# Permutation patterns, Stanley symmetric functions, and the Edelman-Greene correspondence

Sara Billey, Brendan Pawlowski

*FPSAC 2013 Paris, France* DMTCS proc. **AS**, 2013, 205–216

# *Permutation patterns, Stanley symmetric functions, and the Edelman-Greene correspondence*

Sara Billey and Brendan Pawlowski†

*Department of Mathematics, University of Washington, Seattle, WA, USA*

**Abstract.** Generalizing the notion of a vexillary permutation, we introduce a filtration of $S_\infty$ by the number of Edelman-Greene tableaux of a permutation, and show that each filtration level is characterized by avoiding a finite set of patterns. In doing so, we show that if $w$ is a permutation containing $v$ as a pattern, then there is an injection from the set of Edelman-Greene tableaux of $v$ to the set of Edelman-Greene tableaux of $w$ which respects inclusion of shapes. We also consider the set of permutations whose Edelman-Greene tableaux have distinct shapes, and show that it is closed under taking patterns.

**Résumé.** Généralisant la notion d'une permutation vexillaire, nous introduisons une filtration de $S_\infty$ par le nombre de tableaux d'Edelman-Greene d'une permutation, et nous montrons que chaque niveau de la filtration se caractérise par un ensemble fini des motifs exclus. Ce faisant, nous montrons que si $w$ est une permutation qui inclut le motif $v$, il existe une injection de l'ensemble des tableaux d'Edelman-Greene de $v$ dans l'ensemble des tableaux d'Edelman-Greene de $w$ qui respecte l'inclusion de formes. Nous considérons aussi l'ensemble des permutations dont les tableaux d'Edelman-Greene ont des formes distinctes, et nous montrons que c'est clos pour l'inclusion de motifs.

**Keywords:** Edelman-Greene correspondence, Stanley symmetric functions, Specht modules, pattern avoidance

## 1 Introduction

Stanley (1984) defined a symmetric function $F_w$ depending on a permutation $w$, with the property that the coefficient of $x_1 \cdots x_\ell$ in $F_w$ is the number of reduced words of $w$. Therefore if $F_w = \sum_\lambda a_{w\lambda} s_\lambda$ is written in terms of Schur functions, then

$$|\operatorname{Red}(w)| = \sum_\lambda a_{w\lambda} f^\lambda, \tag{1}$$

where $f^\lambda$ is the number of standard Young tableaux of shape $\lambda$ and $\operatorname{Red}(w)$ the set of reduced words of $w$.

Edelman and Greene (1987) gave an algorithm which realizes (1) bijectively and shows that the $a_{w\lambda}$ are nonnegative.

†The authors were supported by DMS-1101017 from the NSF.

**Theorem 1.1** *Given a permutation $w$, there is a set $EG(w)$ of semistandard Young tableaux and a bijection*

$$\mathrm{Red}(w) \leftrightarrow \{(P,Q) : P \in EG(w), Q \text{ a standard tableau of shape } \mathrm{shape}(P)\}. \tag{2}$$

The tableaux $EG(w)$ are those semistandard tableaux whose column word—gotten by reading up columns starting with the leftmost—is a reduced word for $w$. The shapes of these tableaux precisely give the Schur function expansion of $F_w$:

$$F_w = \sum_{P \in EG(w)} s_{\mathrm{shape}(P)}. \tag{3}$$

Stanley also characterized those $w$ for which $F_w$ is a single Schur function, or equivalently for which $|EG(w)| = 1$. These are the *vexillary* permutations, those avoiding the pattern 2143. Our main results can be viewed as generalizations of this characterization. The first main theorem shows that $EG(w)$ is well-behaved with respect to pattern containment.

**Theorem 1.2** *Let $v, w$ be permutations with $w$ containing $v$ as a pattern. There is an injection $\iota$ : $EG(v) \hookrightarrow EG(w)$ such that if $P \in EG(v)$, then* $\mathrm{shape}(P) \subseteq \mathrm{shape}(\iota(P))$. *Moreover, if $P$, $P'$ have the same shape, so do $\iota(P), \iota(P')$.*

An immediate corollary is that the sets $\{w \in \bigcup_{n\geq 0} S_n : |EG(w)| \leq k\}$ respect pattern containment, in the sense that if $|EG(w)| \leq k$ and $w$ contains $v$, then $|EG(v)| \leq k$. Our second main result is a sort of converse.

**Definition 1.3** *Given a positive integer $k$, a permutation $w \in S_n$ is* $k$-vexillary *if $|EG(w)| \leq k$.*

**Theorem 1.4** *For each integer $k \geq 1$, there is a finite set $V_k$ of permutations such that $w$ is $k$-vexillary if and only if $w$ avoids all patterns in $V_k$.*

Stembridge (2001) gives a criterion for the product of Schur functions $s_\lambda s_\mu$ to be *multiplicity-free*, i.e. a sum of distinct Schur functions, which Thomas and Yong (2010) generalize by answering the analogous question for Schubert classes on Grassmannians. Call a permutation $w$ multiplicity-free if $F_w$ is multiplicity-free. Theorem 1.2 shows that the set of multiplicity-free $w$ is closed under patterns.

**Theorem 1.5** *If a permutation $w$ contains the pattern $v$, and $w$ is multiplicity-free, then so is $v$.*

We have not been able to prove that the property of being multiplicity-free is equivalent to avoiding a finite set of patterns. However, an explicit computation shows that if $w$ in $S_{12}$ is not multiplicity-free, then $w$ properly contains a pattern $v$ which is not multiplicity-free.

**Conjecture 1** *A permutation $w$ is multiplicity-free if and only if $w$ avoids every non-multiplicity-free pattern in $S_m$ for $m \leq 11$.*

In Section 2, we recall the connection between Stanley symmetric functions and the representation theory of the symmetric group, along with the Lascoux-Schützenberger recurrence for computing Stanley symmetric functions. We also recall the definitions of pattern avoidance and containment. Section 3 introduces the notion of a James-Peel tree for a general diagram, following James and Peel (1979). In Section 4, we specialize these ideas to permutation diagrams, with the Lascoux-Schützenberger tree as a key tool, and prove Theorem 1.2. In Section 5 we analyze in more detail the relationship between $|EG(w)|$ and $|EG(v)|$ for $v$ a pattern in $w$, and prove Theorem 1.4. Section 6 is devoted to open problems.

# 2 Background

## 2.1 Permutation patterns

We first recall the definitions of pattern avoidance and containment for permutations.

**Definition 2.1** *Let $v, w$ be two permutations. We say $w$* contains *$v \in S_m$ if there are $i_1 < \cdots < i_m$ such that $v(j) < v(k)$ if and only if $w(i_j) < w(i_k)$. If $w$ does not contain $v$, it* avoids *$v$. Often we say that $w$ contains or avoids the* pattern *$v$.*

**Example 2.2** *The permutation* $2513764$ *contains the patterns* $2143$ *(e.g. as the subsequence* $2174$*) and* $23154$*. It avoids* $1234$*.*

## 2.2 Specht modules

Our proof of Theorem 1.2 goes via the representation theory of $S_n$, specifically the interpretation of $F_w$ as the Frobenius character of a certain *Specht module*, which we discuss next.

**Definition 2.3** *A* diagram *is a finite subset of* $\mathbb{N} \times \mathbb{N}$.

We refer to the elements of a diagram as *cells*. The diagrams of greatest interest for us will be *permutation diagrams* (sometimes called Rothe diagrams, from Rothe (1800)). Given $w \in S_n$, define

$$D(w) = \{(i, w(j)) : 1 \leq i < j \leq n, w(i) > w(j)\}. \tag{4}$$

We'll draw $D(w)$ using matrix coordinates:

$$D(42153) = \begin{array}{ccccc} \circ & \circ & \circ & \times & \cdot \\ \circ & \times & \cdot & \cdot & \cdot \\ \times & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \circ & \cdot & \times \\ \cdot & \cdot & \times & \cdot & \cdot \end{array}$$

Members of a diagram will be represented by $\circ$. We'll often augment $D(w)$ by adding $\times$ at the points $(i, w(i))$.

A *filling* of a diagram $D$ is a bijection $T : D \to \{1, \ldots, n\}$, where $n = |D|$. There is a natural left action of $S_n$ on fillings of $D$ by permuting entries. The *row group* $R(T)$ of a filling $T$ is the subgroup of $S_n$ consisting of permutations $\sigma$ which act on $T$ by permuting entries within their row; the *column group* $C(T)$ is defined analogously. The *Young symmetrizer* of a filling $T$ is an element of $\mathbb{C}[S_n]$, defined by

$$c_T = \sum_{p \in R(T)} \sum_{q \in C(T)} \operatorname{sgn}(q) qp. \tag{5}$$

**Definition 2.4** *Given a diagram $D$ and a choice of filling $T$, the* Specht module *$S^D$ is the $S_n$-module $\mathbb{C}[S_n]c_T$, where $n = |D|$. The* Schur function *$s_D$ of $D$ is the Frobenius characteristic of $S^D$.*

The isomorphism type of $S^D$ is independent of the choice of $T$, and is also unaffected by permuting rows or columns of $D$.

**Definition 2.5** *If a diagram $D$ is gotten from a diagram $D'$ by permuting rows and columns, say $D$ and $D'$ are* equivalent*, and write $D \simeq D'$. This includes inserting or deleting empty rows and columns.*

Over $\mathbb{C}$, the Specht modules of partition diagrams form complete sets of irreducible $S_n$-representations: see Sagan (2001) or Fulton (1997). In general, it is an open problem to find a reasonable combinatorial algorithm for decomposing $S^D$ into irreducibles. The Littlewood-Richardson rule handles the case where $D$ is a skew shape, and Reiner and Shimozono (1998) and Liu (2009) treat other classes of diagrams.

**Definition 2.6** *Given two diagrams $D_1, D_2$, where $D_1 \subseteq [r] \times [c]$, define their* product *$D_1 \cdot D_2$ to be the diagram*

$$D_1 \cup \{(i+r, j+c) : (i,j) \in D_2\}.$$

One can check that $s_{D_1 \cdot D_2} = s_{D_1} s_{D_2}$. We will use this operation in Section 3.

## 2.3 Stanley symmetric functions

The *Stanley symmetric function* of a permutation $w$ of length $\ell$ is

$$F_w = \sum_{a \in \mathrm{Red}(w)} \sum_i x_{i_1} x_{i_2} \cdots x_{i_\ell}, \tag{6}$$

where for each $a \in \mathrm{Red}(w)$, $i$ runs over all integer sequences $1 \le i_1 \le \cdots \le i_\ell$ such that $i_j < i_{j+1}$ if $a_j > a_{j+1}$.

Write $D^t$ for the transpose of $D$. For a permutation $w$, let $1^m \times w = 12\cdots m(w(1)+m)(w(2)+m)\cdots$. The results of Billey et al. (1993) show that $F_w = \lim_{m\to\infty} \mathfrak{S}_{1^m \times w^{-1}}$, where $\mathfrak{S}_v$ is a Schubert polynomial. Theorem 31 in Reiner and Shimozono (1995b) and Theorem 20 in Reiner and Shimozono (1998) then imply the following result, which is also implicit in Kraśkiewicz (1995).

**Theorem 2.7** *For any permutation $w$, $F_w = s_{D(w)^t} = s_{D(w^{-1})}$.*

Stanley symmetric functions can be decomposed into Schur functions using a recursion introduced in Lascoux and Schützenberger (1985). Given a permutation $w$, let $r$ be maximal with $w(r) > w(r+1)$. Then let $s > r$ be maximal with $w(s) < w(r)$. Let $t_{ij}$ denote the transposition $(i\,j)$, and define

$$T(w) = \{w t_{rs} t_{rj} : \ell(w t_{rs} t_{rj}) = \ell(w)\}; \tag{7}$$

or, if the set on the right-hand side is empty, set $T(w) = T(1\times w)$ where $1\times w = 1(w(1)+1)(w(2)+1)\cdots$ in one-line notation. The members of $T(w)$ are called *transitions* of $w$. The *Lascoux-Schützenberger tree* (L-S tree for short) is the finite rooted tree of permutations with root $w$ where the children of a vertex $v$ are:

- None, if $v$ is vexillary (avoids 2143).
- $T(v)$ otherwise.

Monk's rule for Schubert polynomials and the identity $F_w = \lim_{m\to\infty} \mathfrak{S}_{1^m\times w^{-1}}$ lead to the recurrence $F_w = \sum_{v \in T(w)} F_v$. This, together with the finiteness of the Lascoux-Schützenberger tree and Stanley's result that $F_v$ is a Schur function exactly when $v$ is vexillary, imply that

$$F_w = \sum_v s_{\mathrm{shape}(v)^t}, \tag{8}$$

or equivalently

$$s_{D(w)} = \sum_{v} s_{\mathrm{shape}(v)}, \tag{9}$$

where $v$ runs over the leaves of the Lascoux-Schützenberger tree, and $\mathrm{shape}(v)$ denotes the partition whose shape is equivalent to $D(v)$. Here we use the fact that $D(v)$ is equivalent to a partition diagram if and only if $v$ is vexillary.

**Remark 2.8** The reduced words of $1 \times w$ are exactly those of $w$ with all letters shifted up by $1$, so the same is true of the tableaux in $EG(1 \times w)$ compared to the tableaux in $EG(w)$. In particular, the shapes are the same and $F_w = F_{1\times w}$. Since the Lascoux-Schützenberger tree is finite, there is some $m$ such that in constructing the tree for $1^m \times w$, we never need to make the replacement of $v$ by $1 \times v$. Thus we will ignore this possible step in what follows.

# 3 James-Peel moves

Let $D$ be a diagram. Given two positive integers $a, b$, let $R_{a\to b}D$ be the diagram which contains a cell $(i, j)$ if and only if one of the following cases holds:

- $i \neq a, b$ and $(i, j) \in D$.
- $i = b$, and either $(a, j) \in D$ or $(b, j) \in D$.
- $i = a$, and $(a, j), (b, j) \in D$.

That is, $R_{a\to b}D$ is gotten by moving cells in row $a$ to row $b$ if the appropriate position is empty. Similarly, we define $C_{c\to d}D$ by moving cells of $D$ in column $c$ to column $d$ if possible.

The operators $R_{a\to b}$ and $C_{c\to d}$ were introduced in James and Peel (1979), so we will call them *James-Peel moves*. The next theorem uses James-Peel moves to find irreducible factors inside $S^D$, and can be viewed as a generalization of Pieri's rule.

**Definition 3.1** *A subset $D'$ of a diagram $D$ is a* subdiagram *if it is the intersection of some rows and columns with $D$. That is, there are sets $X, Y \subseteq \mathbb{N}$ such that $D' = (X \times Y) \cap D$.*

Let $\delta_p$ denote the partition $(p-1, p-2, \dots, 1)$.

**Theorem 3.2** *Suppose $D$ contains $\delta_p \cdot (1)$ as a subdiagram in rows $\alpha(1) < \cdots < \alpha(p)$ and columns $\beta(1) < \cdots < \beta(p)$. There is a filtration*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_p = S^D$$

*of $S^D$ by $S_{|D|}$-submodules such that for each $1 \leq j \leq p$, there is a surjection*

$$M_j/M_{j-1} \twoheadrightarrow S^{R_{\alpha(p)\to\alpha(p-j+1)}C_{\beta(p)\to\beta(j)}D}.$$

The case $p = 1$ of Theorem 3.2 is Theorem 2.4 of James and Peel (1979). Theorem 3.2 is valid over any field, but for convenience, we will work over $\mathbb{C}$. In this case, complete irreducibility of $S_n$-modules gives an inclusion

$$\bigoplus_{j=1}^{p} S^{R_{\alpha(p)\to\alpha(p-j+1)}C_{\beta(p)\to\beta(j)}D} \hookrightarrow S^D.$$

**Example 3.3** *Take*

$$D = \begin{array}{ccccc} \bullet & \circ & \bullet & \cdot & \cdot \\ \bullet & \circ & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \circ & \bullet \end{array} \simeq D(4317256).$$

*The subdiagram $\delta_2 \cdot (1)$ appears in rows $1,2,3$ and columns $1,3,5$, and is shaded in the picture above. Then*

$$R_{3\to 3}C_{5\to 1}D = \begin{array}{cccc} \bullet & \circ & \bullet & \cdot \\ \bullet & \circ & \cdot & \cdot \\ \bullet & \circ & \cdot & \circ \end{array} \quad R_{3\to 2}C_{5\to 3}D = \begin{array}{cccc} \bullet & \circ & \bullet & \cdot \\ \bullet & \circ & \bullet & \circ \\ \cdot & \circ & \cdot & \cdot \end{array} \quad R_{3\to 1}C_{5\to 5}D = \begin{array}{ccccc} \bullet & \circ & \bullet & \circ & \bullet \\ \bullet & \circ & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot & \cdot \end{array}$$

*The second and third diagrams here are equivalent to the diagrams of the partitions $(4,3,1)$ and $(5,2,1)$ respectively. Hence if $D_1$ is the first diagram, Theorem 3.2 gives $S^{D_1} \oplus S^{(4,3,1)} \oplus S^{(5,2,1)} \hookrightarrow S^D$. Another application of Theorem 3.2 to the subdiagram of $D_1$ in rows $1,3$ and columns $3,4$ gives $S^{(4,2,2)} \oplus S^{(3,3,2)} \hookrightarrow S^{D_1}$.*

*In fact, both these inclusions are isomorphisms, as one can check with the Littlewood-Richardson rule since $D$ is equivalent to the skew shape $(5,3,3)/(2,1)$. Alternatively, one can compute the Edelman-Greene tableaux of $4317256$ and look at their (transposed) shapes:*

$$\begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 6 & \end{array} \quad \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & & \\ 6 & & \end{array} \quad \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & 5 & \\ 6 & & \end{array} \quad \begin{array}{ccc} 1 & 2 & 4 \\ 2 & 3 & \\ 3 & & \\ 5 & & \\ 6 & & \end{array} .$$

Even without the conditions of Theorem 3.2, applying a James-Peel move always gives an inclusion of Specht modules (over $\mathbb{C}$).

**Lemma 3.4** *For any positive integers $a, b$, $S^{R_{a\to b}D} \hookrightarrow S^D$ and $S^{C_{a\to b}D} \hookrightarrow S^D$.*

James-Peel moves and Theorem 3.2 present one possible way to decompose a Specht module into irreducibles. In general it is not known if an arbitrary Specht module can be decomposed by finding some appropriate tree of James-Peel moves, as the surjections in Theorem 3.2 may be not be isomorphisms. The way we prove Theorem 1.2 is to find such a tree for the case of $D(w)$. The usefulness of James-Peel moves for us comes from the fact that they are well-behaved with respect to subdiagram inclusion, and pattern inclusion for permutations corresponds to subdiagram inclusion on the level of permutation diagrams.

To be more precise about this, we make the following definition.

**Definition 3.5** *A* James-Peel tree *for a diagram $D$ is a rooted tree $\mathcal{T}$ with vertices labeled by diagrams and edges labeled by sequences of James-Peel moves, satisfying the following conditions:*

- *The root of $\mathcal{T}$ is $D$.*
- *If $B$ is a child of $A$ with a sequence $\mathbf{JP}$ of James-Peel moves labeling the edge $A$—$B$, then $B = \mathbf{JP}(A)$.*

- *If $A$ has more than one child, these children arise as a result of applying Theorem 3.2 to $A$. That is, $A$ contains $\delta_p \cdot (1)$ as a subdiagram in rows $\alpha(1) < \cdots < \alpha(p)$ and columns $\beta(1) < \cdots < \beta(p)$, and each edge leading down from $A$ is labeled $R_{\alpha(p)\to\alpha(p-j+1)}C_{\beta(p)\to\beta(j)}$ for some distinct values $1 \leq j \leq p$ (perhaps not all such $j$ appear).*

Theorem 3.2 and Lemma 3.4 immediately imply the following statement.

**Lemma 3.6** *If $D$ has a James-Peel tree $\mathcal{T}$ with leaves $A_1, \ldots, A_n$, then $\bigoplus_i S^{A_i} \hookrightarrow S^D$.*

**Definition 3.7** *A James-Peel tree $\mathcal{T}$ for $D$ is* complete *if its leaves $A_1, \ldots, A_n$ are equivalent to Ferrers diagrams of partitions and $S^D \simeq \bigoplus_i S^{A_i}$.*

In James and Peel (1979), an algorithm is given which constructs a complete James-Peel tree when $D$ is a skew shape. More generally, Reiner and Shimozono (1995a) construct a complete James-Peel tree for any *column-convex* diagram: a diagram $D$ for which $(a, x), (b, x) \in D$ with $a < b$ implies $(i, x) \in D$ for all $a < i < b$. In the next section we construct a complete James-Peel tree for the diagram of a permutation, so it's worth noting that neither of these classes of diagrams contains the other. For example, $D(37154826)$ is not equivalent to any column-convex or row-convex diagram, while the column-convex diagram

$$\begin{matrix} \circ & \circ & \cdot & \cdot \\ \circ & \cdot & \circ & \cdot \\ \circ & \cdot & \cdot & \circ \end{matrix}$$

is not equivalent to the diagram of any permutation.

Notice that if $w$ contains a pattern $v$, then $D(v)$ is (up to reindexing) a subdiagram of $D(w)$. Specifically, if $v$ appears in positions $i_1, \ldots, i_k$ of $w$, then $D(v)$ is the subdiagram of $D(w)$ induced by the rows $i_1, \ldots, i_k$ and columns $w(i_1), \ldots, w(i_k)$.

Let $I(D)$ denote the multiset of partitions of $n = |D|$ such that $S^D \simeq \bigoplus_{\lambda \in I(D)} S^\lambda$.

**Lemma 3.8** *Suppose $D'$ is a subdiagram of $D$ and that $D'$ has a complete James-Peel tree. Then there is an injection $\iota : I(D') \hookrightarrow I(D)$ such that if $\lambda \in I(D')$, then $\lambda \subseteq \iota(\lambda)$. Moreover, $\iota$ depends only on shape, in the sense that if $\lambda$ appears $m$ times in $I(D')$, then $\iota(\lambda)$ appears at least $m$ times in $I(D')$.*

In particular, taking $D = D(w)^t$ and $D' = D(v)^t$ for $v$ a pattern in $w$, Lemma 3.8 together with the equalities

$$s_{D(w^{-1})} = F_w = \sum_{P \in EG(w)} s_{\mathrm{shape}(P)} \tag{10}$$

will immediately imply Theorem 1.2 once we show that $D(w)$ always has a complete James-Peel tree.

The main idea of the proof of Lemma 3.8 is to view the James-Peel tree $\mathcal{T}'$ for $D'$ as a James-Peel tree $\mathcal{T}$ for $D$, via the inclusion $D' \subseteq D$. One can write down an explicit formula for $\iota$, although in general there will be many possible choices.

**Example 3.9** *Take $D = D(w)^t$ where $w = 4317256$ as in Example 3.3, so $F_w = s_{D(w)^t} = s_{332} + s_{3311} + s_{3221} + s_{32111}$. Write $w^i = fl(w(1) \cdots \widehat{w(i)} \cdots w(n))$. We give $F_{w^i} = s_{D(w^i)^t}$ for several $i$, and line up each term $s_\lambda$ in $F_{w^i}$ with the corresponding term $s_{\iota(\lambda)}$ in $F_w$, for a particular choice of $\iota$, so $D' = D(w^i)^t$.*

$$\begin{array}{rcccccccc} F_w &=& s_{332} &+& s_{3311} &+& s_{3221} &+& s_{32111} \\ F_{w^1} &=& s_{221} &+& && s_{2111} && \\ F_{w^4} &=& s_{221} &&&&&& \\ F_{w^5} &=& s_{32} &+& s_{311} &+& s_{221} &+& s_{2111} \end{array}$$

*Note that if $F_w$ is multiplicity-free, as in this example, an injection $\iota : I(D(v)^t) \hookrightarrow I(D(w)^t)$ uniquely defines an injection $EG(v) \hookrightarrow EG(w)$.*

# 4 Transitions as James-Peel moves

Recall the following notation from Section 2.3. Given a permutation $w$, take $r$ maximal with $w(r) > w(r+1)$, then $s > r$ maximal with $w(s) < w(r)$. The set of transitions of $w$ is $T(w) = \{wt_{rs}t_{rj} : \ell(wt_{rs}t_{rj}) = \ell(w)\}$, or else $T(1 \times w)$ if the set on the right is empty.

Upon taking diagrams of permutations, one finds that transitions correspond to certain sequences of James-Peel moves.

**Lemma 4.1** *Given a permutation $w$, let $r, s$ be as above and take $v = wt_{rs}t_{rj} \in T(w)$. Then*

$$D(v) = R_{r\to j}C_{w(s)\to w(j)}D(w) = C_{w(s)\to w(j)}R_{r\to j}D(w). \tag{11}$$

**Theorem 4.2** *For a permutation $w$, the diagram $D(w)$ has a complete James-Peel tree.*

**Proof sketch:** If $w$ has $p$ transitions, then $D(w)$ contains $\delta_p \cdot (1)$ as a certain subdiagram. Lemma 4.1 shows that the diagrams arising from applying Theorem 3.2 to this subdiagram are, up to equivalence, exactly the diagrams $D(v)$ as $v$ runs over $T(w)$. This implies that replacing permutations by diagrams in the Lascoux-Schützenberger tree and labeling edges with James-Peel moves according to Lemma 4.1 yields a James-Peel tree for $D(w)$. This tree is complete because $s_{D(w)} = \sum_v s_{D(v)}$ for $v$ running over the leaves of the L-S tree. □

Lemma 3.8 and the discussion following it now imply Theorem 1.2.

**Corollary 4.3** *If a permutation $w$ is $k$-vexillary and $v$ is a pattern in $w$, then $v$ is $k$-vexillary.*

**Remark 4.4** Theorem 1.2 shows the existence of an injection $EG(v) \hookrightarrow EG(w)$ which respects inclusion of shapes for $v$ a pattern contained in $w$, but an explicit map on tableaux is lacking. The Edelman-Greene correspondence shows that this is equivalent to an injection $\mathrm{Red}(v) \hookrightarrow \mathrm{Red}(w)$ which is an inclusion on the shapes of Edelman-Greene insertion tableaux. The characterization in Tenner (2006) of vexillary permutations yields an explicit injection in the case where $v$ is vexillary.

# 5 $k$-vexillary permutations

In this section we show that the property of $k$-vexillarity is characterized by avoiding a finite set of patterns for any $k$. The key step is to remove some inessential moves from the James-Peel tree for $D(w)$, namely those which only permute rows or columns.

If $D$ is an arbitrary diagram, and $\sigma, \tau$ are permutations, let $(\sigma,\tau)D$ be the diagram $\{(\sigma(i), \tau(j)) : (i,j) \in D\}$. Given a James-Peel tree $\mathcal{T}$ for $D$, let $(\sigma,\tau)\mathcal{T}$ denote the James-Peel tree for $(\sigma,\tau)D$ gotten by replacing every James-Peel move $R_{x\to y}$ labeling an edge of $\mathcal{T}$ by $R_{\sigma(x)\to\sigma(y)}$, and every move $C_{x\to y}$ by $C_{\tau(x)\to\tau(y)}$, and relabeling vertices accordingly. Whenever a move labeling an edge $e$ of a James-Peel tree just permutes rows or columns, we can eliminate that move from the tree at the cost of relabeling rows and columns of James-Peel moves below $e$.

**Definition 5.1** *Given a James-Peel tree $\mathcal{T}$ of a diagram $D$, the* reduced James-Peel tree $\operatorname{red}(\mathcal{T})$ *of $D$ is defined inductively as follows.*

- *If $D$ has no children in $\mathcal{T}$, then $\operatorname{red}(\mathcal{T}) = \mathcal{T}$.*
- *If $D$ has just one child $F$, and $D = (\sigma,\tau)F$ for some $\sigma,\tau \in S_\infty$, let $\mathcal{T}_1$ be the subtree of $\mathcal{T}$ below $F$ with root $F$. Then $\operatorname{red}(\mathcal{T}) = (\sigma,\tau)\operatorname{red}(\mathcal{T}_1)$.*
- *If $D$ has at least two children $F_1, F_2, \dots, F_p$ or $D$ has one child $F_1$ not equivalent to $D$, let $\mathcal{T}_i$ be the subtree of $\mathcal{T}$ below $F_i$ with root $F_i$. Then $\operatorname{red}(\mathcal{T})$ is $\mathcal{T}$ with each $\mathcal{T}_i$ replaced by $\operatorname{red}(\mathcal{T}_i)$.*

Note that $\operatorname{red}(\mathcal{T})$ is still a James-Peel tree for $D$. As equivalent diagrams have isomorphic Specht modules, if $\mathcal{T}$ is complete then so is $\operatorname{red}(\mathcal{T})$.

**Definition 5.2** *A rooted tree is* bushy *if every non-leaf vertex has at least two children.*

If a James-Peel tree has a vertex $A$ with just one child $B$, but $A$ and $B$ are not equivalent, the tree cannot be complete. This implies:

**Lemma 5.3** *If $\mathcal{T}$ is a complete James-Peel tree, then $\operatorname{red}(\mathcal{T})$ is bushy.*

A bushy tree cannot have too many more edges than leaves.

**Lemma 5.4** *The largest number of edges possible in a bushy tree with $k$ leaves is $2k-2$.*

Let $JP(w)$ be the James-Peel tree for $D(w)$ constructed in Theorem 4.2, and $RJP(w) = \operatorname{red}(JP(w))$. Suppose $\mathcal{T}$ is a subtree of $RJP(w)$ with root $D(w)$. Let $R(\mathcal{T})$ be the union of $\{a,b\}$ over all moves $R_{a\to b}$ appearing in $\mathcal{T}$, and $C(\mathcal{T})$ the union of $\{c,d\}$ over all $C_{c\to d}$ appearing in $\mathcal{T}$. Write $R(\mathcal{T}) \cup w^{-1}C(\mathcal{T}) = \{i_1 < \cdots < i_r\}$, and define a permutation

$$w_{\mathcal{T}} = fl(w(i_1)\cdots w(i_r)).$$

**Remark 5.5** *In Section 2 we noted that, for convenience, $w$ could be replaced by $1^m \times w$ to remove the necessity of sometimes replacing $v$ by $1 \times v$ in the Lascoux-Schützenberger tree. The definition of $w_{\mathcal{T}}$ above is then an abuse of notation, since we are really taking a subsequence of $1^m \times w$. However, rows and columns $1,\dots,m$ of $D(w)$ are empty, so are not affected at all by the James-Peel moves in $RJP(w)$ or $\mathcal{T}$. This means that the subsequence defining $w_{\mathcal{T}}$ occurs entirely after the $m$th position of $1^m \times w$, so we are free to shift it down by $m$ and consider it as a subsequence of $w$. This applies also to Theorems 5.8 and 5.9 below.*

Each edge of $JP(w)$ has a label $RC$ for some row and column moves $R, C$. Some of these moves end up being equivalences, and so are lost in $RJP(w)$. Thus, $RJP(w)$ has edges with labels of the form $RC$, $R$, or $C$—in fact, each internal vertex always has one $R$-edge, one $C$-edge, and the remaining edges are $RC$-edges. We are interested in controlling the number of letters in $w_{\mathcal{T}}$, which is at most $|R(\mathcal{T})|+|C(\mathcal{T})|$. This motivates the next definition.

**Definition 5.6** *A subtree $\mathcal{T}$ of $RJP(w)$ with root $D(w)$ is* colorful *if each non-leaf vertex of $\mathcal{T}$ has at least the two children corresponding to its $R$-edge and its $C$-edge. Thus, colorful implies bushy.*

**Lemma 5.7** *Say $\mathcal{T}$ is a subtree of $RJP(w)$ rooted at $D(w)$ with $k$ leaves. Then $k \leq |EG(w_{\mathcal{T}})| \leq |EG(w)|$. If $\mathcal{T}$ is colorful, then $w_{\mathcal{T}} \in S_m$ for some $m \leq 4k-4$.*

**Proof sketch:** Up to relabeling rows and columns to account for flattening, the tree $\mathcal{T}$ is a James-Peel tree for $D(w_{\mathcal{T}})$ (not necessarily complete), so $k \leq EG(w_{\mathcal{T}})$. Theorem 1.2 implies $|EG(w_{\mathcal{T}})| \leq |EG(w)|$.

If $F$ is a vertex of $\mathcal{T}$ with $p$ children, a careful count shows that the edge labels leading down from $F$ contribute at most $p$ elements to each of $|R(\mathcal{T})|$ and $|C(\mathcal{T})|$. Summing over all vertices leads to the result. □

In particular, taking $\mathcal{T} = RJP(w)$ in Lemma 5.7 gives the following result.

**Theorem 5.8** *Any permutation $w$ contains a pattern $v \in S_m$ such that $|EG(w)| = |EG(v)|$, for some $m \leq 4|EG(w)| - 4$.*

More generally, Lemma 5.7 lets us show that $k$-vexillarity is characterized by avoiding a *finite* set of patterns.

**Theorem 5.9** *Say $w$ is a permutation with $|EG(w)| > k$. Then $w$ contains a pattern $v \in S_m$ such that $|EG(v)| > k$, for some $m \leq 4k$.*

**Proof:** By Lemma 5.7, it is enough to exhibit a colorful subtree of $RJP(w)$ with $k+1$ leaves. Such a tree is not difficult to construct by choosing one edge at a time. □

**Corollary 5.10** *A permutation $w$ is $k$-vexillary if and only if it avoids all non-$k$-vexillary patterns in $S_m$ for $1 \leq m \leq 4k$.*

For $k = 2$, we can explicitly find all non-2-vexillary patterns in $S_m$ for $1 \leq m \leq 8$ and eliminate those containing a smaller non-2-vexillary pattern to find a minimal list.

**Theorem 5.11** *A permutation $w$ is 2-vexillary if and only if it avoids all of the following 35 patterns.*

| | | | | | | |
|---|---|---|---|---|---|---|
| 21543 | 231564 | 315264 | 5271436 | 26487153 | 54726183 | 64821537 |
| 32154 | 241365 | 426153 | 5276143 | 26581437 | 54762183 | 64872153 |
| 214365 | 241635 | 2547163 | 5472163 | 26587143 | 61832547 | 65821437 |
| 214635 | 312645 | 4265173 | 25476183 | 51736284 | 61837254 | 65827143 |
| 215364 | 314265 | 5173264 | 26481537 | 51763284 | 61873254 | 65872143 |

This process is also feasible for $k = 3$, in which case we need to look at non-3-vexillary patterns up through $S_{12}$. Here we find that the bound in Corollary 5.10 is not sharp.

**Theorem 5.12** *A permutation $w$ is 3-vexillary if and only if it avoids a list of 91 patterns in $S_6 \cup S_7 \cup S_8$.*

Searching through all non-4-vexillary permutation in $S_{16}$ is currently beyond our computational capabilities. However, one does find that every non-4-vexillary permutation in $S_{13}$ contains a proper non-4-vexillary pattern.

**Conjecture 2** *A permutation $w$ is $4$-vexillary if and only if it avoids a list of $2346$ patterns in $S_6 \cup S_7 \cup \cdots \cup S_{12}$.*

Unless $n$ is large compared to $k$, our pattern characterizations are less efficient for checking that $w \in S_n$ is $k$-vexillary than the Lascoux-Schützenberger tree. On the other hand, pattern characterizations give an easy way to compare theorems. As an example, the *essential set* $Ess(w)$ of a permutation $w$ is the set of southeast corners of connected components of $D(w)$. Fulton (1992) introduced the essential set and showed that the rank conditions for the Schubert variety indexed by $w$ need only be checked at cells in the essential set. Making $Ess(w)$ into a poset by $(i_1, j_1) \leq (i_2, j_2)$ if $i_1 \geq i_2$ and $j_1 \leq j_2$, Fulton also showed that $w$ is vexillary if and only if $Ess(w)$ is a chain.

Similarly, one can check that the property that $Ess(w)$ is a union of two chains is characterized by $w$ avoiding a specific set of patterns. All of these patterns turn out to be non-3-vexillary, and then Theorem 1.2 shows that if $w$ is 3-vexillary, $Ess(w)$ is a union of two chains.

# 6 Future work

We were led to Theorem 1.2 by trying to prove Lemma 3.8 for arbitrary diagrams and subdiagrams. Lemma 3.8 holds when the subdiagram is (isomorphic to) a permutation diagram, a skew shape, or a column-convex diagram, since these diagrams all admit complete James-Peel trees. The algorithm given by Reiner and Shimozono in Reiner and Shimozono (1998) for decomposing Specht modules shows that the conclusion of Lemma 3.8 also holds when $D$ is percent-avoiding and $D' = D \cap \{i : a \leq i \leq b\} \times \{j : c \leq j \leq d\}$ for some $a, b, c, d$.

We have no simple characterizations of the lists of patterns arising from Corollary 5.10 and Theorems 5.11 and 5.12. One necessary condition for $w$ to be non-$k$-vexillary but contain only $k$-vexillary patterns is that every $w(i)$ participates in some 2143 pattern. Otherwise, the $i$th row and $w(i)$th column of $D(w)$ are contained in or contain every other row and column, and so they do not participate in the James-Peel moves of $RJP(w)$. This is far from sufficient, however.

In Billey and Lam (1998), vexillary signed permutations of types $B, C, D$ in the hyperoctahedral group are defined as those whose Stanley symmetric function is equal to a single Schur $P$- or $Q$-function ($P$ in types $B, D$, and $Q$ in type $C$), and it is shown that the vexillary signed permutations are again characterized by avoiding a finite set of patterns. Computer calculations show that Corollary 4.3 with $k = 2$ holds in $B_9$ for types $B, C$ and in $D_8$; moreover, the 2-vexillary patterns in $B_9$ of types $B, C$ are characterized by avoiding sets of patterns in $B_3 \cup \cdots \cup B_8$. The main obstacle to extending our proofs to these other root systems is the apparent lack of an analogue of the Specht module of a diagram. In a recent preprint, Anderson and Fulton (2012) give a different definition of vexillary permutations in types $B, C, D$, and one might ask if there is a reasonable notion of $k$-vexillary in their setting.

# Acknowledgements

We would like to thank Dave Anderson, Mark Shimozono, Ricky Liu, and Aaron Pihlman for helpful discussions, and Eric Peterson for the term “bushy”.

# References

D. Anderson and W. Fulton. Degeneracy loci, pfaffians, and vexillary signed permutations in types B, C, and D. arXiv:1210.2066v1 [math.AG], 2012.

S. Billey and T. K. Lam. Vexillary elements in the hyperoctahedral group. *Journal of Algebraic Combinatorics*, 8:139–152, 1998.

S. Billey, W. Jockusch, and R. P. Stanley. Some combinatorial properties of Schubert polynomials. *Journal of Algebraic Combinatorics*, 2:345–374, 1993.

P. Edelman and C. Greene. Balanced tableaux. *Advances in Mathematics*, 1:42–99, 1987.

W. Fulton. Flags, Schubert polynomials, degeneracy loci, and determinantal formulas. *Duke Mathematical Journal*, 65:381–420, 1992.

W. Fulton. *Young Tableaux: With Applications to Representation Theory and Geometry*. Cambridge University Press, 1997.

G. James and M. Peel. Specht series for skew representations of symmetric groups. *Journal of Algebra*, 56:343–364, 1979.

W. Kraśkiewicz. Reduced decompositions in Weyl groups. *European Journal of Combinatorics*, 16: 293–313, 1995.

A. Lascoux and M.-P. Schützenberger. Schubert polynomials and the Littlewood-Richardson rule. *Letters in Mathematical Physics*, 10:111–124, 1985.

R. Liu. Matching polytopes and Specht modules. *Transactions of the American Mathematical Society*, 364:1089–1107, 2009.

V. Reiner and M. Shimozono. Specht series for column-convex diagrams. *Journal of Algebra*, 174: 489–522, 1995a.

V. Reiner and M. Shimozono. Plactification. *Journal of Algebraic Combinatorics*, 4:331–351, 1995b.

V. Reiner and M. Shimozono. Percentage-avoiding, northwest shapes, and peelable tableaux. *Journal of Combinatorial Theory, Series A*, 82:1–73, 1998.

H. Rothe. Ueber Permutationen, in Beziehung auf die Stellen ihrer Elemente. Anwendung der daraus abgeleiteten Satze auf das Eliminationsproblem. In C. Hindenburg, editor, *Sammlung Combinatorisch-Analytischer Abhandlungen*, pages 263–305. Bey G. Fleischer dem jüngern, 1800.

B. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Springer-Verlag, 2001.

R. P. Stanley. On the number of reduced decompositions of elements of Coxeter groups. *European Journal of Combinatorics*, 5:359–372, 1984.

J. R. Stembridge. Multiplicity-free products of Schur functions. *Annals of Combinatorics*, 5(2):113–121, 2001.

B. E. Tenner. Reduced decompositions and permutation patterns. *Journal of Algebraic Combinatorics*, 24:263–284, 2006.

H. Thomas and A. Yong. Multiplicity-free Schubert calculus. *Canadian Mathematical Bulletin. Bulletin Canadien de Mathématiques*, 53(1):171–186, 2010.