On the interaction problem between a compressible fluid and a Saint-Venant Kirchhoff elastic structure

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Abstract

In this paper, we consider an elastic structure immersed in a compressible viscous fluid. The motion of the fluid is described by the compressible Navier-Stokes equations whereas the motion of the structure is given by the nonlinear Saint-Venant Kirchhoff model. For this model, we prove the existence and uniqueness of regular solutions defined locally in time. To do so, we first rewrite the nonlinearity in the elasticity equation in an adequate way. Then, we introduce a linearized problem and prove that this problem admits a unique regular solution. To obtain time regularity on the solution, we use energy estimates on the unknowns and their successive derivatives in time and to obtain spatial regularity, we use elliptic estimates. At last, to come back to the nonlinear problem, we use a fixed point theorem.

AMS subject classification: 74F10, 76N10, 74B20

1 Introduction

1.1 Statement of problem

In this paper, we deal with a fluid-solid interaction problem where the fluid is governed by the compressible Navier-Stokes equations and the solid is an hyperelastic structure which fulfills the Saint-Venant Kirchhoff nonlinear model.

Let $T > 0$ be given. We suppose that the structure and the fluid move in a fixed connected bounded domain $\Omega \subset \mathbb{R}^3$. At time t, we denote by $\Omega_S(t)$ the solid domain and by $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ the fluid domain. We suppose that the boundaries of $\Omega_S(0)$ and Ω are smooth (C^4 for instance) and that $\Omega_S(0)$ does not touch the external boundary. The fluid velocity u and the fluid density ρ satisfy the compressible Navier-Stokes equations: $\forall\, t \in (0,T)$, $\forall\, x \in \Omega_F(t)$,

$$\begin{cases} \big(\partial_t \rho + \nabla \cdot (\rho u)\big)(t,x) = 0, \\ \big(\rho \partial_t u + \rho (u \cdot \nabla) u\big)(t,x) - \nabla \cdot \big(2\mu \epsilon(u) + \mu'(\nabla \cdot u)\mathrm{Id} - p\mathrm{Id}\big)(t,x) = 0, \end{cases} \tag{1}$$

where $(\epsilon(u))_{ij} = \frac{1}{2}(\nabla u + \nabla u^t)_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ denotes the symmetric part of the gradient and $\mathrm{Id} \in \mathcal{M}_3(\mathbb{R})$ stands for the identity matrix. We assume that the viscosity coefficients (μ, μ') belong to $\mathbb{R}_+^* \times \mathbb{R}_+$ and that the pressure p only depends on ρ and is given by $p = P(\rho) - P(\overline{\rho})$, for some $P \in C^\infty(\mathbb{R}_+^*)$ and some constant $\overline{\rho} > 0$.

For results concerning the well-posedness and regularity of the Navier-Stokes compressible equations, we refer to the books [28] and [16] and the references therein.

As long as the structure is concerned, its elastic displacement ξ satisfies the Saint-Venant Kirchhoff model (see, for instance, [8]):

$$\partial_t^2 \xi - \nabla \cdot \sigma(\xi) = 0 \quad \text{in } (0,T) \times \Omega_S(0), \tag{2}$$

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where the first Piola-Kirchhoff tensor $\sigma(\xi)$ is given by :

$$\sigma(\xi) := (\mathrm{Id} + \nabla\xi)\left(\lambda(\nabla\xi + \nabla\xi^t + \nabla\xi^t\nabla\xi) + \frac{\lambda'}{2}(2\nabla\cdot\xi + |\nabla\xi|^2)\mathrm{Id}\right).$$

We also assume that the viscosity coefficients (λ, λ') belong to $\mathbb{R}_+^* \times \mathbb{R}_+$. These equations were considered, for instance, in [30] for Neumann boundary conditions and in [17] for Dirichlet boundary conditions.

We now introduce the flow $\chi(t, \cdot) : \Omega_F(0) \to \mathbb{R}^3$ which associates to the lagrangian coordinate of a fluid particle its eulerian coordinate. For all $y \in \Omega_F(0)$, the flow $\chi(\cdot, y)$ satisfies

$$\begin{cases} \partial_t\chi(t, y) = u(t, \chi(t, y)) \quad t \in (0, T), \\ \chi(0, y) = y. \end{cases} \tag{3}$$

Then, we set $\Omega_F(t) := \chi(t, \Omega_F(0))$. Notice that this time-dependent domain is implicitly defined since $u(t, \cdot)$ itself satisfies an equation on $\Omega_F(t)$. This definition allows to make the link between the lagrangian point of view on the structure and the eulerian point of view on the fluid.
The structure and fluid motions are coupled on the interface. Since the fluid is viscous, the velocity at the interface is supposed to be continuous. Moreover, due to the law of reciprocal actions, the normal component of the stress tensors is also supposed to be continuous. Using the flow χ, we can write the normal component of the fluid stress tensor on $\partial\Omega_S(0)$. This way, on $(0, T) \times \partial\Omega_S(0)$, we have

$$\begin{cases} u \circ \chi = \partial_t\xi \\ \mathbb{T}(u, \rho) \circ \chi \operatorname{cof} \nabla\chi\, \mathrm{n} = \sigma(\xi)\mathrm{n}, \end{cases} \tag{4}$$

where n is the outward unit normal defined on $\partial\Omega_S(0)$ and we have denoted

$$\mathbb{T}(u, \rho) := \big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - (P(\rho) - P(\overline{\rho}))\mathrm{Id}\big). \tag{5}$$

Here, in order to simplify the writing, we have used the classical notation

$$(f \circ \chi)(t, y) := f(t, \chi(t, y)) \quad \forall (t, y) \in (0, T) \times \Omega_F(0),$$

for a function f defined in $(0, T) \times \Omega_F(t)$.

The system is complemented with a Dirichlet condition on the external boundary:

$$u = 0 \text{ on } (0, T) \times \partial\Omega. \tag{6}$$

Observe that $(\overline{\rho}, 0, 0)$ is a stationary solution of system (1), (2) and (4)-(6).

Finally, we introduce the initial conditions

$$\rho(0, \cdot) = \rho_0 \text{ in } \Omega_F(0),\ u(0, \cdot) = u_0 \text{ in } \Omega_F(0) \tag{7}$$

and

$$\xi(0, \cdot) = 0 \text{ in } \Omega_S(0),\ \partial_t\xi(0, \cdot) = \xi_1 \text{ in } \Omega_S(0) \tag{8}$$

which satisfy

$$\rho_0 \in H^3(\Omega_F(0)),\ \rho_0 \geqslant \rho_{min} > 0 \text{ in } \Omega_F(0),\ u_0 \in H^6(\Omega_F(0)),\ \xi_1 \in H^3(\Omega_S(0)). \tag{9}$$

To summarize, the system we consider in this paper is the following :

$$\begin{cases} \big(\partial_t \rho + \nabla\cdot(\rho u)\big)(t,x) = 0 & \text{in } \Omega_F(t), \\ \big(\rho\partial_t u + \rho(u\cdot\nabla)u\big)(t,x) - \nabla\cdot\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - p\mathrm{Id}\big)(t,x) = 0 & \text{in } \Omega_F(t), \\ \partial_t^2\xi - \nabla\cdot\sigma(\xi) = 0 & \text{in } \Omega_S(0), \\ u = 0 & \text{on } \partial\Omega, \\ u\circ\chi = \partial_t\xi & \text{on } \partial\Omega_S(0), \\ \mathbb{T}(u,\rho)\circ\chi\,\mathrm{cof}\,\nabla\chi\,\mathrm{n} = \sigma(\xi)\mathrm{n} & \text{on } \partial\Omega_S(0), \\ \rho(0,\cdot) = \rho_0, \quad u(0,\cdot) = u_0 & \text{in } \Omega_F(0), \\ \xi(0,\cdot) = 0, \quad \partial_t\xi(0,\cdot) = \xi_1 & \text{in } \Omega_S(0), \end{cases} \tag{10}$$

where χ is defined by (3).

To deal with the above system, we are going to rewrite the elasticity part in the same spirit as in [17]. For this purpose, let us set

$$c_{i\alpha j\beta}(\nabla\xi) := \lambda(\delta_{\beta i}\delta_{\alpha j} + \delta_{\alpha\beta}\delta_{ij}) + \lambda'\delta_{i\alpha}\delta_{j\beta} + c^{\ell}_{i\alpha j\beta}(\nabla\xi) + c^{q}_{i\alpha j\beta}(\nabla\xi), \tag{11}$$

where $c^{\ell}_{i\alpha j\beta}(\nabla\xi)$ stands for the linear part

$$\begin{aligned} c^{\ell}_{i\alpha j\beta}(\nabla\xi) := {} & \lambda(\delta_{ij}\partial_\beta\xi_\alpha + \delta_{\alpha j}\partial_\beta\xi_i + \delta_{ij}\partial_\alpha\xi_\beta + \delta_{\alpha\beta}\partial_j\xi_i + \delta_{i\beta}\partial_\alpha\xi_j + \delta_{\alpha\beta}\partial_i\xi_j) \\ & + \lambda'(\delta_{i\alpha}\partial_\beta\xi_j + \delta_{\alpha\beta}\delta_{ij}(\nabla\cdot\xi) + \delta_{j\beta}\partial_\alpha\xi_i) \end{aligned} \tag{12}$$

and $c^{q}_{i\alpha j\beta}(\nabla\xi)$ is the quadratic part

$$c^{q}_{i\alpha j\beta}(\nabla\xi) := \lambda(\delta_{ij}(\partial_\beta\xi\cdot\partial_\alpha\xi) + \partial_\beta\xi_i\partial_\alpha\xi_j + \delta_{\alpha\beta}(\nabla\xi_j\cdot\nabla\xi_i)) + \lambda'\left(\frac{1}{2}\delta_{ij}\delta_{\alpha\beta}|\nabla\xi|^2 + \partial_\alpha\xi_i\partial_\beta\xi_j\right). \tag{13}$$

Here and in what follows, ∂_k for $k = i,\alpha,j,\beta \in \{1,2,3\}$ represents the partial derivative with respect to the spatial variable y_k and ∂_t and ∂_s represents the partial derivative with respect to the time variable. We remark that the coefficients $c_{i\alpha j\beta}$ satisfy the following symmetry property :

$$c_{i\alpha j\beta} = c_{j\beta i\alpha},\ \forall i,j,\alpha,\beta = 1,2,3. \tag{14}$$

Then, one can prove that

$$\partial_r(\sigma(\xi))_{i\alpha} = \sum_{j,\beta=1}^{3} c_{i\alpha j\beta}(\nabla\xi)\partial^2_{r\beta}\xi_j \quad \forall i,\alpha = 1,2,3,$$

where r can represent either the time derivative or a spatial derivative. In particular, one deduces

$$(\nabla\cdot\sigma(\xi))_i = \sum_{\alpha,j,\beta=1}^{3} c_{i\alpha j\beta}(\nabla\xi)\partial^2_{\alpha\beta}\xi_j \quad \forall i = 1,2,3.$$

and

$$\sum_{\alpha=1}^{3}(\sigma(\xi))_{i\alpha}n_\alpha = \sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t c_{i\alpha j\beta}(\nabla\xi)\partial^2_{s\beta}\xi_j\,ds\right)n_\alpha \quad \text{on } \partial\Omega_S(0),\ \forall i = 1,2,3,$$

where we have used that $\sigma(0,\cdot) = 0$ on $\partial\Omega_S(0)$.

Taking into account the above considerations, we get the following system :

$$\begin{cases} \big(\partial_t\rho + \nabla\cdot(\rho u)\big)(t,x) = 0 & \text{in } \Omega_F(t), \\ \big(\rho\partial_t u + \rho(u\cdot\nabla)u\big)(t,x) - \nabla\cdot\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - p\mathrm{Id}\big)(t,x) = 0 & \text{in } \Omega_F(t), \\ \partial_t^2\xi_i - \displaystyle\sum_{\alpha,j,\beta=1}^{3} c_{i\alpha j\beta}(\nabla\xi)\partial^2_{\alpha\beta}\xi_j = 0, \quad i=1,2,3 & \text{in } \Omega_S(0), \\ u = 0 & \text{on } \partial\Omega, \\ u\circ\chi = \partial_t\xi & \text{on } \partial\Omega_S(0), \\ \mathbb{T}(u,\rho)\circ\chi\,\mathrm{cof}\,\nabla\chi\,\mathrm{n} = \displaystyle\sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t c_{i\alpha j\beta}(\nabla\xi)\partial^2_{s\beta}\xi_j\,ds\right) n_\alpha & \text{on } \partial\Omega_S(0), \end{cases} \tag{15}$$

complemented with the initial conditions (7)-(8).

Observe that, contrarily to system (10), in system (15) the boundary conditions of the elasticity part do not combine nicely with the elasticity equation. Indeed, in view of the elasticity equation $(15)_3$, it would be natural to have

$$\sum_{\alpha,j,\beta=1}^{3} c_{i\alpha j\beta}(\nabla\xi)\partial_\beta\xi_j\, n_\alpha$$

in the right-hand side of $(15)_6$. In fact, this re-writing of the elasticity equation is the only way we have found to perform a fixed-point argument on the elasticity equation

$$\partial_t^2\xi - \nabla\cdot\sigma(\xi) = 0,$$

regardless of the boundary conditions. This strategy allows us to overcome the difficulties coming from the nonlinearities in $\sigma(\xi)$ and the hyperbolic character of the equation.

Due to this discordance between the boundary conditions and the elasticity equation, we will need to consider an auxiliary problem (see (45) below, where the boundary conditions are the natural ones).

1.2 Compatibility conditions

We will also assume that the following compatibility conditions on the initial data hold :

$$\begin{cases} u_0 = 0 & \text{on } \partial\Omega, \\ u_0 = \xi_1 & \text{on } \partial\Omega_S(0), \\ \mathbb{T}(u_0,\rho_0)n = 0 & \text{on } \partial\Omega_S(0), \\ \nabla\cdot(\mathbb{T}(u_0,\rho_0)) = 0 & \text{on } \partial\Omega_F(0), \\ \mathcal{S}_1 n = (2\lambda\epsilon(\xi_1) + \lambda'(\nabla\cdot\xi_1)\mathrm{Id})n & \text{on } \partial\Omega_S(0), \\ \nabla\cdot(\mathbb{T}_1(U_1) + P'(\rho_0)\rho_0\nabla\cdot u_0\,\mathrm{Id}) = 0 & \text{on } \partial\Omega, \\ U_3 = \nabla\cdot(2\lambda\epsilon(\xi_1) + \lambda'\nabla\cdot\xi_1\mathrm{Id}) & \text{on } \partial\Omega_S(0), \\ \mathcal{S}_2 n = \sigma_1(\xi_1)n & \text{on } \partial\Omega_S(0). \end{cases} \tag{16}$$

In the above identities, we have denoted $\mathbb{T}_1(u) := 2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id}$,

$$\begin{aligned}
&\mathcal{S}_1 := \mathbb{T}_1\left(U_1\right) + (u_0\cdot\nabla)\mathbb{T}(u_0,\rho_0) + P'(\rho_0)\nabla\cdot(\rho_0 u_0)\mathrm{Id} - \mathbb{T}(u_0,\rho_0)\nabla u_0^t,\\
&U_1 := \frac{\nabla\cdot\mathbb{T}(u_0,\rho_0)}{\rho_0} - (u_0\cdot\nabla)u_0,\\
&\mathcal{S}_2 := \mathbb{T}_1(U_2) + (U_1\cdot\nabla)\mathbb{T}(u_0,\rho_0) + 2(u_0\cdot\nabla)(\mathbb{T}_1(U_1) + P'(\rho_0)\nabla\cdot(\rho_0u_0)\mathrm{Id})\\
&\qquad + (u_0\cdot\nabla)[(u_0\cdot\nabla)\mathbb{T}(u_0,\rho_0)] - P''(\rho_0)(\nabla\cdot(\rho_0u_0))^2\mathrm{Id} + P'(\rho_0)\nabla\cdot(\rho_0U_1 - \nabla\cdot(\rho_0u_0)u_0)\mathrm{Id}\\
&\qquad + 2[\mathbb{T}_1(U_1) + P'(\rho_0)\nabla\cdot(\rho_0u_0)\mathrm{Id} + (u_0\cdot\nabla)\mathbb{T}(u_0,\rho_0)]((\nabla\cdot u_0)\mathrm{Id} - \nabla u_0^t)\\
&\qquad + \mathbb{T}(u_0,\rho_0)\left(-\nabla\left(\frac{\nabla\cdot\mathbb{T}(u_0,\rho_0)}{\rho_0}\right)^t + 2\mathrm{cof}\,(\nabla u_0)\right)\\
&U_2 := \frac{1}{\rho_0}\nabla\cdot(\mathbb{T}_1(U_1) + P'(\rho_0)\nabla\cdot(\rho_0u_0)\mathrm{Id}) - (u_0\cdot\nabla)U_1 - (U_1\cdot\nabla)u_0,\\
&U_3 := U_2 + (U_1\cdot\nabla)u_0 + (u_0\cdot\nabla)U_1 + (u_0\cdot\nabla)\left(\frac{\nabla\cdot\mathbb{T}(u_0,\rho_0)}{\rho_0}\right)\\
&\sigma_1(\xi_1) := 2\lambda(\nabla\xi_1^t\nabla\xi_1 + (\nabla\xi_1)^2 + \nabla\xi_1\nabla\xi_1^t) + \lambda'(2(\nabla\cdot\xi_1)\nabla\xi_1 + |\nabla\xi_1|^2\mathrm{Id}).
\end{aligned}$$

Observe that $\left\{\partial_t^i[\mathbb{T}(u,\rho)\circ\chi\,\mathrm{cof}\,\nabla\chi]\,n\right\}|_{t=0}$ coincides with $\mathcal{S}_i n$ on $\{0\}\times\partial\Omega_S(0)$ $(i=1,2)$ and that $\left\{\partial_t^i u\right\}|_{t=0}$ coincides with U_i in $\Omega_F(0)$ $(i=1,2)$. To show that, we have used

$$\partial_t\mathrm{cof}\,(\nabla\chi)|_{t=0} = (\nabla\cdot u_0)\mathrm{Id} - \nabla u_0^t \quad \text{in } \Omega_F(0)$$

and

$$\partial_t^2\mathrm{cof}\,(\nabla\chi)|_{t=0} = \nabla\cdot\left(\frac{\nabla\cdot\mathbb{T}(u_0,\rho_0)}{\rho_0}\right)\mathrm{Id} - \nabla\left(\frac{\nabla\cdot\mathbb{T}(u_0,\rho_0)}{\rho_0}\right)^t + 2\mathrm{cof}\,(\nabla u_0) \quad \text{in } \Omega_F(0).$$

Let us briefly explain how these compatibility conditions are obtained. The first three conditions correspond to (4) and (6) taken at $t=0$. The fourth one corresponds to applying the time derivative to $(4)_1$ and (6) and taking $t=0$ and the fifth one corresponds to applying the time derivative to $(4)_2$ and taking $t=0$. As for the sixth (respectively seventh and eighth) condition, it is obtained by applying the second time derivative to (6) (respectively $(4)_1$ and $(4)_2$) and taking $t=0$.

1.3 State of the art and statement of the main result

Let us present some of the main results concerning the existence and uniqueness of solutions of the Navier-Stokes compressible equations. A first result of existence and uniqueness of local regular solutions was proved in [33]. In the case of isentropic fluids (i.e. when $P(\rho)=\rho^\gamma$ with $\gamma>0$), the papers [23] for $\gamma=1$ and [24] for $\gamma>1$ show the global existence of a weak solution for small initial data. The first global existence result for large data was proved in [27] with $\gamma\geqslant 9/5$ for dimension $N=3$ and with $\gamma>N/2$ for $N\geqslant 4$. The conditions on the coefficient γ have been relaxed in [14] where it is assumed that $\gamma>N/2$ for $N\geqslant 3$. We refer to the books [28] and [16] for additional references on compressible fluids.

Let us also cite several works on the existence and uniqueness of solutions of the Saint-Venant Kirchhoff equations (2). In [30], the author considers these equations in dimension 2 complemented with nonlinear Neumann boundary conditions and establishes a local existence result for small data with a loss of derivatives from the boundary data. In [17], the author proves an existence and uniqueness result of local solutions for a general 3-dimensional system of thermoelasticity with right-hand sides and with homogeneous Dirichlet boundary conditions. In this last reference, there is no loss of regularity with respect to the right-hand side of the elasticity equation.

Different kinds of fluid-structure interaction problems have been studied in the literature.
A large number of studies deal with an incompressible fluid modeled by the incompressible Navier-Stokes equations. For the coupling of an incompressible fluid with a rigid structure, we mention [20] which shows the local in time existence of weak solutions and papers [9] and [12] (with variable density) which prove the

global existence of weak solutions. By 'global existence', we mean that the solution exists until collisions between the structure and the external boundary or between two structures. Paper [31] proves the global existence of weak solutions beyond collisions and [32] proves the existence and uniqueness of strong solutions (global in 2D and local in 3D). At last, [21] and [22] study the lack of collision in 2D or 3D.

For the coupling between an incompressible fluid and an elastic structure, the existence of global weak solutions is proved in [13] when the elastic structure is given by a finite sum of modes and in [4] with a regularizing term in the structure motion. These two results give the existence of solutions defined as long as there is no collision between the structure and the boundary and as long as no interpenetration occurs in the structure. The local existence of regular solutions is proved in [10]. Moreover, the coupling with an elastic plate has also been studied: we quote [1] where the existence of local strong solution is obtained, [7] which proves the existence of global weak solution with a regularizing term in the plate equation and [19] which proves the same result without regularizing term in 2D. Recently, two local existence results of regular solutions have been proved in [29] whenever Ω, $\Omega_S(0)$ and $\Omega_F(0)$ are parallelepipeds and in [26] in the general case. Moreover, the two works [18] and [2] study the existence and uniqueness of steady solutions of incompressible Navier-Stokes equations coupled with the nonlinear Saint-Venant Kirchhoff model.

Concerning compressible fluids, the global existence of weak solutions for the interaction with a rigid structure is obtained in [12] (for $P(\rho) = \rho^\gamma$ and $\gamma \geqslant 2$) and in [15] (for $\gamma > N/2$). Moreover, in [5], the existence of global regular solutions is proved for small initial data.

At last, for the interaction between a compressible fluid and an elastic structure, [3] proves the global existence of a weak solution in $3D$ for $\gamma > 3/2$. The result is obtained for an elastic structure described by a regularized elasticity equation. The local existence and uniqueness of a regular solution of the linear version of our problem (15)-(7)-(8) has been proved in [6] and later in [25].

In the present paper, we prove the local existence and uniqueness of regular solutions for system (15) complemented with the initial conditions (7)-(8).

Definition 1 *Let us introduce some spaces :*

$$X_m^T := L^\infty(0,T;H^m(\Omega_S(0))) \cap W^{m,\infty}(0,T;L^2(\Omega_S(0))),\ 0 \leqslant m \leqslant 4.$$

$$Y_1^T := L^\infty(0,T;L^2(\Omega_F(0))) \cap L^2(0,T;H^1(\Omega_F(0))),$$

$$Y_2^T := L^\infty(0,T;H^2(\Omega_F(0))) \cap H^1(0,T;H^1(\Omega_F(0))) \cap W^{1,\infty}(0,T;L^2(\Omega_F(0))),$$

$$Y_4^T := L^\infty(0,T;H^4(\Omega_F(0))) \cap W^{2,\infty}(0,T;H^2(\Omega_F(0))) \cap W^{3,\infty}(0,T;L^2(\Omega_F(0))) \cap H^3(0,T;H^1(\Omega_F(0))).$$

Remark 2 *Observe that the spaces X_m^T correspond to the hyperbolic scaling. As long as the Y_m^T are concerned, one would expect them to correspond to the parabolic scaling but the strong coupling between the elastic displacement and the velocity of the fluid makes the velocity not as regular as usually.*

More precisely, we will prove the following theorem

Theorem 3 *Let (ρ_0, u_0, ξ_1) satisfy (9) and (16). Then, there exists $T^* > 0$ such that system (15) complemented with the initial conditions (7)-(8) admits a unique solution (ρ, u, ξ) defined in $(0, T^*)$ such that*

$$(\rho\circ\chi, u\circ\chi, \xi) \in Z^{T^*} := (L^\infty(0,T^*;H^3(\Omega_F(0))) \cap W^{3,\infty}(0,T^*;L^2(\Omega_F(0)))) \times Y_4^{T^*} \times X_4^{T^*}$$

and

$$\chi \in W^{1,\infty}(0,T^*;H^4(\Omega_F(0))) \cap W^{4,\infty}(0,T^*;L^2(\Omega_F(0))).$$

Moreover, there exists a function $g : \mathbb{R}_+^3 \to \mathbb{R}_+$ increasing in each variable and satisfying $g(0,0,0) = 0$ such that

$$\|(\rho\circ\chi, u\circ\chi, \xi)\|_{Z^{T^*}} \leqslant g(\|\rho_0 - \overline{\rho}\|_{H^3(\Omega_F(0))}, \|u_0\|_{H^6(\Omega_F(0))}, \|\xi_1\|_{H^3(\Omega_S(0))}).$$

Remark 4 *Observe that we assume that $u_0 \in H^6(\Omega_F(0))$ (see Remark 14) while we are not able to prove that $u\circ\chi \in C^0([0,T^*];H^6(\Omega_F(0)))$. This gap is due to the coupling between equations of different nature.*

To prove this result, we will partially linearize our problem, prove a regularity result for this problem and then use a fixed point argument. In the next subsection, we introduce the intermediate problem which is partially linearized with the help of a given fluid velocity and a given elastic deformation. Comparing our strategy with [11] (which considers a coupling between the incompressible Navier-Stokes equations and a quasilinear elasticity system), we do not need to regularize the elastic displacement equations. Indeed, in that reference the authors add an artificial viscosity term so that the global elasticity-velocity system is parabolic.

1.4 A partial linear problem

Let (ρ_0, u_0, ξ_1) satisfy (9) and (16). We introduce the following notations: for all $t > 0$, we define

$$Q_t = (0,t) \times \Omega_F(0),\ \Sigma_t = (0,t) \times \partial\Omega_S(0).$$

For all $p, r \geqslant 0$ and $q, s \in [1, +\infty]$, we denote by $W^{p,q}(W^{r,s})$ the space $W^{p,q}(0,T; W^{r,s}(\Omega_F(0)))$.

Let us also introduce the following vector fields :

$$u_1 := \frac{\nabla \cdot \mathbb{T}(u_0, \rho_0)}{\rho_0}, \tag{17}$$

$$\begin{aligned} u_2 := \nabla \cdot \big[&\mathbb{T}(u_0,\rho_0)(\nabla \cdot u_0 \mathrm{Id} - \nabla u_0^t) + \mathbb{T}_1(u_1) - \mu((\nabla u_0)^2 + (\nabla u_0^t)^2) - \mu'((\nabla u_0)^2 : \mathrm{Id})\mathrm{Id} \\ &- P'(\rho_0)\rho_0 \nabla \cdot u_0 \mathrm{Id}\big]. \end{aligned} \tag{18}$$

Recall that $\mathbb{T}$ was defined in (5) and $\mathbb{T}_1$ was defined right after (16).

Then, we define the following fixed point space, for all $M > 0$ and all $T > 0$:

$$\begin{aligned} A_M^T = \Big\{ &(v,\xi) \in Y_4^T \times X_4^T,\ v = 0 \text{ on } (0,T) \times \partial\Omega,\ \partial_t^j v(0,\cdot) = u_j(\cdot) \text{ in } \Omega_F(0)\ (j = 0,1,2), \\ &\xi(0,\cdot) = 0,\ \partial_t \xi(0,\cdot) = \xi_1(\cdot) \text{ in } \Omega_S(0) \text{ and } \|v\|_{Y_4^T} \leqslant M,\ \|\xi\|_{X_4^T} \leqslant M \Big\} := (A_M^T)_1 \times (A_M^T)_2. \end{aligned} \tag{19}$$

Let $0 < T < 1$ and let $(\hat{v}, \hat{\xi}) \in A_M^T$ be given with $M > 0$ and $T > 0$ specified later. We will use this data to partially linearize our problem. Let us now define the flow $\hat{\chi}$ by

$$\hat{\chi}(t,y) = y + \int_0^t \hat{v}(s,y)\, ds \quad \forall y \in \Omega_F(0). \tag{20}$$

Direct computations allow to prove several estimates on $\hat{\chi}$ which we present in the following lemma :

Lemma 5 *There exists $C > 0$ and $\kappa > 0$ such that for all $\hat{v} \in (A_M^T)_1$ and all T sufficiently small with respect to M, we have:*

$$\|\hat{\chi}\|_{W^{1,\infty}(H^4) \cap W^{3,\infty}(H^2) \cap W^{4,\infty}(L^2) \cap H^4(H^1)} \leqslant C(1+M) \tag{21}$$

$$\|\nabla\hat{\chi} - Id\|_{W^{1,\infty}(H^3) \cap W^{3,\infty}(H^1) \cap H^4(L^2)} \leqslant CM \tag{22}$$

$$\|cof(\nabla\hat{\chi}) - Id\|_{L^\infty(H^3)} + \|(\nabla\hat{\chi})^{-1} - Id\|_{L^\infty(H^3)} \leqslant CT^\kappa M. \tag{23}$$

Here, and in the following, C represents a constant which only depends on the domains $\Omega_F(0)$ and $\Omega_S(0)$.

Remark 6 *By T small with respect to M, we mean that there exists $\varepsilon > 0$ and $n_0 > 0$ such that $T \leqslant T_0$ with*

$$T_0 := \min\left\{\varepsilon, \frac{\varepsilon}{M^{n_0}}\right\}.$$

In Lemma 5 and all through the paper $\kappa > 0$ denotes a generic constant whose value can change from line to line.

In the sequel we denote $\hat{c}_{i\alpha j\beta}$ instead of $c_{i\alpha j\beta}(\nabla\hat{\xi})$ (see (11) for the definition of $c_{i\alpha j\beta}$). From the definition of $(A_M^T)_2$, it is not difficult to see that the following estimates hold :

Lemma 7 *Let $M > 0$, $T > 0$ and $\hat{\xi}$ be given in $(A_M^T)_2$. Then, there exists $C > 0$ such that for all i, α, j, $\beta \in \{1,2,3\}$, we have*

$$\|c^{\ell}_{i\alpha j\beta}(\nabla\hat{\xi}) + c^{q}_{i\alpha j\beta}(\nabla\hat{\xi})\|_{X_3^T} \leqslant C(M + M^2), \tag{24}$$

where $c^{\ell}_{i\alpha j\beta}(\nabla\xi)$ and $c^{q}_{i\alpha j\beta}(\nabla\xi)$ were defined in (12) and (13), respectively. In particular, for all $B \in \mathcal{M}_3(\mathbb{R})$ we have

$$\sum_{i,\alpha,j,\beta=1}^{3} \hat{c}_{i\alpha j\beta} B_{j\beta} B_{i\alpha} \geqslant \frac{\lambda}{2}|B + B^t|^2 + \lambda'|trB|^2 - CT(M + M^2)|B|^2. \tag{25}$$

Observe that from Lemma 5, $\hat{\chi}(t,\cdot)$ is invertible from $\Omega_F(0)$ onto $\widehat{\Omega}_F(t) = \hat{\chi}(t, \Omega_F(0))$ for all $t \in (0,T)$, for T small enough. Let us state a partially linearized system on the reference domains $\Omega_F(0)$ and $\Omega_S(0)$. First we define, for all $(t,y) \in Q_T$

$$v(t,y) := u(t, \hat{\chi}(t,y)),\ \gamma(t,y) = \rho(t,\hat{\chi}(t,y)) - \overline{\rho}. \tag{26}$$

The first equation in (15) is replaced by

$$\partial_t\gamma + \gamma(\nabla\hat{v}(\nabla\hat{\chi})^{-1} : \mathrm{Id}) + \overline{\rho}(\nabla\hat{v}(\nabla\hat{\chi})^{-1} : \mathrm{Id}) = 0 \text{ in } Q_T. \tag{27}$$

Next, the second equation of system (15) becomes

$$(\overline{\rho} + \gamma)\det\nabla\hat{\chi}\partial_t v - \nabla\cdot\widehat{\mathbb{T}}(v,\gamma) = 0 \text{ in } Q_T, \tag{28}$$

where

$$\widehat{\mathbb{T}}(v,\gamma) := \big(\mu(\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t) + \mu'(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} - (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}. \tag{29}$$

Next, the elasticity equation that we consider is

$$\partial_t^2\xi_i - \sum_{\alpha,j,\beta=1}^{3} \hat{c}_{i\alpha j\beta}\partial^2_{\alpha\beta}\xi_j = 0 \quad i = 1,2,3, \text{ in } (0,T)\times\Omega_S(0). \tag{30}$$

As long as the boundary conditions are concerned, we have

$$v = 0 \quad \text{on } (0,T)\times\partial\Omega \tag{31}$$

and

$$\begin{cases} v = \partial_t\xi, \\ (\widehat{\mathbb{T}}(v,\gamma)\,\mathrm{n})_i = \displaystyle\sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t \hat{c}_{i\alpha j\beta}\partial^2_{s\beta}\xi_j ds\right) n_\alpha,\ i = 1,2,3, \end{cases} \tag{32}$$

on Σ_T.

At last, the initial conditions satisfied by (γ, v) are

$$\gamma(0,\cdot) = \gamma_0 := \rho_0 - \overline{\rho} \text{ in } \Omega_F(0),\ v(0,\cdot) = u_0 \text{ in } \Omega_F(0). \tag{33}$$

We observe that, from the definition of u_1 and u_2 (see (17) and (18) above), we have that

$$\partial_t^j v(0,\cdot) = u_j(\cdot) \text{ in } \Omega_F(0)\,(j = 1,2).$$

In order to prove this, we have used the equations of γ and v and the identities :

$$\partial_t(\det(\nabla\hat{\chi}))(0,\cdot) = \nabla\cdot u_0, \quad \partial_t((\overline{\rho}+\gamma)\det(\nabla\hat{\chi}))(0,\cdot) = 0 \quad \text{and} \quad \partial_t((\nabla\hat{\chi})^{-1})(0,\cdot) = -\nabla u_0 \text{ in } \Omega_F(0).$$

We introduce the following fixed point mapping:

$$\Lambda : \quad (\hat{v},\hat{\xi}) \in A_M^T \quad \to \quad (v,\xi) \tag{34}$$

where (v, ξ), together with γ, is solution of system (27)-(32) with the initial conditions (8) and (33).

Notice that a fixed-point of Λ provides a solution (ρ, u, ξ) of (15) complemented with the initial conditions (7)-(8).

First, we will prove that Λ goes from A_M^T to A_M^T for some $M > 0$ and for some $T > 0$ small enough. This is the main purpose of Section 2. Next, in Section 3 we prove the existence of a Banach space Z such that A_M^T is closed in Z and Λ is a contraction for the Z-norm. This will imply the existence of a unique fixed-point for Λ, which achieves the proof of Theorem 3.

2 Regularity results for the partially linearized problem

In what follows, we denote by C_0 a constant of the type

$$C_0 = g(\|\gamma_0\|_{H^3(\Omega_F(0))}, \|u_0\|_{H^6(\Omega_F(0))}, \|\xi_1\|_{H^3(\Omega_S(0))}), \tag{35}$$

where $g : \mathbb{R}_+^3 \to \mathbb{R}_+$ is increasing in each variable and $g(0,0,0) = 0$.

2.1 Regularity of the density

Since the equation (27) satisfied by γ is decoupled from the other variables v and ξ, we can obtain a first regularity result independently from the other equations.

Lemma 8 *Let $\hat{v} \in (A_M^T)_1$. For T small enough with respect to M and for all $\gamma_0 \in H^3(\Omega_F(0))$, there exists a unique solution γ of (27) and (33)$_1$ $\gamma \in W^{k,\infty}(H^{3-k})$, $0 \leqslant k \leqslant 3$. Moreover, there exists $C_0 > 0$ and $\kappa > 0$ such that*

$$\|\gamma\|_{W^{k,\infty}(H^{3-k})} \leqslant C_0 + T^\kappa M, \forall 0 \leqslant k \leqslant 3. \tag{36}$$

Furthermore, for T small enough with respect to M, there exists $\gamma_{min} > -\overline{\rho}$ such that

$$\gamma \geqslant \gamma_{min} \quad \text{in } Q_T. \tag{37}$$

Proof : Equation (27) can be written as

$$\partial_t \gamma + \gamma \hat{z} = -\overline{\rho} \hat{z} \text{ in } Q_T \tag{38}$$

where $\hat{z} = \nabla \hat{v} (\nabla \hat{\chi})^{-1} : \mathrm{Id}$. Thus, γ is explicitly given by, for all $t \in (0, T)$

$$\gamma(t) = -\overline{\rho} \int_0^t \hat{z}(s) \exp\left(\int_t^s \hat{z}(r)\, dr \right) ds + \gamma(0) \exp\left(-\int_0^t \hat{z}(s)\, ds \right) \text{ in } \Omega_F(0). \tag{39}$$

First, from (23) we deduce that

$$\|\hat{z}\|_{L^\infty(H^3)} \leqslant \|[\nabla \hat{v}((\nabla \hat{\chi})^{-1} - \mathrm{Id})] : \mathrm{Id}\|_{L^\infty(H^3)} + \|\nabla \hat{v}\|_{L^\infty(H^3)} \leqslant M(1 + CT^\kappa M) \leqslant CM.$$

Then, coming back to (39) we see that

$$\|\gamma\|_{L^\infty(H^3)} \leqslant C_0 + T^\kappa M. \tag{40}$$

Finally, we are going to estimate $\partial_t^3 \gamma$ using the following equation :

$$\partial_t^3 \gamma = (\overline{\rho} + \gamma)(-\hat{z}^3 + 3\partial_t \hat{z} \hat{z} - \partial_t^2 \hat{z}). \tag{41}$$

From the definition of $\hat{z}$ and using Lemma 5, we deduce

$$\|\partial_t \hat{z}\|_{L^\infty(H^2)} + \|\partial_t^3 \hat{z}\|_{L^2(L^2)} \leqslant C(M + M^2), \tag{42}$$

Using this inequality, we find

$$\|\hat{z}\|_{L^\infty(H^2)} \leqslant \|\hat{z}(0,\cdot)\|_{H^2(\Omega_F(0))} + T\|\partial_t \hat{z}\|_{L^\infty(H^2)} \leqslant C_0 + T^\kappa M. \tag{43}$$

Now, from the definition of $\hat{z}$ and the definition of $(A_M^T)_1$ and using the identities

$$\partial_t((\nabla\hat{\chi})^{-1})(0,\cdot) = -\nabla u_0, \quad \partial_t^2((\nabla\hat{\chi})^{-1})(0,\cdot) = 2(\nabla u_0)^2 - \nabla u_1 \text{ in } \Omega_F(0),$$

we find

$$\partial_t \hat{z}(0,\cdot) = (\nabla u_1 - (\nabla u_0)^2) : \mathrm{Id}, \quad \partial_t^2 \hat{z}(0,\cdot) = \nabla\cdot u_2 - 3\nabla u_1 \nabla u_0 : Id + 2(\nabla u_0)^3 : Id \text{ in } \Omega_F(0).$$

In particular, we obtain the following from (42) :

$$\begin{cases} \|\partial_t^2 \hat{z}\|_{L^\infty(L^2)} \leqslant \|\partial_t^2 \hat{z}(0,\cdot)\|_{L^2(\Omega_F(0))} + T^{1/2}\|\partial_t^3 \hat{z}\|_{L^2(L^2)} \leqslant C_0 + T^\kappa M, \\ \|\partial_t \hat{z}\|_{L^\infty(L^2)} \leqslant \|\partial_t \hat{z}(0,\cdot)\|_{L^2(\Omega_F(0))} + T\|\partial_t^2 \hat{z}\|_{L^\infty(L^2)} \leqslant C_0 + T^\kappa M. \end{cases} \tag{44}$$

Coming back to (41) and using (40), (43) and (44), we deduce

$$\|\partial_t^3 \gamma\|_{L^\infty(L^2)} \leqslant C_0 + T^\kappa M.$$

This, together with (40) readily implies (36). Finally, taking into account that

$$\gamma(t,\cdot) = \rho_0(\cdot)\exp\left(-\int_0^t \hat{z}(s)\,ds\right) - \overline{\rho} \geqslant \rho_0 \exp(-CTM) - \overline{\rho} \text{ in } \Omega_F(0),$$

(37) follows from the fact that $\rho_0 \geqslant \rho_{min} > 0$ (see (9)) by taking T small enough with respect to M.

2.2 Existence and uniqueness for an auxiliary problem

Let us consider an auxiliary problem which will be useful for establishing the existence of solution of our system via a fixed-point argument. Let us take $g \in H_\ell^1(0,T; L^2(\partial\Omega_S(0)))$, where

$$H_\ell^1(0,T) := \{\theta \in H^1(0,T) : \theta(0) = 0\}.$$

We consider the following problem :

$$\begin{cases} (\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_t v - \nabla\cdot\widehat{\mathbb{T}}(v,\gamma) = 0 & \text{in } Q_T, \\ \partial_t^2 \xi_i - \displaystyle\sum_{\alpha,j,\beta=1}^3 \hat{c}_{i\alpha j\beta}\partial_{\alpha\beta}^2 \xi_j = 0, \quad i=1,2,3, & \text{in } (0,T)\times\Omega_S(0), \\ v = 0 & \text{on } (0,T)\times\partial\Omega, \\ v = \partial_t \xi & \text{on } \Sigma_T, \\ \left[\widehat{\mathbb{T}}(v,\gamma)\mathrm{n}\right]_i = \displaystyle\sum_{\alpha,j,\beta=1}^3 \hat{c}_{i\alpha j\beta}\partial_\beta \xi_j n_\alpha + g_i, \ i=1,2,3, & \text{on } \Sigma_T, \\ v(0,\cdot) = u_0 & \text{in } \Omega_F(0), \\ \xi(0,\cdot) = 0, \ \partial_t\xi(0,\cdot) = \xi_1 & \text{in } \Omega_S(0), \end{cases} \tag{45}$$

where γ is the solution of (27) and $(33)_1$. Recall that $\widehat{\mathbb{T}}(v,\gamma)$ was defined in (29). We also denote

$$\widehat{\mathbb{T}}_1(v) := \big(\mu(\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t) + \mu'(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}. \tag{46}$$

Lemma 9 *Let $(\hat{v},\hat{\xi}) \in A_M^T$, $u_0 \in L^2(\Omega_F(0))$, $\xi_1 \in L^2(\Omega_S(0))$, $\gamma_0 \in H^3(\Omega_F(0))$ and $g \in H_\ell^1(0,T; L^2(\partial\Omega_S(0)))$. For T small enough with respect to M and the initial conditions (see (48)), there exists a unique solution $(v,\xi) \in Y_1^T \times X_1^T$ of (45) (recall that Y_1 and X_1 have been defined in Definition 1). Moreover, there exists $C>0$ and $C_0>0$ such that*

$$\|v\|_{Y_1^T} + \|\xi\|_{X_1^T} \leqslant C_0 + C\|\gamma\|_{L^\infty(L^2)} + C\|g\|_{H^1(0,T;L^2(\partial\Omega_S(0)))}. \tag{47}$$

Remark 10 *By T small enough with respect to M and the initial conditions, we mean that there exist $\varepsilon > 0$, $n_0 > 0$ and $f : \mathbb{R}_+^3 \to \mathbb{R}_+$ increasing in each variable such that $T \leqslant T_0$ with*

$$T_0 := \min \left\{ \varepsilon, \frac{\varepsilon}{M^{n_0}}, \frac{\varepsilon}{f(\|\gamma_0\|_{H^3}, \|u_0\|_{H^6}, \|\xi_1\|_{H^3})} \right\}. \tag{48}$$

Proof of Lemma 9:

• *Step 1.* Galerkin approximation of system (45).

Let $\{w_\ell\}_{\ell\in\mathbf{N}^*} \in H_0^1(\Omega_F(0))$ and $\{z_\ell\}_{\ell\in\mathbf{N}^*} \in H^1(\Omega_S(0))$ two orthogonal basis in L^2 and $\{\tilde{z}_\ell\}_{\ell\in\mathbf{N}^*}$ an extension on $H_0^1(\Omega)$ of $\{z_\ell\}_{\ell\in\mathbf{N}^*}$. The initial conditions ξ_1 and u_0 can be decomposed on these basis:

$$\xi_1 = \sum_{\ell=1}^{\infty} \alpha_\ell^1 z_\ell \text{ and } u_0 = \sum_{\ell=1}^{\infty} \alpha_\ell^1 \tilde{z}_\ell + \sum_{\ell=1}^{\infty} \beta_\ell^0 w_\ell$$

We try to find (v^n, ξ^n) satisfying

$$\left\{ \begin{aligned} & \int_{\Omega_F(0)} \widehat{\mathbb{T}}(v^n,\gamma) : \nabla w^n \, dy + \int_{\Omega_S(0)} \partial_t^2 \xi^n \cdot \partial_t z^n \, dy + \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_S(0)} \hat{c}_{i\alpha j\beta} \partial_\beta \xi_j^n \partial_t \partial_\alpha z_i^n dy \\ & + \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_S(0)} \partial_\alpha \hat{c}_{i\alpha j\beta} \partial_\beta \xi_j^n \partial_t z_i^n dy + \int_{\Omega_F(0)} (\overline{\rho}+\gamma) \det(\nabla\hat{\chi}) \partial_t v^n \cdot w^n \, dy = \int_{\partial\Omega_S(0)} g \cdot \partial_t z^n \, d\sigma, \end{aligned} \right. \tag{49}$$

for $t \in (0,T)$, where

$$w^n(t,y) = \sum_{\ell=1}^{n+1} \chi_\ell'(t) \tilde{z}_\ell(y) + \sum_{\ell=1}^{n+1} \kappa_\ell(t) w_\ell(y), \quad t \in (0,T),\, y \in \Omega_F(0)$$

and

$$z^n(t,y) = \sum_{\ell=1}^{n+1} \chi_\ell(t) z_\ell(y), \quad t \in (0,T),\, y \in \Omega_S(0)$$

for χ_ℓ, $\kappa_\ell \in C^\infty([0,T])$ $(1 \leqslant \ell \leqslant n+1)$.

We look for (v^n, ξ^n) in the form

$$v^n(t,y) = \sum_{\ell=1}^{n+1} \alpha_\ell'(t) \tilde{z}_\ell(y) + \sum_{\ell=1}^{n+1} \beta_\ell(t) w_\ell(y) \quad (t,y) \in (0,T) \times \Omega_F(0)$$

and

$$\xi^n(t,y) = \sum_{\ell=1}^{n+1} \alpha_\ell(t) z_\ell(y) \quad (t,y) \in (0,T) \times \Omega_S(0)$$

This yields the system

$$A(t) \frac{d}{dt} \begin{pmatrix} \alpha_i \\ \alpha_i' \\ \beta_i \end{pmatrix} = M(t) \begin{pmatrix} \alpha_i \\ \alpha_i' \\ \beta_i \end{pmatrix} + B(t), \quad t \in (0,T),$$

complemented by the initial conditions:

$$\begin{pmatrix} \alpha_i \\ \alpha_i' \\ \beta_i \end{pmatrix} (0) = \begin{pmatrix} 0 \\ \alpha_i^1 \\ \beta_i^0 \end{pmatrix}.$$

The matrix $A(t) := (A_{ij}(t))_{1\leqslant i,j\leqslant 3}$ for $A_{ij}(t) \in \mathcal{M}_{n+1}(\mathbb{R})$ is given by $A_{11} := \mathrm{Id}$, $A_{1j} \equiv 0$ for $j = 2,3$, $A_{i1} \equiv 0$ for $i = 2,3$,

$$A_{22}(t) := \left(\delta_{k\ell} + \int_{\Omega_F(0)} (\overline{\rho}+\gamma) \det(\nabla\hat{\chi}) \tilde{z}_k \cdot \tilde{z}_\ell \, dy \right)_{1\leqslant k,\ell\leqslant n+1},$$

$$A_{23}(t) := \left(\int_{\Omega_F(0)} (\overline{\rho} + \gamma) \det(\nabla \hat{\chi}) \tilde{z}_k \cdot w_\ell \, dy \right)_{1 \leqslant k, \ell \leqslant n+1},$$

$A_{32} := A_{23}^t$ and

$$A_{33}(t) := \left(\int_{\Omega_F(0)} (\overline{\rho} + \gamma) \det(\nabla \hat{\chi}) w_k \cdot w_\ell \, dy \right)_{1 \leqslant k, \ell \leqslant n+1}.$$

Next, $M(t) := (M_{ij}(t))_{1 \leqslant i,j \leqslant 3}$, where $M_{ij}(t) \in \mathcal{M}_{n+1}(\mathbb{R})$ are given by $M_{1j} \equiv 0$ for $j = 1, 3$, $M_{12} := \mathrm{Id}$, $M_{31} \equiv 0$,

$$M_{21}(t) := - \left(\sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_S(0)} \left(\hat{c}_{i\alpha j\beta} (\partial_\beta z_\ell)_j (\partial_\alpha z_k)_i + \partial_\alpha \hat{c}_{i\alpha j\beta} (\partial_\beta z_\ell)_j (z_k)_i \right) dy \right)_{1 \leqslant k, l \leqslant n+1},$$

$$M_{22}(t) := - \left(\int_{\Omega_F(0)} \widehat{\mathbb{T}}_1(\tilde{z}_\ell) : \nabla \tilde{z}_k \, dy \right)_{1 \leqslant k, \ell \leqslant n+1}, M_{23}(t) := - \left(\int_{\Omega_F(0)} \widehat{\mathbb{T}}_1(w_\ell) : \nabla \tilde{z}_k \, dy \right)_{1 \leqslant k, \ell \leqslant n+1}$$

and

$$M_{32}(t) := - \left(\int_{\Omega_F(0)} \widehat{\mathbb{T}}_1(\tilde{z}_\ell) : \nabla w_k \, dy \right)_{1 \leqslant k, \ell \leqslant n+1}, M_{33}(t) := - \left(\int_{\Omega_F(0)} \widehat{\mathbb{T}}_1(w_\ell) : \nabla w_k \, dy \right)_{1 \leqslant k, \ell \leqslant n+1}.$$

On the other hand, $B(t) := (B_i(t))_{1 \leqslant i \leqslant 3}$ with $B_i(t) \in \mathbb{R}^{n+1}$ given by $B_1(t) \equiv 0$,

$$B_2(t) = \left(\int_{\Omega_F(0)} (P(\overline{\rho} + \gamma) - P(\overline{\rho})) \mathrm{cof}\,(\nabla \hat{\chi}) : \nabla \tilde{z}_\ell \, dy + \int_{\partial \Omega_S(0)} g \cdot z_\ell \, d\sigma \right)_{1 \leqslant \ell \leqslant n+1}$$

and

$$B_3(t) = \left(\int_{\Omega_F(0)} (P(\overline{\rho} + \gamma) - P(\overline{\rho})) \mathrm{cof}\,(\nabla \hat{\chi}) : \nabla w_\ell \, dy \right)_{1 \leqslant \ell \leqslant n+1}.$$

One can easily see that $A(t)$ is positive definite thanks to the fact that $\overline{\rho} + \gamma \geqslant \overline{\rho} + \gamma_{min} > 0$ (see (37)) and $\det(\nabla \hat{\chi})(t) \geqslant C > 0$ (see (22)) for T small enough with respect to M. Moreover, $A^{-1}, M, B \in L^\infty(0,T)$. This gives the existence of a unique solution

$$(v^n, \xi^n) \in W^{1,\infty}(0,T; H^1(\Omega_F(0))) \times W^{2,\infty}(0,T; H^1(\Omega_S(0))).$$

• *Step 2.* Estimate of (v^n, ξ^n).

Let us prove an energy estimate of the form

$$\begin{aligned} \|v^n\|_{L^\infty(L^2)} + \|v^n\|_{L^2(H^1)} + \|\xi^n\|_{L^\infty(0,T;H^1(\Omega_S(0)))} + \|\xi^n\|_{W^{1,\infty}(L^2(0,T;\Omega_S(0)))} \\ \leqslant C_0 + C\|\gamma\|_{L^\infty(L^2)} + C\|g\|_{H^1(0,T;L^2(\partial\Omega_S(0)))}. \end{aligned} \tag{50}$$

In order to do this, we take $w^n := v^n$ and $z^n := \xi^n$ in (49) and we integrate between 0 and t. This yields:

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_F(0)} (\overline{\rho}+\gamma)(t)|v^n(t)|^2 \det\nabla\hat{\chi}(t)\,dy - \frac{1}{2}\int_{\Omega_F(0)} (\overline{\rho}+\gamma_0)|u_0^n|^2\,dy - \frac{1}{2}\iint_{Q_t} |v^n|^2\partial_s((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\,dy\,ds\\
&+\iint_{Q_t}\Big(\frac{\mu}{2}|\nabla v^n(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}(\nabla v^n)^t|^2 + \mu'|\nabla v^n(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2\Big)\det\nabla\hat{\chi}\,dy\,ds\\
&-\iint_{Q_t}(P(\overline{\rho}+\gamma) - P(\overline{\rho}))\,\mathrm{cof}\,\nabla\hat{\chi} : \nabla v^n\,dy\,ds + \frac{1}{2}\int_{\Omega_S(0)} |\partial_t\xi^n(t)|^2\,dy - \frac{1}{2}\int_{\Omega_S(0)} |\xi_1^n|^2\,dy\\
&+\frac{1}{2}\sum_{i,\alpha,j,\beta=1}^{3}\int_{\Omega_S(0)}[\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j^n\partial_\alpha\xi_i^n](t)dy - \frac{1}{2}\sum_{i,\alpha,j,\beta=1}^{3}\int_0^t\int_{\Omega_S(0)}\partial_s\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j^n\partial_\alpha\xi_i^n\,dyds\\
&+\sum_{i,\alpha,j,\beta=1}^{3}\int_0^t\int_{\Omega_S(0)}\partial_\alpha\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j^n\partial_s\xi_i^n\,dyds = \int_0^t\int_{\partial\Omega_S(0)} g\cdot\partial_s\xi^n\,d\sigma ds.
\end{aligned}\tag{51}$$

Here, we have employed the notation

$$\xi_1^n = \sum_{\ell=1}^{n+1}\alpha_\ell^1 z_\ell \text{ and } u_0^n = \sum_{\ell=1}^{n+1}\alpha_\ell^1\tilde{z}_\ell + \sum_{\ell=1}^{n+1}\beta_\ell^0 w_\ell,$$

and we have used $\xi^n_{|t=0} = 0$ and the symmetry of the coefficients $c_{i\alpha j\beta}$ (see (14)).

For the first term, according to (37) and (22)

$$\int_{\Omega_F(0)} (\overline{\rho}+\gamma)(t)|v^n(t)|^2\det\nabla\hat{\chi}(t)\,dy \geqslant (\overline{\rho}+\gamma_{min})(1-CT^\kappa M)\int_{\Omega_F(0)}|v^n(t)|^2\,dy. \tag{52}$$

The second term is bounded by

$$\frac{1}{2}\int_{\Omega_F(0)}(\overline{\rho}+\gamma_0)|u_0^n|^2\,dy \leqslant C(\|\gamma_0\|^2_{L^\infty(\Omega_F(0))} + \|u_0\|^2_{L^2(\Omega_F(0))} + \|u_0\|^4_{L^2(\Omega_F(0))}) = C_0. \tag{53}$$

The third term is estimated by

$$\begin{aligned}
\iint_{Q_t}|v^n|^2|\partial_s((\overline{\rho}+\gamma)\det\nabla\hat{\chi})|\,dy\,ds &\leqslant \iint_{Q_t}|v^n|^2(|\partial_s\gamma|\,|\det\nabla\hat{\chi}| + |\overline{\rho}+\gamma|\,|\partial_s(\det\nabla\hat{\chi})|)\,dy\,ds\\
&\leqslant CT\|v^n\|^2_{L^\infty(L^2)}(\|\gamma\|_{W^{1,\infty}(L^\infty)} + M(\overline{\rho} + \|\gamma\|_{L^\infty(L^\infty)})).
\end{aligned}$$

Here we have used (22) and the fact that T is small with respect to M. Thus, according to (36), we have

$$\iint_{Q_t}|v^n|^2|\partial_s((\overline{\rho}+\gamma)\det\nabla\hat{\chi})|\,dy\,ds \leqslant CT(M + C_0 + C_0M)\|v^n\|^2_{L^\infty(L^2)}. \tag{54}$$

We consider now the viscosity term corresponding to the second line of (51). The term in μ' is estimated, thanks to (23), in the following way :

$$\begin{aligned}
&\mu'\iint_{Q_t}|\nabla v^n(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2\det\nabla\hat{\chi}\,dy\,ds \geqslant \frac{\mu'}{2}\iint_{Q_t}|\nabla\cdot v^n|^2\,dy\,ds - \mu'\iint_{Q_t}|\nabla v^n((\nabla\hat{\chi})^{-1} - \mathrm{Id}) : \mathrm{Id}|^2\,dy\,ds\\
&+\mu'\iint_{Q_t}|\nabla v^n(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2(\det\nabla\hat{\chi} - 1)\,dy\,ds \geqslant \frac{\mu'}{2}\iint_{Q_t}|\nabla\cdot v^n|^2\,dy\,ds - CT^\kappa M\|v^n\|^2_{L^2(H^1)}.
\end{aligned}\tag{55}$$

In the same way, we prove

$$\frac{\mu}{2}\iint_{Q_t}|\nabla v^n(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}(\nabla v^n)^t|^2\det\nabla\hat{\chi}\,dy\,ds \geqslant \mu\iint_{Q_t}|\epsilon(v^n)|^2\,dy\,ds - CT^\kappa M\|v^n\|^2_{L^2(H^1)}. \tag{56}$$

For the first term in the third line of (51), we notice that, for any $\delta > 0$, there exists a positive constant C such that

$$\left| \iint_{Q_t} (P(\overline{\rho}+\gamma) - P(\overline{\rho})) \operatorname{cof} \nabla \hat{\chi} : \nabla v^n \, dy \, ds \right| \leqslant \delta \|\nabla v^n\|^2_{L^2(L^2)} + C \iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho})|^2 \, dy \, ds.$$

According to Lemma 8

$$0 < a = \overline{\rho} + \gamma_{min} \leqslant \overline{\rho} + \gamma \leqslant C + C_0 = b,$$

for T small with respect to M. Thus, there exists an interval $I \subset \mathbb{R}^*_+$ such that $\overline{\rho} \in I$ and $[a,b] \subset I$. Then, since $\|P'\|_{L^\infty(I)}$ is increasing with respect to $\|\gamma_0\|_{H^3}$, $\|u_0\|_{H^6}$ and $\|\xi_1\|_{H^3}$, we obtain

$$\iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho})|^2 \, dy \, ds \leqslant \|P'\|^2_{L^\infty(I)} \|\gamma\|^2_{L^2(L^2)} \leqslant f(\|\gamma_0\|_{H^3}, \|u_0\|_{H^6}, \|\xi_1\|_{H^3}) T \|\gamma\|^2_{L^\infty(L^2)} \leqslant \|\gamma\|^2_{L^\infty(L^2)},$$

for T small enough with respect to the initial conditions (see (48)). This implies that

$$\left| \iint_{Q_t} (P(\overline{\rho}+\gamma) - P(\overline{\rho})) \operatorname{cof} \nabla \hat{\chi} : \nabla v^n \, dy \, ds \right| \leqslant \delta \|\nabla v^n\|^2_{L^2(L^2)} + C \|\gamma\|^2_{L^\infty(L^2)}. \tag{57}$$

For the first term in the fourth line of (51), we use estimate (25) and we find

$$\begin{aligned} \frac{1}{2} \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_S(0)} [\hat{c}_{i\alpha j\beta} \partial_\beta \xi^n_j \partial_\alpha \xi^n_i](t) dy \geqslant \lambda \int_{\Omega_S(0)} |\epsilon(\xi^n)|^2(t) dy + \frac{\lambda'}{2} \int_{\Omega_S(0)} |\nabla \cdot \xi^n|^2(t) dy \\ - C T^\kappa M \|\nabla \xi^n(t)\|^2_{L^2(\Omega_S(0))}. \end{aligned} \tag{58}$$

For the next two terms of (51), we use estimate (24) and we have

$$\begin{aligned} &\left| -\frac{1}{2} \sum_{i,\alpha,j,\beta=1}^{3} \int_0^t \int_{\Omega_S(0)} \partial_s \hat{c}_{i\alpha j\beta} \partial_\beta \xi^n_j \partial_\alpha \xi^n_i dy ds + \sum_{i,\alpha,j,\beta=1}^{3} \int_0^t \int_{\Omega_S(0)} \partial_\alpha \hat{c}_{i\alpha j\beta} \partial_\beta \xi^n_j \partial_s \xi^n_i dy ds \right| \\ &\leqslant CT(M+M^2)(\|\nabla \xi^n\|^2_{L^\infty(0,T;L^2(\Omega_S(0)))} + \|\partial_t \xi^n\|^2_{L^\infty(0,T;L^2(\Omega_S(0)))}). \end{aligned} \tag{59}$$

Finally,

$$\begin{aligned} &\left| \int_0^t \int_{\partial\Omega_S(0)} g \cdot \partial_s \xi^n \, d\sigma ds \right| = \left| \int_{\partial\Omega_S(0)} g(t) \cdot \xi^n(t) \, d\sigma - \int_0^t \int_{\partial\Omega_S(0)} \partial_s g \cdot \xi^n \, d\sigma ds \right| \\ &\leqslant \delta \|\xi^n\|^2_{L^\infty(0,T;H^1(\Omega_S(0)))} + C_\delta \|g\|^2_{H^1(0,T;L^2(\partial\Omega_S(0)))}, \end{aligned} \tag{60}$$

where we have used that $\|g\|_{L^\infty(0,T;L^2(\partial\Omega_S(0)))} \leqslant T^{1/2} \|g\|_{H^1(0,T;L^2(\partial\Omega_S(0)))}$ (recall that $g(0,\cdot) \equiv 0$ on $\partial\Omega_S(0)$).

Thus, we can reassemble inequalities (52) to (60). Taking the supremum of (51) in $t \in (0,T)$, using Korn's inequality and taking δ small enough and T small with respect to M and C_0, we deduce (50).

Thanks to (50), one can pass to the limit as $n \to \infty$ in (49) and show the existence and uniqueness of $(v,\xi) \in Y_1^T \times X_1^T$ a weak solution of (45). Consequently, we have proved Lemma 9.

2.3 Existence of solution of a linear system

Let us come back to the problem given by equations (28)-(32) complemented by the initial conditions (8) and $(33)_2$:

$$\begin{cases} (\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_t v - \nabla\cdot\widehat{\mathbb{T}}(v,\gamma) = 0 & \text{in } Q_T, \\ \partial_t^2\xi_i - \displaystyle\sum_{\alpha,j,\beta=1}^{3}\hat{c}_{i\alpha j\beta}\partial^2_{\alpha\beta}\xi_j = 0, \quad i=1,2,3, & \text{in } (0,T)\times\Omega_S(0), \\ v=0 & \text{on } (0,T)\times\partial\Omega, \\ v=\partial_t\xi & \text{on } \Sigma_T, \\ \left[\widehat{\mathbb{T}}(v,\gamma)\mathrm{n}\right]_i = \displaystyle\sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t \hat{c}_{i\alpha j\beta}\partial^2_{s\beta}\xi_j ds\right) n_\alpha,\ i=1,2,3, & \text{on } \Sigma_T, \\ v(0,\cdot)=u_0 & \text{in } \Omega_F(0), \\ \xi(0,\cdot)=0,\ \partial_t\xi(0,\cdot)=\xi_1 & \text{in } \Omega_S(0). \end{cases} \tag{61}$$

Observe that this system corresponds to the auxiliary problem (45) with g_i given by

$$-\sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t \partial_s\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j ds\right) n_\alpha,\ i=1,2,3.$$

Proposition 11 *Let $(\hat{v},\hat{\xi})\in A_M^T$, $u_0\in H^1(\Omega_F(0))$, $\xi_1\in H^1(\Omega_S(0))$ and $\gamma_0\in H^3(\Omega_F(0))$ satisfying $(16)_1$-$(16)_2$. For T small enough with respect to M and the initial conditions (see (48)), there exists a unique solution $(v,\xi)\in Y_2\times X_2$ (recall that Y_2^T and X_2^T have been defined in Definition 1) of (61). Moreover, there exists $C_0>0$ and $\kappa>0$ such that*

$$\|v\|_{Y_2^T}+\|\xi\|_{X_2^T}\leqslant C_0+T^\kappa M. \tag{62}$$

Proof:

We intend to prove the existence and uniqueness of solution of (61) through a fixed point argument. We thus define Λ_0 which, to each $\tilde{\xi}\in X_2$, associates ξ which is, together with some v, the solution of problem (45) with $g_i=\tilde{h}_i$, where

$$\tilde{h}_i := -\sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t \partial_s\hat{c}_{i\alpha j\beta}\partial_\beta\tilde{\xi}_j ds\right) n_\alpha,\ i=1,2,3. \tag{63}$$

We notice that $\tilde{h}\in H^1_\ell(0,T;L^2(\partial\Omega_S(0)))$ and, using (24), we find

$$\|\tilde{h}\|_{H^1(0,T;L^2(\partial\Omega_S(0)))}\leqslant T^\kappa M\|\tilde{\xi}\|_{X_2^T}.$$

Then, $\tilde{\xi}$ being fixed, the existence and uniqueness of (v,ξ) comes from Lemma 9 and we have

$$\|(v,\xi)\|_{Y_1^T\times X_1^T}\leqslant C_0+C\|\gamma\|_{L^\infty(L^2)}+T^\kappa M\|\tilde{\xi}\|_{X_2^T}.$$

We are going to prove that Λ_0 maps from X_2^T to X_2^T and that it is a contraction. We divide the proof in three steps :

- *Step 1.* Estimates on $\partial_t v$ and $\partial_t\xi$.

  Let us differentiate the first equation in (61) with respect to time. We obtain

$$(\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_t^2 v+\partial_t((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_t v-\partial_t\nabla\cdot\widehat{\mathbb{T}}(v,\gamma)=0 \text{ in } Q_T. \tag{64}$$

Next, we multiply this equation by $\partial_t v$ and we integrate on Q_t for any $t \in (0,T)$. For the first two terms of (64), we have:

$$\iint_{Q_t} \big((\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_s^2 v + \partial_s((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_s v\big)\partial_s v\,dy\,ds$$
$$= \frac{1}{2}\int_{\Omega_F(0)} (\overline{\rho}+\gamma)(t)\det\nabla\hat{\chi}(t)|\partial_t v(t)|^2\,dy - \frac{1}{2}\int_{\Omega_F(0)} \rho_0|\partial_t v(0)|^2\,dy + \frac{1}{2}\iint_{Q_t} |\partial_s v|^2\partial_s((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\,dy\,ds.$$

Thus, arguing exactly as in the proof of Lemma 9, we have, for T small enough with respect to M

$$\begin{aligned}&\iint_{Q_t} \big((\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_s^2 v + \partial_s((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_s v\big)\partial_s v\,dy\,ds\\ &\geqslant \frac{\rho_{min}}{4}\int_{\Omega_F(0)} |\partial_t v(t)|^2\,dy - \frac{1}{2}\int_{\Omega_F(0)} \rho_0|\partial_t v(0)|^2\,dy - CT(M+C_0+C_0M)\|v\|^2_{W^{1,\infty}(L^2)}.\end{aligned} \tag{65}$$

Now, the remaining terms of (64) are

$$\iint_{Q_t} \partial_s\widehat{\mathbb{T}}(v,\gamma) : \partial_s\nabla v\,dy\,ds + \iint_{\Sigma_t}\sum_{i=1}^3 \partial_s\Big[\sum_{\alpha,j,\beta=1}^3 \hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha + \tilde{h}_i\Big]\partial^2_{ss}\xi_i\,d\sigma\,ds. \tag{66}$$

We notice that

$$\begin{aligned}&\mu'\iint_{Q_t} \partial_s[(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}] : \partial_s\nabla v\,dy\,ds = \mu'\iint_{Q_t} |\partial_s\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2\det\nabla\hat{\chi}\,dy\,ds\\ &+\mu'\iint_{Q_t} (\nabla v\partial_s((\nabla\hat{\chi})^{-1}) : \mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi} : \partial_s\nabla v\,dy\,ds + \mu'\iint_{Q_t} (\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\partial_s(\mathrm{cof}\,\nabla\hat{\chi}) : \partial_s\nabla v\,dy\,ds.\end{aligned} \tag{67}$$

Arguing again as in Lemma 9 (see (55)), the first term in the right-hand side is estimated by

$$\mu'\iint_{Q_t} |\partial_s\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2\det\nabla\hat{\chi}\,dy\,ds \geqslant \frac{\mu'}{2}\iint_{Q_t} |\partial_s\nabla\cdot v|^2\,dy\,ds - CT^\kappa M\|v\|^2_{H^1(H^1)}.$$

To bound the second line of (67), we use that $\|\partial_t(\nabla\hat{\chi})^{-1}\|_{L^2(L^\infty)} + \|\partial_t\mathrm{cof}\,\nabla\hat{\chi}\|_{L^2(L^\infty)} \leqslant CT^{1/2}M$ (see (22)). Thus, for the term in μ' in (66), we have

$$\begin{aligned}&\mu'\iint_{Q_t} \partial_s[(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}] : \partial_s\nabla v\,dy\,ds \geqslant \frac{\mu'}{2}\iint_{Q_t} |\partial_s\nabla\cdot v|^2\,dy\,ds - CT^\kappa M(\|v\|^2_{H^1(H^1)} + \|v\|^2_{L^\infty(H^1)})\\ &\geqslant \frac{\mu'}{2}\iint_{Q_t} |\partial_s\nabla\cdot v|^2\,dy\,ds - CT^\kappa M\|v\|^2_{H^1(H^1)} - C\|u_0\|^2_{H^1}.\end{aligned} \tag{68}$$

The term in μ in (66) can be estimated in the same way as follows :

$$\begin{aligned}&\mu\iint_{Q_t} \partial_s\big[\big(\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t\big)\mathrm{cof}\,\nabla\hat{\chi}\big] : \partial_s\nabla v\,dy\,ds\\ &\geqslant \mu\iint_{Q_t} |\partial_s\epsilon(v)|^2\,dy\,ds - CT^\kappa M\|v\|^2_{H^1(H^1)} - C\|u_0\|^2_{H^1}.\end{aligned} \tag{69}$$

Next, for the pressure term in (66), we see that, for any $\delta > 0$, there exists a positive constant C such that

$$\begin{aligned}\iint_{Q_t} \big|\partial_s[(P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}]\big|\,|\partial_s\nabla v|\,dy\,ds &\leqslant \delta\|\partial_t\nabla v\|^2_{L^2(L^2)} + C\iint_{Q_t} |P'(\overline{\rho}+\gamma)|^2|\partial_s\gamma|^2\,dy\,ds\\ &+C\iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho}))|^2|\partial_s\mathrm{cof}\,\nabla\hat{\chi}|^2\,dy\,ds.\end{aligned}$$

With the same arguments as in Lemma 9, we have

$$\iint_{Q_t} |P'(\overline{\rho}+\gamma)|^2 |\partial_s \gamma|^2 \, dy \, ds \leqslant T \|P'\|^2_{L^\infty(I)} \|\gamma\|^2_{W^{1,\infty}(L^2)}$$

and, since $\|\partial_t \mathrm{cof}\, \nabla \hat{\chi}\|_{L^2(L^\infty)} \leqslant C T^{1/2} M$ (see (22)),

$$\iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho}))|^2 |\partial_s \mathrm{cof}\, \nabla \hat{\chi}|^2 \, dy \, ds \leqslant C T M^2 \|P'\|^2_{L^\infty(I)} \|\gamma\|^2_{L^\infty(L^2)}.$$

Thus, we get, for T small with respect to M and the initial conditions,

$$\iint_{Q_t} \left|\partial_s [(P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{cof}\, \nabla \hat{\chi}]\right| |\partial_s \nabla v| \, dy \, ds \leqslant \delta \|\partial_t \nabla v\|^2_{L^2(L^2)} + C \|\gamma\|^2_{W^{1,\infty}(L^2)}. \tag{70}$$

Combining identity (64) with estimates (65), (66) and (68)-(70), we obtain

$$\begin{aligned}
&\frac{\rho_{min}}{4} \int_{\Omega_F(0)} |\partial_t v(t)|^2 dy + \frac{\mu'}{2} \iint_{Q_t} |\partial_s \nabla \cdot v|^2 dy ds + \mu \iint_{Q_t} |\partial_s \epsilon(v)|^2 dy ds \\
&+ \iint_{\Sigma_t} \sum_{i=1}^{3} \partial_s \Big[\sum_{\alpha,j,\beta=1}^{3} \hat{c}_{i\alpha j\beta} \partial_\beta \xi_j n_\alpha + \tilde{h}_i \Big] \partial^2_{ss} \xi_i \, d\sigma \, ds \leqslant \frac{1}{2} \int_{\Omega_F(0)} \rho_0 |\partial_t v(0)|^2 dy \\
&+ C(T(M + C_0 + C_0 M) \|v\|^2_{W^{1,\infty}(L^2)} + \|u_0\|^2_{H^1} + \|\gamma\|^2_{W^{1,\infty}(L^2)}) + (\delta + T^\kappa M) \|v\|^2_{H^1(H^1)}.
\end{aligned} \tag{71}$$

Let us now differentiate in time the second equation in (61), multiply by $\partial_t^2 \xi_i$ and integrate in $(0,t) \times \Omega_S(0)$ for any $t \in (0,T)$. This yields

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega_S(0)} |\partial_t^2 \xi|^2 (t) dy + \sum_{i,\alpha,j,\beta=1}^{3} \frac{1}{2} \int_{\Omega_S(0)} [\hat{c}_{i\alpha j\beta} \partial^2_{t\beta} \xi_j \partial^2_{t\alpha} \xi_i](t) \, dy \\
&- \iint_{\Sigma_t} \sum_{i,\alpha,j,\beta=1}^{3} \hat{c}_{i\alpha j\beta} \partial^2_{s\beta} \xi_j \, \partial^2_{ss} \xi_i \, n_\alpha \, d\sigma \, ds = \int_{\Omega_S(0)} \Big(\lambda |\epsilon(\xi_1)|^2 + \frac{\lambda'}{2} |\nabla \cdot \xi_1|^2 \Big) dy \\
&+ \sum_{i,\alpha,j,\beta=1}^{3} \int_0^t \int_{\Omega_S(0)} \partial_s \hat{c}_{i\alpha j\beta} \partial^2_{\alpha\beta} \xi_j \partial^2_{ss} \xi_i \, dy ds - \sum_{i,\alpha,j,\beta=1}^{3} \int_0^t \int_{\Omega_S(0)} \partial_\alpha \hat{c}_{i\alpha j\beta} \partial^2_{s\beta} \xi_j \partial^2_{ss} \xi_i \, dy ds \\
&+ \frac{1}{2} \sum_{i,\alpha,j,\beta=1}^{3} \int_0^t \int_{\Omega_S(0)} \partial_s \hat{c}_{i\alpha j\beta} \partial^2_{s\beta} \xi_j \partial^2_{s\alpha} \xi_i \, dy ds.
\end{aligned} \tag{72}$$

For the second term in the left-hand side of (72), we use (25) and we have

$$\frac{1}{2} \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_S(0)} [\hat{c}_{i\alpha j\beta} \partial_{t\beta} \xi_j \partial_{t\alpha} \xi_i](t) dy \geqslant \lambda \int_{\Omega_S(0)} |\partial_t \epsilon(\xi)|^2 (t) dy + \frac{\lambda'}{2} \int_{\Omega_S(0)} |\partial_t \nabla \cdot \xi|^2 (t) dy - C T^\kappa M \|\xi\|^2_{X_2^T}. \tag{73}$$

On the other hand, using (24), we have that the last three terms of (72) are estimated by

$$C T (M + M^2) \|\xi\|^2_{X_2^T}.$$

Taking these two facts into account and combining (71) and (72), we deduce

$$\begin{aligned}
&\frac{\rho_{min}}{4}\int_{\Omega_F(0)}|\partial_t v(t)|^2dy+\frac{1}{2}\int_{\Omega_S(0)}|\partial_t^2\xi|^2(t)dy+\frac{\mu'}{2}\iint_{Q_t}|\partial_s\nabla\cdot v|^2dyds+\mu\iint_{Q_t}|\partial_s\epsilon(v)|^2dyds\\
&+\iint_{\Sigma_t}\sum_{i=1}^3\partial_s\Big[\sum_{\alpha,j,\beta=1}^3\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha+\tilde{h}_i\Big]\,\partial^2_{ss}\xi_i\,d\sigma\,ds-\iint_{\Sigma_t}\sum_{i,\alpha,j,\beta=1}^3\hat{c}_{i\alpha j\beta}\partial^2_{s\beta}\xi_j\,\partial^2_{ss}\xi_i\,n_\alpha\,d\sigma\,ds\\
&+\lambda\int_{\Omega_S(0)}|\partial_t\epsilon(\xi)|^2(t)dy+\frac{\lambda'}{2}\int_{\Omega_S(0)}|\partial_t\nabla\cdot\xi|^2(t)dy\leqslant C(\|u_0\|^2_{H^2}+f(\|\gamma_0\|_{H^3})\|\gamma_0\|^2_{H^1}+\|\xi_1\|^2_{H^1})\\
&+C\|\gamma\|^2_{W^{1,\infty}(L^2)}+\delta(\|v\|^2_{W^{1,\infty}(L^2)}+\|v\|^2_{H^1(H^1)})+CT^\kappa M\|\xi\|^2_{X_2^T},
\end{aligned}\tag{74}$$

for T small enough with respect to M and the initial conditions. Here, we have denoted by $f:\mathbb{R}^+\to\mathbb{R}^+$ an increasing function.

Let us now deal with the boundary terms in (74). We have

$$\begin{aligned}
&\iint_{\Sigma_t}\sum_{i=1}^3\partial_s\Big[\sum_{\alpha,j,\beta=1}^3\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha+\tilde{h}_i\Big]\,\partial^2_{ss}\xi_i\,d\sigma\,ds-\iint_{\Sigma_t}\sum_{i,\alpha,j,\beta=1}^3\hat{c}_{i\alpha j\beta}\partial^2_{s\beta}\xi_j\,\partial^2_{ss}\xi_i\,n_\alpha\,d\sigma\,ds\\
&=\iint_{\Sigma_t}\sum_{i,\alpha,j,\beta=1}^3\partial_s\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j\,\partial^2_{ss}\xi_i\,n_\alpha\,d\sigma\,ds+\iint_{\Sigma_t}\sum_{i=1}^3\partial_s\tilde{h}_i\partial^2_{ss}\xi_i\,d\sigma\,ds:=A_1+A_2.
\end{aligned}$$

For A_1, we use that $\partial_t^2\xi=\partial_t v$ on $(0,T)\times\partial\Omega_S(0)$ and we have that

$$|A_1|\leqslant CT^{1/2}\|\partial_t\hat{c}\|_{L^\infty(0,T;L^\infty(\partial\Omega_S(0)))}\|\nabla\xi\|_{L^\infty(0,T;L^4(\partial\Omega_S(0)))}\|\partial_t v\|_{L^2(0,T;L^4(\partial\Omega_S(0)))}.$$

Using now (24), we deduce

$$|A_1|\leqslant CT^\kappa M\|\xi\|_{X_2^T}\|v\|_{H^1(H^1)}.$$

From the definition of $\tilde{h}$ (see (63)), an analogous computation shows that

$$|A_2|\leqslant CT^\kappa M\|\tilde{\xi}\|_{X_2^T}\|v\|_{H^1(H^1)}.$$

Coming back to (74) and using Korn's inequality, we get

$$\begin{aligned}
&\|v\|_{W^{1,\infty}(L^2)}+\|v\|_{H^1(H^1)}+\|\xi\|_{W^{2,\infty}(0,T;L^2(\Omega_S(0)))}+\|\xi\|_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))}\\
&\leqslant C_0+C\|\gamma\|_{W^{1,\infty}(L^2)}+CT^\kappa M(\|\xi\|_{X_2^T}+\|\tilde{\xi}\|_{X_2^T}).
\end{aligned}\tag{75}$$

- *Step 2.* Spatial regularity of v and ξ.

We recall that v solves the stationary elliptic problem : for all $t\in(0,T)$

$$\begin{cases}
-\nabla\cdot\widehat{\mathbb{T}}(v,\gamma)=-(\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_t v & \text{in } \Omega_F(0),\\
v=0 & \text{on } \partial\Omega,\\
\Big[\widehat{\mathbb{T}}(v,\gamma)\mathrm{n}\Big]_i=\displaystyle\sum_{\alpha,j,\beta=1}^3\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha+\tilde{h}_i,\ i=1,2,3, & \text{on } \partial\Omega_S(0),
\end{cases}$$

where $\tilde{h}_i$ was defined in (63). We can rewrite this system as follows :

$$\begin{cases}
-\nabla\cdot(\mu(\nabla v+\nabla v^t)+\mu'(\nabla\cdot v)\mathrm{Id})=F & \text{in } \Omega_F(0),\\
v=0 & \text{on } \partial\Omega\\
(\mu(\nabla v+\nabla v^t)+\mu'\nabla\cdot v)n=\widetilde{G} & \text{on } \partial\Omega_S(0),
\end{cases}\tag{76}$$

with

$$
\begin{aligned}
F := &-(\gamma+\overline{\rho})\det(\nabla\hat{\chi})\partial_t v - \nabla\cdot((P(\gamma+\overline{\rho})-P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}))\\
&+\mu\nabla\cdot((\nabla v((\nabla\hat{\chi})^{-1}-\mathrm{Id})+((\nabla\hat{\chi})^{-t}-\mathrm{Id})(\nabla v)^t)\mathrm{cof}\,(\nabla\hat{\chi}))+\mu\nabla\cdot((\nabla v+\nabla v^t)(\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id}))\\
&+\mu'\nabla\cdot(\nabla v(((\nabla\hat{\chi})^{-1}-\mathrm{Id}):\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}))+\mu'\nabla\cdot((\nabla v:\mathrm{Id})(\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id}))
\end{aligned}
\tag{77}
$$

and, for $i=1,2,3$,

$$
\begin{aligned}
\widetilde{G}_i := &-\mu\left[(\nabla v((\nabla\hat{\chi})^{-1}-\mathrm{Id})+((\nabla\hat{\chi})^{-t}-\mathrm{Id})(\nabla v)^t)\mathrm{cof}\,(\nabla\hat{\chi})n\right]_i-\mu\left[(\nabla v+(\nabla v)^t)(\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id})n\right]_i\\
&-\mu'\left[(\nabla v((\nabla\hat{\chi})^{-1}-\mathrm{Id}):\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi})n\right]_i-\mu'\left[(\nabla v:\mathrm{Id})(\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id})n\right]_i\\
&+\left[(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi})n\right]_i+\sum_{\alpha,j,\beta=1}^{3}\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha+\tilde{h}_i.
\end{aligned}
\tag{78}
$$

Let us show that $F\in L^\infty(L^2)$ and $\widetilde{G}\in L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))$ with suitable estimates. In order to estimate F, we use (22), (23) and (36) :

$$
\|F\|_{L^\infty(L^2)}\leqslant C(\overline{\rho}+C_0+T^\kappa M)\|\partial_t v\|_{L^\infty(L^2)}+f(\|\gamma_0\|_{H^3},\|u_0\|_{H^6},\|\xi_1\|_{H^3})\|\gamma\|_{L^\infty(H^1)}+\|v\|_{L^\infty(H^2)}T^\kappa M.
$$

Here, we have also used that

$$
\|P(\overline{\rho}+\gamma)-P(\overline{\rho})\|_{L^\infty(H^1)}\leqslant\|P'\|_{L^\infty(I)}\|\gamma\|_{L^\infty(H^1)}\leqslant f(\|\gamma_0\|_{H^3},\|u_0\|_{H^6},\|\xi_1\|_{H^3})\|\gamma\|_{L^\infty(H^1)},
$$

where $I\subset\mathbb{R}_+$ is an interval satisfying $\overline{\rho}\in I$ and $I\supset[\overline{\rho}+\gamma_{min},\overline{\rho}+1+C_0]$.

Using the same estimates as for F, we get

$$
\begin{aligned}
\|\widetilde{G}\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))}\leqslant &f(\|\gamma_0\|_{H^3},\|u_0\|_{H^6},\|\xi_1\|_{H^3})\|\gamma\|_{L^\infty(H^1)}+\|v\|_{L^\infty(H^2)}T^\kappa M\\
&+\Big\|\sum_{\alpha,j,\beta=1}^{3}\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha+\tilde{h}_i\Big\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))}.
\end{aligned}
$$

For the last term we use (24) and we obtain

$$
\Big\|\sum_{\alpha,j,\beta=1}^{3}\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha+\tilde{h}_i\Big\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))}\leqslant C(1+T^\kappa M)\|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))}+CT^\kappa M\|\tilde{\xi}\|_{L^\infty(0,T;H^2(\Omega_S(0)))}
$$

Using the elliptic regularity for (76), we obtain

$$
\begin{aligned}
\|v\|_{L^\infty(H^2)}\leqslant &f(\|\gamma_0\|_{H^3},\|u_0\|_{H^6},\|\xi_1\|_{H^3})\|\gamma\|_{L^\infty(H^1)}+C(\overline{\rho}+C_0+T^\kappa M)\|\partial_t v\|_{L^\infty(L^2)}\\
&+\|v\|_{L^\infty(H^2)}T^\kappa M+C(1+T^\kappa M)\|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))}+CT^\kappa M\|\tilde{\xi}\|_{L^\infty(0,T;H^2(\Omega_S(0)))}.
\end{aligned}
\tag{79}
$$

Let us take a look now at the equation satisfied by ξ:

$$
\left\{
\begin{array}{ll}
-\nabla\cdot(2\lambda\epsilon(\xi)+\lambda'(\nabla\cdot\xi)\mathrm{Id})=H & \text{in } \Omega_S(0),\\
\xi(t,\cdot)=\displaystyle\int_0^t v & \text{on } \partial\Omega_S(0).
\end{array}
\right.
\tag{80}
$$

Here, we have denoted

$$
H_i:=-\partial^2_{tt}\xi_i+\sum_{\alpha,j,\beta=1}^{3}(c^\ell_{i\alpha j\beta}(\nabla\hat{\xi})+c^q_{i\alpha j\beta}(\nabla\hat{\xi}))\partial^2_{\alpha\beta}\xi_j,
\tag{81}
$$

for $i = 1, 2, 3$. We have to estimate this term in $L^\infty(0, T; L^2(\Omega_S(0)))$. Observe that the term in c^ℓ and c^q is estimated thanks to (24). Using again classical elliptic estimates, we obtain

$$\begin{aligned}\|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))} &\leqslant C(\|\partial_t^2\xi\|_{L^\infty(0,T;L^2(\Omega_S(0)))} + T^\kappa M\|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))} + \|\int_0^t v\|_{L^\infty(H^2)}),\\ &\leqslant C(\|\partial_t^2\xi\|_{L^\infty(0,T;L^2(\Omega_S(0)))} + T^\kappa M\|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))} + T\|v\|_{L^\infty(H^2)}).\end{aligned}$$

Combining this estimate with (79) and taking T small enough with respect to M, we get

$$\begin{aligned}&\|v\|_{L^\infty(H^2)} + \|\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant f(\|\gamma_0\|_{H^3}, \|u_0\|_{H^6}, \|\xi_1\|_{H^3})\|\gamma\|_{L^\infty(H^1)}\\ &+\|\partial_t^2\xi\|_{L^\infty(0,T;L^2(\Omega_S(0)))} + C(\overline{\rho} + C_0 + T^\kappa M)\|\partial_t v\|_{L^\infty(L^2)} + CT^\kappa M\|\tilde{\xi}\|_{L^\infty(0,T;H^2(\Omega_S(0)))}.\end{aligned} \tag{82}$$

Coming back to (75), we deduce

$$\|v\|_{Y_2^T} + \|\xi\|_{X_2^T} \leqslant C_0 + T^\kappa M\|\tilde{\xi}\|_{X_2} + (C + C_0 + f(\|\gamma_0\|_{H^3}, \|u_0\|_{H^6}, \|\xi_1\|_{H^3}))\|\gamma\|_{Y_2^T}, \tag{83}$$

for T small with respect to M and the initial conditions.

Remark 12 *Looking at the computations made above, we observe that the term $\|\gamma\|_{Y_2^T}$ in (83) only comes from the pressure term*

$$(P(\overline{\rho} + \gamma) - P(\overline{\rho}))cof(\nabla\hat{\chi}).$$

• *Step 3.* Fixed point argument.

Here we are going to prove that Λ_0, which was defined at the beginning of the proof, is a contraction. Let $\tilde{\xi}^a, \tilde{\xi}^b \in X_2^T$. For $c = a$, b, we denote by (v^c, ξ^c) the solution of (45) with

$$\tilde{h}_i^c := -\sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t \partial_s\hat{c}_{i\alpha j\beta}\partial_\beta\tilde{\xi}_j^c ds\right)n_\alpha,\ i = 1, 2, 3,$$

instead of g_i, that is to say, $\xi^c = \Lambda_0(\tilde{\xi}^c)$. Observe that $(v, \xi) := (v^a - v^b, \xi^a - \xi^b)$ satisfy

$$\left\{\begin{array}{ll} (\overline{\rho} + \gamma)\det\nabla\hat{\chi}\partial_t v - \nabla\cdot\widehat{\mathbb{T}}_1(v) = 0 & \text{in } Q_T,\\ \partial_t^2\xi_i - \displaystyle\sum_{\alpha,j,\beta=1}^{3}\hat{c}_{i\alpha j\beta}\partial^2_{\alpha\beta}\xi_j = 0, \quad i = 1, 2, 3, & \text{in } (0, T)\times\Omega_S(0),\\ v = 0 & \text{on } (0, T)\times\partial\Omega,\\ v = \partial_t\xi & \text{on } \Sigma_T,\\ \left[\widehat{\mathbb{T}}_1(v)\mathrm{n}\right]_i = \displaystyle\sum_{\alpha,j,\beta=1}^{3}\hat{c}_{i\alpha j\beta}\partial_\beta\xi_j n_\alpha + \tilde{h}_i^a - \tilde{h}_i^b,\ i = 1, 2, 3, & \text{on } \Sigma_T,\\ v(0,\cdot) = 0 & \text{in } \Omega_F(0),\\ \xi(0,\cdot) = 0,\ \partial_t\xi(0,\cdot) = 0 & \text{in } \Omega_S(0). \end{array}\right. \tag{84}$$

Recall that $\widehat{\mathbb{T}}_1(v)$ was defined in (46).

Let us apply estimate (83) to (v, ξ). Taking into account the definition of C_0 (see (35)) and Remark 12, we obtain in particular that

$$\|\xi\|_{X_2^T} \leqslant T^\kappa M\|\tilde{\xi}^a - \tilde{\xi}^b\|_{X_2^T}.$$

Taking T small enough with respect to M, we find that Λ_0 is a contraction from X_2^T into itself. This gives the existence and uniqueness of a fixed point $\xi \in X_2^T$ which is, together with v, a solution of (61).

Finally, we apply estimate (83) to the fixed point. Here, we estimate $\|\gamma\|_{Y_2^T}$ using Lemma 8, we take T small enough with respect to M and the initial conditions and we deduce (62).

2.4 Regularity of the solution of the linear system

In this subsection, we will prove the following proposition which gives a regularity result for the solution of system (61).

Proposition 13 *Let $(\hat{v},\hat{\xi}) \in A_M^T$, $u_0 \in H^6(\Omega_F(0))$, $\xi_1 \in H^3(\Omega_S(0))$ and $\gamma_0 \in H^3(\Omega_F(0))$ satisfying (16). For T small enough with respect to M and the initial conditions (see (48)), the solution (v,ξ) of (61) belongs to $Y_4^T \times X_4^T$ (recall that Y_4^T and X_4^T have been defined in Definition 1). Moreover, there exists $C_0 > 0$ and $\kappa > 0$ such that*

$$\|v\|_{Y_4^T} + \|\xi\|_{X_4^T} \leqslant C_0 + T^\kappa M. \tag{85}$$

Proof:

• *Step 1.* Time regularity of v and ξ.

Let us differentiate three times with respect to time the first equation of system (61). We obtain in Q_T

$$\begin{aligned}(\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_t^4 v + 3\partial_t((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_t^3 v + 3\partial_t^2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_t^2 v \\ +\partial_t^3((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_t v - \nabla\cdot\partial_t^3\left[\widehat{\mathbb{T}}(v,\gamma)\right] = 0.\end{aligned} \tag{86}$$

Then, we multiply this equation by $\partial_s^3 v$ and integrate on Q_t. For the first four terms, we argue as in the proof of Lemma 9 and Proposition 11 and we obtain that, for T small enough with respect to M,

$$\begin{aligned}&\iint_{Q_t}\left[(\overline{\rho}+\gamma)\det\nabla\hat{\chi}\partial_s^4 v + 3\partial_s((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_s^3 v + 3\partial_s^2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_s^2 v\right.\\ &\left.+\partial_s^3((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\partial_s v\right]\partial_s^3 v\,dy\,ds \geqslant \frac{\rho_{min}}{4}\int_{\Omega_F(0)}|\partial_t^3 v(t)|^2\,dy - \frac{1}{2}\int_{\Omega_F(0)}\rho_0|\partial_t^3 v(0)|^2\,dy \\ &\qquad - CT\|v\|_{Y_4^T}^2\sum_{k=1}^3\|\partial_t^k((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\|_{L^\infty(H^{3-k})}.\end{aligned} \tag{87}$$

Using Lemmas 5 and 8, we have

$$\sum_{k=1}^3\|\partial_t^k((\overline{\rho}+\gamma)\det\nabla\hat{\chi})\|_{L^\infty(H^{3-k})} \leqslant C(C_0 + M + C_0 M) \tag{88}$$

for T small enough with respect to M.

For the last term in (86), we have

$$\begin{aligned}-\iint_{Q_t}\partial_s^3\nabla\cdot\widehat{\mathbb{T}}(v,\gamma)\partial_s^3 v\,dy\,ds &= \iint_{Q_t}\partial_s^3\widehat{\mathbb{T}}(v,\gamma) : \partial_s^3\nabla v\,dy\,ds \\ &+ \sum_{i=1}^3\iint_{\Sigma_t}\sum_{\alpha,j,\beta=1}^3\partial_s^2(\hat{c}_{i\alpha j\beta}\partial_{s\beta}\xi_j)n_\alpha\partial_s^4\xi_i\,d\sigma\,ds.\end{aligned} \tag{89}$$

For the first integral in the right-hand side of (89), we first estimate the term corresponding to $\widehat{\mathbb{T}}_1(v)$:

$$\begin{aligned}\iint_{Q_t}\partial_s^3\widehat{\mathbb{T}}_1(v) : \partial_s^3\nabla v\,dy\,ds &\geqslant C\iint_{Q_t}|\partial_s^3\epsilon(v)|^2\,dy\,ds - CT^\kappa M\|\partial_t^3 v\|_{L^2(H^1)}^2 \\ &- CT^{1/2}\|v\|_{Y_4}^2\sum_{k=1}^3\left(\|\partial_t^k(\nabla\hat{\chi})^{-1}\|_{L^\infty(L^6)} + \|\partial_t^k(\operatorname{cof}\nabla\hat{\chi})\|_{L^\infty(L^6)}\right).\end{aligned} \tag{90}$$

From (22) and taking T small with respect to M, we have

$$\sum_{k=1}^3\left(\|\partial_t^k(\nabla\hat{\chi})^{-1}\|_{L^\infty(L^6)} + \|\partial_t^k(\operatorname{cof}\nabla\hat{\chi})\|_{L^\infty(L^6)}\right) \leqslant CM. \tag{91}$$

For the resting term, we prove thanks to Lemmas 5 and 8 that

$$\|\partial_s^3[(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}]\|_{L^\infty(L^2)} \leqslant C(C_0+M+C_0M),$$

for T small with respect to the initial conditions. Then, we get

$$\begin{aligned}\iint_{Q_t} \partial_s^3[(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}] : \partial_s^3\nabla v\,dy\,ds &\leqslant \delta\|\partial_t^3\nabla v\|^2_{L^2(L^2)} + CT(C_0+M^2+C_0M)\\ &\leqslant \delta\|\partial_t^3\nabla v\|^2_{L^2(L^2)} + CT^\kappa(C_0+M),\end{aligned} \tag{92}$$

for T small with respect to M and the initial conditions. Taking into account (87)-(92), we deduce

$$\begin{aligned}&\frac{\rho_{min}}{4}\int_{\Omega_F(0)}|\partial_t^3 v(t)|^2dy + \sum_{i,\alpha,j,\beta=1}^3\iint_{\Sigma_t}\partial_s^2(\hat{c}_{i\alpha j\beta}\partial_{s\beta}\xi_j)n_\alpha\partial_s^4\xi_i\,d\sigma\,ds + C\iint_{Q_t}|\partial_s^3\epsilon(v)|^2dy\,ds\\ &\leqslant \frac{1}{2}\int_{\Omega_F(0)}\rho_0|\partial_t^3 v(0)|^2\,dy + CT^\kappa(C_0+M)\|v\|^2_{Y_4^T} + \delta\|\partial_t^3\nabla v\|^2_{L^2(L^2)} + CT^\kappa(C_0+M^2).\end{aligned} \tag{93}$$

Let us now estimate the boundary term in (93) :

$$\sum_{i,\alpha,j,\beta=1}^3\iint_{\Sigma_t}\partial_s^2(\hat{c}_{i\alpha j\beta}\partial_{s\beta}\xi_j)n_\alpha\partial_s^4\xi_i\,d\sigma\,ds = \sum_{i,\alpha,j,\beta=1}^3\iint_{\Sigma_t}\hat{c}_{i\alpha j\beta}\partial_s^3\partial_\beta\xi_j n_\alpha\partial_s^4\xi_i\,d\sigma\,ds + A_3, \tag{94}$$

where

$$|A_3| \leqslant \iint_{\Sigma_t}\sum_{i,\alpha,j,\beta=1}^3\left(|\partial_s^2\hat{c}_{i\alpha j\beta}||\partial_{s\beta}\xi_j| + 2|\partial_s\hat{c}_{i\alpha j\beta}||\partial_s^2\partial_\beta\xi_j|\right)|\partial_s^3 v_i|\,d\sigma\,ds.$$

Here, we have used that $\partial_s\xi = v$ on Σ_t.

From (24) and the definition of X_4^T and Y_4^T we deduce that

$$|A_3| \leqslant CT^{1/2}(M+M^2)\|\xi\|_{X_4}\|v\|_{Y_4}.$$

Combining this with (94), we deduce from (93) :

$$\begin{aligned}&\frac{\rho_{min}}{4}\int_{\Omega_F(0)}|\partial_t^3 v(t)|^2dy + \sum_{i,\alpha,j,\beta=1}^3\iint_{\Sigma_t}\hat{c}_{i\alpha j\beta}\partial_s^3\partial_\beta\xi_j n_\alpha\partial_s^4\xi_i\,d\sigma\,ds + C\iint_{Q_t}|\partial_s^3\epsilon(v)|^2dy\,ds\\ &\leqslant \frac{1}{2}\int_{\Omega_F(0)}\rho_0|\partial_t^3 v(0)|^2\,dy + CT^\kappa(C_0+M)(\|v\|^2_{Y_4^T}+\|\xi\|^2_{X_4^T}) + CT^\kappa(C_0+M^2) + \delta\|\partial_t^3\nabla v\|^2_{L^2(L^2)}.\end{aligned} \tag{95}$$

Remark 14 *Observe that, thanks to the assumptions $u_0 \in H^6(\Omega_F(0))$ and $\rho_0 \in H^3(\Omega_F(0))$ (see (9)), we have that*

$$\int_{\Omega_F(0)}\rho_0|\partial_t^3 v(0)|^2\,dy = C_0.$$

In order to deal with the remaining boundary term in (95), we differentiate three times with respect to t the equation satisfied by ξ (see (61)), we multiply it by $\partial_s^4\xi$ and integrate on Q_t. We obtain

$$\frac{1}{2}\int_{\Omega_S(0)}|\partial_t^4\xi(t)|^2dy - \sum_{i,\alpha,j,\beta=1}^3\int_0^t\!\!\int_{\Omega_S(0)}\hat{c}_{i\alpha j\beta}\partial_s^3\partial^2_{\alpha\beta}\xi_j\partial_s^4\xi_i\,dy\,ds + B_1 = \frac{1}{2}\int_{\Omega_S(0)}|\partial_t^4\xi(0)|^2dy, \tag{96}$$

where we can estimate B_1, thanks to (24), as follows :

$$|B_1| \leqslant CT(M+M^2)\|\xi\|^2_{X_4^T}. \tag{97}$$

We integrate by parts in the second term :

$$\int_0^t\int_{\Omega_S(0)} \hat{c}_{i\alpha j\beta}\partial_s^3\partial^2_{\alpha\beta}\xi_j\partial_s^4\xi_i\,dy\,ds = -\int_0^t\int_{\Omega_S(0)} \hat{c}_{i\alpha j\beta}\partial_s^3\partial_\beta\xi_j\partial_s^4\partial_\alpha\xi_i\,dy\,ds$$
$$-\int_0^t\int_{\Omega_S(0)} \partial_\alpha\hat{c}_{i\alpha j\beta}\partial_s^3\partial_\beta\xi_j\partial_s^4\xi_i\,dy\,ds + \iint_{\Sigma_t} \hat{c}_{i\alpha j\beta}\partial_s^3\partial_\beta\xi_j\partial_s^4\xi_i\,n_\alpha\,d\sigma\,ds.$$

We integrate by parts in time in the second term and we use (14). This yields :

$$\int_0^t\int_{\Omega_S(0)} \hat{c}_{i\alpha j\beta}\partial_s^3\partial^2_{\alpha\beta}\xi_j\partial_s^4\xi_i\,dy\,ds = -\frac{1}{2}\int_{\Omega_S(0)} \left[(\hat{c}_{i\alpha j\beta}\partial_t^3\partial_\beta\xi_j\partial_t^3\partial_\alpha\xi_i)(t) - (\hat{c}_{i\alpha j\beta}\partial_t^3\partial_\beta\xi_j\partial_t^3\partial_\alpha\xi_i)(0)\right]\,dy$$
$$-\int_0^t\int_{\Omega_S(0)} (\partial_\alpha\hat{c}_{i\alpha j\beta}\partial_s^4\xi_i - \frac{1}{2}\partial_s\hat{c}_{i\alpha j\beta}\partial_s^3\partial_\alpha\xi_i)\partial_s^3\partial_\beta\xi_j\,dy\,ds + \iint_{\Sigma_t} \hat{c}_{i\alpha j\beta}\partial_s^3\partial_\beta\xi_j\partial_s^4\xi_i\,n_\alpha\,d\sigma\,ds.$$

One can easily prove that the first term in the second line is estimated like in (97). Combining this with (96) and taking into account (97), we get

$$\begin{aligned}&\frac{1}{2}\int_{\Omega_S(0)} |\partial_t^4\xi(t)|^2dy + \frac{1}{2}\sum_{i,\alpha,j,\beta=1}^{3}\int_{\Omega_S(0)} (\hat{c}_{i\alpha j\beta}\partial_t^3\partial_\beta\xi_j\partial_t^3\partial_\alpha\xi_i)(t)\,dy\\ &- \sum_{i,\alpha,j,\beta=1}^{3}\iint_{\Sigma_t} \hat{c}_{i\alpha j\beta}\partial_s^3\partial_\beta\xi_j\partial_s^4\xi_i\,n_\alpha\,d\sigma\,ds \leqslant C_0 + CT(M+M^2)\|\xi\|^2_{X_4^T}.\end{aligned} \tag{98}$$

Here, we have used that

$$\frac{1}{2}\int_{\Omega_S(0)} |\partial_t^4\xi(0)|^2dy + \frac{1}{2}\int_{\Omega_S(0)} (\hat{c}_{i\alpha j\beta}\partial_t^3\partial_\beta\xi_j\partial_t^3\partial_\alpha\xi_i)(0)\,dy \leqslant C\|\xi_1\|^2_{H^3} = C_0.$$

Combining (98) and (95), we see that the boundary terms simplify. On the other hand, using Körn's inequality and (25) we obtain

$$\begin{aligned}\|\partial_t^3 v\|^2_{L^\infty(L^2)} + \|\partial_t^3 v\|^2_{L^2(H^1)} + \|\partial_t^4\xi\|^2_{L^\infty(0,T;L^2(\Omega_S(0)))} + \|\partial_t^3\xi\|^2_{L^\infty(0,T;H^1(\Omega_S(0)))}\\ \leqslant C_0 + T^\kappa(C_0+M)(\|v\|^2_{Y_4^T} + \|\xi\|^2_{X_4^T}) + T^\kappa M^2.\end{aligned}$$

Using that T is small, we also have

$$\begin{aligned}\|v\|^2_{W^{3,\infty}(L^2)} + \|v\|^2_{W^{2,\infty}(H^1)} + \|v\|^2_{H^3(H^1)} + \|\xi\|^2_{W^{4,\infty}(0,T;L^2(\Omega_S(0)))} + \|\xi\|^2_{W^{3,\infty}(0,T;H^1(\Omega_S(0)))}\\ \leqslant C_0 + T^\kappa(\|v\|^2_{Y_4^T} + \|\xi\|^2_{X_4^T}) + T^\kappa M^2.\end{aligned} \tag{99}$$

• *Step 2.* Regularity in space of v and ξ.

We divide this step in two parts :

- *Step 2.1.* Let us first prove that $v \in W^{2,\infty}(H^2)$ and $\xi \in W^{2,\infty}(0,T;H^2(\Omega_S(0)))$.

We first consider the stationary system satisfied by v :

$$\begin{cases} -\nabla\cdot(\mu(\nabla v + \nabla v^t) + \mu'(\nabla\cdot v)\mathrm{Id}) = F & \text{in } \Omega_F(0),\\ v = 0 & \text{on } \partial\Omega\\ (\mu(\nabla v + \nabla v^t) + \mu'\nabla\cdot v)n = G & \text{on } \partial\Omega_S(0).\end{cases} \tag{100}$$

In this system, F is given by (77) and G is given by (78) where the last two terms are replaced by

$$\sum_{\alpha,j,\beta=1}^{3}\int_0^t (\hat{c}_{i\alpha j\beta}\partial_\beta\partial_s\xi_j n_\alpha)\,ds.$$

Using (36) and

$$\|\partial_t^k \det(\nabla\hat{\chi})\|_{L^\infty(H^{3-k})} + \|\partial_t^k \mathrm{cof}\,(\nabla\hat{\chi})\|_{L^\infty(H^{3-k})} \leqslant C_0 + T^\kappa M \ \ (k=1,2), \tag{101}$$

we deduce that the first two terms of F are estimated as follows :

$$\|(\overline{\rho}+\gamma)\det(\nabla\hat{\chi})\partial_t v\|_{W^{2,\infty}(L^2)} \leqslant (C + C_0 + T^\kappa M)(\|v\|_{W^{3,\infty}(L^2)} + \|v\|_{W^{2,\infty}(H^1)}) \tag{102}$$

and

$$\|\nabla\cdot((P(\gamma+\overline{\rho}) - P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}))\|_{W^{2,\infty}(L^2)} \leqslant C_0 + T^\kappa M, \tag{103}$$

thanks to (48).

Using (101), the first term in the second line of (77) is estimated in $L^\infty(L^2)$ by

$$C(\|\nabla v\|_{W^{2,\infty}(H^1)}\|\mathrm{cof}\,(\nabla\hat{\chi})\|_{W^{2,\infty}(H^2)}\|(\nabla\hat{\chi})^{-1} - \mathrm{Id}\|_{L^\infty(H^3)} + (C_0 + T^\kappa M)\|v\|_{W^{1,\infty}(H^2)}).$$

From (23), (101) and the interpolation inequality

$$\|v\|_{W^{1,\infty}(H^2)} \leqslant \|v\|^{1/2}_{W^{1,\infty}(H^1)}\|v\|^{1/2}_{W^{1,\infty}(H^3)} \leqslant \|v\|^{1/2}_{W^{1,\infty}(H^1)}\|v\|^{1/2}_{Y_4^T},$$

we find that the first term in the second line of (77) is estimated in $L^\infty(L^2)$ by

$$\delta\|v\|_{Y_4^T} + (C_0 + T^\kappa M)\|v\|_{W^{1,\infty}(H^1)}.$$

The other terms in (77) can be estimated analogously.

Combining this with (102), (103) and (99), we obtain

$$\|F\|_{W^{2,\infty}(L^2)} \leqslant C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}). \tag{104}$$

Concerning the term G, we get

$$\|G\|_{W^{2,\infty}(H^{1/2}(\partial\Omega_S(0)))} \leqslant C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}) + \left\| \sum_{\alpha,j,\beta=1}^{3} \int_0^t (\hat{c}_{i\alpha j\beta}\partial_\beta\partial_s\xi_j n_\alpha)\, ds \right\|_{W^{2,\infty}(0,T;H^1(\Omega_S(0)))}, \tag{105}$$

where we still denote n a regular extension of the normal vector to all $\Omega_S(0)$. First, noticing that

$$\|\hat{c}_{i\alpha j\beta}\|_{W^{2,\infty}(0,T;H^1(\Omega_S(0)))} + \|\hat{c}_{i\alpha j\beta}\|_{W^{1,\infty}(0,T;H^2(\Omega_S(0)))} \leqslant C(1 + M + M^2)$$

for all $i,\alpha,j,\beta \in \{1,2,3\}$ (see (24)),

$$\|\hat{c}_{i\alpha j\beta}\nabla\partial_t\xi\|_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))} + \|\hat{c}_{i\alpha j\beta}\nabla\partial_t\xi\|_{W^{2,\infty}(0,T;L^2(\Omega_S(0)))} \leqslant C(1 + M + M^2)\|\xi\|_{X_4^T}$$

and so

$$\sum_{\alpha,j,\beta=1}^{3} \left\| \int_0^t \hat{c}_{i\alpha j\beta}\partial_\beta\partial_s\xi_j n_\alpha\, ds \right\|_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))\cap W^{2,\infty}(0,T;L^2(\Omega_S(0)))} \leqslant C_0 + T^\kappa\|\xi\|_{X_4^T}. \tag{106}$$

Then, we observe that

$$\partial_t\nabla(\hat{c}_{i\alpha j\beta}\partial_t\nabla\xi) = \hat{c}_{i\alpha j\beta}\partial_t^2\nabla\nabla\xi + R^1_{i\alpha j\beta}$$

where

$$\|\partial_t R^1_{i\alpha j\beta}\|_{L^\infty(0,T;L^2(\Omega_S(0)))} \leqslant C(M + M^2)\|\xi\|_{X_4^T}.$$

Then, taking into account that $c^\ell_{i\alpha j\beta}(\nabla\hat{\xi})_{|t=0} = 0$, $c^q_{i\alpha j\beta}(\nabla\hat{\xi})_{|t=0} = 0$ and using (24), we find

$$\|\partial_t\nabla(\hat{c}_{i\alpha j\beta}\partial_t\nabla\xi)\|_{L^\infty(L^2)} \leqslant C\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} + CT(M + M^2)\|\xi\|_{W^{2,\infty}(H^2)} + C_0 + T^\kappa\|\xi\|_{X_4^T}.$$

Using (106) and this last inequality, we find from (105)

$$\|G\|_{W^{2,\infty}(0,T;H^{1/2}(\partial\Omega_S(0)))} \leqslant C\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} + C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}).$$

Using regularity estimates for system (100), we deduce

$$\|v\|_{W^{2,\infty}(H^2)} \leqslant C\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} + C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}). \tag{107}$$

Next, we consider the stationary system (80) where H is given by (81). Then, we have

$$\partial_t^2 H_i = -\partial_t^4 \xi_i + \sum_{\alpha,j,\beta=1}^{3} \partial_t^2[(c^\ell_{i\alpha j\beta}(\nabla\hat{\xi}) + c^q_{i\alpha j\beta}(\nabla\hat{\xi}))\partial^2_{\alpha\beta}\xi_j]. \tag{108}$$

Let $L_{i\alpha j\beta} := \partial_t^2[(c^\ell_{i\alpha j\beta}(\nabla\hat{\xi}) + c^q_{i\alpha j\beta}(\nabla\hat{\xi}))\partial^2_{\alpha\beta}\xi_j]$ for $1 \leqslant i,\alpha,j,\beta \leqslant 3$. Using (24) and the fact that $|\partial_t c^\ell_{i\alpha j\beta}(\nabla\hat{\xi})| \leqslant C|\xi_1|$ in $\Omega_S(0)$, we obtain

$$\begin{aligned}
\|L_{i\alpha j\beta}\|_{L^\infty(0,T;L^2(\Omega_S(0)))} &\leqslant \|L_{i\alpha j\beta}(0,\cdot)\|_{L^2(\Omega_S(0))} + \left\|\int_0^t \partial_s L_{i\alpha j\beta}(s,\cdot)ds\right\|_{L^\infty(0,T;L^2(\Omega_S(0)))} \\
&\leqslant C(\|\xi_1\|^2_{H^3} + T(M+M^2)\|\xi\|_{X_4^T}) + \left\|\int_0^t (c^\ell_{i\alpha j\beta}(\nabla\hat{\xi}) + c^q_{i\alpha j\beta}(\nabla\hat{\xi}))\partial_s^3\partial^2_{\alpha\beta}\xi_j ds\right\|_{L^\infty(0,T;L^2(\Omega_S(0)))}.
\end{aligned}$$

Integrating by parts in time in the last term and using (24), we deduce

$$\|L_{i\alpha j\beta}\|_{L^\infty(0,T;L^2(\Omega_S(0)))} \leqslant C(\|\xi_1\|^2_{H^3} + T(M+M^2)\|\xi\|_{X_4^T}),\ 1 \leqslant i,\alpha,j,\beta \leqslant 3. \tag{109}$$

From (108)-(109), we find

$$\|H\|_{W^{2,\infty}(0,T;L^2(\Omega_S(0)))} \leqslant C_0 + T^\kappa\|\xi\|_{X_4^T} + \|\xi\|_{W^{4,\infty}(0,T;L^2(\Omega_S(0)))}.$$

Using elliptic regularity for system (80), we deduce

$$\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} \leqslant C_0 + T^\kappa\|\xi\|_{X_4} + \|\xi\|_{W^{4,\infty}(0,T;L^2(\Omega_S(0)))} + \left\|\int_0^t v\right\|_{W^{2,\infty}(H^2)}.$$

Using the definition of Y_4^T and (99), we get

$$\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} \leqslant C_0 + T^\kappa M + T^\kappa(\|\xi\|_{X_4^T} + \|v\|_{Y_4^T}).$$

Combining this estimate with (107), we obtain

$$\|v\|_{W^{2,\infty}(H^2)} + \|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} \leqslant C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}). \tag{110}$$

- *Step 2.2.* Next we prove that $v \in L^\infty(H^4)$ and $\xi \in L^\infty(0,T;H^4(\Omega_S(0)))$.

We first estimate v in $L^\infty(H^4)$. In order to do this, we consider again system (100) and we estimate $\|F\|_{L^\infty(H^2)}$ and $\|G\|_{L^\infty(0,T;H^{5/2}(\partial\Omega_S(0)))}$. Using (21)-(23), (36) and (110), we find

$$\|F\|_{L^\infty(H^2)} \leqslant C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}).$$

Analogously, for G we get

$$\|G\|_{L^\infty(0,T;H^{5/2}(\partial\Omega_S(0)))} \leqslant C_0 + T^\kappa M + T^\kappa\|v\|_{Y_4^T} + \left\|\sum_{\alpha,j,\beta=1}^{3}\int_0^t (\hat{c}_{i\alpha j\beta}\partial_\beta\partial_s\xi_j n_\alpha)\,ds\right\|_{L^\infty(0,T;H^3(\Omega_S(0)))}. \tag{111}$$

In order to estimate this last term we first observe that, since $\|\hat{c}_{i\alpha j\beta}\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant C(1+M+M^2)$ for all $i,\alpha,j,\beta \in \{1,2,3\}$ and $\|\nabla\partial_t\xi\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant \|\xi\|_{X_4^T}$, we have

$$\left\| \sum_{\alpha,j,\beta=1}^{3} \int_0^t (\hat{c}_{i\alpha j\beta}\partial_\beta\partial_s\xi_j n_\alpha)\, ds \right\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant CT(1+M+M^2)\|\xi\|_{X_4^T} \leqslant T^\kappa\|\xi\|_{X_4^T}, \tag{112}$$

taking T small with respect to M. On the other hand, for any $\vec{\alpha} \in \mathbb{N}^3$ with $|\vec{\alpha}| = 3$ we have

$$\partial^{\vec{\alpha}}(\hat{c}_{i\alpha j\beta}\partial_\beta\partial_s\xi_j) = \hat{c}_{i\alpha j\beta}\partial^{\vec{\alpha}}(\partial_\beta\partial_s\xi_j) + R^2_{i\alpha j\beta}. \tag{113}$$

Then, from (24) one can prove that

$$\|R^2_{i\alpha j\beta}\|_{L^\infty(0,T;L^2(\Omega_S(0)))} \leqslant C(M+M^2)\|\xi\|_{X_4^T}. \tag{114}$$

We integrate by parts in the first term of (113) and we obtain

$$\int_0^t (\hat{c}_{i\alpha j\beta}\partial^{\vec{\alpha}}(\partial_\beta\partial_s\xi_j))(s)ds = -\int_0^t (\partial_s\hat{c}_{i\alpha j\beta}\partial^{\vec{\alpha}}(\partial_\beta\xi_j))(s)ds + (\hat{c}_{i\alpha j\beta}\partial^{\vec{\alpha}}(\partial_\beta\xi_j))(t). \tag{115}$$

Combining this identity with (112)-(115), we obtain the following from (111) and taking T small with respect to M :

$$\|G\|_{L^\infty(0,T;H^{5/2}(\partial\Omega_S(0)))} \leqslant C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}) + C\|\xi\|_{L^\infty(0,T;H^4(\Omega_S(0)))}.$$

From elliptic estimates for system (100), we get

$$\|v\|_{L^\infty(H^4)} \leqslant C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}) + C\|\xi\|_{L^\infty(0,T;H^4(\Omega_S(0)))}. \tag{116}$$

We consider now the elliptic system satisfied by ξ given by (80) where H is defined by (81). Using here again that

$$\|c^\ell_{i\alpha j\beta}(\nabla\hat{\xi})+c^q_{i\alpha j\beta}(\nabla\hat{\xi})\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant CT\|\partial_t(c^\ell_{i\alpha j\beta}(\nabla\hat{\xi})+c^q_{i\alpha j\beta}(\nabla\hat{\xi}))\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant CT(M+M^2)$$

for $1 \leqslant i,\alpha,j,\beta \leqslant 3$, we directly obtain

$$\|H\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant C\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} + CT(M+M^2)\|\xi\|_{X_4^T} \leqslant C\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} + T^\kappa\|\xi\|_{X_4^T}.$$

Using elliptic estimates for this system, we find

$$\|\xi\|_{L^\infty(0,T;H^4(\Omega_S(0)))} \leqslant C\|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} + T^\kappa\|\xi\|_{X_4^T} + CT\|v\|_{L^\infty(H^4)}.$$

Combining this estimate with (116) and taking into account (110) we obtain

$$\|\xi\|_{L^\infty(0,T;H^4(\Omega_S(0)))} + \|\xi\|_{W^{2,\infty}(0,T;H^2(\Omega_S(0)))} + \|v\|_{L^\infty(H^4)} + \|v\|_{W^{2,\infty}(H^2)} \leqslant C_0 + T^\kappa M + T^\kappa(\|v\|_{Y_4^T} + \|\xi\|_{X_4^T}).$$

Finally, we combine this with (99) and we obtain the desired estimate (85).

3 Fixed point argument

According to Lemma 8 and Proposition 13, there exist $\overline{C}_0 > 0$ and $\overline{\kappa} > 0$ such that, for all $M > 0$ and all $(\hat{v},\hat{\xi}) \in A^T_M$, there exists $T_1 > 0$ such that the solution (γ, v, ξ) of (27), $(33)_1$ and (61) satisfies

$$\|\gamma\|_{W^{k,\infty}(H^{3-k})} + \|v\|_{Y_4^T} + \|\xi\|_{X_4^T} \leqslant \overline{C}_0 + T^{\overline{\kappa}}M$$

for all $T \leqslant T_1$ and all $0 \leqslant k \leqslant 3$.

Let us take $\overline{M} = 2\overline{C}_0$ and let us define $\overline{T} \leqslant T_1$ such that $2\overline{T}^{\overline{\kappa}} \leqslant \frac{1}{2}$. Then, we get

$$\|\gamma\|_{W^{k,\infty}(H^{3-k})} + \|v\|_{Y_4^T} + \|\xi\|_{X_4^T} \leqslant \overline{M} \tag{117}$$

for all $T \leqslant \overline{T}$ and all $0 \leqslant k \leqslant 3$.

We apply the following contraction fixed-point theorem (see [34], p. 17):

Theorem 15 *Let K be a nonempty, closed subset of a Banach space Z and suppose that $\Lambda : K \to K$ satisfies*

$$\|\Lambda(\hat{v}_1) - \Lambda(\hat{v}_2)\|_Z \leqslant \theta \|\hat{v}_1 - \hat{v}_2\|_Z \quad \forall \hat{v}_1, \hat{v}_2 \in K, \tag{118}$$

for some $\theta < 1$. Then, Λ has a unique fixed point.

We set $Z := Y_2^T \times X_2^T$ and $K := A_{\overline{M}}^T$ (see its definition in (19)), where $T \leqslant \overline{T}$ will be fixed at the end of the proof in terms of $\overline{M}$. Then K is a closed subset of Z. Let us define $\Lambda : (\hat{v}, \hat{\xi}) \to (v, \xi)$ where (v, ξ) is the solution (61) with γ the solution of (27) and $(33)_1$. Then, according to (117), $\Lambda(K) \subset K$ for any $T \leqslant \overline{T}$.

Let us now prove inequality (118). In what follows, C will denote a constant which may depend on $\overline{M}$.

We consider $(\hat{v}_1, \hat{\xi}_1)$ (resp. $(\hat{v}_2, \hat{\xi}_2)$) in K and we denote (γ_1, v_1, ξ_1) (resp. (γ_2, v_2, ξ_2)) the corresponding solution of (27), $(33)_1$ and (61) associated to $(\hat{v}_1, \hat{\xi}_1)$ (resp. $(\hat{v}_2, \hat{\xi}_2)$). Then, the function $\gamma_1 - \gamma_2$ satisfies

$$\left\{ \begin{array}{ll} (\gamma_1 - \gamma_2)_t + (\nabla \hat{v}_2 (\nabla \hat{\chi}_2)^{-1} : \mathrm{Id})(\gamma_1 - \gamma_2) = L_0 & \text{in } Q_T, \\ (\gamma_1 - \gamma_2)_{|t=0} = 0 & \text{in } \Omega_F(0), \end{array} \right.$$

with

$$L_0 := (\nabla \hat{v}_2 (\nabla \hat{\chi}_2)^{-1} : \mathrm{Id} - \nabla \hat{v}_1 (\nabla \hat{\chi}_1)^{-1} : \mathrm{Id})(\overline{\rho} + \gamma_1).$$

Since $\hat{v}_1, \hat{v}_2 \in K^1 := (A_{\overline{M}}^T)_1$ and γ_1 satisfies (117), we have that

$$\|L_0\|_{L^\infty(H^1)} \leqslant C \|\hat{v}_1 - \hat{v}_2\|_{Y_2^T}.$$

From the equation satisfied by $\gamma_1 - \gamma_2$, we have that

$$\|\gamma_1 - \gamma_2\|_{W^{1,\infty}(H^1)} \leqslant C \|\hat{v}_1 - \hat{v}_2\|_{Y_2^T}. \tag{119}$$

Let now $w := v_1 - v_2$ and $\zeta := \xi_1 - \xi_2$. Then w satisfies $w(0, \cdot) = 0$ in $\Omega_F(0)$, $w = 0$ on $\partial\Omega$ and the following equation:

$$(\gamma_2 + \overline{\rho}) \det(\nabla \hat{\chi}_2) w_t - \nabla \cdot \widehat{\mathbb{T}}_{1,2}(w, \gamma_1, \gamma_2) = F_0 \quad \text{in } \Omega_F(0), \tag{120}$$

where

$$\begin{aligned} \widehat{\mathbb{T}}_{1,2}(w, \gamma_1, \gamma_2) := {} & (\mu(\nabla w (\nabla \hat{\chi}_2)^{-1} + (\nabla \hat{\chi}_2)^{-t} (\nabla w)^t) + \mu' (\nabla w (\nabla \hat{\chi}_2)^{-1} : \mathrm{Id}) \mathrm{Id}) \mathrm{cof}\,(\nabla \hat{\chi}_2) \\ & - (P(\gamma_2 + \overline{\rho}) - P(\gamma_1 + \overline{\rho})) \mathrm{cof}\,(\nabla \hat{\chi}_2) \end{aligned}$$

and

$$\begin{aligned} F_0 := {} & ((\gamma_2 + \overline{\rho}) \det(\nabla \hat{\chi}_2) - (\gamma_1 + \overline{\rho}) \det(\nabla \hat{\chi}_1)) v_{1,t} - \mu \nabla \cdot [\nabla v_1 ((\nabla \hat{\chi}_2)^{-1} \mathrm{cof}\,(\nabla \hat{\chi}_2) - (\nabla \hat{\chi}_1)^{-1} \mathrm{cof}\,(\nabla \hat{\chi}_1))] \\ & - \mu \nabla \cdot ((\nabla \hat{\chi}_2)^{-t} (\nabla v_1)^t \mathrm{cof}\,(\nabla \hat{\chi}_2) - (\nabla \hat{\chi}_1)^{-t} (\nabla v_1)^t \mathrm{cof}\,(\nabla \hat{\chi}_1)) \\ & - \mu' \nabla \cdot [(\nabla v_1 (\nabla \hat{\chi}_2)^{-1} : \mathrm{Id}) \mathrm{cof}\,(\nabla \hat{\chi}_2) - (\nabla v_1 (\nabla \hat{\chi}_1)^{-1} : \mathrm{Id}) \mathrm{cof}\,(\nabla \hat{\chi}_1)] \\ & - \nabla \cdot [(P(\gamma_1 + \overline{\rho}) - P(\overline{\rho}))(\mathrm{cof}\,(\nabla \hat{\chi}_1) - \mathrm{cof}\,(\nabla \hat{\chi}_2))]. \end{aligned}$$

On the other hand ζ satisfies $\zeta(0, \cdot) = 0$ and $\zeta_t(0, \cdot) = 0$ in $\Omega_S(0)$ and the following equation for $i = 1, 2, 3$:

$$\partial_t^2 \zeta_i - \sum_{\alpha, j, \beta = 1}^{3} c_{i\alpha j\beta}(\nabla \hat{\xi}_2) \partial^2_{\alpha\beta} \zeta_j = H_{0,i} \tag{121}$$

where

$$H_{0,i} := \sum_{\alpha, j, \beta = 1}^{3} (c_{i\alpha j\beta}(\nabla \hat{\xi}_1) - c_{i\alpha j\beta}(\nabla \hat{\xi}_2)) \partial^2_{\alpha\beta} \xi_{1,j}. \tag{122}$$

As long as the boundary conditions are concerned, we have on $\partial\Omega_S(0)$:

$$w_i = \partial_t \zeta_i \quad \text{and} \quad \left(\widehat{\mathbb{T}}_{1,2}(w,\gamma_1,\gamma_2)n\right)_i = \sum_{\alpha,j,\beta=1}^{3} \left(\int_0^t c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{s\beta}\zeta_j ds\right) n_\alpha + G_{0,i}, \tag{123}$$

for all $1 \leqslant i \leqslant 3$, where

$$\begin{aligned} G_{0,i} :=\ &\mu\left(\nabla v_1((\nabla\hat{\chi}_2)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla\hat{\chi}_1)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_1))n\right)_i \\ &+ \mu\left(((\nabla\hat{\chi}_2)^{-t}(\nabla v_1)^t\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla\hat{\chi}_1)^{-t}(\nabla v_1)^t\mathrm{cof}\,(\nabla\hat{\chi}_1))n\right)_i \\ &+ \mu'\left(((\nabla v_1(\nabla\hat{\chi}_2)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla v_1(\nabla\hat{\chi}_1)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}_1))n\right)_i \\ &- ((P(\gamma_1+\overline{\rho}) - P(\overline{\rho}))(\mathrm{cof}\,(\nabla\hat{\chi}_1) - \mathrm{cof}\,(\nabla\hat{\chi}_2))n)_i \\ &+ \sum_{\alpha,j,\beta=1}^{3}\left(\int_0^t (c_{i\alpha j\beta}(\nabla\hat{\xi}_1) - c_{i\alpha j\beta}(\nabla\hat{\xi}_2))\partial^2_{s\beta}\xi_{1,j} ds\right) n_\alpha. \end{aligned} \tag{124}$$

- Let us estimate w in $H^1(H^1) \cap W^{1,\infty}(L^2)$ and ζ in $W^{2,\infty}(0,T;L^2(\Omega_S(0))) \cap W^{1,\infty}(0,T;H^1(\Omega_S(0)))$.

First, we differentiate the equation of w with respect to t. This yields

$$(\gamma_2+\overline{\rho})\det(\nabla\hat{\chi}_2)\partial_t^2 w - \partial_t\nabla\cdot\widehat{\mathbb{T}}_{1,2}(w,\gamma_1,\gamma_2) = \tilde{F}_0 \quad \text{in } \Omega_F(0), \tag{125}$$

and

$$\left(\partial_t\widehat{\mathbb{T}}_{1,2}(w,\gamma_1,\gamma_2)\mathrm{n}\right)_i = \sum_{\alpha,j,\beta=1}^{3} c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{t\beta}\zeta_j n_\alpha + \partial_t G_{0,i} \quad \text{on } \partial\Omega_S(0), \tag{126}$$

where

$$\tilde{F}_0 := \partial_t F_0 - \partial_t\gamma_2\det(\nabla\hat{\chi}_2)\partial_t w - (\gamma_2+\overline{\rho})\partial_t\det(\nabla\hat{\chi}_2)\partial_t w.$$

We multiply equation (125) by $\partial_t w$ and we integrate in $\Omega_F(0)$. After an integration by parts, we obtain :

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_F(0)}(\gamma_2+\overline{\rho})\det(\nabla\hat{\chi}_2)\partial_t((w_t)^2)\,dy + \int_{\Omega_F(0)}\partial_t\widehat{\mathbb{T}}_{1,2}(w,\gamma_1,\gamma_2) : \partial_t\nabla w\,dy \\ &+ \int_{\partial\Omega_S(0)}\partial_t\widehat{\mathbb{T}}_{1,2}(w,\gamma_1,\gamma_2)\partial_t w\,n\,d\sigma + \int_{\Omega_F(0)}\tilde{F}_0\cdot\partial_t w\,dy, \end{aligned} \tag{127}$$

where we have used that $w=0$ on $\partial\Omega$. For the second term, we use (23) and (119) and we obtain

$$\begin{aligned} \int_{\Omega_F(0)}\partial_t\widehat{\mathbb{T}}_{1,2}(w,\gamma_1,\gamma_2) : \partial_t\nabla w\,dy \geqslant (1-CT^\nu)\int_{\Omega_F(0)}|\partial_t\epsilon(w)|^2dy - C\int_{\Omega_F(0)}|\nabla w|^2dy \\ - \delta\int_{\Omega_F(0)}|\partial_t\nabla w|^2dy - C\|\hat{v}_1-\hat{v}_2\|^2_{Y_2^T}. \end{aligned}$$

Recall that $\epsilon(\cdot)$ was defined right after (1). Here and in what follows, $\nu>0$ will denote a constant which may change from line to line.

For the last term in (127), we have

$$\left|\int_{\Omega_F(0)}\tilde{F}_0\cdot\partial_t w\,dy\right| \leqslant C\int_{\Omega_F(0)}|\partial_t w|^2dy + C\|\hat{v}_1-\hat{v}_2\|^2_{Y_2^T}.$$

Here, we have used the definition of Y_2^T given in Definition 1 and estimates (117) and (119).

Then, one deduces from (127)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\Omega_F(0)}(\gamma_2+\overline{\rho})\det(\nabla\hat{\chi}_2)|\partial_t w|^2\,dy + \int_{\partial\Omega_S(0)}\partial_t\widehat{\mathbb{T}}_{1,2}(w,\gamma_1,\gamma_2)\partial_t w\,n\,d\sigma + (1-CT^\nu)\int_{\Omega_F(0)}|\partial_t\epsilon(w)|^2dy \\ &\leqslant C\left(\int_{\Omega_F(0)}|\nabla w|^2dy + \int_{\Omega_F(0)}|\partial_t w|^2dy + \|\hat{v}_1-\hat{v}_2\|^2_{Y_2^T}\right) + \delta\int_{\Omega_F(0)}|\partial_t\nabla w|^2dy \\ &\leqslant C\left(T\|\partial_t\nabla w\|^2_{L^2(L^2)} + \|\partial_t w\|^2_{L^\infty(L^2)} + \|\hat{v}_1-\hat{v}_2\|^2_{Y_2^T}\right) + \delta\int_{\Omega_F(0)}|\partial_t\nabla w|^2dy, \end{aligned}$$

where we have used that $w(0,\cdot)\equiv 0$ in $\Omega_F(0)$. We integrate now between 0 and t. Using (37), (117) and $\partial_t w(0,\cdot)\equiv 0$ in $\Omega_F(0)$ for the first term and replacing the boundary terms thanks to (123), we find

$$\left(\frac{\rho_{min}}{2}-CT^\nu\right)\int_{\Omega_F(0)}|\partial_t w(t)|^2dx+\sum_{i=1}^3\int_0^t\int_{\partial\Omega_S(0)}\left(\sum_{\alpha,j,\beta=1}^3 c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{s\beta}\zeta_j n_\alpha+\partial_s G_{0,i}\right)\partial_s^2\zeta_i\,dy\,ds$$
$$+(1-CT^\nu)\int_0^t\int_{\Omega_F(0)}|\partial_s\epsilon(w)|^2dy\,ds\leqslant C\left((T^2+\delta)\|\partial_t\nabla w\|^2_{L^2(L^2)}+T\|\partial_t w\|^2_{L^\infty(L^2)}+T\|\hat{v}_1-\hat{v}_2\|^2_{Y_2^T}\right).\tag{128}$$

Let us now differentiate with respect to t the equation satisfied by ζ. This yields for $i=1,2,3$:

$$\partial_t^3\zeta_i-\sum_{\alpha,j,\beta=1}^3 c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial_t\partial^2_{\alpha\beta}\zeta_j=\tilde{H}_{0,i},\tag{129}$$

where

$$\tilde{H}_{0,i}:=\partial_t H_{0,i}+\sum_{\alpha,j,\beta=1}^3\partial_t c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{\alpha\beta}\zeta_j.$$

We multiply this equation by $\partial_t^2\zeta_i$ and we integrate in $\Omega_S(0)$:

$$\frac12\frac{d}{dt}\int_{\Omega_S(0)}|\partial_t^2\zeta|^2dy+\frac12\frac{d}{dt}\sum_{i,\alpha,j,\beta=1}^3\int_{\Omega_S(0)}c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{t\beta}\zeta_j\partial^2_{t\alpha}\zeta_i\,dy-\sum_{i,\alpha,j,\beta=1}^3\int_{\partial\Omega_S(0)}c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{t\beta}\zeta_j\partial_t^2\zeta_i\,n_\alpha\,d\sigma$$
$$=\sum_{i,\alpha,j,\beta=1}^3\int_{\Omega_S(0)}\left\{\left(\frac12\partial_t c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{t\alpha}\zeta_i-\partial_\alpha\hat{c}^2_{i\alpha j\beta}\partial_t^2\zeta_i\right)\partial^2_{t\beta}\zeta_j+\partial_t c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{\alpha\beta}\zeta_j\partial_t^2\zeta_i\right\}dy$$
$$+\int_{\Omega_S(0)}\partial_t H_0\cdot\partial_t^2\zeta\,dy\leqslant C\|\zeta\|^2_{X_2^T}+\int_{\Omega_S(0)}|\partial_t H_0|\,|\partial_t^2\zeta|\,dy.\tag{130}$$

Here, we have integrated by parts and used the symmetry of $c_{i\alpha j\beta}(\nabla\hat{\xi}_2)$ (see (14)). In order to estimate the last term, we use that

$$\|\hat{c}^1_{i\alpha j\beta}-\hat{c}^2_{i\alpha j\beta}\|_{L^\infty(0,T;H^1(\Omega_S(0)))}\leqslant C\|\hat{\xi}_1-\hat{\xi}_2\|_{L^\infty(0,T;H^2(\Omega_S(0)))}$$

and

$$\|\partial_t\hat{c}^1_{i\alpha j\beta}-\partial_t\hat{c}^2_{i\alpha j\beta}\|_{L^\infty(0,T;L^2(\Omega_S(0)))}\leqslant C\|\hat{\xi}_1-\hat{\xi}_2\|_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))}\tag{131}$$

and we obtain

$$\int_{\Omega_S(0)}|\partial_t H_0|\,|\partial_t^2\zeta|\,dy\leqslant C(\|\partial_t^2\zeta\|^2_{L^\infty(0,T;L^2(\Omega_S(0)))}+\|\hat{\xi}_1-\hat{\xi}_2\|^2_{L^\infty(0,T;H^2(\Omega_S(0)))\cap W^{1,\infty}(0,T;H^1(\Omega_S(0)))}).$$

Integrating between 0 and t in (130) and using (73) (for $c_{i\alpha j\beta}(\nabla\hat{\xi}_2)$ instead of $\hat{c}_{i\alpha j\beta}$) we deduce :

$$\frac12\int_{\Omega_S(0)}|\partial_t^2\zeta|^2(t)dy+\lambda\int_{\Omega_S(0)}|\partial_t\epsilon(\zeta)|^2(t)\,dy+\frac{\lambda'}{2}\int_{\Omega_S(0)}|\partial_t\nabla\cdot\zeta|^2(t)dy$$
$$-\sum_{i,\alpha,j,\beta=1}^3\int_0^t\int_{\partial\Omega_S(0)}c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{s\beta}\zeta_j\partial_s^2\zeta_i\,n_\alpha\,d\sigma\,ds$$
$$\leqslant CT(\|\zeta\|^2_{X_2^T}+\|\hat{\xi}_1-\hat{\xi}_2\|^2_{L^\infty(0,T;H^2(\Omega_S(0)))}+\|\hat{\xi}_1-\hat{\xi}_2\|^2_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))}).$$

We combine this inequality with (128) and we use Körn's inequality :

$$\|w\|^2_{W^{1,\infty}(L^2)}+\|w\|^2_{H^1(H^1)}+\|\zeta\|^2_{W^{2,\infty}(0,T;L^2(\Omega_S(0)))}+\|\zeta\|^2_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))}\leqslant\int_0^T\int_{\partial\Omega_S(0)}|\partial_t G_0||\partial_t^2\zeta|\,dy\,ds$$
$$+CT(\|\zeta\|^2_{X_2^T}+\|\hat{v}_1-\hat{v}_2\|^2_{Y_2^T}+\|\hat{\xi}_1-\hat{\xi}_2\|^2_{L^\infty 0,T;(H^2(\Omega_S(0)))}+\|\hat{\xi}_1-\hat{\xi}_2\|^2_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))}).\tag{132}$$

Next we observe that

$$\|\partial_t G_0\|_{L^2((0,T)\times\partial\Omega_S(0))} \leqslant CT^{1/2}(\|\nabla\hat{v}_1 - \nabla\hat{v}_2\|_{L^\infty(H^1)} + \|\nabla\hat{\xi}_1 - \nabla\hat{\xi}_2\|_{L^\infty(0,T;H^1(\Omega_S(0)))}).$$

Since $\partial_t^2\zeta = \partial_t w$ on $\partial\Omega_S(0)$, we find from (132) :

$$\begin{aligned} &\|w\|^2_{W^{1,\infty}(L^2)} + \|w\|^2_{H^1(H^1)} + \|\zeta\|^2_{W^{2,\infty}(0,T;L^2(\Omega_S(0)))} + \|\zeta\|^2_{W^{1,\infty}(0,T;H^1(\Omega_S(0)))} \\ &\leqslant CT(\|\zeta\|^2_{X_2^T} + \|\hat{v}_1 - \hat{v}_2\|^2_{Y_2^T} + \|\hat{\xi}_1 - \hat{\xi}_2\|^2_{X_2^T}). \end{aligned} \tag{133}$$

• Let us estimate w in $L^\infty(H^2)$ and ζ in $L^\infty(0,T;H^2(\Omega_S(0)))$.

Let us consider the following elliptic problem satisfied by w (see (120), (123)):

$$\begin{cases} -\nabla\cdot(\mu(\nabla w + (\nabla w)^t) + \mu'\nabla\cdot w) = F_0 + \sum_{j=1}^4 F_j & \text{in } \Omega_F(0), \\ \mu(\nabla w + (\nabla w)^t)n + \mu'(\nabla\cdot w)n = G_0 + \sum_{j=1}^4 G_j & \text{on } \partial\Omega_S(0), \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{134}$$

where the volume terms are given by

$$F_1 := -(\gamma_2 + \overline{\rho})\det(\nabla\hat{\chi}_2)\partial_t w,\ F_2 := \nabla\cdot((P(\gamma_2+\overline{\rho}) - P(\gamma_1+\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}_2)),$$

$$F_3 := -\mu\nabla\cdot(\nabla w(\mathrm{Id} - (\nabla\hat{\chi}_2)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_2)) + ((\nabla w)^t - (\nabla\hat{\chi}_2)^{-1}(\nabla w)^t\mathrm{cof}\,(\nabla\hat{\chi}_2))),$$

and

$$F_4 := -\mu'\nabla\cdot(\nabla w(\mathrm{Id} - ((\nabla\hat{\chi}_2)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}_2))),$$

and the boundary terms are given by

$$G_{1,i} := \sum_{\alpha,j,\beta=1}^3 \left(\int_0^t c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial^2_{s\beta}\zeta_j\,ds\right)n_\alpha\ (1\leqslant i\leqslant 3),\ G_2 := -(P(\gamma_2+\overline{\rho}) - P(\gamma_1+\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}_2)n,$$

$$G_3 := \mu(\nabla w(\mathrm{Id} - (\nabla\hat{\chi}_2)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_2)) + (\nabla w)^t - (\nabla\hat{\chi}_2)^{-t}(\nabla w)^t\mathrm{cof}\,(\nabla\hat{\chi}_2))n$$

and

$$G_4 := \mu'\nabla w(\mathrm{Id} - ((\nabla\hat{\chi}_2)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}_2))n.$$

First, using (119) we have

$$\|F_0\|_{L^\infty(L^2)} \leqslant CT(\|\partial_t(\gamma_1-\gamma_2)\|_{L^\infty(L^2)} + \|\nabla\hat{v}_1 - \nabla\hat{v}_2\|_{L^\infty(H^1)}) \leqslant CT\|\hat{v}_1 - \hat{v}_2\|_{Y_2^T}$$

and

$$\|F_1\|_{L^\infty(L^2)} + \|F_2\|_{L^\infty(L^2)} \leqslant C(\|\partial_t w\|_{L^\infty(L^2)} + T\|\partial_t(\gamma_1-\gamma_2)\|_{L^\infty(H^1)}) \leqslant C(\|^{\partial}_t w\|_{L^\infty(L^2)} + T\|\hat{v}_1 - \hat{v}_2\|_{Y^2}).$$

For F_3 and F_4 we use that $\hat{v}_2 \in (A^T_M)_1$ and we find

$$\|F_3\|_{L^\infty(L^2)} + \|F_4\|_{L^\infty(L^2)} \leqslant CT\|w\|_{L^\infty(H^2)}.$$

Next, from the definition of G_0 (see (124)) we obtain

$$\|G_0\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))} \leqslant CT(\|\hat{v}_1 - \hat{v}_2\|_{L^\infty(H^2)} + \|\nabla\hat{\xi}_1 - \nabla\hat{\xi}_2\|_{L^\infty(0,T;H^1(\Omega_S(0)))}).$$

In order to estimate the last term in G_0 we have used that $H^1(\Omega_S(0)) \hookrightarrow L^6(\Omega_S(0))$. We integrate by parts in G_1 :

$$G_{1,i} := \sum_{\alpha,j,\beta=1}^3 \left(-\int_0^t \partial_s c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\partial_\beta\zeta_j\,ds + c_{i\alpha j\beta}(\nabla\hat{\xi}_2)(t)\partial_\beta\zeta(t)\right)n_\alpha.$$

Since $\|c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\|_{L^\infty(0,T;H^2(\Omega_S(0)))} \leqslant C$, we deduce taking $T \leqslant 1$:

$$\|G_1\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))} \leqslant C\|\zeta\|_{L^\infty(0,T;H^2(\Omega_S(0)))}.$$

Arguing as for F_2, F_3 and F_4, we find

$$\|G_2\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))}+\|G_3\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))}+\|G_4\|_{L^\infty(0,T;H^{1/2}(\partial\Omega_S(0)))} \leqslant CT(\|\hat{v}_1-\hat{v}_2\|_{Y_2^T}+\|w\|_{L^\infty(H^2)}).$$

Using all these estimates we deduce that w, solution of (134), belongs to $L^\infty(H^2)$ and

$$\begin{aligned}&\|w\|_{L^\infty(H^2)} \leqslant C(T(\|\hat{v}_1-\hat{v}_2\|_{Y_2^T}+\|\nabla\hat{\xi}_1-\nabla\hat{\xi}_2\|_{L^\infty(0,T;H^1(\Omega_S(0)))})\\&+\|\zeta\|_{L^\infty(0,T;H^2(\Omega_S(0)))}+\|\partial_t w\|_{L^\infty(L^2)}).\end{aligned} \tag{135}$$

We consider now the following elliptic problem satisfied by ζ :

$$\begin{cases} -\nabla\cdot(\lambda(\nabla\zeta+(\nabla\zeta)^t)+\lambda'\nabla\cdot\zeta) = H_0+H_1+H_2 & \text{in } \Omega_S(0),\\ \zeta(t,\cdot) = \displaystyle\int_0^t w(s,\cdot)\,ds & \text{on } \partial\Omega_S(0),\end{cases} \tag{136}$$

where H_0 was defined in (122),

$$H_1 := -\partial_t^2\zeta$$

and

$$H_{2,i} := -\sum_{\alpha,j,\beta=1}^{3}(c^\ell_{i\alpha j\beta}(\nabla\hat{\xi}_2)+c^q_{i\alpha j\beta}(\nabla\hat{\xi}_2))\partial^2_{\alpha\beta}\zeta_j \ (1\leqslant i\leqslant 3).$$

Using (131) we have

$$\|H_0\|_{L^\infty(L^2(\Omega_S(0)))} \leqslant \sum_{\alpha,j,\beta=1}^{3}\|c_{i\alpha j\beta}(\nabla\hat{\xi}_1)-c_{i\alpha j\beta}(\nabla\hat{\xi}_2)\|_{L^\infty(L^2(\Omega_S(0)))}\|\xi_1\|_{L^\infty(W^{2,\infty}(\Omega_S(0)))} \leqslant CT\|\hat{\xi}_1-\hat{\xi}_2\|_{W^{1,\infty}(H^1(\Omega_S(0)))}.$$

For H_2, we use the fact that $\hat{c}^\ell$ and $\hat{c}^q$ vanish at $t=0$ and (24) and we obtain :

$$\|H_{2,i}\|_{L^\infty(L^2(\Omega_S(0)))} \leqslant \sum_{\alpha,j,\beta=1}^{3} T\|\partial_t(c^\ell_{i\alpha j\beta}(\nabla\hat{\xi}_2)+c^q_{i\alpha j\beta}(\nabla\hat{\xi}_2))\|_{L^\infty(H^2(\Omega_S(0)))}\|\zeta\|_{L^\infty(H^2(\Omega_S(0)))} \leqslant CT\|\zeta\|_{L^\infty(H^2(\Omega_S(0)))}.$$

Then, $\zeta \in L^\infty(H^2(\Omega_S(0)))$ and

$$\|\zeta\|_{L^\infty(H^2(\Omega_S(0)))} \leqslant C(T(\|\hat{\xi}_1-\hat{\xi}_2\|_{W^{1,\infty}(H^1(\Omega_S(0)))}+\|w\|_{L^\infty(H^2(\Omega_F(0)))})+\|\zeta\|_{W^{2,\infty}(L^2(\Omega_S(0)))}),$$

for T small enough.

Combining this with (135), we deduce

$$\begin{aligned}&\|w\|_{L^\infty(H^2)}+\|\zeta\|_{L^\infty(H^2(\Omega_S(0)))} \leqslant C(T(\|\hat{v}_1-\hat{v}_2\|_{Y_2^T}+\|\hat{\xi}_1-\hat{\xi}_2\|_{X_2^T})\\&+\|\partial_t w\|_{L^\infty(L^2)}+\|\zeta\|_{W^{2,\infty}(L^2(\Omega_S(0)))}).\end{aligned}$$

Finally, using the estimate for the time derivatives (133), we find

$$\|w\|_{Y_2^T}+\|\zeta\|_{X_2^T} \leqslant CT^{1/2}(\|\hat{v}_1-\hat{v}_2\|_{Y_2^T}+\|\hat{\xi}_1-\hat{\xi}_2\|_{X_2^T}),$$

for T small enough.

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