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Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory

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Abstract. We develop a new method for studying the asymptotics of symmetric polynomials of representation–theoretic origin as the number of variables tends to infinity. Several applications of our method are presented: We prove a number of theorems concerning characters of infinite–dimensional unitary group and their q–deformations. We study the behavior of uniformly random lozenge tilings of large polygonal domains and find the GUE–eigenvalues distribution in the limit. We also investigate similar behavior for Alternating Sign Matrices (equivalently, six–vertex model with domain wall boundary conditions). Finally, we compute the asymptotic expansion of certain observables in the $O(n = 1)$ dense loop model.

Résumé. Nous développons une nouvelle méthode pour étudier l'asymptotique des polynômes symétriques d'origine représentation théorique quand le nombre de variables tend vers l'infini. Plusieurs applications de notre méthode seront présentées: Nous démontrons un certain nombre de théorèmes concernant les caractères du groupe unitaire de dimension infinie et leurs q–déformations. Nous étudions le comportement des pavages en losange a distribution uniforme et aléatoire de grands domaines polygonaux et nous trouvons la distribution des valeurs propres des GUE à la limite. Nous étudions également le comportement similaire des ASM. Enfin, nous calculons l'expansion asymptotique de certains paramètres observables en $O(n = 1)$ modèle de la boucle dense.

Keywords: Schur functions, asymptotics, GUE, ASM, infinite-dimensional unitary group, dense loop model, lozenge tilings

1 Introduction

In this article we study the asymptotic behavior of symmetric functions as the number of variables tends to infinity. The functions of interest originate in representation theory but have interpretations in combinatorics and statistical mechanics. In this extended abstract we focus on the Schur functions, but most of

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our results hold in a greater generality, e.g. for the characters of the symplectic and orthogonal groups, BC_n characters.

The [rational] Schur function $s_\lambda(x_1,\dots,x_n)$ is a symmetric Laurent polynomial in variables x_1, …, x_n parameterized by N–tuple of integers $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N)$ (we call such N–tuples *signatures*, they from the set $\mathbb{GT}_N$) and given by Weyl's character formula as

$$s_\lambda(x_1,\dots,x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^{N}}{\prod_{i<j}(x_i-x_j)}.$$

Here we study the asymptotic behavior of the following *normalized* symmetric polynomials

$$S_\lambda(x_1,\dots,x_k;N,1) = \frac{s_\lambda(x_1,\dots,x_k,\overbrace{1,\dots,1}^{N-k})}{s_\lambda(\underbrace{1,\dots,1}_{N})} \tag{1}$$

$$S_\lambda(x_1,\dots,x_k;N,q) = \frac{s_\lambda(x_1,\dots,x_k,1,\dots,q^{N-k-1})}{s_\lambda(1,\dots,q^{N-1})} \tag{2}$$

for some $q > 0$. Here $\lambda = \lambda(N)$ is allowed to vary with N, k is any fixed number and $x_1,\dots,x_k$ are complex numbers, which may or may not vary together with N, depending on the context.

The asymptotic analysis of expressions (1), (2) is important because of the various applications in representation theory, statistical mechanics and probability, including:

- Convergence of (1) for any k and any *fixed* $x_1,\dots,x_k$ with $|x_i| = 1$ to some limit and identification of this limit can be put in representation–theoretic framework as the approximation of indecomposable characters of infinite–dimensional unitary group $U(\infty)$ by normalized characters of unitary groups $U(N)$, the latter problem was first studied by Vershik and Kerov [VK].

- Convergence of (2) for any k and any *fixed* $x_1,\dots,x_k$ is similarly related to the *quantization* of characters of $U(\infty)$, see [G].

- Asymptotic behavior of (1) can be put in the context of Random Matrix Theory as the study of Harish–Chandra-Itzykson–Zuber integral

$$\int_{U(N)} \exp(\mathrm{Trace}(AUBU^{-1}))dU, \tag{3}$$

  where A is a fixed Hermitian matrix of finite rank and $B = B(N)$ is an $N \times N$ matrix changing in a regular way as $N \to \infty$. This problem was thoroughly studied by Guionnet and Maïda [GM].

- (1) can be interpreted as the expectation of a certain observable in the probabilistic model of uniformly random lozenge tilings of planar domains. The asymptotical analysis of (1) as $N \to \infty$ with $x_i = \exp(y_i/\sqrt{N})$ and fixed y_i gives a direct way to prove the local convergence of random tilings to a distribution of random matrix origin — GUE–corners process. Informal argument explaining that such convergence should hold was suggested earlier by Okounkov and Reshetikhin [OR1].

- When λ is the *double staircase Young diagram* with $2N$ rows $\lambda(2N) = (N-1, N-1, N-2, N-2, \dots, 1, 1, 0, 0)$, then (1) gives the expectation of a certain observable for the uniformly random configurations of the six–vertex model with domain wall boundary conditions, equivalently, Alternating Sign Matrices. Asymptotic behavior $N \to \infty$ with $x_i = \exp(y_i/\sqrt{N})$ and fixed y_i gives a way to study the local limit of this model near the boundary, equivalently, the positions of 1s and -1s in ASMs near the edges.

- For the same staircase λ the expression involving (1) with $k = 4$ and Schur polynomials replaced by the characters of symplectic group is related to the boundary-to-boundary current for the completely packed $O(n = 1)$ model, see [GNP]. The asymptotic (now with fixed x_i not depending on N) gives the limit behavior of this current, significant for the understanding of this critical model. The problem of finding the asymptotic behavior was presented by Jan de Gier during the MSRI program on Random Spatial Processes and it is solved in the present paper.

We develop a new unified approach to the study of the asymptotics of Schur functions (1), (2) (and also for more general symmetric functions like symplectic characters and polynomials corresponding to the root system BC_n), which gives a way to answer all of the above limit questions. There are 3 main ingredients of our method:

1. We find a simple contour integral representations for the normalized Schur polynomials (1), (2) with $k = 1$, i.e. for
$$\frac{s_\lambda(x, 1, \dots, 1)}{s_\lambda(1, \dots, 1)} \quad \text{and} \quad \frac{s_\lambda(x, 1, \dots, q^{N-2})}{s_\lambda(1, \dots, q^{N-1})}, \tag{4}$$
and also for more general symmetric functions of representation–theoretic origin.

2. We study the asymptotics of the above contour integrals using the *steepest descent* method.

3. We find the formulas expressing (1), (2) as $k \times k$ determinant of expressions involving (4), and combining these formulas with the asymptotics of (4) compute the limits of (1), (2).

In the rest of this abstract we will state our asymptotic results and then explain in more detail how they are applied to solve the limit problems listed above. Full details, background and a complete list of references can be found in our 67-page long paper by the same name.

2 Method and asymptotic results

The main ingredient of our approach to the asymptotic analysis of symmetric functions is the following integral formula.

Theorem 2.1 . *Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N)$, and let x be a complex number other than* 0 *and* 1 *, then*

$$S_\lambda(x; N, 1) = \frac{s_\lambda(x, 1^{N-1})}{s_\lambda(1^N)} = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi\sqrt{-1}} \oint_C \frac{x^z}{\prod_{i=1}^N (z - (\lambda_i + N - i))} dz, \tag{5}$$

where the contour C encloses all the singularities of the integrand.

We also prove various generalizations of formula (5), omitted here for brevity: one can replace 1s by geometric series $1, q, q^2, \dots$, Schur functions can be replaced with characters of symplectic group or, more, generally, with multivariate Jacobi polynomials. In all these cases a normalized symmetric function is expressed as a contour integral with integrand being the product of elementary factors. The only exception is the most general case of Jacobi polynomials, where we have to use certain hypergeometric series. Similar formula appears in [CPZ] and [HJ].

Applying tools from complex analysis to formula (5), mainly the method of steepest descent, we compute the limit behavior of (1) for $k = 1$ under different convergence regimes for λ, as described below.

Suppose that there exists a function $f(t)$ for which the vector $(\lambda_1(N)/N, \dots, \lambda_N(N)/N)$ converges to $(f(1/N), \dots, f(N/N))$ pointwise as $N \to \infty$. Let R_1, R_∞ denote the corresponding norms of the difference of vectors $(\lambda_j(N)/N)$ and $f(j/N))$:

$$R_1(\lambda, f) = \sum_{j=1}^{N} \left| \frac{\lambda_j(N)}{N} - f(j/N) \right|, \quad R_\infty(\lambda, f) = \sup_{j=1\dots,N} \left| \frac{\lambda_j(N)}{N} - f(j/N) \right|.$$

We also introduce the function $\mathcal{F}(w)$ defined as

$$\mathcal{F}(w) = \int_0^1 \ln(w - f(t) - 1 + t) dt. \tag{6}$$

Proposition 2.2 *Suppose that $f(t)$ is piecewise-continuous, $R_\infty(\lambda(N), f)$ is bounded and $R_1(\lambda(N), f)/N$ tends to zero as $N \to \infty$, then we have for any fixed $y \in \mathbb{R} \setminus \{0\}$*

$$\lim_{N\to\infty} \frac{\ln S_{\lambda(N)}(e^y; N, 1)}{N} = y w_0 - F(w_0) - 1 - \ln(e^y - 1),$$

where w_0 is a root of $\mathcal{F}'(w) = y$ (when y is real then w is real also).

Remark 1. This proposition holds for any real y and in many cases when y is complex under some mild technical assumptions on $\mathcal{F}(w)$ which will not be discussed here.

Remark 2. Note that piecewise-continuity of $f(t)$ is a reasonable assumption since f is monotonous.

Remark 3. The solution w to $\partial/\partial w \mathcal{F}(w) = y$ can be interpreted as *inverse Hilbert transform*.

Remark 4. A somewhat similar statement was proven by Guionnet and Maïda, [GM, Theorem 1.2].

Proposition 2.2 can be refined as follows.

Proposition 2.3 *Suppose that the limit shape of λ, given by $f(t)$, is twice-differentiable and*

$$\lim_{N\to\infty} \prod_{j=1}^{N} \left(1 + \frac{f(j/N) - \lambda_j(N)/N}{w - f(j/N) - 1 + j/N}\right) = g(w) \tag{7}$$

uniformly on an open $\mathcal{M}$ set in $\mathbb{C}$, containing w_0. Then as $N \to \infty$ we have for any fixed $y \in \mathbb{R} \setminus \{0\}$

$$S_{\lambda(N)}(e^y; N, 1) = \sqrt{-\frac{w_0 - f(0) - 1}{\mathcal{F}''(w_0)(w_0 - f(1))}}\, g(w_0) \frac{\exp(N(y w_0 - \mathcal{F}(w_0)))}{e^N (e^y - 1)^{N-1}} \Big(1 + o(1)\Big).$$

where w_0 is a root of $\mathcal{F}'(w) = y$. The remainder $o(1)$ is uniform over y belonging to compact subsets of $\mathbb{R} \setminus \{0\}$ and such that $w_0 = w_0(y) \in \mathcal{M}$.

The last crucial ingredient in the asymptotic analysis is expressing the multivariate normalized symmetric functions through the univariate ones.

Theorem 2.4 *: $S_\lambda(x_1,\dots,x_k;N,1)$ can be expressed in terms of $S_\lambda(x_i;N,1)$ as follows*

$$S_\lambda(x_1,\dots,x_k;N,1) = \frac{1}{\prod_{i<j}(x_i-x_j)} \prod_{i=1}^{k} \frac{(N-i)!}{(x_i-1)^{N-k}} \times \det\left[D_{x_i}^{k-j}\right]_{i,j=1}^{k} \left(\prod_{j=1}^{k} S_\lambda(x_j;N,1) \frac{(x_j-1)^{N-1}}{(N-1)!} \right), \quad (8)$$

where D_x is the differential operator $x\frac{\partial}{\partial x}$.

Formula (8) can again be generalized: the 1s can be replaced with geometric series $1, q, q^2, \dots$ (and D is replaced by $D_q(f)(x) = \frac{f(qx)-f(x)}{q-1}$), Schur functions can be replaced with characters of symplectic group or, more, generally, with multivariate Jacobi polynomials. In principle, the formulas similar to (8) can be found in the literature, see e.g. [GP, Proposition 6.2].

Formula (8) allows us to derive the full asymptotics for the multivariate normalized Schur function, $S_\lambda(x_1,\dots,x_k;N,1)$ from the asymptotics for $S_\lambda(x;N,1)$. As a side remark, since we deal with analytic functions and convergence in our formulas is always (at least locally) uniform, differentiation in formula (8) does not introduce any issues.

3 Applications

3.1 Asymptotic representation theory

The asymptotic analysis developed in this paper can be applied to obtain new simpler proofs of some classical theorems in asymptotic representation theory.

Let $U(N)$ denote the group of all $N \times N$ unitary matrices. The infinite dimensional unitary group is defined as the inductive limit of $U(N)$s, where $U(N)$ embeds in $U(N+1)$ by fixing the $N+1$st vector:

$$U(\infty) = \bigcup_{N=1}^{\infty} U(N).$$

A [normalized] *character* of a group G is a continuous function which is constant on conjugacy classes, positive definite and evaluates to 1 at the unit element. *Extreme character* is an extreme point of the convex set of all characters. If G is a compact group, then extreme characters are normalized matrix traces of irreducible representations. Applying this result to $U(N)$, and using the classical fact that irreducible representations of the unitary group $U(N)$ are parameterized by signatures $\lambda \in \mathbb{GT}_N$, and their characters are the Schur functions $s_\lambda(u_1,\dots,u_N)$ we have that normalized characters of $U(N)$ are the normalized Schur functions $s_\lambda(u)/s_\lambda(1^N)$.

For "big groups" such as $U(\infty)$ the situation is more delicate. The classification theorem for the characters of infinite dimensional unitary group is summarized in the Voiculescu-Edrei ([Vo], [Ed]) theorem, which gives the explicit form of the extreme characters. The following *approximation theorem* explains the connection of characters of $U(\infty)$ with limits of normalized Schur functions.

Proposition 3.1 ([VK],[OO]) *Every extreme normalized character χ of $U(\infty)$ is a uniform limit of extreme characters of $U(N)$: for every χ there exists a sequence $\lambda(N) \in \mathbb{GT}_N$ such that for every k*

$$\chi(u_1, \dots, u_k, 1, \dots) = \lim_{N\to\infty} S_\lambda(u_1, \dots, u_k; N, 1)$$

uniformly on the torus $(S_1)^k$.

The sequences of characters of $U(N)$ which approximate characters of $U(\infty)$ has originally been found in [VK] as follows. Let μ be a Young diagram with row lengths μ_i, column lengths μ_i' and the length of main diagonal d. Introduce *modified Frobenius coordinates*:

$$p_i = \mu_i - i + 1/2, \quad q_i = \mu_i' - i + 1/2, \quad i = 1, \dots, d.$$

Note that $\sum_{i=1}^d p_i + q_i = |\mu|$. Let $\lambda \in \mathbb{GT}_N$ be a signature, we associate two Young diagrams λ^+ and λ^- to it: row lengths of λ^+ are positive of λ_i's, while row lengths of λ^- are minus negative ones. In this way we get two sets of modified Frobenius coordinates: p_i^+, q_i^+, $i = 1, \dots, d^+$ and p_i^-, q_i^-, $i = 1, \dots, d^-$. As a direct corollary of our results on asymptotics of normalized Schur polynomials from Section 2 we can reprove the following Theorem.

Theorem 3.2 ([VK], [OO], [BO],[P2]) *Let $\omega = (\alpha^\pm, \beta^\pm, ; \delta^\pm)$, such that*

$$\alpha^\pm = (\alpha_1^\pm \ge \alpha_2^\pm \ge \dots \ge 0) \in \mathbb{R}^\infty, \quad \beta^\pm = (\beta_1^\pm \ge \beta_2^\pm \ge \dots \ge 0) \in \mathbb{R}^\infty,$$

$$\sum_{i=1}^{\infty} (\alpha_i^\pm + \beta_i^\pm) \le \delta^\pm,\ \beta_1^+ + \beta_1^- \le 1.$$

Suppose that the sequence $\lambda(N) \in \mathbb{GT}_N$ is such that

$$p_i^+(N)/N \to \alpha_i^+, \quad p_i^-(N)/N \to \alpha_i^-, \quad q_i^+(N)/N \to \beta_i^+, \quad q_i^-(N)/N \to \beta_i^+,$$

$$|\lambda^+|/N \to \delta^+, \quad |\lambda^-|/N \to \delta^-.$$

Then for every k the normalized character of $U(\infty)$, parameterized by ω, satisfies

$$\chi^\omega(u_1, \dots, u_k, 1, \dots) = \lim_{N\to\infty} S_{\lambda(N)}(u_1, \dots, u_k; N, 1)$$

uniformly on torus $(S_1)^k$.

Note that every normalized character of $U(\infty)$ is in fact parametrized by such ω.

Voiculescu–Edrei's formula for the characters of $U(\infty)$ exhibits a remarkable multiplicavity where the value of character on the matrix is expressed as a product of values on its eigenvalues of a single function. Formula (8) should be viewed as a manifestation of *approximate multiplicativity* for (normalized) characters of $U(N)$, in particular, formula (8) implies informally that

$$S_\lambda(x_1, \dots, x_k; N, 1) = S_\lambda(x_1; N, 1) \cdots S_\lambda(x_k; N, 1) + O(1/N), \tag{9}$$

so normalized characters of $U(N)$ are approximately multiplicative and exactly multiplicative as $N \to \infty$.

A q–deformation of the notion of character of $U(\infty)$ was suggested in [G] . Let $\lambda(N)$ be such that

$$\frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N})}{s_{\lambda(i)}(1,q^{-1},\dots,q^{1-N})} \tag{10}$$

converges uniformly on $\{(x_1,\dots,x_k)\in\mathbb{C}^k \mid |x_i| = q^{1-i}\}$ for every k. The q–analogues of formulas (5) and (8) give a short proof of the second part of the following quantized version of Theorem 3.2.

Theorem 3.3 ([G]) *Let $0<q<1$. Extreme q–characters of $U(\infty)$ are parameterized by the points of set $\mathcal{N}$ of all non-decreasing sequences of integers: $\mathcal{N}=\{\nu_1\le\nu_2\le\nu_3\le\dots\}\subset\mathbb{Z}^\infty$, Suppose that $\lambda(N)\in\mathbb{GT}_N$ is such that for any $j>0$ $\lim\limits_{i\to\infty}\lambda(N)_{N+1-j}=\nu_j$, then for every k*

$$\frac{s_{\lambda(N)}(x_1,\dots,x_k,q^{-k},q^{-k-1},\dots,q^{1-N})}{s_{\lambda(N)}(1,q^{-1},\dots,q^{1-N})} \tag{11}$$

converges uniformly on $\{(x_1,\dots,x_k)\in\mathbb{C}^k \mid |x_i| = q^{1-i}\}$. These limits define the q–character of $U(\infty)$.

3.2 Random lozenge tilings

Consider a tiling of a domain drawn on the regular triangular lattice of the kind shown at Figure 1 with rhombi of 3 types, called *lozenges*, where each rhombus is a union of 2 elementary triangles. The configuration of the domain is encoded by the number N which is its width and N integers $\mu_1>\mu_2>\dots>\mu_N$ which are the positions of *horizontal lozenges* sticking out of the right boundary. If we write $\mu_i=\lambda_i+N-i$, then λ is a partition of size N and we denote the corresponding domain by Ω_λ. Tilings of such domains are in correspondence with tilings of certain polygonal domains, see Figure 1. It is well–known that each lozenge tiling can be identified with a stepped surface in $\mathbb{R}^3$, see e.g. [Ke].

Let Υ_λ denote a uniformly random lozenge tiling of Ω_λ. Lozenge tilings have remarkable asymptotic behavior. When N is large the rescaled stepped surface corresponding to Υ_λ concentrates near a deterministic limit shape (this holds for more general domains as well, see [CKP]). One feature of the limit shape is the formation of so–called *frozen regions*; these are the regions where asymptotically with high probability only a single type of lozenges is observed. For general polygonal domains the frozen boundary is an inscribed algebraic curve, see [KO] and [P1].

In this article we study the local behavior of lozenge tiling near a *turning point* of the frozen boundary, which is the point where the boundary of the frozen region touches (and is tangent to) the boundary of the domain. Okounkov and Reshetikhin gave in [OR1] a non-rigorous argument explaining that the scaling limit of tilings in such situation should be governed by, what we call *GUE–corners process* (introduced and studied in [Bar] and [JN] as *GUE-minors process*) and defined below. In one model of tilings of infinite polygonal domains presented in [OR1], the proof of the convergence is based on the determinantal structure of the correlation functions of the model and on the double–integral representation for the correlation kernel.

The GUE random matrix ensemble is a probability measure on the set of $k\times k$ Hermitian matrix with density proportional to $\exp(-Trace(X^2)/2)$ and GUE–distribution $\mathbb{GUE}_k$ is the distribution of the k (ordered) eigenvalues of such random matrix. The *GUE–corners process* is the joint distribution of the eigenvalues of the k principal submatrices of a $k\times k$ matrix from a GUE ensemble.

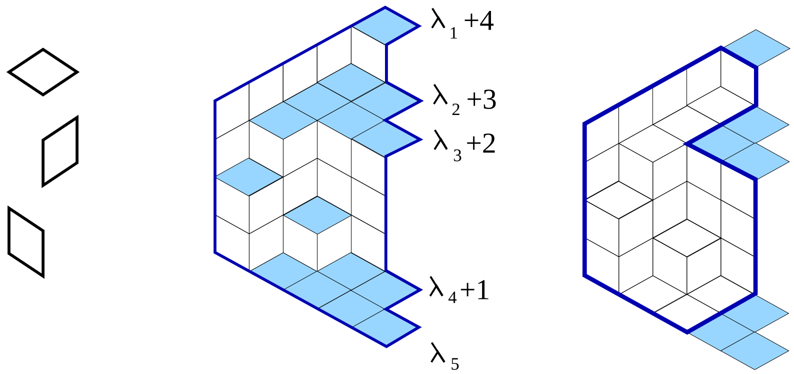

Fig. 1: Left: the 3 types of lozenges, top one is called "horizontal". Middle and right: a lozenge tiling of the domain encoded by a signature λ (here $\lambda = (4,3,3,0,0)$) and of the corresponding polygonal domain (right).

Theorem 3.4 *Let $\lambda(N) \in \mathbb{GT}_N$, $N = 1,2,\dots$ be a sequence of signatures. Suppose that there exist a non-constant piecewise-differentiable weakly decreasing function $f(t)$ such that*

$$\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f(i/N) \right| = o(\sqrt{N})$$

as $N \to \infty$ and also $\sup_{i,N} |\lambda_i(N)/N| < \infty$. Then for every k as $N \to \infty$ we have

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 f(t)^2 dt - E(f)^2 + \int_0^1 f(t)(1-2t)dt.$$

Corollary 3.5 *Under the same assumptions as in Theorem 3.4 the (rescaled) joint distribution of the $k(k+1)/2$ horizontal lozenges (tiles) on the left k lines weakly converges to the GUE–corners process.*

Approach to the proof of Theorem 3.4: The moment generating function for the position of the horizontal lozenges along the kth vertical section of the tiling is essentially given by $S_\lambda(e^{x_1},\dots,e^{x_k};N,1)$. Using the asymptotic results of Section 2 we derive the following asymptotic expansion

$$S_\lambda(e^{\frac{h_1}{\sqrt{N}}},\dots,e^{\frac{h_k}{\sqrt{N}}};N,1) \;=\; \exp\left(\sqrt{N}E(f)(h_1+\dots+h_k) + \frac{1}{2}S(f)(h_1^2+\dots+h_k^2) + o(1)\right),$$

which corresponds to the moment generating function for the GUE–corners process. □

Remark. As $N \to \infty$ our domains may approximate a non–polygonal limit domain, where the results of [KO] describing the limit shape as an algebraic curves do not apply and the exact shape of the frozen boundary is unknown. Even an explicit expression for the position of the point where the frozen boundary touches the left boundary (a side result of Theorem 3.4) seems not to have been present in the literature.

3.3 The six–vertex model and random Alternating Sign Matrices

An *Alternating Sign Matrix* (ASM) of size N is a $N \times N$ matrix filled with 0s 1s and $-1s$ such that the sum along every row and column is 1 and along each row and each column the nonzero entries alternate in sign. ASMs are in bijection with configurations of the six-vertex model with domain-wall boundary conditions as shown at Figure 2 and have been a subject of interest in both combinatorics and statistical mechanics, see [Br] for a review. They are enumerated by a remarkable formula, proven independently by Zeilberger and Kuperberg twenty years ago, but not much more of their refined enumeration or statistics is known, see [BFZ] for some state of the art results.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

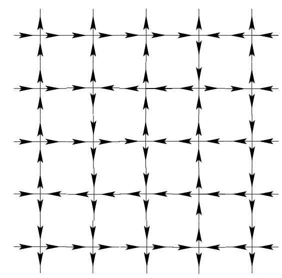

Fig. 2: Alternating sign matrix of size 5 and the corresponding configuration of the 6–vertex model .

We are interested in what the *uniformly random* ASM of size N looks like when N is large. Conjecturally, the features of this model should be similar to those of lozenge tilings: we expect the formation of a limit shape and various connections with random matrices. The existence and properties of the limit shape were studied by Colomo and Pronko [CP2], however their arguments are physical, while a mathematical proof is yet unavailable.

In the present article we prove a partial result toward the following conjecture.

Conjecture 3.6 *Fix any k. As $N \to \infty$ the probability that the number of $(-1)s$ in the first k rows of uniformly random ASM of size N is maximal (i.e. there is one (-1) in second row, two $(-1)s$ in third row, etc) tends to 1, and, thus, the $1s$ in the first k rows are interlacing. After proper centering and rescaling the distribution of the positions of $1s$ tends to GUE–corners process as $N \to \infty$.*

Let $\Psi_k(N)$ denote the sum of horizontal coordinates of 1s minus the sum of horizontal coordinates of (-1)s in the kth row of a uniformly random ASM of size N. We prove that

Theorem 3.7 *For any fixed k the random variable $\frac{\Psi_k(n)-N/2}{\sqrt{N}}$ weakly converges to the normal random variable $\mathcal{N}(0, \sqrt{3/8})$. Moreover, the joint distribution of any collection of such variables converges to the distribution of independent normal random variables $\mathcal{N}(0, \sqrt{3/8})$.*

Our proof of Theorem 3.7 is based on the results of Okada [Ok] and Stroganov [St] that sums of certain quantities over all ASMs (i.e. the partition functions of the 6-vertex model) can be expressed through Schur polynomials and on the asymptotic analysis of these polynomials. In fact, we claim that the precise statement Theorem 3.7 together with additional probabilistic argument implies Conjecture 3.6. However, this argument in itself is unrelated to the asymptotics of symmetric polynomials and is postponed to a future publication.

3.4 The $O(n = 1)$ dense loop model

Recently found parafermionic observables in the *completely packed* $O(n = 1)$ *dense loop model* in a strip are also simply related to symmetric polynomials, see [GNP]. The $O(n = 1)$ dense loop model is one of the representations of the percolation model on the square lattice. For the critical percolation models similar observables and their asymptotic behavior were studied (see e.g. [Sm]), however, the methods involved are usually completely different from ours.

A configuration of the $O(n = 1)$ loop model in a vertical strip consists of two parts: a tiling of the strip on a square grid of width L and infinite height with squares of two types shown in Figure 3, and a choice of one of the two types of boundary conditions for each 1×2 segment along each of the vertical boundaries of the strip, see3. Let $\urcorner_L$ denote the set of all configurations of the model in the strip of width L. As the arcs drawn on squares and boundary segments form closed loops and paths joining the boundaries, the elements of $\urcorner_L$ are interpreted as collections of non-intersecting paths and closed loops.

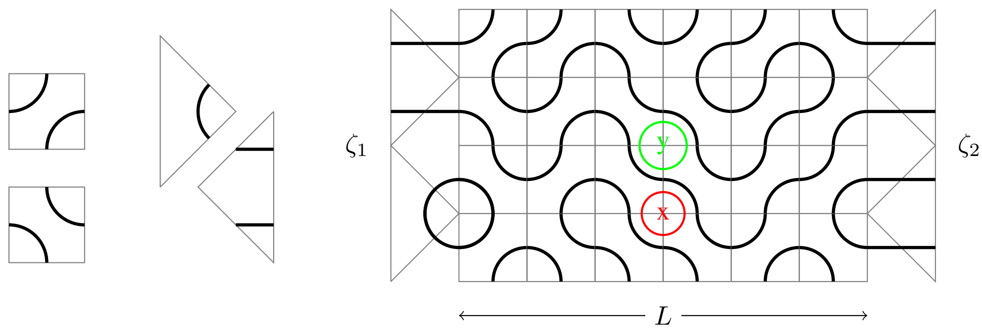

Fig. 3: Left: the two types of squares. Middle: the two types of boundary conditions. Right:A particular configuration of the dense loop model showing a path passing between two vertically adjacent points x and y.

A probability distribution on $\urcorner_L$ is defined by choosing the type of square for each point on the grid according to a weight defined as a certain function of its horizontal coordinate and depending on L parameters $z_1, \dots, z_L$; two other parameters ζ_1, ζ_2 control the probabilities of the boundary conditions and, using a parameter q, the whole configuration is further weighted by its number of closed loops. Setting $z_i = 1$ and $q = \exp(-\sqrt{-1}\pi/6)$ makes the choice of square type for each position an I.I.D. Bernoulli RV with parameter $1/2$. See [GNP] for exact details.

Fix two points x and y and consider a configuration $\omega \in \urcorner_L$. For each path τ passing between x and y, define the current $c(\tau)$ as 1 if τ joins the two boundaries and x lies above τ; -1 if τ joins the two boundaries and x lies below τ; and 0 otherwise. The total current $C^{x,y}(\omega)$ is the sum of $c(\tau)$ over all [necessarily finitely many] paths passing between x and y. The *mean total current* $F^{x,y}$ is defined as the expectation of $C^{x,y}$. As $F^{x,y}$ is skew-symmetric and additive, it can be expressed as a sum of several instances of the mean total current between two horizontally adjacent points, $F^{(i,j),(i,j+1)}$, and two vertically adjacent points, $F^{(j,i),(j+1,i)}$. The authors of [GNP] present a formula for $F^{(i,j),(i,j+1)}$, and $F^{(j,i),(j+1,i)}$, which, based on certain assumptions, expresses them through the symplectic characters $\chi_{\lambda^L}(z_1^2, \dots, z_L^2, \zeta_1^2, \zeta_2^2)$ as certain functions denoted Y_L and $X_L^{(j)}$, where $\lambda^L = (\lfloor \frac{L-1}{2} \rfloor, \lfloor \frac{L-2}{2} \rfloor, \dots, 1, 0, 0)$.

Our approach from Section 2 allows us to compute the asymptotic behavior of the formulas of [GNP] as the lattice width $L \to \infty$ in the homogenous case with $q = \exp(-\sqrt{-1}\pi/6)$ as follows.

Theorem 3.8 *As $L \to \infty$, the formula of [GNP] for the mean total current between two horizontally adjacent points is asymptotically*

$$X_L^{(j)}\Big|_{z_j=z;\, z_i=1,\ i\neq j} = \frac{\sqrt{-3}}{4L}(z^3 - z^{-3}) + o\left(\frac{1}{L}\right).$$

The formula of [GNP] for the mean total current between two vertically adjacent points is asymptotically

$$Y_L\Big|_{z_i=1,\ i=1,\ldots,L;z_{L+1}=w,z_{L+2}=q^{-1}w} = \frac{\sqrt{-3}}{4L}(w^3 - w^{-3}) + o\left(\frac{1}{L}\right).$$

Remark 1. When $z = 1$, $X_L^{(j)}$ is identically zero and so is our asymptotics.
Remark 2. The fully homogeneous case corresponds to $w = \exp(-\sqrt{-1}\pi/6)$, then $Y_L = \frac{\sqrt{3}}{2L} + o\left(\frac{1}{L}\right)$.
Remark 3. The leading asymptotics terms do not depend on the boundary parameters ζ_1 and ζ_2.

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