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Résumé : Ce travail présente une dérivation rigoureuse des équations quasi-géostrophiques (QG) de l'océan grande échelle lorsque le nombre de Rossby tend vers 0. On suit les techniques classiques de dérivation de ces modèles (voir par exemple [?]), mais les équations d'origine sur lesquelles on s'appuie prennent en compte tous les termes de rotation, ainsi que dans [?]. Le modèle ainsi obtenu est légèrement différent du modèle QG traditionnel, et présente une seconde direction verticale, légèrement inclinée. Ceci confirme des simulations numériques et des expérimentations antérieures (voir [?, ?, ?]).

Mots-clés : analyse multi-échelles, force de Coriolis, approximation traditionnelle, modèles de circulation générale océanique, limite quasi-géostrophique

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On the quasi-hydrostatic quasi-geostrophic model

Abstract: This paper introduces a rigorous derivation of the quasi-geostrophic (QG) equations of large scale ocean as the Rossby number goes to zero. We follow classical techniques for the derivation (as in [**?**]), but the primitive equations that we consider account for all the rotating terms, as in [**?**]. We end up with a slightly different QG model with a tilted vertical direction, which has been illustrated in previous works (see [**?**, **?**, **?**]).

Key-words: multiscale analysis, Coriolis force, traditional approximation, ocean gloal circulation models, quasi-geostrophic limit

1 Introduction

The quasi-geostrophic equations are very familiar to oceanographers and meteorologists for they have been extensively used for modeling oceanic and atmospheric circulations ([?, ?]). These equations are obtained from the 3D primitive equations thanks to an asymptotic expansion with respect to the Rossby number. The model is also very familiar to applied mathematicians, and several studies establishing the well-posedness of the corresponding boundary-value problem have been published (see *e.g.* [?, ?, ?]). The primitive equations from which the traditional QG equations are obtained rely on a few well-known hypotheses, among which the so-called *traditional approximation*, which consists in neglecting the rotating terms involving $2\Omega\cos\theta$ that appear in the zonal and vertical components (??) and (??) of the momentum equation. This approximation has also been widely discussed in the literature (see [?] and the correspondence in [?, ?, ?, ?]). One may choose not to do this approximation, as in [?, ?] : in this case, the primitive equations are called quasi-hydrostatic (see [?] for a mathematical study).
Since the QG equations are obtained in the zero-limit of the Rossby number (large rotational effects), one could think of retaining all the rotating terms in the primitive equations, before performing the asymptotic analysis. This is the main objective of this paper. We will see in (??) that the modified QG equations, that we will call quasi-hydrostatic quasi-geostrophic (QHQG) equations, are very similar to the traditional QG ones, except that they raise a new vertical direction (denoted Z hereafter), which differ from the traditional vertical direction z. The tilt between z and Z is proportional to the nondimensional parameter λ introduced in (??), which mesures the ratio between traditional and nontraditional Coriolis terms. Experimental and numerical evidences of this tilted vertical direction can be found in [?, ?, ?] : we provide a first mathematical justification in this paper.

The paper is organized as follows : in Section ?? we perform the rigorous derivation of the QHQG equations, starting from the QH primitive equations. Then, we present in Section ?? some simple physical properties of the QHQG model.

2 Derivation of the QHQG model

In this section we present the derivation of the quasi-hydrostatic quasi-geostrophic (QHQG) equations. The derivation follows classical principles (as in [?]) : scaling, asymptotic expansion with respect to a small parameter, equations satisfied at order zero and one. Here, the small parameter (denoted ε in the sequel) is the Rossby number, so that we underline the effect of rotating terms. In order to account for the complete Coriolis force (see *e.g.* [?] and references therein), we retain all the rotating terms in the primitive equations, including the terms that are usually neglected in the so-called *traditionnal approximation*[1] : see the discussion in [**?**, **?**, **?**, **?**]).

2.1 Scaling Parameters and Scaled Equations

We consider a three-dimensional domain with periodic boundary conditions in the horizontal directions, rigid lid and flat bottom in the vertical. The governing primitive equations, including the complete Coriolis force, read :

$$\frac{Du}{Dt} - fv + f^* w = -\frac{\partial \varphi}{\partial x}, \tag{1a}$$

$$\frac{Dv}{Dt} + fu = -\frac{\partial \varphi}{\partial y}, \tag{1b}$$

$$\frac{Dw}{Dt} - f^* u + \frac{g\,\varrho}{\varrho_0} = -\frac{\partial \varphi}{\partial z}, \tag{1c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1d}$$

$$\frac{D\varrho}{Dt} = 0. \tag{1e}$$

Here (u, v, w) and ϱ are respectively the three-dimensional velocity and density of the fluid, and φ is the renormalized pressure, $\varphi = p/\varrho_0$. The scalars $f = 2\Omega \sin(\theta)$ and $f^* = 2\Omega \cos(\theta)$ are the Coriolis parameters where Ω stands for the angular velocity of the earth and θ is the latitude ; g is the universal gravity constant, and ϱ_0 stands for the averaged density of the fluid.
Equations (**??**) to (**??**) describe the conservation of momentum, where D/Dt is the material derivative $D/Dt = \partial/\partial_t + u\partial_x + v\partial_y + w\partial_z$, and Equation (**??**) corresponds to the conservation of mass. Finally, Equation (**??**) describes the advection of tracers (here the density ϱ). The density and the pressure may be classically decomposed as

$$\varrho(x, y, z, t) = \overline{\rho}(z) + \rho(x, y, z, t) \text{ and } \varphi(x, y, z, t) = \overline{\varphi}(z) + \phi(x, y, z, t),$$

where $\overline{\rho}$ and $\overline{\varphi}$ are the (known) background density and potential, depending only on the vertical variable. We also denote by $N^2(z) = -1/\overline{\rho}'(z)$ the buoyancy frequency, assuming that $\overline{\rho}'(z)$ is bounded away from zero.

Before going further in the derivation of the corresponding QG model, let us insist on the fact that we keep in equations (**??**) and (**??**) the Coriolis terms $f^* w$ and $f^* u$. This is actually the novelty of this study and will finally lead to a slightly modified QG model (see (**??**)). We think that it is a relevant modification, since the QG approximation aims at underlying the earth's rotation effects : one should thus include every rotation term in the primitive equations prior to an asymptotic expansion with respect to the Rossby number.
In the context of the β-plane approximation, we have[2], with θ_0 the average

1. The denomination of *traditional approximation* was introduced by Carl Eckart in his book [**?**].
2. For simplicity we do not consider any Taylor expansion of f^* with respect to θ. In fact we only keep the 0^{th} order term, as suggested in [**?**].

latitude :

$$f = f_0(1 + \varepsilon\beta_0 y) = 2\Omega \sin\theta_0(1 + \varepsilon\beta_0 y), \qquad f^* = f_0^* = 2\Omega\cos\theta_0.$$

We now introduce the following dimensionless variables, as it is classically done in QG modeling :

$$\begin{aligned} (x, y) = L\,(x', y'), \qquad & z = H\,z', \qquad t = \frac{L}{U}t', \\ u = U\,u', \qquad & v = U\,v', \qquad w = \frac{UH}{L}w', \\ \overline{\rho} = P\overline{\rho}', \qquad & \rho = \tfrac{\varrho_0 f_0 U L}{gH}\rho', \qquad \phi = f_0 U L \phi'. \end{aligned}$$

The Rossby number $\varepsilon = U/f_0 L$ is the fundamental ordering parameter in the following asymptotic expansion. A secondary ordering parameter is the scale ratio of ρ to ϱ : this ratio is assumed to be ε, that is, we assume :

$$\frac{\varrho_0 f_0 U L}{gH} = P\varepsilon.$$

Finally, the density may be expressed in terms of nondimensionnal quantities :

$$\varrho = P\Big(\overline{\rho}'(z) + \varepsilon\rho'\Big).$$

Another usual nondimentional number is the aspect ratio $\delta = H/L$; δ also appears in the ratio between the two Coriolis terms in the zonal momentum equation (**??**), which scales as

$$\lambda = \delta\cot[\theta_0]. \tag{2}$$

When we considered the scaling numbers introduced above, we have implicitely assumed that the leading term at the left-hand-side of Equation (**??**) was fv, which means that $\delta\cot[\theta_0]$ should not be too large :

$$\lambda \lesssim 1. \tag{3}$$

Fortunately, because the aspect ratio $\delta = H/L$ is rather small in large ocean models, the condition (**??**) is easily satisfied. However, the objective of the present work is to draw the reader's attention on the fact that λ is not necessarily small, and that it may have some physical repercussions (see Section **??**).

We end this section with the scaled equations (we naturally drop the primes) :

$$\begin{aligned} \varepsilon\frac{Du}{Dt} - (1 + \varepsilon\beta_0 y)v + \lambda w &= -\frac{\partial\phi}{\partial x}, && \text{(4a)} \\ \varepsilon\frac{Dv}{Dt} + (1 + \varepsilon\beta_0 y)u &= -\frac{\partial\phi}{\partial y}, && \text{(4b)} \\ \varepsilon\delta^2\frac{Dw}{Dt} - \lambda u + \rho &= -\frac{\partial\phi}{\partial z}, && \text{(4c)} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, && \text{(4d)} \\ \varepsilon\frac{D\varrho}{Dt} + w\overline{\rho}_z &= 0, && \text{(4e)} \end{aligned}$$

where we recall that ε is the Rossby number (meant to go to zero), $\delta = H/L$ is the domain aspect ratio, and $\lambda = \delta\cot(\theta_0)$.

2.2 Geostrophic Balance

We now consider an asymptotic expansion of all variables with respect to the Rossby number : for every unknown function ξ, we write the formal asymptotic expansion

$$\xi = \xi^{(0)} + \varepsilon\xi^{(1)} + \varepsilon^2\xi^{(2)} + ...$$

where $(\xi^{(j)})_{j\geq 0}$ behave as $O(1)$ as ε goes to zero. Equations (**??**)-(**??**) give, keeping only the order zero terms in ε :

$$\begin{aligned} -v^{(0)} + \lambda w^{(0)} &= -\phi_x^{(0)}, && (5)\\ u^{(0)} &= -\phi_y^{(0)}, && (6)\\ -\lambda u^{(0)} + \rho^{(0)} &= -\phi_z^{(0)}. && (7) \end{aligned}$$

The incompressibility condition reads $w_z^{(0)} = -u_x^{(0)} - v_y^{(0)}$ and this *traditionally* leads to $w^{(0)} = 0$, thanks to equations (**??**), (**??**) and boundary conditions on w (see [**?**]). Here, the incompressibility condition does not provide $w_z^{(0)} = 0$, but we have, denoting $\partial_Z = \partial_z + \lambda\partial_y$:

$$\begin{aligned} w_Z^{(0)} &= w_z^{(0)} + \lambda w_y^{(0)}\\ &= -u_x^{(0)} - v_y^{(0)} + \lambda w_y^{(0)}\\ &= \mathrm{curl}(\phi_y, \phi_x)\\ w_Z^{(0)} &= 0. && (8) \end{aligned}$$

Thanks to (**??**) and to homogeneous boundary conditions on $w^{(0)}$, we finally obtain[3] that $w^{(0)} = 0$.

The geostrophic equations read :

$$\begin{aligned} -v^{(0)} &= -\phi_x^{(0)}, && (9a)\\ u^{(0)} &= -\phi_y^{(0)}, && (9b)\\ \rho^{(0)} &= -\phi_z^{(0)} - \lambda\phi_y^{(0)} = -\phi_Z^0, && (9c)\\ w^{(0)} &= 0. && (9d) \end{aligned}$$

2.3 Quasi-Geostrophic Equations

Now we need the first order equations in order to determine the evolution of $\phi^{(0)}$. We denote by d_g the zero-order material derivative :

$$d_g = \partial_t + u^{(0)}\partial_x + v^{(0)}\partial_y.$$

The first order equations are :

$$\begin{aligned} d_g u^{(0)} - \beta_0 y v^{(0)} - v^{(1)} + \lambda w^{(1)} &= -\phi_x^{(1)}, && (10a)\\ d_g v^{(0)} + \beta_0 y u^{(0)} + u^{(1)} &= -\phi_y^{(1)}, && (10b)\\ -\lambda u^{(1)} + \rho^{(1)} &= -\phi_z^{(1)}, && (10c)\\ u_x^{(1)} + v_y^{(1)} + w_z^{(1)} &= 0, && (10d)\\ d_g\rho^{(0)} + w^{(1)}\overline{\rho}_z = 0. && && (10e) \end{aligned}$$

3. Alternatively, we have equation (**??**) which (written to the order zero and since $\overline{\rho}(z)$ never vanishes) leads to $w^{(0)} = 0$.

We now take the curl of equations (**??**)-(**??**) to obtain, thanks to Equation (**??**)

$$d_g(v_x^{(0)} - u_y^{(0)}) - w_z^{(1)} - \lambda w_y^{(1)} + \beta_0 v^{(0)} = 0. \tag{11}$$

We notice, as for the traditional QG equations, that $\beta_0 v^{(0)} = d_g(\beta_0 y)$. We thus try to express $-w_z^{(1)} - \lambda w_y^{(1)} = w_Z^1$ as $d_g(\Gamma)$ where Γ is a function to be defined. To this aim, we will extensively make use of Equation (**??**) that we reformulate :

$$w^{(1)} = N^2 d_g \rho^{(0)} = d_g(N^2 \rho^{(0)}). \tag{12}$$

Given (**??**), we may compute the required quantity

$$w_z^{(1)} + \lambda w_y^{(1)} = \Big(d_g(N^2\rho^{(0)}\Big)_z + \lambda\Big(d_g(N^2\rho^{(0)})\Big)_y. \tag{13}$$

We remark that for any function ξ and any variable $*$ we have the identity

$$\Big(d_g(\xi)\Big)_* = d_g(\xi_*) + u_*^{(0)}\partial_x\xi + v_*^{(0)}\partial_y\xi,$$

so that we can write

$$w_z^{(1)} + \lambda w_y^{(1)} = d_g\Big((N^2\rho^{(0)})_z\Big) + \lambda d_g\Big((N^2\rho^{(0)})_y\Big) + R, \tag{14}$$

where the rest R, according to the remark above, writes

$$\begin{aligned} R &= u_z^{(0)}(N^2\rho^{(0)})_x + v_z^{(0)}(N^2\rho^{(0)})_y + \lambda u_y^{(0)}(N^2\rho^{(0)})_x + \lambda v_y^{(0)}(N^2\rho^{(0)})_y \\ &= (N^2\rho^{(0)})_x\,(u_z^{(0)} + \lambda u_y^{(0)}) + (N^2\rho^{(0)})_y\,(v_z^{(0)} + \lambda v_y^{(0)}). \end{aligned} \tag{15}$$

Using (**??**)-(**??**) again, we have

$$R = N^2\rho_x^{(0)}\rho_y^{(0)} - N^2\rho_y^{(0)}\rho_x^{(0)} = 0,$$

which simplifies Equation (**??**) as follows :

$$w_Z^{(1)} = w_z^{(1)} + \lambda w_y^{(1)} = d_g\Big((N^2\rho^{(0)})_z + \lambda(N^2\rho^{(0)})_y\Big). \tag{16}$$

Back to Equation (**??**), we obtain the quasi-hydrostatic quasi-geostrophic equation :

$$d_g\Big(v_x^{(0)} - u_y^{(0)} - (N^2\rho^{(0)})_z - \lambda(N^2\rho^{(0)})_y + \beta_0 y\Big) = 0. \tag{17}$$

The potential vorticity

$$v_x^{(0)} - u_y^{(0)} - (N^2\rho^{(0)})_z - \lambda(N^2\rho^{(0)})_y + \beta_0 y$$

is thus conserved along material paths.

Let us now rewrite Equation (**??**), expressing everything in terms of $\phi^{(0)}$. We have

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big)\Big(\Delta\phi^{(0)} + N^2(\partial_z + \lambda\partial_y)^2\phi^{(0)} + N_z^2(\partial_z + \lambda\partial_y)\phi^{(0)} + \beta_0 y\Big) = 0, \tag{18}$$

which we could also write, with $\partial_Z = \partial_z + \lambda\partial_y$,

$$\Big(\partial_t - \phi_y^{(0)}\partial_x + \phi_x^{(0)}\partial_y\Big)\Big(\Delta\phi^{(0)} + (N^2\phi_Z^{(0)})_Z + \beta_0 y\Big) = 0. \quad (19)$$

One can thus easily recognize the traditional QG equation (see Equation (2.23) in [?]), except that the differential operator ∂_z is replaced by $\partial_Z = \partial_z + \lambda\partial_y$. We recall here that $\lambda = \delta\cot\theta_0$ is proportional to the domain aspect ratio. In particular, we recover the traditional QG equation when setting $\delta = 0$ in equations (**??**), (**??**) or (**??**).

3 Simple physical properties of the QH-QG model

We reformulate the QHQG equation as follows :

$$D[q + \beta y] = 0, \quad (20)$$

$$\Big(\partial_x^2 + \partial_y^2 + \partial_Z[\frac{1}{N^2(z)}\partial_Z]\Big)\phi = q, \quad (21)$$

where $D = \partial_t + J_{xy}[\Phi,]$, $\partial_Z = \partial_z + \lambda\partial_y$, and $\lambda = \frac{H}{L}\cot\theta_0$.

3.1 Separable solutions

Unlike in QG this system, when linearized by neglecting $J[\phi, q]$, does not have vertically separable solutions for general $N(z)$. One could think about whether the system could be justified with a background stratification of $N(Z)$, which would have Z-separable solutions. However, if N is constant, then vertically separable solutions do exist.

3.2 Ranges for the non-traditional terms

Assuming a wavenumber-space characterization of the solution in terms of vertical and horizontal wavenumbers, kv and kh, or equivalently local values of H and L, we can ask when the λ value is not small.
In QG the common view (sometimes called Charney's stretched isotropy) is that the Burger number, $Bu = NH/2\Omega L = kh/(2\Omega kv/N)$, is order one while $2\Omega/N$ is small. If $H/L \sim 2\Omega/N \ll 1$, then $\lambda \sim (2\Omega/N)\cot\theta_0$, which will be small except when θ_0 is very close to the Equator.
Alternatively, if we consider flow patterns with $2\Omega kv/Nkh \sim r \ll 1$, then $\lambda \sim (2\Omega/Nr)\cot\theta_0$. For small enough r, λ need not be small, and the QH correction to QG will be important. This happens when the aspect ratio H/L is large, the stratification is weak, and/or θ_0 is small. Large aspect ratio can be described from the QG perspective as atypically "tall" flows.

3.3 Rossby wave modes

One class of simple solutions is Rossby wave modes. They satisfy a linearized PDE usually justified by an assumption of small amplitude flow,

$$\frac{\partial q}{\partial t} + \beta \frac{\partial \phi}{\partial x} = 0. \tag{22}$$

With constant N in either a vertically bounded or unbounded domain, eigenmodes are

$$\phi \propto e^{i(kx+ly+mz-\omega t)}, \tag{23}$$

with a dispersion relation,

$$\omega = \frac{-\beta k}{K^2}, \quad K^2 = k^2 + l^2 + \frac{1}{N^2}(m + \lambda l)^2. \tag{24}$$

Thus, $\lambda \neq 0$ implies lower frequency and slower phase speed. We can also consider the group velocity for wave energy propagation, $c_g = \partial_k \omega$. Its components are

$$\begin{aligned} \partial_k \omega &= \frac{\beta(2k^2 - K^2)}{K^4}, \\ \partial_l \omega &= \frac{2\beta k(l + \lambda(m + \lambda l)/N^2)}{K^4} \\ \partial_m \omega &= \frac{2\beta k(m + \lambda l)/N^2}{K^4} \end{aligned} \tag{25}$$

This usually has a more westward zonal propagation due to $\lambda \neq 0$, and its (y, z) propagation is altered as well. These effects also occur for the barotropic mode ($m = 0$) where the vertical propagation is nonzero because of the QH correction.

3.4 Vortex solutions

A simple vortex solution is a nonlinear stationary state when $\beta = 0$ and N is constant. In QG this occurs for any axisymmetric profile, $\phi(r, z)$, where $r = \sqrt{x^2 + y^2}$ is the radial coordinate, which is sufficient to make $J[\phi, q] = 0$ even when the arguments have large amplitude. Physical interest usually lies in profiles that are localized in r, e.g., a monopole with $\phi(r)$ decaying away from a central extremum. In QH nonlinear stationary solutions exist for any profile $\phi(\xi, z)$,where $\xi^2 = x^2 + y^2/(1 + \lambda^2/N^2)$; ξ is an elliptical "radius" associated with the coordinate transformation,

$$x = \xi cos[v], y = \sqrt{1 + \lambda^2/N^2}\, \xi sin[v]. \tag{26}$$

Thus, QH vortices are meridionally elongated rather than horizontally round. Notice that in the vortex solutions the relevant parameter for QH differences from QG is λ/N rather than λ alone. This gives further emphasis to situations with weak stratification (small N).

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