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Transition matrices for symmetric and quasisymmetric Hall-Littlewood polynomials (Extended Abstract)

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Abstract We introduce explicit combinatorial interpretations for the coefficients in some of the transition matrices relating to skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x;t)$ and Hivert's quasisymmetric Hall-Littlewood polynomials $G_\gamma(x;t)$. More specifically, we provide the following:

1. G_γ-expansions of the P_λ, the monomial quasisymmetric functions, and Gessel's fundamental quasisymmetric functions F_α, and
2. an expansion of the $P_{\lambda/\mu}$ in terms of the F_α.

The F_α expansion of the $P_{\lambda/\mu}$ is facilitated by introducing the set of *starred tableaux*. In the full version of the article we also provide G_γ-expansions of the quasisymmetric Schur functions and the peak quasisymmetric functions of Stembridge.

Résumé. Nous introduisons des interprétations combinatoires explicites pour les coefficients dans l'expansion de quelques matrices de transition en relation avec les polynômes skew de Hall-Littlewood $P_{\lambda/\mu}(x;t)$ et les fonctions quasisimmetriques de Hall-Littlewood $G_\gamma(x;t)$. Plus spécifiquement, nous donnons les suivants:

1. G_γ-expansions pour le P_λ, les fonctions monomiales quasisimmetriques et les fonctions fondamentales quasisymmetriques de Gessel's F_α et
2. une expansion des $P_{\lambda/\mu}$ en termes des F_α.

L'expansion des $P_{\lambda/\mu}$ en termes des F_α est facilitée grâce a l'introduction de l'ensemble de *tableaux étoilés*. Dans la version complete de cette article, nous donnons aussi G_γ-expansions pour les fonctions quasisymmetriques de Schur et les fonctions quasisymmetriques de pic de Stembridge.

Keywords: symmetric functions, quasisymmetric functions, Hall-Littlewood polynomials, standardization, Young tableaux, noncommutative symmetric functions

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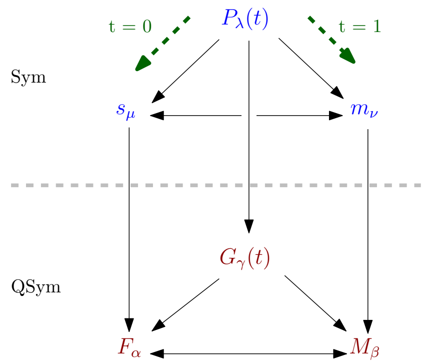

Fig. 1: Prism of bases and transitions.

1 Introduction

The ring of symmetric functions Sym and the ring of quasisymmetric functions QSym both play important roles in algebra and combinatorics. Much of the combinatorial richness arising from these rings stems from their various distinguished bases and the relationships between these bases. The goal of this paper is to present explicit, combinatorial descriptions of several such transition matrices relating to the Hall-Littlewood polynomials. Figure 1 illustrates the bases discussed.

In the top triangle in Figure 1 are included two classical bases for the ring of symmetric functions: the *Schur functions* s_μ and the *monomial symmetric functions* m_ν. The s_μ and m_ν are closely related to a third, one-parameter family of symmetric functions $P_\lambda(x;t)$, known as *Hall-Littlewood polynomials*. More specifically, P_λ equals s_λ at $t = 0$, and it equals m_λ at $t = 1$. The P_λ arose out of a problem studied by P. Hall. Hall had used his eponymous algebra (isomorphic to the algebra of symmetric functions) to encode the structure of finite abelian p-groups. However, at the time there was no known explicit basis of symmetric functions with the same structure constants as that of the natural basis for Hall's algebra. D. E. Littlewood [11] solved this problem in 1961 with his introduction of the $P_\lambda(x;t)$.

The bottom triangle of Figure 1 consists of quasisymmetric analogues of the above bases. In the context of quasisymmetric functions, the *monomial quasisymmetric functions*, M_β, are a very natural analogue of the m_ν. Moreover, there do exist quasisymmetric Schur functions [6]. However, for reasons described in the next paragraph, we anchor the lower-left portion of the bottom triangle in Figure 1 by Gessel's *fundamental quasisymmetric functions*, denoted here by F_α. By defining an action of the Hecke algebra on polynomials which leaves the quasisymmetric functions invariant, Hivert [7] has constructed the *quasisymmetric Hall-Littlewood polynomials* $G_\gamma(x;t)$. (See also work of Lascoux, Novelli, and Thibon [8] for constructions of quasisymmetric and noncommutative symmetric functions with extra parameters.) Similarly to what happens in the top triangle, specialization of the G_γ at $t = 0$ (which corresponds to the southwest-pointing arrow in Figure 1) yields F_γ, while specialization at $t = 1$ yields M_γ.

We now motivate our choice of the F_α as the desired quasisymmetric analogue of the Schur functions. The Schur functions are the prototypical example of a symmetric function with combinatorial expansions in terms of both a collection of *semistandard objects* (i.e., semistandard Young tableaux) and of *standard objects* (i.e., standard Young tableaux). The first case is that of the classical expansion in terms of monomials weighted by the Kostka numbers. The second expansion (due to Gessel [3]) expresses the Schur functions in terms of fundamental quasisymmetric functions F_α. This expansion, which follows from the technique of *standardization*, is indicated by the vertical line connecting s_μ and F_α in Figure 1. Such standardizations have been used recently to give F-expansions of various symmetric functions including plethysms of Schur functions [14], the modified Macdonald polynomials [4, 5], the Lascoux-Leclerc-Thibon (LLT) polynomials [10], and (conjecturally) the image of a Schur function under the Bergeron-Garsia nabla operator [13].

Given Hivert's construction, the following question arises. Is there an expansion of the P_λ in terms of the G_γ which would interpolate between the F-expansion of the s_λ at $t = 0$ and the M-expansion of the m_μ at $t = 1$? The main purpose of this paper is to provide such an expansion, and also to provide other change-of-basis matrices between different bases of the Hall algebra and the algebra of quasisymmetric functions, as explained below. In terms of Figure 1, we provide the middle vertical edge as well as the two downward directed edges in the bottom face (namely, from $G_\gamma(t)$ to both F_α and M_β).

G-expansion of the P Basis. In Theorem 5.6 we give an explicit combinatorial expansion of the Hall-Littlewood polynomials $P_\lambda(x;t)$ in terms of the Hivert quasisymmetric Hall-Littlewood polynomials $G_\gamma(x;t)$. This provides the desired t-interpolation between Gessel's F-expansion of Schur polynomials (i.e., $t = 0$) and the obvious expansion of m_λ's into M_α's (i.e., $t = 1$).

F-expansion of the P Basis. One of the main tools for our calculations is the definition of a new class of tableaux, called *starred tableaux*. With these, we give in Theorem 4.1 a combinatorial expansion of the skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x;t)$ in terms of the fundamental quasisymmetric functions F_α. A minor variation to our method gives a corresponding expansion for the dual Hall-Littlewood polynomials $Q_{\lambda/\mu}$.

G-expansion of the F and M Bases. In Theorems 5.1 and 5.3 we give explicit combinatorial expansions for the F_α and the M_β in terms of the G_γ. These are inverse matrices to those found in [7].

The structure of this extended abstract is as follows. The bases discussed are defined in §2 while the known transition matrices are summarized in §3. The expansions of the Hall-Littlewood polynomials in terms of the F_α and $G_\gamma(x;t)$ are presented in §4 and §5, respectively.

This text is an extended abstract of the preprint [12], where complete proofs can be found. Furthermore, in [12] we give explicit combinatorial expansions for the peak quasisymmetric functions K_α and the quasisymmetric Schur functions $\mathcal{S}_\beta$ in terms of the G_γ.

2 Review of Symmetric and Quasisymmetric Bases

This section reviews the definitions of the symmetric and quasisymmetric functions appearing in Figure 1. Logically, the precise definitions of the various bases are not needed in this paper, as the expansions found in §4 and §5 are derived from the known transition matrices of §3. However, the material of this section is included for completeness.

2.1 Compositions and Partitions

Given $n \in \mathbb{N}$, a *composition of* n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers (called *parts*) with $\alpha_1 + \cdots + \alpha_k = n$. Define the *length* $\ell(\alpha)$ to be the number of parts of α, and the *size* $|\alpha|$ to be the sum of its parts. For example, the composition $\alpha = (2, 4, 1)$ has $\ell(\alpha) = 3$ and $|\alpha| = 7$. We may abbreviate the notation, writing α as 241, when no confusion can arise. Let Comp_n be the set of compositions of n, and let Comp be the set of all compositions. A composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathrm{Comp}_n$ is called a *partition of* n iff $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We write Par_n for the set of partitions of n and Par for the set of all partitions.

For $n \in \mathbb{N}^+$, there are 2^{n-1} compositions of n and 2^{n-1} subsets of $[n-1] = \{1, 2, \ldots, n-1\}$. One can define natural bijections between these sets of objects as follows. Given $\alpha \in \mathrm{Comp}_n$ as above, let

$$\mathrm{sub}(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\} \subseteq [n-1].$$

The inverse bijection sends any subset $T = \{t_1 < t_2 < \cdots < t_m\} \subseteq [n-1]$ to

$$\mathrm{comp}(T) = (t_1, t_2 - t_1, t_3 - t_2, \ldots, t_m - t_{m-1}, n - t_m) \in \mathrm{Comp}_n .$$

Given $\alpha, \beta \in \mathrm{Comp}_n$, we say β is *finer* than α, denoted $\beta \succeq \alpha$, iff $\mathrm{sub}(\alpha) \subseteq \mathrm{sub}(\beta)$. Informally, β is finer than α if we can chop up some of the parts of α into smaller pieces (without reordering anything) and obtain β. For example, $(1,1,1,1) \succeq (1,2,1) \succeq (3,1) \succeq (4)$.

2.2 Symmetric Polynomials

Let K be a field of characteristic zero, and let $\mathfrak{S}_N$ denote the symmetric group on N letters. A polynomial $f \in K[x_1, \ldots, x_N]$ is called *symmetric* iff

$$f(x_{w(1)}, x_{w(2)}, \ldots, x_{w(N)}) = f(x_1, x_2, \ldots, x_N) \text{ for all } w \in \mathfrak{S}_N.$$

Write Sym_N for the ring of symmetric polynomials in N variables. For each $n \geq 0$, let Sym_N^n be the subspace of Sym_N consisting of zero and the homogeneous polynomials of degree n. For $N \geq n$, bases of the vector space Sym_N^n are naturally indexed by partitions of n.

Given $\lambda \in \mathrm{Par}_n$ of length $k \leq N$, the *monomial symmetric polynomial* $m_\lambda(x_1, \ldots, x_N)$ is the sum of all distinct monomials that can be obtained by permuting subscripts in $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$. For $N \geq n$, $\{m_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}_n\}$ is readily seen to be a basis of Sym_N^n.

Now suppose $N \geq n$ and $\nu \in \mathrm{Par}_n$ is a partition with *distinct* parts. If necessary, we append parts of size zero to the end of ν to make ν have length N. The *monomial antisymmetric polynomial indexed by* ν *in* N *variables* is

$$a_\nu(x_1, \ldots, x_N) = \sum_{w \in \mathfrak{S}_N} \prod_{i=1}^{N} \mathrm{sgn}(w) x_{w(i)}^{\nu_i} = \det ||x_i^{\nu_j}||_{1 \leq i,j \leq N}.$$

Letting $\delta_N = (N-1, N-2, \ldots, 2, 1, 0)$, $a_{\delta_N}(x_1, \ldots, x_N) = \prod_{1 \leq i < j \leq N}(x_i - x_j)$ is the Vandermonde determinant. Given $\lambda \in \mathrm{Par}_n$, the *Schur symmetric polynomial indexed by* λ *in* N *variables* is

$$s_\lambda(x_1, \ldots, x_N) = \frac{a_{\lambda+\delta_N}(x_1, \ldots, x_N)}{a_{\delta_N}(x_1, \ldots, x_N)}.$$

It can be shown that this rational function is both a polynomial and symmetric. Moreover $\{s_\lambda : \lambda \in \mathrm{Par}_n\}$ is a basis of Sym_N^n [15, §I.3, p. 40].

For the rest of the paper, let t be an indeterminate, and let K be any field containing $\mathbb{Q}(t)$ as a subfield. Following [15, §III.1, pp. 204–7], we define the *Hall-Littlewood symmetric polynomials* as follows. Fix $\lambda \in \mathrm{Par}_n$ and $N \geq n$. Extend λ to have length N by appending parts of size zero if needed. Define

$$R_\lambda(x_1, \ldots, x_N; t) = \frac{\sum_{w \in \mathfrak{S}_N} \mathrm{sgn}(w) x_{w(1)}^{\lambda_1} \cdots x_{w(N)}^{\lambda_N} \prod_{1 \leq i < j \leq N}(x_{w(i)} - t x_{w(j)})}{\prod_{1 \leq i < j \leq N}(x_i - x_j)}.$$

Define $[m]_t = 1+t+t^2+\cdots+t^{m-1}$, $[0]_t = 0$, $[m]!_t = \prod_{i=1}^{m}[i]_t$, and $[0]!_t = 1$. Given that λ has m_0 parts equal to 0, m_1 parts equal to 1, and so on, it can be shown that R_λ is divisible by $[m_0]!_t[m_1]!_t \cdots [m_N]!_t$. We then define the Hall-Littlewood polynomial

$$P_\lambda(x_1,\ldots,x_N;t) = \frac{R_\lambda(x_1,\ldots,x_N;t)}{[m_0]!_t[m_1]!_t\cdots[m_N]!_t}.$$

It can be shown [15, §III.2, p. 209] that for $N \geq n$, the set $\{P_\lambda(x_1,\ldots,x_N;t) : \lambda \in \mathrm{Par}_n\}$ is a basis for Sym_N^n. Moreover, setting $t = 0$ in P_λ gives s_λ, whereas setting $t = 1$ in P_λ gives m_λ. Thus, the Hall-Littlewood basis "interpolates" between the Schur basis and the monomial basis. One can define Schur polynomials and Hall-Littlewood polynomials more concretely by giving combinatorial descriptions of their expansions in terms of monomial symmetric polynomials. See §3.1 below.

2.3 Quasisymmetric Polynomials

A polynomial $f \in K[x_1,\ldots,x_N]$ is called *quasisymmetric* iff for every composition $\alpha = (\alpha_1,\ldots,\alpha_k)$ with at most N parts and every $1 \leq i_1 < i_2 < \cdots < i_k \leq N$, the monomials $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$ and $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ have the same coefficient in f. Write QSym_N for the ring of quasisymmetric polynomials in N variables. For each $n \geq 0$, let QSym_N^n be the subspace of QSym_N consisting of zero and the homogeneous polynomials of degree n. For $N \geq n$, linear bases of QSym_N^n are naturally indexed by compositions of n. Symmetric polynomials are quasisymmetric, so Sym_N^n is a subspace of QSym_N^n.

For $\alpha \in \mathrm{Comp}_n$ of length $k \leq N$, the *monomial quasisymmetric polynomial* $M_\alpha(x_1,\ldots,x_N)$ is the sum of all monomials $x_{i_1}^{\alpha_1}x_{i_2}^{\alpha_2}\cdots x_{i_k}^{\alpha_k}$ for which $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. For $N \geq n$, $\{M_\alpha(x_1,\ldots,x_N) : \alpha \in \mathrm{Comp}_n\}$ is readily seen to be a basis of QSym_N^n.

For $\alpha \in \mathrm{Comp}_n$ of length at most N, define Gessel's *fundamental quasisymmetric polynomial* [3] by

$$F_\alpha(x_1,\ldots,x_N) = \sum x_{w_1}x_{w_2}\cdots x_{w_n},$$

where we sum over all subscript sequences $w = w_1w_2\cdots w_n$ such that $1 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq N$ and for all $j \in \mathrm{sub}(\alpha)$, $w_j < w_{j+1}$. In other words, strict increases in the subscripts are required in the "breaks" between parts of the composition α. A routine inclusion-exclusion argument shows that for $N \geq n$, $\{F_\alpha(x_1,\ldots,x_N) : \alpha \in \mathrm{Comp}_n\}$ is a basis of QSym_N^n. Note that some authors index fundamental quasisymmetric polynomials by pairs n, T where $T \subseteq [n-1]$. Additionally, various letters (F, L, Q, etc.) have been used to denote these polynomials.

As in the symmetric case, we would like to have quasisymmetric Hall-Littlewood polynomials (depending on a parameter t) that interpolate between F_α (when $t = 0$) and M_α (when $t = 1$). We sketch the definition of one such family of polynomials, introduced and studied by Hivert [7]. Quasisymmetric functions arise as the invariants of a certain action of $\mathfrak{S}_n$ on polynomials. From this action, one can define divided difference operators in a degenerate Hecke algebra $H_n(0)$ which can then be lifted to $H_n(q)$. Hivert's quasisymmetric Hall-Littlewood polynomials thereby arise from a corresponding t-analogue $\boxdot_\omega$ of the Weyl symmetrizer. For a composition α of length $k \leq N$, define

$$G_\alpha(x_1,\ldots,x_N;t) = \frac{1}{[k]!_t[N-k]!_t}\boxdot_\omega(x_1^{\alpha_1}\cdots x_k^{\alpha_k}).$$

As in the case of symmetric Hall-Littlewood polynomials, there is a more concrete combinatorial definition of G_α giving its expansion into monomials. We discuss this definition in §3.2.

3 Known Transition Matrices

In the theory of symmetric and quasisymmetric polynomials, much combinatorial information is encoded in the transition matrices between various bases. Given two bases $B = \{B_\lambda : \lambda \in \mathrm{Par}_n\}$ and $C = \{C_\lambda : \lambda \in \mathrm{Par}_n\}$ of Sym_N^n, the *transition matrix* $\mathcal{M}(B,C)$ is the unique matrix (with entries in K and rows and columns indexed by partitions of n) such that

$$B_\lambda = \sum_{\mu \in \mathrm{Par}_n} \mathcal{M}(B,C)_{\lambda,\mu} C_\mu.$$

Given a third basis D, it follows readily that $\mathcal{M}(B,D) = \mathcal{M}(B,C)\,\mathcal{M}(C,D)$ and $\mathcal{M}(C,B) = \mathcal{M}(B,C)^{-1}$. We define $\mathcal{M}(B,C)$ similarly if B and C are bases of QSym_N^n, but here the rows and columns of the matrix are indexed by compositions of n. Finally, if B is a basis of Sym_N^n and C is a basis of QSym_N^n, then $\mathcal{M}(B,C)$ is a rectangular matrix expressing each B_λ as a K-linear combination of the C_α's.

This section gives formulas for previously known matrices associated to some of the edges in Figure 1.

3.1 $\mathcal{M}(s,m)$, $\mathcal{M}(s,P)$, and $\mathcal{M}(P,m)$

The expansion of Schur polynomials into monomials uses semistandard tableaux. For later work, we will also need tableaux of skew shape. Suppose $\lambda, \mu \in \mathrm{Par}$ satisfy $\mu \subseteq \lambda$, i.e., $\mu_i \leq \lambda_i$ for all i. Define the *skew diagram*

$$\lambda/\mu = \{(i,j) \in \mathbb{N}^+ \times \mathbb{N}^+ : 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}.$$

We will draw skew diagrams using the English convention where the longest rows are at the top. For $N \in \mathbb{N}^+$, a *semistandard tableau* (SSYT) of shape λ/μ with entries in $[N] = \{1, 2, \ldots, N\}$ is a function $T : \lambda/\mu \to [N]$ that is weakly increasing along rows and strictly increasing down columns. Writing $n = |\lambda/\mu|$, a *standard tableau* (SYT) of shape λ/μ is a bijection $S : \lambda/\mu \to [n]$ that is also a SSYT. Let $\mathrm{SSYT}_N(\lambda/\mu)$ be the set of all SSYT of shape λ/μ with entries in $[N]$, and let $\mathrm{SYT}(\lambda/\mu)$ be the set of all SYT of shape λ/μ. For any $T \in \mathrm{SSYT}_N(\lambda/\mu)$, the *content monomial* x^T is defined to be $\prod_{c \in \lambda/\mu} x_{T(c)}$.

The *skew Schur polynomial* in N variables can now be defined as

$$s_{\lambda/\mu}(x_1, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\lambda/\mu)} x^T.$$

The ordinary Schur polynomial s_λ is obtained by taking $\mu = (0)$. For $\lambda, \nu \in \mathrm{Par}_n$ and $N \geq n$, it follows that $\mathcal{M}(s,m)_{\lambda,\nu}$ is the Kostka number $K_{\lambda,\nu}$, namely the number of SSYT of shape λ and content ν.

Lascoux and Schützenberger [9] first discovered a combinatorial formula for the *t-Kostka matrix* $\mathcal{M}(s,P)$ involving the famous *charge* statistic. Given a permutation $w = w_1 w_2 \cdots w_n$ of $[n]$, let $\mathrm{IDes}(w)$ be the set of $k < n$ such that $k+1$ appears to the left of k in w, and let $\mathrm{chg}(w) = \sum_{k \in \mathrm{IDes}(w)} (n-k)$.

Next, let v be a word of partition content (i.e., for all $k \geq 1$, the number of $(k+1)$'s in v is no greater than the number of k's). Extract one or more permutations from v as follows. Scan v from left to right marking the first 1, then the first 2 after that, etc., returning to the beginning of v when the right end is reached. Do this until the largest symbol has been marked. Remove the marked symbols from v (in the order they appear) to get the first permutation. Continue to extract permutations in this way until all symbols of v have been used, and let $\mathrm{chg}(v)$ be the sum of the charges of the associated permutations. Finally, given a SSYT T of partition content, let $w(T)$ be the word obtained by reading symbols row by row from top to bottom, reading each row from right to left. Then define $\mathrm{chg}(T) = \mathrm{chg}(w(T))$.

Theorem 3.1 *[9] For all $\lambda, \mu \in \mathrm{Par}_n$, $\mathcal{M}(s,P)_{\lambda,\mu} = \sum t^{\mathrm{chg}(T)}$ summed over all $T \in \mathrm{SSYT}_n(\lambda)$ of content μ.*

Macdonald [15, §III.5, p. 229] gives a formula for the monomial expansion of *skew Hall-Littlewood polynomials* $P_{\lambda/\mu}(x_1,\ldots,x_N;t)$, which yields $\mathcal{M}(P,m)$ and $\mathcal{M}(P,M)$ by taking $\mu = (0)$. We introduce the following combinatorial model for Macdonald's formula.

Let λ/μ be a skew shape with $N \geq \ell(\lambda)$. For $T \in \mathrm{SSYT}_N(\lambda/\mu)$, define the set of *special cells* as

$$\mathrm{Sp}(T) = \{(i,j) \in \lambda/\mu : j > 1 \text{ and for all } u \text{ with } (u,j-1) \in \lambda/\mu, T((u,j-1)) \neq T((i,j))\}.$$

Define the *weight* of a special cell (i,j) to be

$$\begin{aligned} \mathrm{wt}((i,j)) \;&=\; |\{(u,j-1) \in \lambda/\mu : u \geq i \text{ and } T((u,j-1)) < T((i,j))\}| \\ &\quad +|\{(u,j-1) \in \mu/(0) : u \geq i\}|. \end{aligned}$$

In other words, a cell c with entry $v = T(c)$ is special for T iff c is not in column 1 and there are no v's in the column of T just left of c's column. In this case, the weight of c is the number of cells weakly below c in the column just left of c that either have entries less than v or are part of the diagram for μ. Now define the set of *starred semistandard tableaux*

$$\mathrm{SSYT}^*_N(\lambda/\mu) = \{(T,E) : T \in \mathrm{SSYT}_N(\lambda/\mu) \text{ and } E \subseteq \mathrm{Sp}(T)\}.$$

A starred tableau $T^* = (T,E)$ has *sign* $\mathrm{sgn}(T^*) = (-1)^{|E|}$, *t-weight* $\mathrm{tstat}(T^*) = \sum_{c\in E}\mathrm{wt}(c)$, *x-weight* $x^{T^*} = x^T$, and *overall weight* $\mathrm{sgn}(T^*)t^{\mathrm{tstat}(T^*)}x^{T^*}$.

For $T \in \mathrm{SSYT}_N(\lambda/\mu)$, Macdonald defines $\psi_T(t) = \prod_{c\in\mathrm{Sp}(T)}(1-t^{\mathrm{wt}(c)})$. Then Macdonald's monomial expansion of the skew Hall-Littlewood polynomials is

$$P_{\lambda/\mu}(x_1,\ldots,x_N;t) = \sum_{T\in\mathrm{SSYT}_N(\lambda/\mu)} \psi_T(t)x^T.$$

Expanding the product in $\psi_T(t)$ using the distributive law, we get $\sum_{E\subseteq\mathrm{Sp}(T)}\prod_{c\in E}(-t^{\mathrm{wt}(c)})$. Comparing to the overall weight of starred tableaux, we find that

$$P_{\lambda/\mu}(x_1,\ldots,x_N;t) = \sum_{T^*\in\mathrm{SSYT}^*_N(\lambda/\mu)} \mathrm{sgn}(T^*)t^{\mathrm{tstat}(T^*)}x^{T^*}. \tag{1}$$

Example 3.2 *Let $\lambda = (8,6,5,4)$, $\mu = (0)$, $N \geq 8$, and*

$$T = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \underline{2} & 2 & \underline{4} & \underline{5} & 5 \\ \hline 2 & 2 & 3 & 3 & \underline{6} & \underline{8} \\ \cline{1-6} 3 & 3 & \underline{4} & 4 & \underline{7} \\ \cline{1-5} 5 & 5 & 5 & 5 \\ \cline{1-4} \end{array} \qquad T^* = \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2^* & 2 & 4^* & 5 & 5 \\ \hline 2 & 2 & 3 & 3 & 6 & 8^* \\ \cline{1-6} 3 & 3 & 4 & 4 & 7 \\ \cline{1-5} 5 & 5 & 5 & 5 \\ \cline{1-4} \end{array}.$$

In T, the special cells are indicated by the underlined entries. Specifically,

$$\mathrm{Sp}(T) = \{(1,4),(1,6),(1,7),(2,5),(2,6),(3,3),(3,5)\}.$$

These special cells have respective weights $1,1,1,3,2,1,2$. So T contributes the term $(1-t)^4(1-t^2)^2(1-t^3)x^T$ to P_λ. A typical starred tableau is $T^* = (T,\{(1,4),(1,6),(2,6)\})$. The overall weight of the object T^* is $(-1)^3t^{1+1+2}x_1^3x_2^4x_3^4x_4^3x_5^6x_6x_7x_8 = -t^4x^T$.

3.2 $\mathcal{M}(s,F)$, $\mathcal{M}(G,F)$, and $\mathcal{M}(G,M)$

The fundamental quasisymmetric expansion of Schur polynomials is a sum over standard tableaux, rather than semistandard tableaux. Given $\lambda \in \mathrm{Par}_n$ and $S \in \mathrm{SYT}(\lambda)$, define the *descent set* $\mathrm{Des}(S)$ to be the set of $k < n$ such that $k+1$ appears in a lower row of S than k. Define the descent composition $\mathrm{Des}'(S) = \mathrm{comp}(\mathrm{Des}(S))$ to be the composition associated to this subset of $[n-1]$. Gessel first proved [3] that for $N \geq n = |\lambda|$,

$$s_\lambda(x_1,\ldots,x_N) = \sum_{S\in\mathrm{SYT}(\lambda)} F_{\mathrm{Des}'(S)}(x_1,\ldots,x_N). \tag{2}$$

In other words, $\mathcal{M}(s,F)_{\lambda,\alpha}$ is the number of standard tableaux with shape λ and descent set $\mathrm{sub}(\alpha)$.

Let $\alpha,\beta \in \mathrm{Comp}_n$ with β finer than α. Say $\ell(\alpha)=k$ and $\ell(\beta)=m$. By definition, there exist indices $0 = i_0 < i_1 < \cdots < i_k = m$ such that $\alpha_j = \beta_{i_{j-1}+1} + \cdots + \beta_{i_j}$ for $1 \leq j \leq k$. The *refining composition* $\mathrm{Bre}(\beta,\alpha) = (i_1 - i_0, i_2 - i_1, \ldots, i_k - i_{k-1})$ records the number of parts of β derived from each part of α. Define $s(\alpha,\beta) = \sum_{j=1}^{k} j(i_j - i_{j-1} - 1)$. Note that in the notation $\mathrm{Bre}(\beta,\alpha)$ from [7], the finer composition is listed *first*, but in the function s (and g, ξ defined in §5.1), we list the finer composition *second*. This ordering is more convenient when working with transition matrices.

Theorem 3.3 *[7, Theorem 6.6] For all $N \geq n$ and $\alpha \in \mathrm{Comp}_n$,*

$$G_\alpha(x_1,\ldots,x_N;t) = \sum_{\beta\succeq\alpha} (-1)^{\ell(\beta)-\ell(\alpha)} t^{s(\alpha,\beta)} F_\beta(x_1,\ldots,x_N).$$

In other words, $\mathcal{M}(G,F)_{\alpha,\beta} = (-1)^{\ell(\beta)-\ell(\alpha)} t^{s(\alpha,\beta)}$ if $\beta \succeq \alpha$ and 0 otherwise.

Example 3.4 *Take $\beta = (1,2,2,1,4,3,1,2,1,1)$ and $\alpha = (5,5,3,1,4)$. Then $\mathrm{Bre}(\beta,\alpha) = (3,2,1,1,3)$ and $s(\alpha,\beta) = 1\cdot 2 + 2\cdot 1 + 3\cdot 0 + 4\cdot 0 + 5\cdot 2 = 14$. So $\mathcal{M}(G,F)_{\alpha,\beta} = (-1)^5 t^{14}$.*

Using $\mathcal{M}(G,M) = \mathcal{M}(G,F)\,\mathcal{M}(F,M)$, one can prove the following result giving the monomial expansion of Hivert's quasisymmetric Hall-Littlewood polynomials.

Theorem 3.5 *[7, eq. (105)] For all $\alpha \in \mathrm{Comp}_n$ and $N \geq n$,*

$$G_\alpha(x_1,\ldots,x_N;t) = \sum_{\beta\succeq\alpha} M_\beta(x_1,\ldots,x_N;t) \prod_{i=1}^{\ell(\mathrm{Bre}(\beta,\alpha))} (1-t^i)^{\mathrm{Bre}(\beta,\alpha)_i - 1}.$$

4 F-expansion of Skew Hall-Littlewood Polynomials

Recall from §3.1 the combinatorial formula (1) for the monomial expansion of the skew Hall-Littlewood polynomials $P_{\lambda/\mu}(x_1,\ldots,x_N;t)$. This section converts this formula to an expansion of these polynomials in terms of the fundamental quasisymmetric basis. In particular, this provides a combinatorial interpretation for the entries of $\mathcal{M}(P,F)$. We remark that one can also obtain $\mathcal{M}(P,F) = \mathcal{M}(P,s)\,\mathcal{M}(s,F)$ by combining Carbonara's combinatorial formula for $\mathcal{M}(P,s)$ in terms of special tournaments [1] with Gessel's formula (2) for $\mathcal{M}(s,F)$. However, this produces a quite complicated interpretation for the coefficients in $\mathcal{M}(P,F)$ as signed combinations of standard tableaux and special tournaments. The new interpretation developed below is much simpler.

To state our result, we need a few more definitions. Given a skew diagram λ/μ with n cells, let $\mathrm{SYT}^*(\lambda/\mu)$ be the set of starred standard tableaux $S^* = (S, E)$ such that S is a standard tableau of shape λ/μ. In this case, observe that $\mathrm{Sp}(S)$ consists of all cells in the diagram not in column 1. So E can be an arbitrary subset of cells of λ/μ not in column 1. Define the *ascent set* of S^*, denoted $\mathrm{Asc}(S^*)$, to be the set of all $k < n$ such that either (a) $k+1$ appears in S in a lower row than k, or (b) there exist u, i, j with $S((u, j-1)) = k$, $S((i, j)) = k+1$, and $(i, j) \in E$. The second alternative says that $k+1$ appears in a cell of E located in the next column after the column containing k. Define $\mathrm{Asc}'(S^*) = \mathrm{comp}(\mathrm{Asc}(S^*))$ to be the associated composition.

Theorem 4.1 *For all skew shapes λ/μ with $n \leq N$ cells,*

$$P_{\lambda/\mu}(x_1, \ldots, x_N; t) = \sum_{S^* \in \mathrm{SYT}^*(\lambda/\mu)} \mathrm{sgn}(S^*) t^{\mathrm{tstat}(S^*)} F_{\mathrm{Asc}'(S^*)}(x_1, \ldots, x_N).$$

We derive a similar formula for the skew Hall-Littlewood polynomials $Q_{\lambda/\mu}$ in [12].

Example 4.2 *Using Theorem 4.1, we can make the following calculation. Each term corresponds to the starred standard tableau shown below it:*

$$P_{21}(t) \;=\; \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}}{F_{21}} \;-\; \underset{\begin{array}{|c|c|}\hline 1 & 2^* \\ \hline 3 \\ \cline{1-1}\end{array}}{tF_{111}} \;+\; \underset{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}}{F_{12}} \;-\; \underset{\begin{array}{|c|c|}\hline 1 & 3^* \\ \hline 2 \\ \cline{1-1}\end{array}}{t^2F_{111}}.$$

Remark 4.3 *Carbonara [1] expressed the entries of the inverse t-Kostka matrix $\mathcal{M}(P, s)$ as signed, weighted sums of special tournament matrices. An alternative description can be obtained by following $\mathcal{M}(P, F)$ by the projection from QSym to Sym given in [2]. The entry of $\mathcal{M}(P, s)_{\lambda,\mu}$ is again described as a sum of signed, weighted objects. However, in this description the objects are pairs (S^*, T) where $S^* \in \mathrm{SYT}^*(\lambda)$ and T is a "flat special rim-hook tableau" of shape μ and content $\mathrm{Asc}'(S^*)$.*

In addition to working for skew Hall-Littlewood polynomials, this new description may have computational advantages. For $n = 4$, there are 37 special tournament matrices that contribute to the calculation of $\mathcal{M}(P, s)$. However, only 23 pairs (S^*, T) are now needed. We note that these pairs do not correspond to a subclass of special tournament matrices in any simple way. Carbonara's description computes the value $\mathcal{M}(P, s)_{4,22} = 0$ via the fact that there are no special tournament matrices with parameters $\lambda = (4)$ and $\mu = (2, 2)$. There are two such pairs (S^*, T), albeit of opposite sign and equal weight.

5 New Transition Matrices involving the Hivert G-basis

This section discusses combinatorial formulas for the transition matrices $\mathcal{M}(F, G)$, $\mathcal{M}(M, G)$, and $\mathcal{M}(P, G)$.

5.1 $\mathcal{M}(F, G)$

Let $\alpha, \beta \in \mathrm{Comp}_n$ with β finer than α. Define $\xi_{\alpha,\beta}(j)$ to be j if β_j and β_{j+1} are formed from the same part of α and 0 otherwise. Set $g(\alpha, \beta) = \sum_{j=1}^{\ell(\beta)-1} \xi_{\alpha,\beta}(j)$.

Theorem 5.1 *For all $\alpha, \beta \in \mathrm{Comp}_n$,*

$$\mathcal{M}(F, G)_{\alpha,\beta} = \begin{cases} t^{g(\alpha,\beta)}, & \text{if } \beta \succeq \alpha; \\ 0, & \text{otherwise}. \end{cases}$$

The idea of the proof is to show that $\mathcal{M}(G,F)\,\mathcal{M}(F,G) = I$, where $\mathcal{M}(F,G)$ is defined above and $\mathcal{M}(G,F)$ is defined in Theorem 3.3. Equivalently, we must show that for all compositions $\beta \succeq \alpha$,

$$\sum_{\gamma:\,\beta\succeq\gamma\succeq\alpha} (-1)^{\ell(\gamma)-\ell(\alpha)} t^{s(\alpha,\gamma)} t^{g(\gamma,\beta)}$$

is 1 for $\beta = \alpha$ and 0 otherwise. We prove this in [12] using a sign-reversing involution that cancels all negative objects.

Example 5.2 *Using Theorem 5.1, we calculate* $F_3 = G_3 + tG_{21} + tG_{12} + t^3 G_{111}$, $F_{21} = G_{21} + tG_{111}$, $F_{12} = G_{12} + t^2 G_{111}$, *and* $F_{111} = G_{111}$.

5.2 $\mathcal{M}(M,G)$

Theorem 5.3 *For all* $\alpha, \beta \in \mathrm{Comp}_n$ *with* $\beta \succeq \alpha$,

$$\mathcal{M}(M,G)_{\alpha,\beta} = (-1)^{\ell(\beta)-\ell(\alpha)} \prod_{j:\,\xi_{\alpha,\beta}(j)=j} (1-t^j).$$

For other α, β, $\mathcal{M}(M,G)_{\alpha,\beta} = 0$.

The idea of the proof is to use $\mathcal{M}(M,G) = \mathcal{M}(M,F)\,\mathcal{M}(F,G)$ to see that the α,β-entry of $\mathcal{M}(M,G)$ is

$$(-1)^{\ell(\beta)-\ell(\alpha)} \sum_{\gamma:\beta\succeq\gamma\succeq\alpha} (-1)^{\ell(\gamma)-\ell(\beta)} t^{g(\gamma,\beta)}.$$

We then use a counting argument to rewrite the sum as the product $\prod_{j:\xi_{\alpha,\beta}(j)=j}(1-t^j)$.

Example 5.4 *Consider* $\alpha = 22$ *and* $\beta = 1111$. *Then* $\xi_{\alpha,\beta}(1) = 1$, $\xi_{\alpha,\beta}(2) = 0$ *and* $\xi_{\alpha,\beta}(3) = 3$. *So* $\mathcal{M}(M,G)_{\alpha,\beta} = (-1)^2(1-t)(1-t^3)$.

Example 5.5 *We calculate* $M_3 = G_3 - (1-t)G_{21} - (1-t)G_{12} + (1-t)(1-t^2)G_{111}$, $M_{21} = G_{21} - (1-t)G_{111}$, $M_{12} = G_{12} - (1-t^2)G_{111}$, *and* $M_{111} = G_{111}$.

5.3 $\mathcal{M}(P,G)$

By multiplying $\mathcal{M}(P,F)$ (as given in §4) and $\mathcal{M}(F,G)$ (as given in §5.1), we obtain the formula

$$\mathcal{M}(P,G)_{\lambda,\beta} = \sum_{\substack{S^*=(S,E)\in\mathrm{SYT}^*(\lambda)\\ \mathrm{Asc}'(S^*)\preceq\beta}} (-1)^{|E|} t^{\mathrm{tstat}(S^*)+g(\mathrm{Asc}'(S^*),\beta)}. \tag{3}$$

However, this can be simplified. In order to do so, we introduce some new notation.

For $S \in \mathrm{SYT}(\lambda)$, define $\mathrm{Sp}(S)$ and $\mathrm{wt}(c)$ as in §3.1. For $E \subseteq \mathrm{Sp}(S)$, define $\mathrm{Asc}(S^*) = \mathrm{Asc}((S,E))$ as in §4. We define the following subset of $\mathrm{Sp}(S)$:

$$\mathrm{Esp}(S) = \{c \in \mathrm{Sp}(S) :\ \mathrm{Asc}((S,\{c\})) \neq \mathrm{Asc}((S,\emptyset))\}.$$

For each $j \in \mathrm{sub}(\beta)$, let $n_j = n_j(\beta)$ be the number of elements of $\mathrm{sub}(\beta)$ that are at most j. Let $c_j = c_j(S)$ be the unique cell of S in which j appears. Let $n'_j = n_j$ if $j \in \mathrm{sub}(\beta) \setminus \mathrm{Des}(S)$ and 0 otherwise. Recall $\mathrm{Des}(S)$ was defined in §3.2.

Theorem 5.6 *For all $\lambda \in \mathrm{Par}_n$ and $\beta \in \mathrm{Comp}_n$,*

$$\mathcal{M}(P,G)_{\lambda,\beta} = \sum_{\substack{S\in\mathrm{SYT}(\lambda)\\ \mathrm{Des}(S)\subseteq\mathrm{sub}(\beta)}} \prod_{\substack{j\in\mathrm{sub}(\beta):\\ c_{j+1}\in\mathrm{Esp}(S)}} (t^{n_j} - t^{\mathrm{wt}(c_{j+1})}) \prod_{j: c_{j+1}\in\mathrm{Sp}(S)\setminus\mathrm{Esp}(S)} t^{n'_j}(1-t^{\mathrm{wt}(c_{j+1})}). \quad (4)$$

The idea of the proof is to group together summands in (3) indexed by the starred tableaux $S^* = (S, E)$ with the same underlying standard tableau S. A careful case analysis leads to the sum of products in (4).

Remark 5.7 *If $n_j = \mathrm{wt}(c_{j+1})$ for some $j \in \mathrm{sub}(\beta)$ with $c_{j+1} \in \mathrm{Esp}(S)$, then S can be omitted from the sum in* (4)*.*

Example 5.8 *Let $\lambda = 32$ and $\beta = 1211$. Note that $\mathrm{sub}(\beta) = \{1,3,4\}$ and so $n_1 = 1$, $n_3 = 2$, and $n_4 = 3$. In Table 1 we list the five elements of $\mathrm{SYT}(32)$ (referred to from left to right as $S_1,\ldots,S_5$) along with pertinent data. The row labeled $\prod_1$ (resp. $\prod_2$) gives the contributions from the first (resp. second) product in* (4)*. Since $\mathrm{Des}(S_2), \mathrm{Des}(S_3) \not\subseteq \mathrm{sub}(\beta)$, $\prod_1$ and $\prod_2$ have been left blank for these two tableaux. (For reference, the corresponding products for these tableaux are $(t-t)\cdot t^2(1-t)(1-t)$ and $(t-t)(t^2-t)(t^3-t^2)\cdot 1$, respectively.) Note that Remark 5.7 applies to S_1 with $j = n_j = 1$. So the only contributions are from the last two columns and we find that*

$$\mathcal{M}(P,G)_{32,1211} = (t^2-t)(1-t) + (t^3-t^2)(1-t) = -t^4 + t^3 + t^2 - t.$$

Tab. 1: Computation of $\mathcal{M}(P,G)_{32,1211}$.

S	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2&5\\\cline{1-2}\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$
$\mathrm{Des}(S)$	$\{3\}$	$\{2,4\}$	$\{2\}$	$\{1,4\}$	$\{1,3\}$
$\mathrm{Sp}(S)$	$\{c_2,c_3,c_5\}$	$\{c_2,c_4,c_5\}$	$\{c_2,c_4,c_5\}$	$\{c_3,c_4,c_5\}$	$\{c_3,c_4,c_5\}$
$\mathrm{Esp}(S)$	$\{c_2,c_3,c_5\}$	$\{c_2\}$	$\{c_2,c_4,c_5\}$	$\{c_3,c_4\}$	$\{c_3,c_5\}$
$\prod_1$	$(t-t)(t^3-t)$			(t^2-t)	(t^3-t^2)
$\prod_2$	1			$(1-t)$	$(1-t)$

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