t+4}{3}$. So only $2p = 6t$ disjoint STS($6t+3$) are needed. $\square$

6.4.2 Addition Process: Construction B

We can extend Proposition 10 to get well balanced families for more values of b either by deleting or adding blocks. We just present here the addition process for the case $v = 10$ and $v = 16$. For the general case, we need the existence of a second disjoint KTS($6t+3$) K_B and of other disjoint STS (which is ensured if there exists an LKTS(v) which is the case for $v = 9$ and 15). We conjecture that such a structure always exists.

Conjecture 5 *For $v = 6t+3$, there exists an LSTS(v) such that two of its STS(v) are KTS(v).*

We start with the well balanced family obtained in Construction A for b_{2p} by using K_A and by choosing the other $2p-1$ STS$(6t+3)$ to be different from K_B and do repetitions of the following construction B.

Construction B. Choose a class C of K_B, replace a block $\{x, y, z\}$ with the block $\{x, y, \alpha\}$ and add some of the other $2t$ blocks of this class. This construction can be combined with Construction A as long as $\{x, y\}$ is not a pair appearing in a modified block of K_A (otherwise the block $\{x, y, \alpha\}$ will be repeated).

For $v = 10$, we use the two disjoint KTS(9) K_A and K_B given in Example 1. We do Construction A with p classes of K_A, getting a solution for $b_{2p} = 30p$. We first add the blocks of the 4th class of K_B. Now, λ_α is 1 behind the rest. Then we replace the block $\{x_1, y_1, z_1\} = \{0, 6, 4\}$ of the first class C_1 of K_B with $\{0, 6, \alpha\}$ and add this modified block and the two other blocks $\{1, 7, 8\}$ and $\{2, 3, 5\}$. Now λ_α and λ_4 are 1 behind the rest. Note that the pair $\{x_1, y_1\} = \{0, 6\}$ appears in the block $\{0, 1, 6\}$ of the 4th class of K_A (class C_A). Therefore we will not modify this class in Construction A. We have $z_1 = 4$, which appears in the block $\{4, 5, 8\}$ of the 4th class of K_A. So, we choose $x_2 = 8$ and replace the block $\{x_2, y_2, z_2\} = \{8, 4, 2\}$ of the second class C_2 of K_B with $\{4, 8, \alpha\}$ and add the two other blocks $\{1, 3, 6\}$ and $\{0, 5, 7\}$; here $z_2 = 2$. Note that the pair $\{x_2, y_2\} = \{4, 8\}$ appears in the block $\{4, 5, 8\}$ of the 4th class C_A of K_A. At that point we have got well balanced families for $30p \le b \le 30p+9$, for $p = 0, 1, 2, 3$. We cannot go further when $p = 3$.

For $p < 3$ we get a solution for $b = 30p + 10$ as follows. We add to the solution obtained for $b = 30p$: the blocks of the first class C_1 of K_B replacing the block $\{x_1, y_1, z_1\} = \{0, 6, 4\}$ by $\{0, 6, \alpha\}$; the blocks of the second class C_2 of K_B replacing the block $\{x_2, y_2, z_2\} = \{8, 4, 2\}$ by $\{4, 8, \alpha\}$; the blocks of the third class C_3 of K_B replacing the block $\{x_3, y_3, z_3\} = \{2, 7, 6\}$ by $\{2, 7, \alpha\}$ (here $z_3 = y_1 = 6$). Note that we are lucky, as the pairs $\{x_1, y_1\} = \{0, 6\}$, $\{x_2, y_2\} = \{4, 8\}$ appear in the unmodified 4th class C_A of K_A, but also the pair $\{2, 7\}$ (in the triple $\{2, 3, 7\}$). Furthermore, we add the block $\{z_1, z_2, z_3\} = \{2, 4, 6\}$, which appears in the STS $K_C = K_A + 4$, different from K_A (it is obtained by adding 4 to $\{0, 2, 5\}$). That works as $p < 3$ and so we can choose the $2p$ STS$(6t+3)$ used in the proof of Proposition 10 to be K_A and $2p-1$ STSs different from K_C. Note that all the λ_x are equal. Finally, we can add the blocks of the 4th class of K_B to get solutions $b_{2p} = 30p + 11 \le b \le 30p + 13 = b_{2p} + 13$.

Construction B has allowed us to cover values of b such that $b_{2p} = 30p \le b \le 30p + 13 = b_{2p} + 13$ with $p = 0, 1, 2$, or $30p \le b \le 30p + 9$ with $p = 3$, and by complementation the values $30p' - 9 \le b \le 30p'$ and, for $p' \ge 1$, $30p' - 13 \le b \le 30p'$.

In summary, we get all the values except $b \equiv 14, 15, 16$ *(mod* 30*), which we know by Proposition 6 no well balanced family can exist and* $b = 17, 18, 19, 20$ *(and* $b = 100, 101, 102, 103$*), for which we will prove the existence of a well balanced family later (Construction C).*

6.4.3 Construction C

We take the blocks of an STS(v), $v \equiv 1$ or 3 (mod 6). We choose $\frac{v+1}{2}$ pairs $\{x_i, y_i\}$ $(0 \le i \le \frac{v-1}{2})$ covering all the elements. So, as v is odd, each element is covered once, except one x_0 which is covered twice. Then, we add the $\frac{v+1}{2}$ blocks $\{x_i, y_i, \alpha\}$. Doing so we get a well balanced family for $v+1$ and $b = \frac{v(v-1)}{6} + \frac{v+1}{2}$; indeed $\lambda_x = \frac{v+1}{2}$ except $\lambda_{x_0} = \frac{v+1}{2} + 1$ and $\lambda_{x,y} = 1$ except for the $\frac{v+1}{2}$ chosen pairs and $\{x_0, \alpha\}$ for which the value is 2. Then we can continue adding h disjoint blocks $(1 \le h \le \frac{v}{3})$ as long as they are not in the STS(v), do not contain x_0 and do not contain one of the pairs for which the value $\lambda_{x,y} = 2$. We can continue the process as long as we keep the balance.

More generally, when $v + 1 = 6t + 4$, we apply Construction C starting with some STS(v) and choosing the $\frac{v+1}{2} = 3t + 2$ covering pairs in a small number of classes (only 2 if possible) of another KTS$(6t+3)$. Then we can add the h blocks of a non used class replacing the block $\{x_0, y_0, z_0\}$ containing the x_0 which is repeated twice by the block $\{\alpha, y_0, z_0\}$. We get all the values of b such that $(3t+1)(2t+1) + 3t + 2 \le b \le (3t+1)(2t+1) + 5t + 3 = (6t+4)(t+1)$. Note that we have, for $b = (6t+4)(t+1)$: $\lambda_x = 3(t+1)$ and $\lambda_{x,y} = 1$ or 2. We can also mix Construction C with Construction A as long as the pairs containing α are not in a modified class of K_A. We can then continue adding a new class with a block modified and so on like we did in Construction B.

Let us now show how Construction C gives the missing values $b \in \{17, 18, 19, 20\}$ for $v = 10$.

We choose as first STS(9) K_A and pick the pairs in the KTS K_B given in Example 1. We add the triples $\{1, 8, \alpha\}$, $\{3, 5, \alpha\}$, $\{0, 6, \alpha\}$, obtained with pairs appearing in the first class of K_B. We also add $\{2, 4, \alpha\}$, $\{0, 7, \alpha\}$ using pairs appearing in the second class of K_B. We get a well balanced family for $b = 12 + 5 = 17$. Here $\lambda_x = 5$ except $\lambda_0 = 6$, as 0 appears in two added blocks; $\lambda_{x,y} = 1$ except for the 6 pairs $\{1, 8\}$, $\{3, 5\}$,

$\{0,6\}$, $\{2,4\}$, $\{0,7\}$, and $\{0,\alpha\}$. Then, we can add the 2 blocks of the 3rd class $\{1,4,5\}$ and $\{2,6,7\}$ and the block $\{3,8,\alpha\}$. Therefore we get the missing values $17 \leq b \leq 20$. Note that for $b = 20$, $\lambda_x = 6$ and $\lambda_{x,y} = 1$ or 2, as a pair appears in exactly one block of K_B.

So, we have completely solved the case $v = 10$, as summarized in the following proposition:

Proposition 11 *For $v = 10$ conjecture 2 is verified; that is there exists a well balanced family for all b, except $b \equiv 14, 15, 16$ (mod 30) for which such a family cannot exist.*

We also completely solve the case $v = 16$. The proof of Proposition 12 is given in Appendix 1.

Proposition 12 *For $v = 16$, Conjecture 2 is verified; that is there exists a well balanced family for all b except $b \equiv 38, 39, 40, 41, 42$ (mod 80) for which such a family cannot exist.*

6.5 Other tools

We can also obtain results for other congruences of v.

Proposition 13 *Let $k = 3$ and $v = 6t > 6$ (resp. $v = 6t + 2$). There exists a well balanced family for $b = ht(6t - 2)$ (resp. $b = ht(6t + 2)$).*

Proof. Take, as $v + 1 \equiv 1$ or $3 \pmod 6$, the blocks of a set of h disjoint STS$(v + 1)$ and delete all the $h\frac{v}{2}$ blocks containing the element $v + 1$. □

We can extend this construction to other values. As an example, consider $v = 8$ and $b = 12$. We start with the solution obtained before for $b = 8$ by deleting the blocks containing element 8 in KTS(9) K_A of Example 1. Note that $\lambda_{x,y} = 1$ except for the 4 pairs $\{0,7\}$, $\{1,3\}$, $\{2,6\}$, $\{4,5\}$ which are missing. We can add now 4 blocks taken from another KTS(9), for example K_B of Example 1, containing these pairs; namely the blocks $\{0,5,7\}$, $\{1,3,6\}$, $\{2,6,7\}$, $\{1,4,5\}$.

We can also use, instead of triple systems, **packing or covering** with triples. For example, it is known (see [9]), that when $v \equiv 5 \pmod 6$, $K_v - H$, where H is a 2-regular graph can be decomposed into triples when the number of edges is a multiple of 3. In particular, if we take a cycle $H = C_{3h+1}$, $3h + 1 \leq v$, we get a well balanced family for $b = \frac{v(v-1)-6h-2}{6}$. We get more values by taking decompositions of $\lambda K_v - H$, where H is a 2-regular graph (see [5, 6], but one needs to check that there are no repeated triples). Similarly (see [10]), for $v \equiv 5 \pmod 6$, $K_v + H$, where H is a 2-regular graph can be decomposed into triples if the number of edges is a multiple of 3. In particular if we take $H = C_{3h'+2}$, $3h' + 2 \leq v$ we get a well balanced family for $b = \frac{v(v-1)+6h'+4}{6}$. For example, for $v = 11$ we get a well balanced family for $b = 15, 16, 17$ and $b = 20, 21, 22$.

Proposition 14 *Let $k = 3$ and $v = 6t + 5$. Then there exists a well balanced family for $b = \frac{v(v-1)-6h-2}{6}$ with $3h + 1 \leq v$ and $b = \frac{v(v-1)+6h'+4}{6}$ with $3h' + 2 \leq v$.*

Similarly, when $v \equiv 0, 2 \pmod 6$, K_v minus a perfect matching can be decomposed into triples (delete one vertex from an STS$(v+1)$) and so we get a well balanced family for $b = \frac{v(v-2)}{6}$ and when $v \equiv 0 \pmod 6$, K_v plus a perfect matching can be decomposed into triples and so we get a well balanced family for $b = \frac{v^2}{6}$.

6.6 Small values of v

We can apply the preceding techniques and other tools to deal with the small values of v, verifying Conjecture 2 for $v \leq 11$. We give the proofs and technical details in Appendix 2.

Proposition 15 *Let $k = 3$, for $v \leq 11$, there exists a well balanced family for the values of v and b different from that excluded by Propositions 6 and 7. In particular for $v = 7, 9$ there exists a well balanced family for any b.*

7 Case $k > 3$

We can generalize Proposition 6 in different ways. The first one concerns the non existence of 2-balanced families.

Proposition 16 *Let* $\lambda(v-1) = q(k-1) + r$ *with* $0 < r \le k-2$. *If* $\lambda v(v-1) - rv < k(k-1)b < \lambda v(v-1) + (k-1-r)v$, *then there does not exist a 2-balanced family.*

Proof. Note that the number of possible pairs is $\frac{v(v-1)}{2}$ and that a block contains $\frac{k(k-1)}{2}$ pairs. We distinguish 3 cases.

- $k(k-1)b = \lambda v(v-1)$. In that case a 2-balanced family will verify $\lambda_{x,y} = \lambda$ for all pairs $\{x, y\}$ and then we should have $\lambda_x = \lambda\frac{v-1}{k-1}$ which is impossible as $r \neq 0$ (non existence of a (v, k, λ)-design).
- $k(k-1)b < \lambda v(v-1)$. In that case, we cannot have all the $\lambda_{x,y} \ge \lambda$. So we have one of the $\lambda_{x,y} \le \lambda - 1$ and if the family is 2-balanced all the $\lambda_{x,y} \le \lambda$. But, then $\lambda_x \le \lambda\frac{v-1}{k-1}$ and according to the definition of r, $\lambda_x \le \lambda\frac{v-1}{k-1} - \frac{r}{k-1}$. Using Equation 1, $kb = \sum_x \lambda_x \le \lambda\frac{v(v-1)}{k-1} - \frac{rv}{k-1}$. Therefore there does not exist a 2-balanced family if $\lambda v(v-1) - rv < k(k-1)b < \lambda v(v-1)$.
- The case $\lambda v(v-1) < k(k-1)b < \lambda v(v-1) + (k-1-r)v$ can be handled exactly as the preceding one.

□

We can also generalize Proposition 6 to ensure the non existence of p-balanced families $p > 2$. We give the result for $p = 3$.

Proposition 17 *Let* $\lambda_3(v-2) = q(k-2) + r$ *with* $0 < r \le k-3$. *If* $\lambda_3 v(v-1)(v-2) - rv(v-1) < k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2) + (k-2-r)v(v-1)$, *then there does not exist a 3-balanced family.*

Proof. Note that the number of possible triples is $\frac{v(v-1)(v-2)}{6}$ and that a block contains $\frac{k(k-1)(k-2)}{6}$ triples. We distinguish 3 cases.

- $k(k-1)(k-2)b = \lambda_3 v(v-1)(v-2)$. In that case a 3-balanced family will verify $\lambda_{x,y,z} = \lambda_3$ for all triples $\{x, y, z\}$ and then we should have $\lambda_{x,y} = \lambda_3\frac{v-2}{k-2}$ which is impossible as $r \neq 0$ (non existence of a (v, k, λ_3) 3-design).
- $k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2)$. In that case we cannot have all the $\lambda_{x,y,z} \ge \lambda_3$. So we have one of the $\lambda_{x,y,z} \le \lambda_3 - 1$ and if the family is 3-balanced all the $\lambda_{x,y,z} \le \lambda_3$. But, then $\lambda_{x,y} \le \lambda_3\frac{v-2}{k-2}$ and according to the definition of r, $\lambda_{x,y} \le \lambda_3\frac{v-2}{k-2} - \frac{r}{k-2}$. Using Equation 1, $\frac{k(k-1)}{2}b = \sum_{xy} \lambda_{x,y} \le \lambda_3\frac{v(v-1)(v-2)}{2(k-2)} - \frac{rv(v-1)}{2(k-2)}$. Therefore there does not exist a 3-balanced family if $\lambda_3 v(v-1)(v-2) - rv(v-1) < k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2)$.
- The case $\lambda_3 v(v-1)(v-2) < k(k-1)(k-2)b < \lambda_3 v(v-1)(v-2) + (k-2-r)v(v-1)$ can be handled exactly as the preceding one.

□

We could also get a similar result for a 3-balanced family by using the values of λ_x but the result is in fact a consequence of Propositions 16 and 17.

Consider for example $k = 4$ and $v = 9$. By Proposition 16, with $\lambda = 1$ there does not exist a 2-balanced family for $b = 5, 6$ and with $\lambda = 2$ for $b = 12, 13$; and more generally for $b \equiv 5, 6, 12, 13 \pmod{18}$. By Proposition 17, with $\lambda_3 = 1$ there does no exist a 3-balanced family for $b = 19, 20, 21, 22, 23$ and with $\lambda_3 = 3$ for $b = 61, 62, 63, 64, 65$ and more generally for $b \equiv 19, 20, 21, 22, 23 \pmod{42}$.

On the constructive side we have seen in Section 2 that a $(v, k, \lambda)(k-1)$-design is a well balanced family. Recall that a (v, k, λ) t-design is a family of blocks of size k such that each t-element subset appears in exactly λ blocks. When $t = k-1$ and $\lambda = 1$ a $(v, k, 1)$ $(k-1)$-design is also called a Steiner System $S(k-1, k, n)$. For $k = 3$ we have the classical STS(v).

For $k = 4$ it has been proved that a $(v, 4, 1)$ 3-design also called a quadruple system SQS(v) exists if and only if $v \equiv 2$ or $4 \pmod 6$ [12]. For larger values of λ see for example Table 4.37 on page 82 of [8]. For $k \ge 5$

only few Steiner systems are known (see Chapter II.5 of [8]), such as the $(12, 6, 1)$ 5-design and the $(11, 5, 1)$ 4-design obtained by deleting an element, the $(24, 6, 1)$ 5-design and the $(23, 5, 1)$ 4-design.

Similar techniques as those used for $k = 3$ can be used for small values of v to obtain well balanced families for $k = 4$. We can also use resolvable designs. For $k = 4$ and $v \equiv 4$ or 8 (mod 12), there exist resolvable Kirkman Quadruple Systems, that is $(v, 4, 1)$ 3-design such that the quadruples can themselves be partitioned into $\frac{(v-1)(v-2)}{6}$ parallel classes, each consisting of $\frac{v}{4}$ blocks forming a partition of the v elements. We can also use disjoint SQS(v). Two SQS(v) are said to be disjoint if they have no quadruple in common. Similarly to STS(v), a set of $v-3$ disjoint SQS(v) is called a *large set of disjoint SQS(v)* and briefly denoted by LSQS(v). Unfortunately no such system has been shown to exist. However in [11] $v-5$ disjoint quadruple systems have been exhibited when $v = 5 \cdot 2^p$.

Finally, it will be nice to prove a lemma analogous to that used for $k = 2$ in Section 4 and $k = 3$ (see [19]), showing that one can modify a 4 and 3 and 2-balanced family to obtain a well balanced family and more generally that one can modify a 4 and 3-balanced family to obtain a well balanced family.

8 Conclusion

In this article we attack a conjecture (Conjecture 1) coming from a data placement problem. In this process, we introduce a new class of combinatorial objects, called well balanced families, which generalize classical designs. We give constructions of well balanced families of triples and propose Conjecture 2 which has been recently proved to be true. In some cases direct constructions will follow from some conjectures on disjoint Steiner Triple Systems which are of interest in themselves (Conjectures 3, 4, 5) and we hope that this paper will motivate new research in design theory.

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Appendix 1: case $v = 16$ (proof of Proposition 12)

For $v = 16$ we will use the two disjoint KTS(15) of Example 2, denoted respectively K_A and K_B. Recall that an LKTS(15) is obtained by developing K_A modulo 13, that is, applying the automorphism fixing 13 and 14 and mapping i to $i + 1$, therefore getting 13 disjoint KTS(15).

Example 2 *Two disjoint Kirkman Triple Systems for $v = 15$:*

K_A :	$\{0,1,9\}$	$\{0,2,7\}$	$\{0,3,11\}$	$\{0,4,6\}$	$\{0,5,8\}$	$\{0,10,12\}$	$\{1,4,5\}$
	$\{2,4,12\}$	$\{3,4,8\}$	$\{1,7,12\}$	$\{1,8,11\}$	$\{1,2,3\}$	$\{3,5,9\}$	$\{2,6,11\}$
	$\{5,10,11\}$	$\{5,6,12\}$	$\{6,8,10\}$	$\{2,9,10\}$	$\{6,7,9\}$	$\{4,7,11\}$	$\{3,7,10\}$
	$\{7,8,13\}$	$\{9,11,13\}$	$\{2,5,13\}$	$\{3,12,13\}$	$\{4,10,13\}$	$\{1,6,13\}$	$\{8,9,12\}$
	$\{3,6,14\}$	$\{1,10,14\}$	$\{4,9,14\}$	$\{5,7,14\}$	$\{11,12,14\}$	$\{2,8,14\}$	$\{0,13,14\}$

K_B :	$\{1,2,10\}$	$\{1,3,8\}$	$\{1,4,12\}$	$\{1,5,7\}$	$\{1,6,9\}$	$\{0,1,11\}$	$\{2,5,6\}$
	$\{0,3,5\}$	$\{4,5,9\}$	$\{0,2,8\}$	$\{2,9,12\}$	$\{2,3,4\}$	$\{4,6,10\}$	$\{3,7,12\}$
	$\{6,11,12\}$	$\{0,6,7\}$	$\{7,9,11\}$	$\{3,10,11\}$	$\{7,8,10\}$	$\{5,8,12\}$	$\{4,8,11\}$
	$\{8,9,13\}$	$\{10,12,13\}$	$\{3,6,13\}$	$\{0,4,13\}$	$\{5,11,13\}$	$\{2,7,13\}$	$\{0,9,10\}$
	$\{4,7,14\}$	$\{2,11,14\}$	$\{5,10,14\}$	$\{6,8,14\}$	$\{0,12,14\}$	$\{3,9,14\}$	$\{1,13,14\}$

Construction B. We start with the solution obtained for $b = 80p$ in the proof of Proposition 10 by using Construction A with the new element α. We suppose here that the 7th class of K_A is not modified in construction A. We first add the blocks of the third class of K_B. Now λ_α is 1 behind the rest. Then we apply Construction B, by replacing the block $\{8, 9, 13\}$ of the first class of K_B by $\{8, 9, \alpha\}$ ($x_1 = 8$, $y_1 = 9$, and $z_1 = 13$) and adding the other blocks of this class. Now λ_α and λ_{13} are 1 behind the rest. Then we add the blocks of the 4th class of K_B replacing the block $\{0, 4, 13\}$ by $\{0, 13, \alpha\}$ ($x_2 = 0$, $z_2 = 4$). So we get a well balanced family for $80p \leq b \leq 80p + 15$ for $p \leq 6$. For $p = 6$ we cannot get further.

For $p < 6$, we get a solution for $b = 80p + 16$ as follows. We start with the solution obtained for $b = 80p$. We add the blocks of the first class of K_B replacing the block $\{8, 9, 13\}$ by $\{8, 9, \alpha\}$; the blocks of the 4th class of K_B replacing the block $\{0, 4, 13\}$ by $\{0, 13, \alpha\}$; the blocks of the second class of K_B replacing the block $\{4, 5, 9\}$ by $\{4, 5, \alpha\}$, ($x_3 = 5$, $z_3 = 9$). Note that the pairs $\{8, 9\}$, $\{0, 13\}$, $\{4, 5\}$ are in the same class of K_A, namely the 7th class. Finally, we add the block $\{z_1, z_2, z_3\} = \{4, 9, 13\}$ which appears in the KTS $K_A + 3$ as translated from the block $\{1, 6, 13\}$ (recall that 13 is invariant).

Furthermore we can add the blocks of the third class of K_B to obtain a well balanced family for $80p+17 \leq b \leq 80p + 21$ for $p < 6$.

At that point we use a variant of Construction B; indeed instead of choosing the pair $\{x_4, y_4\}$ in a class of K_B different from one already used, we can choose it in a modified class and add all the blocks of another class to keep the balance. More precisely, we replace the block $\{3, 10, 11\}$ of the 4th class by $\{3, 10, \alpha\}$ ($z_4 = 11$). At that point λ_{11} is one behind the rest. Then we add the blocks of the 5th class of K_B, but starting with the block containing 11. At that point λ_{11} and λ_α are one behind the rest. We now replace $\{6, 11, 12\}$ of the first class by $\{6, 11, \alpha\}$ ($z_5 = 12$). The advantage is that the pairs $\{3, 10\}$ and $\{6, 11\}$ are still in the 7th class of K_A. So we get a well balanced family for all $80p \leq b \leq 80p + 26$ (we have to choose in K_A not to modify the 7th class in Construction A). At that point λ_{11} and λ_{12} are one behind the rest. Then we can add the block $\{z_4, z_5, z_6\} = \{3, 11, 12\}$ which appears in $K_A + 8$ as translated from the block $\{8, 3, 4\}$. So now λ_3 is one ahead of the rest. We can then replace the block $\{3, 7, 12\}$ of the 7th class of K_B by $\{7, 12, \alpha\}$ ($z_6 = 3$) and then add all the other blocks of the 7th class of K_B. We should not modify in K_A the class containing $\{7, 12\}$ namely the 3rd one. That is possible; indeed, as $p \leq 5$, we can leave 2 classes unmodified in K_A (the 3rd and 7th). Finally we add the blocks of the 6th class of K_B.

In summary we get a well balanced family for $80p \leq b \leq 80p + 37$ for $0 \leq p \leq 5$ and for $p = 6$, only $480 \leq b \leq 495$. Using Proposition 5 and the fact that $\binom{16}{3} = 560$ we get also the values $65 \leq b \leq 80$ and for $p' \geq 2$, $80p' - 37 \leq b \leq 80p'$. So, we get all the values except $b \equiv 38, 39, 40, 41, 42 \pmod{80}$, for which we know by Proposition 6 that no well balanced family can exist, and $43 \leq b \leq 64$ (and $496 \leq b \leq 517$) for which we will use Construction C.

Construction C. We use the Construction C by choosing the STS(15) K_A and by picking the pairs in the KTS K_B. We add the triples $\{0, 5, \alpha\}$, $\{11, 12, \alpha\}$, $\{8, 9, \alpha\}$ and $\{4, 7, \alpha\}$ obtained with pairs appearing in the first class of K_B. We also add $\{1, 3, \alpha\}$, $\{0, 6, \alpha\}$, $\{10, 13, \alpha\}$ and $\{2, 14, \alpha\}$ with pairs appearing in the second class of K_B. We get a well balanced family for $b = 35 + 8 = 43$. Here $\lambda_x = 8$ except $\lambda_0 = 9$, as 0 appears in two pairs. Then, we can add the blocks of the 4th class replacing $\{0, 4, 13\}$ with $\{4, 13, \alpha\}$ and so we get the missing values $43 \leq b \leq 48$. Note that for $b = 48$, $\lambda_x = 9$ and $\lambda_{x,y} = 1$ or 2 as a pair appears exactly in one block of K_B. We then add the blocks of the 3rd class of K_B,with $\{7, 9, 11\}$ replaced by $\{7, 11, \alpha\}$. Then we add the blocks of the 5th class with $\{2, 3, 4\}$ replaced by $\{2, 3, \alpha\}$ starting with the block containing 9. We add the block $\{4, 5, 9\}$ which appears in the second class of K_B which has not been modified. Finally we add the blocks of the 6th class with first $\{5, 8, 12\}$ replaced by $\{8, 12, \alpha\}$. Note that $\lambda_x = 12$ and no pairs appears 3 times as each element appears with α once or twice (case of $0, 2, 3, 4, 7, 8, 11, 12, 13$). Therefore, we get all the values $43 \leq b \leq 64$. So we have completely solved the case $v = 16$.

Appendix 2: Small cases (proof of Proposition 15)

The case $v = 9$ was settled in Proposition 9 and $v = 10$ in Proposition 11. For the other values, by Proposition 3 and Proposition 5 we can consider only the values of $b \leq \frac{1}{2}\binom{v}{3}$.

v = 5. For $v = 5$, as $\binom{5}{3} = 10$ we have to consider only the values of $b \leq 5$.

We have well balanced families for $b = 1$ (one block) and $b = 2$ (two blocks $\{1, 2, 3\}$ and $\{1, 4, 5\}$), but not for $b = 3$ as we have seen in the example of the introduction (see also Proposition 7). However there exists an optimal solution $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{3, 4, 5\}$ 1-balanced but not 2-balanced ($\lambda_{1,2} = 2$ but $\lambda_{1,5} = \lambda_{2,5} = 0$). By Proposition 7, there is no well balanced solution for $b = 4$; an optimal one consists of the blocks $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{3, 4, 5\}$. For $b = 5$ there exists a well balanced solution with $\lambda_x = 3$ and $\lambda_{x,y} = 1$ or 2 and consisting of the 5 blocks $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 5\}$, $\{3, 4, 5\}$, $\{2, 4, 5\}$.

v = 6. For $v = 6$, as $\binom{6}{3} = 20$ we need to consider only the values of $b \le 10$.

For $b = 5$ (and so $b = 15$), there does not exist a well balanced family (Proposition 6). An optimal solution $\mathcal{F}^*$ consists of the 5 blocks: $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 5, 6\}$, $\{2, 5, 6\}$, $\{3, 4, 5\}$ ($\lambda_x = 2$ or 3 and $\lambda_{1,2} = \lambda_{5,6} = 2$ but $\lambda_{3,6} = \lambda_{4,6} = 0$) with $P(\mathcal{F}^*, x) = 4x^2 + 16x$ as associated polynomial. The proof is obtained by inspection of the different possible cases. Proposition 4 in [13] also allows us to conclude directly for this case.

For the other values of b, we can construct well balanced families as follows. Let $B_1 = \{1, 2, 3\}$, $B_2 = \{4, 5, 6\}$; $C_1 = \{1, 2, 4\}$, $C_2 = \{1, 3, 5\}$, $C_3 = \{2, 3, 6\}$; $D_1 = \{1, 4, 6\}$, $D_2 = \{2, 5, 6\}$, $D_3 = \{3, 4, 5\}$ and $C'_1 = \{1, 2, 5\}$, $C'_2 = \{1, 3, 6\}$, $C'_3 = \{2, 3, 4\}$. Note that the C_i and C'_i (resp. D_i) intersect B_1 (resp. B_2) in three different pairs and B_2 (resp. B_1) in 3 different elements. Solutions are obtained by taking: for $b = 1$, B_1; for $b = 2$, B_1, B_2; for $b = 3$, C_1, C_2, C_3; for $b = 4$, C_1, C_2, C_3, B_2; for $b = 6$, C_1, C_2, C_3, D_1, D_2, D_3; for $b = 7$, C_1, C_2, C_3, D_1, D_2, D_3, B_1; for $b = 8$, C_1, C_2, C_3, D_1, D_2, D_3, B_1, B_2; for $b = 9$, C_1, C_2, C_3, D_1, D_2, D_3, C'_1, C'_2, C'_3; for $b = 10$, C_1, C_2, C_3, D_1, D_2, D_3, C'_1, C'_2, C'_3, B_2.

v = 7. For $v = 7$, as $\binom{7}{3} = 35$, we have to consider only the values of $b \le 17$. Kirkman proved that there exist two disjoint STS(7). The first one consists of the 7 blocks $C_i = \{i, i + 1, i + 3\}$, for $0 \le i \le 6$ and the second one of the 7 blocks $D_i = \{i, i + 2, i + 3\}$, for $0 \le i \le 6$ (indices modulo 7). Let $B_1 = \{1, 2, 3\}$, $B_2 = \{4, 5, 6\}$, $B_3 = \{0, 1, 4\}$, $B_4 = \{0, 2, 5\}$, $B_5 = \{0, 3, 6\}$. Note that these 5 blocks are disjoint from the blocks C_i and D_i. For $b = j$, $1 \le j \le 5$, take the blocks B_i, $1 \le i \le j$. For $b = 7$ take the first STS(7) (that is all the C_i). For $b = 6$ delete one block from the STS(7). For $b = 7 + j$, $1 \le j \le 5$ add to the STS(7) the blocks B_i, $1 \le i \le j$. For $b = 14$ take the two disjoint STS(7) (that is all the C_i and D_i). For $b = 13$ delete one block from one STS(7). For $b = 14 + j$, $1 \le j \le 5$ add to the two disjoint STS(7) the blocks B_i, $1 \le i \le j$.

v = 8. For $v = 8$, as $\binom{8}{3} = 56$ we have to consider only the values of $b \le 28$. By Proposition 6 and 7 there do not exist well balanced families for $b = 9, 10, 18, 19, 27, 28$. For the other values let us construct a well balanced family.

Cases $1 \le b \le 8$. By Proposition 13, we have a solution for $b = 8$, consisting of the 8 blocks obtained by deleting element 8 in the KTS(9) K_A (see Section 6.4) namely: $B_1 = \{1, 2, 4\}$, $B_2 = \{3, 5, 6\}$, $B_3 = \{0, 2, 5\}$, $B_4 = \{4, 6, 7\}$, $B_5 = \{0, 3, 4\}$, $B_6 = \{1, 5, 7\}$, $B_7 = \{0, 1, 6\}$, $B_8 = \{2, 3, 7\}$. For $b = 2q$, $q = 1, 2, 3$, we have a well balanced family by taking the blocks B_j, $1 \le j \le 2q$. For $b = 3$ (resp. $b = 5$) add to B_1, B_2 (resp. B_1, B_2, B_3, B_4) the block $\{0, 1, 7\}$. For $b = 7$ take the blocks B_j, $1 \le j \le 7$.

Cases $11 \le b \le 17$ and $b = 20$. We apply Construction C starting from the STS(7) with the 7 blocks $C_i = \{i, i + 1, i + 3\}$, for $0 \le i \le 6$ (values modulo 7) and adding a new element $\alpha = 7$. For $b = 11$, we consider the 4 covering pairs $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, $\{0, 6\}$ and add to the STS(7), the 4 blocks $E_1 = \{0, 1, \alpha\}$, $E_2 = \{2, 3, \alpha\}$, $E_3 = \{4, 5, \alpha\}$, $E_4 = \{0, 6, \alpha\}$. Note that $\lambda_x = 4$ except $\lambda_0 = 5$ and $\lambda_{x,y} = 1$ except for the covering pairs $\{0, 1\}$, $\{2, 3\}$, $\{4, 5\}$, $\{0, 6\}$, and $\{0, \alpha\}$. Then we can add successively $E_5 = \{1, 3, 5\}$, $E_6 = \{2, 4, 6\}$. At that point $\lambda_x = 5$, except $\lambda_\alpha = 4$ and $\lambda_{x,y} \le 2$. We can still add $E_7 = \{1, 2, \alpha\}$, $E_8 = \{0, 3, 4\}$, $E_9 = \{5, 6, \alpha\}$, and $E_{10} = \{0, 2, 5\}$ getting solutions for $11 \le b \le 17$. For $b = 20$ we add furthermore the 3 blocks $E_{11} = \{1, 4, 6\}$, $E_{12} = \{3, 6, \alpha\}$, $E_{13} = \{0, 4, \alpha\}$. One can note that all these blocks are disjoint from those of the STS.

Cases $21 \le b \le 26$. We will use again Construction C, starting with the two disjoint STS(7) with blocks $C_i = \{i, i+1, i+3\}$, for $0 \le i \le 6$ and the second one with blocks $D_i = \{i, i+2, i+3\}$, for $0 \le i \le 6$ (indices modulo 7). Add the 7 blocks $F_i = \{i, i + 1, \alpha\}$, $0 \le i \le 6$. We get a solution for $b = 21$. Note that $\lambda_x = 8$ except $\lambda_\alpha = 7$ and $\lambda_{x,y} = 2$ except for the pairs $\{i, i + 1\}$ for which it is 3. Then add the blocks $\{0, 4, \alpha\}$, $\{2, 6, \alpha\}$, $\{1, 3, 5\}$ (at that point for $b = 24$, $\lambda_x = 9$) and $\{0, 2, 5\}$, $\{1, 4, 6\}$.

v = 11. We only need to consider $b \le \lfloor \binom{11}{3}/2 \rfloor = 82$. By Proposition 7 there are no solutions for $b = 18, 19, 36, 37, 73, 74$. Solutions for all the other values will be constructed below.

Cases $1 \le b \le 10$. We take the following blocks (in given order): $\{0, 1, 2\}$, $\{3, 4, 5\}$, $\{6, 7, 8\}$, $\{0, 9, 10\}$, $\{2, 5, 8\}$, $\{3, 6, 9\}$, $\{4, 7, 10\}$, $\{1, 5, 9\}$, $\{1, 8, 10\}$, and $\{2, 4, 6\}$.

Case $b = 11$. A solution is obtained with all blocks of the form $\{i, i+1, i+3\}$ (mod 11) for $0 \le i \le 10$.

Cases $12 \le b \le 17$ and $b = 20, 21$. Solutions for $12 \le b \le 17$ are obtained using the results of Section 6.5. However to be complete, we give here explicit solutions. The number of edges in any $K_{11} - C_4 - C_3 - C_3$ is 55 - 10 = 45 hence a multiple of 3. The graph can therefore be decomposed into 15 K_3. For instance with $C_4 = (1, 2, 3, 4, 1)$ and $C_3 = \{5, 6, 9\}$ and $\{7, 8, 10\}$, one such decomposition is $\{0, 1, 7\}$, $\{0, 2, 5\}$, $\{0, 3, 10\}$, $\{0, 4, 9\}$, $\{0, 6, 8\}$, $\{1, 3, 6\}$, $\{1, 5, 10\}$, $\{1, 8, 9\}$, $\{2, 4, 8\}$, $\{2, 6, 7\}$, $\{2, 9, 10\}$, $\{3, 5, 8\}$, $\{3, 7, 9\}$, $\{4, 5, 7\}$, and $\{4, 6, 10\}$. It provides a solution for $b = 15$. Removing successively blocks $\{0, 1, 7\}$, $\{2, 9, 10\}$ and $\{3, 5, 8\}$ yields solutions for $b = 14$, 13 and 12. Solutions for $b = 16$ and $b = 17$ are obtained by adding back the blocks $\{5, 6, 9\}$ and $\{7, 8, 10\}$ to the solution for $b = 15$. Adding the three blocks $\{1, 2, 4\}$, $\{3, 4, 7\}$, and $\{2, 3, 9\}$, then block $\{0, 1, 5\}$, gives solutions for $b = 20$ and 21. In fact, for $b = 20$, this solution is a K_3-covering of $K_{11} + C_5$, where the 5-cycle is $(2, 4, 7, 3, 9)$.

For larger values of b, we adapt the constructions introduced in Section 6.4. We will use the two disjoint KTS(9) K_A and K_B of Example 1 as given at the beginning of Section 6.2.

K_A :	$\{0,7,8\}$	$\{0,2,5\}$	$\{0,3,4\}$	$\{0,1,6\}$	K_B :	$\{1,7,8\}$	$\{1,3,6\}$	$\{1,4,5\}$	$\{0,1,2\}$
	$\{1,2,4\}$	$\{1,3,8\}$	$\{1,5,7\}$	$\{2,3,7\}$		$\{2,3,5\}$	$\{2,4,8\}$	$\{2,6,7\}$	$\{3,4,7\}$
	$\{3,5,6\}$	$\{4,6,7\}$	$\{2,6,8\}$	$\{4,5,8\}$		$\{0,4,6\}$	$\{0,5,7\}$	$\{0,3,8\}$	$\{5,6,8\}$

Note that each column forms a parallel class of the system. Together here $b = 24$, $\lambda_x = 8$ and $\lambda_{xy} = 2$.

Cases $22 \le b \le 33$. We use a construction similar to Construction C. We start with K_B and add the following 10 blocks: $\{1, 7, \alpha\}$, $\{1, 8, \beta\}$, $\{2, 3, \alpha\}$, $\{2, 5, \beta\}$, $\{0, 4, \alpha\}$, $\{0, 6, \beta\}$, $\{5, 6, \alpha\}$, $\{3, 4, \beta\}$, $\{\alpha, \beta, 7\}$ and $\{\alpha, \beta, 8\}$ ($\alpha = 9, \beta = 10$). That gives a solution for $b = 22$. Here $\lambda_x = 6$ and $\lambda_{x,y} = 1$ or 2 (11 pairs). We use the solution to construct solutions for some other values of b: (i) adding the block(s) $\{0, 1, 5\}$, $\{2, 4, 7\}$, and $\{3, 6, 8\}$ gives the solutions for $b = 23, 24$, and 25 (ii) adding the 4 blocks $\{0, 1, 5\}$, $\{2, 4, 7\}$, $\{6, 8, \alpha\}$, and $\{3, 5, \beta\}$ gives a solution for $b = 26$, and then adding $\{1, 4, 6\}$ and $\{0, 2, 8\}$ results solutions for $b = 27, 28$.

Consider the solution for $b = 25$: (i) adding the 4 blocks $\{4, 6, \alpha\}$, $\{5, 8, \alpha\}$, $\{1, 2, \beta\}$, $\{0, 3, \beta\}$, we have a solution for $b = 29$; then adding $\{3, 5, 7\}$ we obtain a solution for $b = 30$, and adding $\{0, 7, 8\}$ a solution for $b = 31$ (ii) adding the 7 blocks: $\{0, 7, 8\}$, $\{1, 2, \alpha\}$, $\{4, 6, \alpha\}$, $\{3, 5, \alpha\}$, $\{1, 6, \beta\}$, $\{0, 2, \beta\}$, and $\{3, 7, \beta\}$ gives a solution for $b = 32$. Adding the block $\{4, 5, 8\}$, we get a solution for $b = 33$.

Cases $33 \le b \le 35$ and $38 \le b \le 44$. We use a construction similar to Construction A. We add two new vertices α and β and replace each block $\{x, y, z\}$ in the first parallel class of K_A with three blocks: $\{x, y, \alpha\}$, $\{x, z, \alpha\}$, and $\{y, z, \alpha\}$, and repeat this operation to the second parallel class of K_A with β. There are $b = 36$ blocks in total now. Now, $\lambda_x = 10$, $\lambda_\alpha = \lambda_\beta = 9$, and $\lambda_{xy} = \lambda_{x\beta} = \lambda_{x\alpha} = 2$ and $\lambda_{\alpha\beta} = 0$. This solution is of course not well balanced (as no well balanced design exists for $b = 36$), but we will use it to construct solutions for $33 \le b \le 35$ and $38 \le b \le 44$: (i) delete the two blocks $\{0, 7, \alpha\}$ and $\{0, 2, \beta\}$, and add the block $\{0, \alpha, \beta\}$. Now, $\lambda_{x,y} = 2$ except $\lambda_{0,7}, \lambda_{0,2}, \lambda_{7,\alpha}, \lambda_{2,\beta}, \lambda_{\alpha,\beta} = 1$, and $\lambda_x = 10$ except λ_0, λ_2, λ_7, $\lambda_\alpha, \lambda_\beta = 9$. This gives a solution for $b = 35$. Furthermore, deleting block $\{1, 3, 6\}$ and then $\{4, 5, 8\}$ from the solution for $b = 35$ gives solutions for $b = 34, 33$. (ii) adding two blocks $\{\alpha, \beta, 0\}$ and $\{\alpha, \beta, 3\}$, we have a solution for $b = 38$. Here $\lambda_x = 10$ except $\lambda_0, \lambda_3, \lambda_\alpha, \lambda_\beta = 11$ and $\lambda_{x,y} = 2$ except $\lambda_{0,\alpha}, \lambda_{0,\beta}, \lambda_{3,\alpha}, \lambda_{3,\beta} = 3$. Now adding blocks $\{1, 2, 5\}$, $\{4, 6, 8\}$, $\{3, 6, 7\}$, $\{0, 1, 8\}$, $\{2, 4, 7\}$, and $\{\alpha, \beta, 5\}$ we have the solutions for $38 \le b \le 44$.

Cases $45 \le b \le 72$ and $b = 75, 76$. The solution for $b = 55$ can be obtained from a $(11, 3, 3)$-design. Here $\lambda_{xy} = 3$ and $\lambda_x = 15$. A solution consists of the 5 classes $\{i, i+1, i+2\}, \{i, i+2, i+4\}, \{i, i+3, i+6\}, \{i, i+4, i+8\}$ $\{i, i+5, i+10\}$ (the values are taken modulo 11).

Let us now introduce a device which is useful to quickly identify pairs in proposed solutions. To a given block $\{a, b, c\}$ we associate its "difference family", the (unordered) list made of the three "smallest" differences between values of a block (a pair $\{a, b\}$ has two possible differences $a - b$ and $b - a$ modulo 11). Note that all the blocks of the class obtained by translating a given block, that is the blocks $\{a+i, b+i, c+i\}$ (values are taken modulo 11), have the same difference family. The converse is not true; for example blocks with difference family 123 can be in the class $\{i, i+1, i+3\}$ or $\{i, i+2, i+3\}$.

The solution above is then generated by the 5 difference families: 112, 224, 335, 443, 551. Note that each difference occurs three times. This solution is now used to obtain solutions for the following values of b.

(i) for $45 \le b \le 54$, just deleting some or all of the 10 blocks for $b = 10$ (note that the difference families of these blocks are in the following set: 112, 224, 335, and 443),

(ii) adding the following 10 blocks gives solutions for $56 \le b \le 65$: $\{0,1,3\}$, $\{4,5,7\}$, $\{2,8,10\}$, $\{5,6,9\}$, $\{3,4,6\}$, $\{0,7,10\}$, $\{1,2,9\}$, $\{0,8,9\}$, $\{2,3,5\}$, and $\{1,7,8\}$ as all these blocks have difference families with no repetition,

(iii) adding a class of 11 blocks with difference family 123, for example the blocks $\{i, i+1, i+3\}$ gives a solution for $b = 66$.

Finally, observe that the blocks in the solutions for $12 \le b \le 17$ and $b = 20, 21$, have difference families different from $jj(2j)$ $(1 \le j \le 5)$, whereas in the solution for $b = 55$, only blocks with difference families $jj(2j)$ are used. Therefore, combining the above solutions with $b = 55$, we have solutions for $67 \le b \le 72$ and $b = 75, 76$.

Cases $77 \le b \le 82$. For $b = 77$, take the following 7 classes: $(i, i+1, i+2)$, $(i, i+2, i+4)$, $(i, i+3, i+6)$, $(i, i+4, i+8)$, $(i, i+5, i+10)$, $(i, i+1, i+3)$, $(i, i+1, i+5)$. The corresponding difference families are $112, 224, 335, 443, 551, 123, 145$. Hence $\lambda_x = 21$, $\lambda_{xy} = 5$ for pairs with difference 1, and 4 for all the pairs with other differences. Now adding some or all the blocks: $\{0,2,5\}$, $\{1,3,7\}$, $\{4,6,9\}$, $\{2,8,10\}$, $\{3,5,8\}$ gives solution for $78 \le b \le 82$.