Stability of the Kalman Filter for Output Error Systems

Boyi Ni, Qinghua Zhang

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Boyi Ni * **Qinghua Zhang** **

* *SAP Labs China, 1001 Chenhui Rd., Pudong New District, Shanghai 201203, China*
** *INRIA Rennes Bretagne Atlantique, Campus de Beaulieu, 35042 Rennes Cedex, France (email: qinghua.zhang@inria.fr)*

Abstract: Optimality and numerical efficiency are well known properties of the Kalman filter, whereas its stability property, though equally classical and important in practice, is less often mentioned in the recent literature. The stability of the Kalman filter is usually ensured by the uniform complete controllability *regarding the process noise* and the uniform complete observability of linear time varying systems. Such classical results cannot be applied to *output error systems*, in which the process noise is totally absent. It is shown in this paper that the uniform complete observability is sufficient to ensure the stability of the Kalman filter applied to time varying output error systems, regardless of the stability of the considered system itself.

Keywords: Stability, Kalman filter, Output Error System.

1. INTRODUCTION

The well known Kalman filter has been extensively studied and is being applied in many different fields (Anderson and Moore (1979); Jazwinski (1970); Zarchan and Musof (2005); Grewal and Andrews (2008)). The purpose of the present paper is to study the stability of the Kalman filter in a particular case rarely covered in the literature: the absence of process noise in the state equation of a linear time varying (LTV) system. Such systems are known as *output error* (OE) systems. Though typically process noise and output noise are both considered in Kalman filter applications, the case with *no* process noise is of particular interest when state equations come from physical laws that are believed accurate enough. It is also important for the application of the Kalman filter to OE system identification (Goodman and Dudley (1987); Forssell and Ljung (2000)).

While the optimal property of the Kalman filter is frequently recalled, its stability property is less often mentioned in the recent literature. The classical stability analysis is based both on the uniform complete controllability *regarding the process noise* and on the uniform complete observability of the considered system (Kalman (1963); Jazwinski (1970)). In the case of OE systems, there is no process noise at all in the state equation, hence the controllability regarding the process noise cannot be fulfilled, and the classical stability results are not applicable. The present paper aims at completing this missing case.

The optimal state estimation realized by the Kalman filter is usually viewed as a trade-off between the uncertainties in the state equation and in the output equation. In an OE system, the state equation is assumed noise-free. This point of view suggests that the state estimation should solely rely on the state equation, provided that the initial state of the OE system is *exactly* known. In practice the Kalman filter remains useful when the initial state is not exactly known or when the OE system is unstable. Of course, if the state of an unstable system diverges, so does its state estimate by the Kalman filter. Typically in practice, unstable systems are stabilized by feedback controllers so that the system state remains bounded. The Kalman filter can be applied either to the controlled system itself or to the entire closed loop system. In the latter case, the controller must be linear and completely known, excluding the saturation protection and any other nonlinearities.

The classical *optimality* results of the Kalman filter are also valid in the case of OE systems (Jazwinski, 1970, chapter 7). Nevertheless, it remains to complete the stability analysis, as the classical results are not applicable here.

The main results presented in this paper are as follows. Under the uniform complete observability condition, the dynamics of the Kalman filter applied to a LTV OE system is asymptotically stable, *regardless of the stability of the system itself.* The boundedness of the solution of the Riccati equation, which ensures the boundedness of the Kalman gain, is also proved under the same condition. These results are quite similar to the classical results (Kalman (1963); Jazwinski (1970)), which exclude the case of OE systems.

For linear time invariant (LTI) systems, it is a common practice to design the Kalman filter by solving an algebraic Riccati equation (in contrast to dynamic differential Riccati equation for general LTV systems as considered in the present paper). In this case, the controllability and observability conditions can be replaced by the weaker stabilizability and detectability conditions (Laub (1979); Arnold and Laub (1984)). Some preliminary results about

⋆ This work has been partly supported by the ITEA2 MODRIO project.

LTI OE systems have been presented in (Ni and Zhang (2013)).

The rest of the paper is organized as follows. Some preliminary elements are introduced in Section 2. The problem considered in this paper is formulated in Section 3. The properties of the solution of the Riccati equation are analyzed in Section 4. The stability of the Kalman filter for OE systems is established in Section 5. Some numerical examples are presented in Section 6. Finally, concluding remarks are drawn in Section 7.

2. DEFINITIONS

Let us shortly recall some definitions about LTV systems, which are necessary for the following sections.

Let m and n be any two positive integers. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. For a matrix $A \in \mathbb{R}^{m\times n}$, $\|A\|$ denotes the matrix norm induced by the Euclidean vector norm, which is equal to the largest singular value of A and known as the spectral norm when $m = n$. Then $\|Ax\| \le \|A\|\|x\|$ for all $A \in \mathbb{R}^{m\times n}$ and all $x \in \mathbb{R}^n$. For two real square symmetric positive definite matrices A and B, $A > B$ means $A-B$ is positive definite.

Let $A(t) \in \mathbb{R}^{m\times n}$ be defined for $t \in \mathbb{R}$. It is said (upper) bounded if $\|A(t)\|$ is bounded.

Consider the homogeneous LTV system

$$\frac{dx(t)}{dt} = A(t)x(t) \tag{1}$$

with $x(t) \in \mathbb{R}^n$ and $A(t) \in \mathbb{R}^{n\times n}$, and let $\Phi(t, t_0)$ be the associated state transition matrix such that, for all $t, t_0 \in \mathbb{R}$, $d\Phi(t, t_0)/dt = A(t)\Phi(t, t_0)$ and $\Phi(t, t) = I_n$ with I_n denoting the $n \times n$ identity matrix.

Definition 1. System (1) is *Lyapunov stable* if there exists a positive constant γ such that, for all $t, t_0 \in \mathbb{R}$ satisfying $t \ge t_0$, the following inequality holds

$$\|\Phi(t, t_0)\| \le \gamma. \tag{2}$$

□

This definition concerns the boundedness of the state vector, whereas the following definition ensures its convergence to zero.

Definition 2. System (1) is *asymptotically stable* if it is Lyapunov stable and if the following limiting behavior holds

$$\lim_{t\to+\infty} \|x(t)\| = 0. \tag{3}$$

□

The last concept to be recalled here is about the observability of LTV systems, following (Kalman (1963)).

Definition 3. The matrix pair $[A(t), C(t)]$ with $A(t) \in \mathbb{R}^{n\times n}$ and $C(t) \in \mathbb{R}^{m\times n}$ is *uniformly completely observable* if there exist positive constants τ, ρ_1 and ρ_2 such that, for all $t \in \mathbb{R}$, the following inequalities hold

$$\rho_1 I_n \le \int_{t-\tau}^{t} \Phi^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi(s,t)ds \tag{4}$$

$$\le \rho_2 I_n \tag{5}$$

with some bounded symmetric positive definite matrix $R(s) \in \mathbb{R}^{m\times m}$ (typically the covariance matrix of the output noise in a stochastic state space system). □

3. PROBLEM FORMULATION AND ASSUMPTIONS

In this section is first recalled the classical Kalman filter under its usual assumptions, before the presentation of the particular case of OE systems considered in this paper.

3.1 Kalman filter in the usual case

The Kalman filter in continuous-time is usually applied to LTV systems modeled by

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt + Q^{\frac{1}{2}}(t)d\omega(t) \tag{6a}$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t) \tag{6b}$$

where $t \in \mathbb{R}$ represents the time, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^l$ the bounded input, $y(t) \in \mathbb{R}^m$ the output, $\omega(t) \in \mathbb{R}^n, \eta(t) \in \mathbb{R}^m$ are two independent Brownian processes with identity covariance matrices, $A(t), B(t), C(t), Q(t), R(t)$ are real matrices of appropriate sizes. The matrix $Q(t)$ is symmetric positive semi-definite, and $R(t)$ is symmetric positive definite. The notations $Q^{\frac{1}{2}}(t)$ and $R^{\frac{1}{2}}(t)$ denote respectively the symmetric positive (semi)-definite matrix square roots of $Q(t)$ and $R(t)$. The initial state $x(t_0) \in \mathbb{R}^n$ is a random vector following the Gaussian distribution $x(t_0) \sim \mathcal{N}(x_0, P_0)$ with $x_0 \in \mathbb{R}^n$ and $P_0 \in \mathbb{R}^{n\times n}$.

The Kalman filter for this LTV system writes

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + B(t)u(t)dt + K(t)(dy(t) - C(t)\hat{x}(t)dt) \tag{7a}$$

$$K(t) = P(t)C^T(t)R^{-1}(t) \tag{7b}$$

$$\frac{d}{dt}P(t) = A(t)P(t) + P(t)A^T(t) - P(t)C(t)^T R^{-1}(t)C(t)P(t) + Q(t) \tag{7c}$$

$$\hat{x}(t_0) = x_0, \quad P(t_0) = P_0 \tag{7d}$$

where the solution of the Riccati equation (7c) is a matrix function $P(t) \in \mathbb{R}^{n\times n}$ and the Kalman gain $K(t) \in \mathbb{R}^{n\times m}$.

The optimal properties of the Kalman filter are well known. It is less well known, though equally classical and important, that the solution $P(t)$ of the Riccati equation is bounded and that the dynamics of the Kalman filter is stable, provided the matrix pair $[A(t), Q^{\frac{1}{2}}(t)]$ is uniformly completely controllable and the matrix pair $[A(t), C(t)]$ is uniformly completely observable (Kalman (1963); Jazwinski (1970)). These are obviously crucial properties in practice for online applications.

It is worth mentioning that, in (Kalman (1963); Jazwinski (1970)), these classical results are based on an important lemma, which turns out to be incorrect, as pointed out independently by the authors of (Delyon (2001); Pengov et al. (2001)). Fortunately, the mistake has been repaired in these more recent references so that the main classical results remain correct.

3.2 Output error systems and Kalman filter

In the case of OE systems, the process noise is absent from the state equation, hence the general LTV system (6) becomes

$$dx(t) = A(t)x(t)dt + B(t)u(t)dt \tag{8a}$$

$$dy(t) = C(t)x(t)dt + R^{\frac{1}{2}}(t)d\eta(t). \tag{8b}$$

An OE system can be seen as a particular LTV system (6) with $Q(t) \equiv 0$. The Kalman filter (7) then becomes

$$d\hat{x}(t) = A(t)\hat{x}(t)dt + B(t)u(t)dt + K(t)(dy(t) - C(t)\hat{x}(t)dt) \tag{9a}$$

$$K(t) = P(t)C^T(t)R^{-1}(t) \tag{9b}$$

$$\frac{d}{dt}P(t) = A(t)P(t) + P(t)A^T(t) - P(t)C(t)^T R^{-1}(t)C(t)P(t) \tag{9c}$$

$$\hat{x}(t_0) = x_0, \quad P(t_0) = P_0 \tag{9d}$$

In this case, the uniform complete controllability condition cannot be satisfied by the matrix pair $[A(t), Q^{\frac{1}{2}}(t)]$, as $Q^{\frac{1}{2}}(t) \equiv 0$. Consequently, the classical results on the stability of the Kalman filter cannot be applied here. The present paper is for the purpose of studying the Kalman filter stability in this particular case.

3.3 Assumptions

The assumptions stated here are required throughout this paper.

The considered OE system (8) is defined with bounded and piecewise continuous real matrices $A(t), B(t), C(t), R(t)$ of appropriate sizes, among which $R(t)$ is positive definite. It is also assumed that $R^{-1}(t)$ is bounded.

The initial state $x(t_0) \in \mathbb{R}^n$ is a random vector following the Gaussian distribution

$$x(t_0) \sim \mathcal{N}(x_0, P_0). \tag{10}$$

with some known mean vector $x_0 \in \mathbb{R}^n$ and symmetric positive definite covariance matrix $P_0 \in \mathbb{R}^{n\times n}$.

It is further assumed that the matrix pair $[A(t), C(t)]$ is uniformly completely observable (see Definition 3).

4. PROPERTIES OF THE SOLUTION OF THE RICCATI EQUATION

Let us first establish the positive definiteness and the upper boundedness of $P(t)$, which are of crucial importance in practice, before studying the stability of the Kalman filter.

It will be shown that the solution of the Riccati equation (9c) is closely related to the solution of the Lyapunov equation

$$\frac{d\Omega(t)}{dt} + A^T(t)\Omega(t) + A(t)\Omega(t) = C^T(t)R^{-1}(t)C(t) \tag{11a}$$

$$\Omega(t_0) = P_0^{-1} \tag{11b}$$

with the same matrices $A(t), C(t), R(t), P_0$ as in (9c) and (9d).

Proposition 1. The solution $\Omega(t) \in \mathbb{R}^{n\times n}$ of the Lyapunov equation (11) is a symmetric positive definite matrix for all $t \geq t_0$.

Proof. It can be directly checked that

$$\Omega(t) = \Phi^T(t_0,t)\Omega(t_0)\Phi(t_0,t) + \int_{t_0}^{t} \Phi^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi(s,t)ds \tag{12}$$

satisfies the Lyapunov equation (11). As $\Omega(t_0) = P_0^{-1}$ is positive definite, $\Phi(t_0,t)$ is an invertible matrix, and for all $t \geq t_0$ the integral in (12) is positive semidefinite, therefore, $\Omega(t)$ is positive definite for all $t \geq t_0$. □

Now it is certain that, for all $t \geq t_0$, $\Omega(t)$ is positive definite, thus *invertible*. By applying the matrix differentiation rule

$$\frac{d\Omega^{-1}(t)}{dt} = -\Omega^{-1}(t)\frac{d\Omega(t)}{dt}\Omega^{-1}(t),$$

the following result can be directly checked.

Proposition 2. Let $\Omega(t)$ be the solution the Lyapunov equation (11) for $t \geq t_0$, then $P(t) \triangleq \Omega^{-1}(t)$ solves the Riccati equation (9c) with the initial condition $P(t_0) = P_0$. □

It is then clear that the properties of $P(t)$ can be studied through those of $\Omega(t)$.

Proposition 3. Under the assumptions stated in Section 3.3, for all $t \geq t_0$, the solution $P(t)$ of the Riccati equation (9c) with the initial condition $P(t_0) = P_0 > 0$ is symmetric positive definite and is upper bounded. □

Proof. The positive definiteness of $P(t)$ is immediate from that of $\Omega(t)$. To show that $P(t)$ is upper bounded, it will be shown that $\Omega(t)$ is lower bounded.

First consider the case $t \in [t_0, t_0 + \tau]$ (remind that τ is the positive constant involved in the observability condition (4)). In this finite time interval, the first term in (12) is positive definite, and the second term is positive semidefinite, then

$$\Omega(t) \geq \rho_0 I_n \tag{13}$$

where $\rho_0 > 0$ is the minimum of the smallest singular value of the matrix $\Phi^T(t_0,t)\Omega(t_0)\Phi(t_0,t)$ for $t \in [t_0, t_0 + \tau]$.

It remains to consider the case $t > t_0 + \tau$. Let

$$N \triangleq \left\lfloor \frac{t - t_0}{\tau} \right\rfloor \tag{14}$$

be the largest integer smaller than or equal to $(t - t_0)/\tau$. For the second term of (12), decompose the integral into the sum of N integrals over intervals of size τ, possibly dropping the part of the integral at the beginning of $[t_0, t_0 + \tau]$ smaller than τ if $(t - t_0)/\tau$ is not exactly an integer. It then yields

$$\begin{aligned}
&\int_{t_0}^{t} \Phi^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi(s,t)ds \\
&\geq \sum_{k=0}^{N-1} \int_{t-(k+1)\tau}^{t-k\tau} \Phi^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi(s,t)ds \\
&= \sum_{k=0}^{N-1} \Phi^T(t-k\tau,t)S_k\Phi(t-k\tau,t)
\end{aligned} \tag{15}$$

with

$$S_k \triangleq \int_{t-(k+1)\tau}^{t-k\tau} \Phi^T(s,t-k\tau)C^T(s)R^{-1}(s)C(s) \cdot \Phi(s,t-k\tau)ds \tag{16}$$

Each S_k (for $k = 0, 1, \ldots, N$–1) corresponds to the integral in the uniform complete observability condition (4), which holds for *all* $t \in \mathbb{R}$. Hence $S_k \geq \rho_1 I_n$ for $k = 0, 1, \ldots, N-1$, with the positive constant ρ_1 as in (4). Then

$$\int_{t_0}^{t} \Phi^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi(s,t)ds \geq \rho_1 \sum_{k=0}^{N-1} \Phi^T(t-k\tau,t)\Phi(t-k\tau,t) \tag{17}$$

Now in the sum of (17) keep only the term with $k = 0$ (the other terms will be useful for proving other results). Then

$$\int_{t_0}^{t} \Phi^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi(s,t)ds \tag{18}$$

$$\geq \rho_1 \Phi^T(t,t)\Phi(t,t) \tag{19}$$

$$\geq \rho_1 I_n \tag{20}$$

It means that the second term of (12) is lower bounded by $\rho_1 I_n$ for $t > t_0 + \tau$. By summarizing the results for the cases $t \in [t_0, t_0 + \tau]$ and $t > t_0 + \tau$, it is thus shown that $\Omega(t)$ is lower bounded by a strictly positive definite matrix for all $t \geq t_0$. To be more specific,

$$\Omega(t) \geq \min(\rho_0, \rho_1) I_n \tag{21}$$

It is then concluded that $P(t) = \Omega^{-1}(t)$ is upper bounded for all $t \geq t_0$. □

5. STABILITY OF THE KALMAN FILTER FOR OUTPUT ERROR SYSTEMS

For the Kalman filter (9), it is already shown that the matrix $P(t)$ is upper bounded, it then follows from the assumptions about the boundedness of $C(t)$ and $R^{-1}(t)$ that the Kalman gain $K(t)$ is also upper bounded. Equation (9a) governing $\hat{x}(t)$ can be viewed as a dynamic system driven by the "exogenous input" terms $B(t)u(t)dt$ and $K(t)dy$, but its intrinsic stability property is only related to its homogeneous part, namely the LTV system

$$\frac{d}{dt} z(t) = (A(t) - K(t)C(t))z(t) \tag{22}$$

with $z(t) \in \mathbb{R}^n$. Like in (Jazwinski (1970)), when the stability of the Kalman filter is talked about in this paper, it is about the stability of this homogeneous LTV system.

Before studying the stability of system (22), let us recall a classical result.

Lemma 1. The matrix pair $[A(t), C(t)]$ with $A(t) \in \mathbb{R}^{n\times n}$ and $C(t) \in \mathbb{R}^{m\times n}$ is uniformly completely observable, if and only if the matrix pair $[A(t)-K(t)C(t), C(t)]$, with any bounded and piecewise continuous $K(t) \in \mathbb{R}^{n\times m}$, is uniformly completely observable. □

Proofs of this lemma can be found in (Anderson et al., 1986, page 38, Lemma 2.3), (Ioannou and Sun, 1996, page 221, Lemma 4.8.1) and (Zhang and Zhang (2015)).

If the state equation (22) was associated with a deterministic observation equation $y(t) = C(t)z(t)$, the term $K(t)C(t)z(t)$ in equation (22) would be equal to $K(t)y(t)$ and could be seen as an output feedback. Lemma 1 means that such an output feedback preserves the observability of an LTV system.

Now it is ready to present the main result of this section.

Theorem 1. Under the assumptions stated in Section 3.3, the homogeneous part of the Kalman filter state estimation equation (9a), as expressed in equation (22), is asymptotically stable. □

As mentioned at the beginning of this section, the stability of the homogeneous part of the Kalman filter is an intrinsic stability property independent of the signals being processed by the filter. It implies that the mathematical expectation of the state estimation error is asymptotically stable. Moreover, the boundedness of the covariance matrix of the state estimation error, namely $P(t)$, is ensued by Proposition 3.

Proof of Theorem 1.
Define the Lyapunov function candidate

$$V(z(t), t) \triangleq z^T(t)\Omega(t)z(t) \tag{23}$$

with the positive definite $\Omega(t)$ as defined in (11).

Compute the derivative of $V(z(t), t)$ along the trajectory of (22),

$$\frac{dV(z(t),t)}{dt} = z^T(t)\Bigg((A(t) - K(t)C(t))^T\Omega(t) + \Omega(t)(A(t) - K(t)C(t)) + \frac{d\Omega(t)}{dt}\Bigg)z(t) \tag{24}$$

Remind that $\Omega(t)$ satisfies (11a), then

$$\frac{dV(z(t),t)}{dt} = z^T(t)\Bigg((-K(t)C(t))^T\Omega(t) + \Omega(t)(-K(t)C(t)) + C^T(t)R^{-1}(t)C(t)\Bigg)z(t)$$

In (9b) replace $P(t)$ by $\Omega^{-1}(t)$, then

$$K(t) = \Omega^{-1}(t)C^T(t)R^{-1}(t),$$

hence

$$\frac{dV(z(t),t)}{dt} = -z^T(t)C^T(t)R^{-1}(t)C(t)z(t) \tag{25}$$

which is negative semidefinite. It is then clear that the value of $V(z(t), t)$ cannot increase with the time $t \geq t_0$, thus

$$V(z(t), t) \leq V(z(t_0), t_0), \quad \forall t \geq t_0. \tag{26}$$

It then follows from the definition of $V(z(t), t)$ that

$$z^T(t)\Omega(t)z(t) \leq z^T(t_0)\Omega(t_0)z(t_0)$$

Let σ_0 be the smallest singular value of P_0, then σ_0^{-1} is the largest singular value of $\Omega(t_0) = P_0^{-1}$. Remind the lower bound of $\Omega(t)$ for all $t \geq t_0$ as shown in (21), then

$$\min(\rho_0, \rho_1)\|z(t)\|^2 \leq z^T(t)\Omega(t)z(t) \leq z^T(t_0)\Omega(t_0)z(t_0) \leq \sigma_0^{-1}\|z(t_0)\|^2$$

Hence

$$\|z(t)\| \leq \frac{1}{\sqrt{\sigma_0 \min(\rho_0, \rho_1)}} \|z(t_0)\| \tag{27}$$

holds for all $z(t_0) \in \mathbb{R}^n$. Therefore the homogeneous system (22) is Lyapunov stable.

To show the asymptotic stability of (22), the limiting behavior of $z(t)$ when $t \to +\infty$ will be analyzed.

Let the state transition matrix of the homogeneous LTV system (22) be denoted by $\Phi_K(s,t)$, then $z(s) = \Phi_K(s,t)z(t)$ for any $s, t \in \mathbb{R}$. It then follows from (25) that

$$\frac{dV(z(s),s)}{ds}$$
$$= -z^T(t)\Phi_K^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi_K(s,t)z(t)$$

and therefore

$$V(z(t),t) - V(z(t-\tau), t-\tau) \tag{28}$$
$$= \int_{t-\tau}^{t} \frac{dV(z(s),s)}{ds} ds \tag{29}$$
$$= -z^T(t)O_K(t,t-\tau)z(t) \tag{30}$$

with

$$O_K(t,t-\tau) \triangleq \int_{t-\tau}^{t} \Phi_K^T(s,t)C^T(s)R^{-1}(s)C(s)\Phi_K(s,t)ds$$

which is the observability Gramian matrix of the matrix pair $[A(t)–K(t)C(t), C(t)]$. The fact that $P(t), C(t), R^{-1}(t)$ are all bounded implies that the Kalman gain $K(t)$ is also bounded. According to Lemma 1, the assumed uniform complete observability of the matrix pair $[A(t), C(t)]$ (see Section 3.3) implies that the matrix pair $[A(t) - K(t)C(t), C(t)]$ is also uniformly completely observable, hence there exists a positive constant ρ_3 such that

$$0 < \rho_3 I_n \leq O_K(t,t-\tau). \tag{31}$$

This result, together with (30), leads to

$$V(z(t),t) \leq V(z(t-\tau), t-\tau) - \rho_3\|z(t)\|^2. \tag{32}$$

It means that, over *each* time interval of size τ, the value of $V(z(t),t)$ is decreased by $\rho_3\|z(t)\|^2 \geq 0$.

Though $V(z(t),t)$ is decreased over *each* time interval of size τ, it may not necessarily tend to zero, as the decrement may be infinitesimal.

It will be shown that $\|z(t)\|$ tends to zero by means of contradiction.

For the purpose of *proof by contradiction*, assume that $\|z(t)\|$ does *not* tend to zero when $t \to +\infty$. Then there exists a constant $\varepsilon > 0$, such that for any (arbitrarily large) $T \in \mathbb{R}$, there exists $t > T$ such that $\|z(t)\| > \varepsilon$, therefore

$$V(z(t),t) \leq V(z(t-\tau), t-\tau) - \rho_3\varepsilon^2. \tag{33}$$

There exist infinitely many such values of t, as T can be arbitrarily large. Moreover, it was already shown that the value of $V(z(t),t)$ cannot increase with the time t. The inequality (33) then says that $V(z(t),t)$ is repeatedly decreased by $\rho_3\varepsilon^2$ for larger and larger values of t. Consequently, $V(z(t),t)$ will become negative for sufficiently large values of t. This is in contradiction with the definition of $V(z(t),t)$ in (23) which is positive definite. Therefore, it is proved that $\|z(t)\|$ tends to zero when $t \to +\infty$.

The asymptotic stability of the homogeneous system (22) is then established. □

6. NUMERICAL EXAMPLES

This section presents some numerical examples confirming the theoretical results established in the previous sections. The examples of LTV systems (the $A(t)$ and $\Phi(t,t_0)$ matrices) have been inspired by (Choi, 2010, Lecture 24). The numerical solutions are computed with the ode45 solver in Matlab.

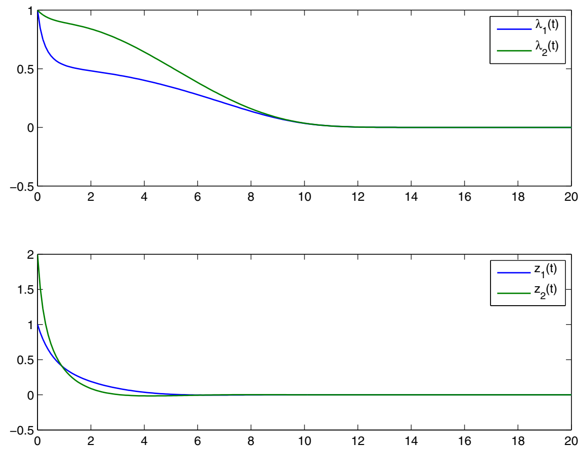

Fig. 1. Example 1. Top: eigenvalues of $P(t)$ solution of (9c), bottom: components of $z(t)$ solution of (22).

Example 1 – Lyapunov stable system.

Consider

$$A(t) = \begin{bmatrix} \cos(0.2t) & \sin(0.2t) \\ -\sin(0.2t) & \cos(0.2t) \end{bmatrix},$$

$$C(t) = \begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix}, \quad R(t) = I_2.$$

The corresponding state transition matrix is

$$\Phi(t,t_0) = \exp(\varphi(t,t_0)) \begin{bmatrix} \cos(\psi(t,t_0)) & \sin(\psi(t,t_0)) \\ -\sin(\psi(t,t_0)) & \cos(\psi(t,t_0)) \end{bmatrix}$$

with

$$\varphi(t,t_0) \triangleq 5(\sin(0.2t) - \sin(0.2t_0))$$
$$\psi(t,t_0) \triangleq 5(\cos(0.2t_0) - \cos(0.2t)).$$

This system is Lyapunov stable (see Definition 1).

The Riccati equation (9c) is initialized with $P(0) = I_2$ and the homogenous system (22) with $z(0) = [1,2]^T$. In Figure 1 are shown the two eigenvalues (equal to the singular values) of $P(t)$, which are bounded and tend to zero. In Figure 1 is also shown the state vector $z(t)$ (solution of (22)), which converges to zero.

Example 2 – unstable system.

Consider

$$A(t) = \begin{bmatrix} 2-e^{-t} & e^{-t}-2 \\ 0 & 2-e^{-t} \end{bmatrix}, \quad C(t) = [1 \ 0], \quad R(t) = 1.$$

The corresponding state transition matrix is

$$\Phi(t,t_0) = \begin{bmatrix} \exp(\psi(t,t_0)) & -\psi(t,t_0)\exp(\psi(t,t_0)) \\ 0 & \exp(\psi(t,t_0)) \end{bmatrix}$$

with $\psi(t,t_0) \triangleq 2(t–t_0)+e^{-t}–e^{-t_0}$. This system is unstable ($\Phi(t,t_0)$ diverges when $t \to +\infty$).

The Riccati equation (9c) is initialized with $P(0) = I_2$ and the homogenous system (22) with $z(0) = [1,2]^T$. In Figure 2 are shown the two eigenvalues (equal to the singular values) of $P(t)$, which are both upper and lower bounded. In Figure 2 is also shown the state vector $z(t)$ (solution of (22)), which converges to zero.

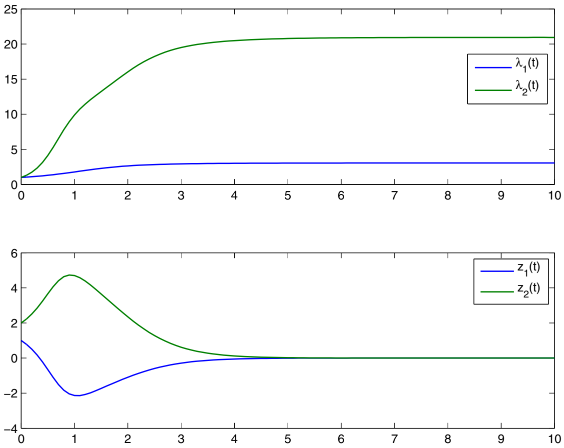

Fig. 2. Example 2. Top: eigenvalues of $P(t)$ solution of (9c), bottom: components of $z(t)$ solution of (22).

7. CONCLUSION

For any recursive algorithm running continuously in real time, the boundedness of all the involved variables is obviously an important property. It is established in this paper that, when the Kalman filter is applied to LTV OE systems, the solution of the Riccati equation and the Kalman gain are both bounded, essentially under the observability condition. It is further shown that the (homogeneous part of the) Kalman filter is asymptotically stable. The case of OE systems studied in this paper is not covered by the classical results requiring a controllability condition regarding the process noise, which is totally absent in OE systems. This difference caused a technical difficulty: unlike in the classical case, here the solution of the Riccati equation does not always have a strictly positive definite lower bound, therefore the associated "natural" Lyapunov function cannot be used in the usual sense in the convergence proof. The results of this paper show that, when the Kalman filter is applied to an OE system, there is no need to artificially introduce a small process noise into the state equation that would alter the results of state estimation.

The asymptotic stability of the Kalman filter for LTV OE systems has been established in this paper, *regardless of the stability of the considered systems*. This result can be further refined, by characterizing the exponential or polynomial convergence rate of the Kalman filter, but such developments depend on the stability or instability properties of the considered systems. Due to space limitation of the present paper, such results will be reported elsewhere.

REFERENCES

Anderson, B.D.O. and Moore, J.B. (1979). *Optimal Filtering*. Prentice-Hall, Inc., New Jersey.

Anderson, B.D., Bitmead, R.R., Johnson, C.R.J., Kokotovic, P.V., Kosut, R.L., Mareels, I.M., Praly, L., and Riedle, B.D. (1986). *Stability of adaptive systems: passivity and averaging analysis*. Signal processing, optimization, and control. MIT Press, Cambridge, Massachusetts.

Arnold, W.F. and Laub, A.J. (1984). Generalized eigenproblem algorithms and software for algebraic Riccati equations. *Proceedings of the IEEE*, 72(12), 1746–1754.

Choi, J. (2010). Linear systems and controls. Lecture notes. Http://www.egr.msu.edu/classes/me851/jchoi /lecture/index.html.

Delyon, B. (2001). A note on uniform observability. *IEEE Transactions on Automatic Control*, 46(8), 1326–1327.

Forssell, U. and Ljung, L. (2000). Identification of unstable systems using output error and box–jenkins model structures. *IEEE Transactions on Automatic Control*, 45(1), 137–141.

Goodman, D.M. and Dudley, D.G. (1987). An output error model and algorithm for electromegnetic system identification. *Circuits Systems Signal Process*, 6(4), 471–505.

Grewal, M.S. and Andrews, A.P. (2008). *Kalman Filtering: Theory and Practice Using Matlab*. Wiley-IEEE Press, 3rd edition.

Ioannou, P. and Sun, J. (1996). *Robust Adaptive Control*. Prentice Hall. Electronic copy http://www-rcf.usc.edu/∼ioannou/Robust_Adaptive_Control.htm.

Jazwinski, A.H. (1970). *Stochastic Processes and Filtering Theory*. Academic Press, New York and London.

Kalman, R.E. (1963). New methods in Wiener filtering theory. In J.L. Bogdanoff and F. Kozin (eds.), *Proceedings of the First Symposium on Engineering Applications of Random Function Theory and Probability*. John Wiley & Sons, New York.

Laub, A.J. (1979). A Schur method for solving algebraic Riccati equations. *IEEE Transactions on Automatic Control*, AC-24(6), 913–921.

Ni, B. and Zhang, Q. (2013). Kalman filter for process noise free systems. In *IFAC International Workshop on Adaptation and Learning in Control and Signal Processing (ALCOSP)*. Caen, France.

Pengov, M., Richard, E., and Vivalda, J.C. (2001). On the boundedness of the solutions of the continuous riccati equation. *Journal of Inequalities and Applications*, 6(6), 641–649.

Zarchan, P. and Musof, H. (2005). *Fundamentals of Kalman filtering: a practical approach*. AIAA.

Zhang, L. and Zhang, Q. (2015). Observability conservation by output feedback and observability Gramian bounds. *Automatica*. Accepted for publication.