On Symmetric Norm Inequalities And Hermitian Block-Matrices

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Abstract

The main purpose of this paper is to englobe some new and known types of Hermitian block-matrices $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ satisfying or not the inequality $\|M\| \leq \|A + B\|$ for all symmetric norms. For positive definite block-matrices another inequality is established and it is shown that it can be sharper (for some symmetric norms) than the following holding inequality $\|M\| \leq \|A\| + \|B\|$.

Keywords: Symmetric norm; Matrix Inequalities; Partial positive transpose matrix; Hermitian Block-matrices.

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1 Introduction and preliminaries

The first section presents some known results relatd to the inequality together with some preliminaries we used in the second section to derive some new and generalization results. Let $\mathbb{M}_n^+$ denote the set of positive and semi-definite part of the space of $n \times n$ complex matrices. For positive semi-definite block-matrix M, we say that M is P.S.D. and we write $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \in \mathbb{M}_{n+m}^+$, with $A \in \mathbb{M}_n^+$, $B \in \mathbb{M}_m^+$.

The modulus of a matrix X stands for $(X^*X)^{\frac{1}{2}}$ and is denoted by $|X|$. A norm $\|.\|$ over the space of matrices is a symmetric norm if $\|UAV\| = \|A\|$ for all A and all unitaries U and V. Let A be an $n \times n$ matrix and F an $m \times m$ matrix, $(m > n)$ written by blocks such that A is a diagonal block and all entries other than those of A are zeros, then the two matrices have the same singular values and $\|A\| = \|F\| = \|A \oplus 0\|$ for all symmetric norms, we say then that the symmetric norm on $\mathbb{M}_m$ induces a symmetric norm on $\mathbb{M}_n$, so for square matrices we may assume that our norms are defined on all spaces $\mathbb{M}_n$, $n \geq 1$. The spectral norm is denoted by $\|.\|_s$, the Frobenius norm by $\|.\|_{(2)}$, and the Ky Fan $k-$norm by $\|.\|_k$.

A positive partial transpose matrix denoted by P.P.T. is a P.S.D. block matrix M such that both $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ and $M' = \begin{pmatrix} A & X^* \\ X & B \end{pmatrix}$ are positive semi definite. Let $Im(X) := \dfrac{X - X^*}{2i}$ respectively $Re(X) := \dfrac{X + X^*}{2}$ be the imaginary part respectively the real part of a matrix X.

Lemma 1.1. *[2] For every matrix in $\mathbb{M}_{2n}^+$ written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + Im(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - Im(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

Lemma 1.2. *[2] For every matrix in $\mathbb{M}_{2n}^+$ written in blocks of the same size, we have the decomposition:*

$$\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} = U \begin{pmatrix} \frac{A+B}{2} + Re(X) & 0 \\ 0 & 0 \end{pmatrix} U^* + V \begin{pmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - Re(X) \end{pmatrix} V^*$$

for some unitaries $U, V \in \mathbb{M}_{2n}$.

Remark 1.3. *The proofs of Lemma 1.1 respectively Lemma 1.2 suggests that we have $A + B \geq -\dfrac{(X - X^*)}{i}$ and $A + B \geq \dfrac{(X - X^*)}{i}$, respectively $A + B \geq -(X + X^*)$ and $A + B \geq (X + X^*)$ since if we let hereafter $M_1 := \dfrac{A+B}{2} + Im(X)$, $M_2 := \dfrac{A+B}{2} - Im(X)$, $N_1 := \dfrac{A+B}{2} + Re(X)$ and $N_2 := \dfrac{A+B}{2} - Re(X)$ then N_1, N_2, M_1, M_2 are P.S.D as diagonal blocks of the P.S.D matrix JMJ^* for some unitary matrix J ([2]).*

Lemma 1.4. *[3] Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be any square matrix written by blocks of same size, if $AC = CA$ then $\det(M) = \det(AD - CB)$.*

2 Main results

2.1 Symmetric norm inequality

It is well known that if $M \in \mathbb{M}_{n+m}^+$ with $M = \begin{pmatrix} A & X \\ X^* & B \end{pmatrix}$ then

$$\|M\| \leq \|A\| + \|B\| \tag{2.1}$$

for all symmetric norms (see [4]). Hereafter our block matrices are such their diagonal blocks are of equal size.

Lemma 2.1. *[2] Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^+$*, if* X *is Hermitian or Skew-Hermitian then*

$$\|M\| \leq \|A + B\| \tag{2.2}$$

for all symmetric norms.

See [6] for another proof of Lemma 2.1 (the case X is Hermitian).

Lemma 2.2. *[5] Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^+$ *be a positive partial transpose matrix then*

$$\|M\| \leq \|A + B\| \tag{2.3}$$

for all symmetric norms.

Proposition 2.3. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \in \mathbb{M}_{2n}^+$ *be a given positive semi-definite matrix. If* X^* *commute with* A *or* B*, then* M *is unitarily congruent to a P.P.T. matrix and*

$$\|M\| \leq \|A + B\|$$

for all symmetric norms. In addition if X *is normal then* M *is a positive partial transpose matrix.*

Proof. We will assume without loss of generality that X^* commute with A, as the other case is similar. Take the polar decomposition of X so $X = U|X|$ and $X^* = |X|U^*$. Since U^* is unitary and X^* commute with A, X and $|X|$ commute with A thus $AU^* = U^*A$. If I_n is the identity matrix of order n, a direct computation shows that

$$\begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A & |X| \\ |X| & B \end{bmatrix},$$

consequently we have $\|M\| \leq \|A + B\|$ for all symmetric norms and that completes the proof. If X is normal then $|X| = |X^*|$. The polar decomposition discussed above and the following known decomposition: $X = U|X| = |X^*|U$, let us write

$$\begin{bmatrix} U^* & 0 \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & |X| \\ |X| & B \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A & X^* \\ X & B \end{bmatrix},$$

which implies that M is a P.P.T. matrix. □

Remark 2.4. *It is easily seen that if X commute with the Hermitian matrix A so is X^* and conversely.*

The following is a slight generalization result:

Lemma 2.5. *Let $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ be a positive semi definite matrix, if $Im(X) = rI_n$ or $Re(X) = rI_n$ for some r, then $\|M\| \leq \|A+B\|$ for all symmetric norms.*

Proof. Let $\sigma_i(H)$ denote the singular values of a matrix H ordered in decreasing order, by Remark 1.3 the matrices $M_1 = \dfrac{A+B}{2} + Im(X)$ and $M_2 = \dfrac{A+B}{2} - Im(X)$ are positive semi definite since $Im(X) = rI_n$ we have:

$$\sum_{i=1}^{k} \sigma_i \left(\frac{A+B}{2} + Im(X) \right) + \sum_{i=1}^{k} \sigma_i \left(\frac{A+B}{2} - Im(X) \right) = \sum_{i=1}^{k} \sigma_i (A+B).$$

In other words by Lemma 1.1 $\|M\|_k \leq \|M_1\|_k + \|M_2\|_k = \|A+B\|_k$ for all Ky-Fan $k-$norms which from the Ky-Fan dominance theorem (see [1] -Sec 10.7-) completes the proof. Using Lemma 1.2 the other case is similarly proven. $\square$

One can ask if Lemma 2.2 gives in a certain way the sharpest inequality satisfied by P.P.T. matrices, and the answer is yes.

Example 2.6. *Let*

$$M = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix},$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Then we have

$$\|A+B\|_s = 3 = \|M\|_s < \|A\|_s + \|B\|_s = 4.$$

Lemma 2.7. *Let*

$$N = \begin{bmatrix} \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} & D \\ D^* & \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix} \end{bmatrix},$$

where $a_1, \cdots, a_n$ respectively $b_1, \cdots, b_n$ are nonnegative respectively negative real numbers, $A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_n \end{pmatrix}$ *and D is any diagonal matrix, then nor N neither $-N$ is positive semi-definite. Set $(d_1, \cdots, d_n)$ as the diagonal entries of D^*D, if $a_i + b_i \geq 0$ and $a_ib_i - d_i < 0$ for all $i \leq n$, then $\|N\| > \|A + B\|$. for all symmetric norms*

Proof. The diagonal of N has negative and positive numbers, thus nor N neither $-N$ is positive semi-definite, now any two diagonal matrices will commute, in particular D^* and A, by applying Lemma 1.4 we get that the eigenvalues of N are the roots of

$$\det((A - \mu I_n)(B - \mu I_n) - D^*D) = 0.$$

Equivalently the eigenvalues are all the solutions of the n equations:

$$\begin{array}{rrcl}
1) & (a_1 - \mu)(b_1 - \mu) - d_1 & = & 0 \\
2) & (a_2 - \mu)(b_2 - \mu) - d_2 & = & 0 \\
3) & (a_3 - \mu)(b_3 - \mu) - d_3 & = & 0 \\
\vdots & & \vdots & \\
i) & (a_i - \mu)(b_i - \mu) - d_n & = & 0 \\
\vdots & & \vdots & \\
n) & (a_n - \mu)(b_n - \mu) - d_n & = & 0
\end{array} \qquad (S)$$

Let us denote by x_i and y_i the two solutions of the i^{th} equation then:

$$\begin{array}{rclcl}
x_1 + y_1 & = & a_1 + b_1 & \geq & 0 \\
x_2 + y_2 & = & a_2 + b_2 & \geq & 0 \\
\vdots & & \vdots & & \\
x_n + y_n & = & a_n + b_n & \geq & 0
\end{array} \qquad
\begin{array}{rclcl}
x_1y_1 & = & a_1b_1 - d_1 & < & 0 \\
x_2y_2 & = & a_2b_2 - d_2 & < & 0 \\
\vdots & & \vdots & & \vdots \\
x_ny_n & = & a_nb_n - d_n & < & 0
\end{array}$$

This implies that each equation of (S) has one negative and one positive solution, their sum is positive, thus the positive root is bigger or equal than the negative one. Since $A + B = \begin{pmatrix} a_1+b_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_n+b_n \end{pmatrix}$, summing over indexes we see that $\|N\|_k > \|A+B\|_k$ for $k = 1, \cdots, n$ which yields to $\|N\| > \|A+B\|$ for all symmetric norms. $\square$

Remark 2.8. *The result of Lemma 2.7 still holds if the diagonal condition of D is replaced by: D commute with A or B and the matrix D^*D is diagonal.*

It seems easy to construct examples of non positive semi definite matrices N, such that $\|N\|_s > \|A+B\|_s$, let us have a look of such inequality for P.S.D. matrices.

Example 2.9. *Let*

$$N_y = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & y & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix},$$

where $A = \begin{bmatrix} 2 & 0 \\ 0 & y \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. *The eigenvalues of N_y are the numbers:* $\lambda_1 = 4,\ \lambda_2 = 1,\ \lambda_3 = y,\ \lambda_4 = 0,$ *thus if $y \geq 0$, N_y is positive semi-definite and for all y such that $0 \leq y < 1$ we have:*

1. $4 = \|N_y\|_s > \|A+B\|_s = 3.$
2. $16 + y^2 + 1 = \|N_y\|_{(2)}^2 > \|A+B\|_{(2)}^2 = 4(3+y) + y^2 + 1.$

2.2 New Inequality

Theorem 2.10. *Let* $M = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$ *and let r_1, r_2 be two nonnegative numbers, if $M_1 \geq r_1 I_n$ and $M_2 \geq r_2 I_n$ or $N_1 \geq r_1 I_n$ and $N_2 \geq r_2 I_n$ then*

$$\|A+B\| \leq \|2(A+B) - (r_1 + r_2) I_n\| \tag{2.4}$$

and

$$\|M\| \leq \|2(A+B) - (r_1 + r_2) I_n\| \tag{2.5}$$

for all symmetric norms. In particular if $M \geq r I_n$ for some $r \geq 0$ then

$$\|M\| \leq 2\|(A+B) - r I_n\| \tag{2.6}$$

for all symmetric norms.

Proof. If we have the case

$$\frac{A+B}{2} + Im(X) \geq r_1 I_n \tag{2.7}$$
$$\frac{A+B}{2} - Im(X) \geq r_2 I_n \tag{2.8}$$

or the case

$$\frac{A+B}{2} + Re(X) \geq r_1 I_n \tag{2.9}$$

$$\frac{A+B}{2} - Re(X) \geq r_2 I_n \tag{2.10}$$

then summing both equations in each case gives $A+B \geq (r_1+r_2)I_n$ and (2.4) follows. Given that $\|M_1\|_k \leq \|A+B-r_2 I_n\|_k$ and $\|M_2\|_k \leq \|A+B-r_1 I_n\|_k$ for all $k \leq n$ we derive the following inequality:

$$\|M\|_k \leq \|M_1\|_k + \|M_2\|_k = \|2(A+B) - (r_1+r_2)I_n\|_k$$

for all Ky-Fan $k-$norms. By replacing M_i by N_i for $i = 1, 2$ Lemma 1.2 gives the same inequality. The particular case can be easily concluded since by the decompositions in Lemma 1.1 and Lemma 1.2, if $M \geq rI_n$ then all of $M_1 - rI_n$, $N_1 - rI_n$, $M_2 - rI_n$ and $N_2 - rI_n$ are positive semi definite matrices. □

Inequality (2.5) can be sharper than (2.1) as these examples show:

Example 2.11. *Let*

$$M = \begin{bmatrix} 4 & 0 & 0 & -3 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ -3 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix},$$

where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$. *$M$ is positive semi-definite,* $r_1 = r_2 = 2.5$ *with*

$$8 = \|A\|_s + \|B\|_s > \|2(A+B) - (r_1+r_2)I_n\|_s = 7 = \|M\|_s > \|A+B\|_s = 6.$$

Example 2.12. *Let*

$$M = \begin{bmatrix} 1 & 0 & 0 & 0.25 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & X \\ X^* & B \end{bmatrix},$$

where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. *It can be verified that M is positive semi-definite,* $r_1 = r_2 = 0.375$ *and we have* $\|M\|_{(2)} = \sqrt{2.125} > \|A+B\|_{(2)} = \sqrt{2}$ *with* $\|A\|_{(2)} + \|B\|_{(2)} = 2$*, for:*

$$2 > \|2(A+B) - (r_1+r_2)I_n\|_{(2)} = \sqrt{3.125} > \|M\|_{(2)} > \|A+B\|_{(2)}.$$

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