Complexity of greedy edge-colouring

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Abstract

The *Grundy index* of a graph $G = (V, E)$ is the greatest number of colours that the greedy edge-colouring algorithm can use on G. We prove that the problem of determining the Grundy index of a graph $G = (V, E)$ is NP-hard for general graphs. We also show that this problem is polynomial-time solvable for caterpillars. More specifically, we prove that the Grundy index of a caterpillar is $\Delta(G)$ or $\Delta(G) + 1$ and present a polynomial-time algorithm to determine it exactly.

1 Introduction

All the graphs considered in this paper are loopless, but may have multiple edges.

A (proper) *k-colouring* of a graph $G = (V, E)$ is a surjective mapping $c : V \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ for any edge $uv \in E$. The *chromatic number* is $\chi(G) = min\{k \mid G$ admits a k-colouring$\}$. On the algorithmic point of view, finding the chromatic number of a graph is a hard problem. For all $k \geq 3$ it is NP-complete to decide whether a graph admits a k-colouring (see [2]). Furthermore, it is NP-hard to approximate the chromatic number within $|V(G)|^{\varepsilon_0}$ for some positive constant ε_0, as shown by Lund and Yannakakis [5].

Hence lots of heuristics have been developed to colour a graph. The most basic and widespread because it works on-line is the greedy algorithm. Given a vertex ordering $\sigma = v_1 < \cdots < v_n$ of $V(G)$, this algorithm colours the vertices in the order $v_1, \ldots, v_n$, assigning to v_i the smallest positive integer not used on its lower-indexed neighbours. A colouring resulting of the greedy algorithm is called a *greedy colouring*. The *Grundy number* $\Gamma(G)$ is the largest k such that G has a greedy k-colouring. Easily, $\chi(G) \leq \Gamma(G) \leq \Delta(G) + 1$.

Zaker [6] showed that for any fixed k, one can decide in polynomial time whether a given graph has Grundy number at most k. However determining the Grundy number of a graph is NP-hard [6], and given a graph G, it is even NP-complete to decide whether $\Gamma(G) = \Delta(G) + 1$ as shown by Havet and Sampaio [3]. In addition, Asté et al. [1] showed that for any constant $c \geq 1$, it is NP-complete to decide whether $\Gamma(G) \leq c \cdot \chi(G)$.

Graph colouring of many graph classes has also been studied. One of the classes is the one of line graphs. The *line graph* of a graph G, denoted $L(G)$, is the graph whose vertices are the edges of G, with $ef \in E(L(G))$ whenever e and f share an endvertex. Colouring line graphs corresponds to edge-colouring. A *k-edge-colouring* of a graph G is a surjective mapping $\phi : E(G) \to \{1, \ldots, k\}$ such that if two edges e and f are adjacent (i.e share an endvertex), then $\phi(e) \neq \phi(f)$. A *k-edge colouring* may also be seen as a partition of the edge set of G into k disjoint *matchings* $M_i = \{e \mid \phi(e) = i\}$, $1 \leq i \leq k$. By edge-colouring we mean either the mapping ϕ or the partition.

The *chromatic index* $\chi'(G)$ of a graph G is the least k such that G admits a k-edge-colouring. It is easy to see that $\chi'(G) = \chi(L(G))$. Obviously, $\Delta(G) \leq \chi'(G)$ and Shannon's and Vizing's theorems state that $\chi'(G) \leq \max\{\frac{3}{2}\Delta(G); \Delta(G) + \mu(G)\}$, where $\mu(G)$ is the maximum number of edges between two vertices of G. Holyer [4] showed that for any $k \geq 3$, it is NP-complete to decide if a k-regular graph has chromatic index k.

Edge colouring naturally arises in modelling some channel assignment problems in wireless network. From such a network, one can construct the communication graph whose vertices are the

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nodes of the network, and two vertices are connected by an edge whenever they communicate. In order to avoid interferences between the different signals arriving at a node, we need to assign distinct frequencies to the communications at each node. This corresponds to finding an edge colouring of the communication graph.

Usually the communications begin at different dates, and we need to assign the frequencies online. Usually, the frequencies are assigned greedily according to the following the greedy algorithm for edge-colouring, which corresponds to the greedy algorithm to colour a line graph. Given a graph $G = (V, E)$ and an edge ordering $\theta = e_1 < \cdots < e_n$, assign to e_i the least positive integer that was not already assigned to lower-indexed edges adjacent to it. An edge-colouring obtained by this process is called a *greedy edge-colouring* and it has the following property:

For every $j < i$, every edge e in M_i is adjacent to an edge in M_j. $\qquad\qquad$ (P)

Note that an edge-colouring satisfying (P) is a greedy edge-colouring relative to any edge ordering in which the edges of M_i precede those of M_j when $i < j$.

The *Grundy index* $\Gamma'(G)$ of a graph G is the largest number of colours of a greedy edge-colouring of G. Notice that $\Gamma'(G) = \Gamma(L(G))$. By definition, $\chi'(G) \leq \Gamma'(G)$. Furthermore, as an edge is incident to at most $2\Delta(G) - 2$ other edges ($\Delta(G) - 1$ at each endvertex), colouring the edges greedily uses at most $2\Delta(G) - 1$ colours. So $\Delta(G) \leq \Gamma'(G) \leq 2\Delta(G) - 1$. There are graphs for which the Grundy index equals the maximum degree: stars for example. On the opposite, for any Δ there is a tree with maximum degree Δ and Grundy index $2\Delta - 1$. Indeed, consider the trees B'_k defined recursively as follows: $B'_1 = P_2$, $B'_2 = P_3$ and the root of P_2 is one of its vertex and the root of P_3 is one of its leaves; B'_k is obtained from the disjoint union of B'_{k-1} and B'_{k-2} by adding an edge between their roots, and the root of B'_k is the root of B'_{k-2}. An easy induction shows that for every positive k, $\Delta(B'_{2k}) = \Delta(B'_{2k+1}) = k + 1$ and that the root of B'_{2k} has degree k and the root of B'_{2k+1} has degree $k + 1$. Now, $\Gamma'(B'_k) = k$ for every k, because one can show easily by induction the following stronger statement.

Proposition 1. *For every positive integer k, there is a greedy k-edge colouring of B'_k such that the colours assigned to the edges incident to the root are all the odd numbers up to k, if k is odd, and all the even numbers up to k if k is even.*

In this paper, we study the complexity of finding the Grundy index of a graph. We prove that it is NP-hard by showing that the following problem is co-NP-complete.

MINIMUM GREEDY EDGE-COLOURING
Instance: A graph G.
Question: $\Gamma'(G) = \Delta(G)$?

The proof is a reduction from 3-EDGE-COLOURABILITY OF CUBIC GRAPHS which was proved to be NP-complete by Holyer [4]. We recall that a *cubic graph* is a 3-regular graph. The reduction also proves that it is co-NP-complete to decide if $\Gamma'(G) = \chi'(G)$.

3-EDGE-COLOURABILITY OF CUBIC GRAPHS
Instance: A cubic graph G.
Question: Is G 3-edge colourable?

We then extend the result to a more general problem.

f-GREEDY EDGE-COLOURING
Instance: A graph G.
Question: $\Gamma'(G) \leq f(\Delta(G))$?

We show that for any function f such that $k \leq f(k) \leq 2k - 2$, the problem f-GREEDY EDGE-COLOURING is co-NP-Complete.

Since determining the Grundy index is NP-hard, a natural question to ask is for which class of graphs it can be done in polynomial time. Obviously it is the case for the class of graphs with maximum degree k. Indeed the Grundy index of a graph G in this class is at most $2k - 1$ and for every $1 \leq i \leq 2k - 1$, one can check in polynomial time whether $\Gamma'(G) \leq j$. So we must look at classes for which the maximum degree is not bounded. In Section 3, we consider *caterpillars* which are trees such that the deletion of all leaves results in a path, called *backbone*. We show that if T is a caterpillar then $\Gamma'(T) \leq \Delta(T) + 1$ and then give a linear-time algorithm to compute the Grundy index of a caterpillar. In view of this result, a natural question is the following:

Problem 2. Can we compute in polynomial time the Grundy index of a given tree?

2 Co-NP-completeness results

The aim of this section is to prove that f-Greedy Edge-Colouring is co-NP-complete for every function f such that $k \leq f(k) \leq 2k - 2$ for all k.

For sake of clarity, we first show that Minimum Greedy Edge-Colouring is co-NP-Complete.

Minimum Greedy Edge-Colouring is clearly in co-NP, because a greedy edge-colouring of a graph G with at least $\Delta(G) + 1$ colours is a certificate that $\Gamma'(G) > \Delta(G)$. We show that it is co-NP-Complete.

Theorem 3. Minimum Greedy Edge-Colouring *is co-NP-Complete.*

We now prove the co-NP-completeness by reduction from 3-Edge-Colourability of Cubic Graphs.

Let H be a cubic graph on n vertices $w_1, \ldots w_n$. Let G be the graph defined by $V(G) = V(H) \cup \{u_1, \ldots, u_n\} \cup \{v, a, b, c\}$ and $E(G) = E(H) \cup \{u_i w_i \mid 1 \leq i \leq n\} \cup \{vu_i \mid 1 \leq i \leq n\} \cup \{av, bv, cv\}$. See Figure 1.
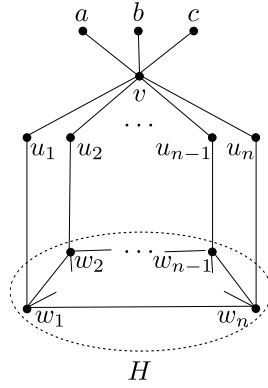

Figure 1: Graph G obtained from a cubic graph H.

In G, $d(v) = n + 3$, while the degree of all other vertices is at most 4. Thus, $\Delta(G) = d(v) = n + 3$ because $n \geq 4$ has H is cubic. Moreover, every edge of G is adjacent to at most $n + 3$ edges so $\Gamma'(G) \leq n + 4 = \Delta(G) + 1$. Hence the Grundy index of G is either $\Delta(G)$ or $\Delta(G) + 1$. The co-NP-completeness of Minimum Greedy Edge-Colouring follows directly from the following claim.

Claim 3.1. $\chi'(H) = 3$ *if and only if* $\Gamma'(G) = \Delta(G) + 1$.

Proof. ($\Rightarrow$) Suppose that there exists a 3-edge-colouring ϕ of H. Let us extend ϕ into a greedy edge-colouring of G with $\Delta(G) + 1 = n + 4$ colours. Set $\phi(av) = 1$, $\phi(bv) = 2$, $\phi(cv) = 3$, and for all $1 \leq i \leq n$, $\phi(u_i w_i) = 4$ and $\phi(u_i v) = i + 4$. Notice that every vertex w_i is incident to an edge of H of each colour in $\{1, 2, 3\}$ since H is cubic. Then it is straightforward to check that ϕ is a greedy $(n + 4)$-edge-colouring of G.

($\Leftarrow$) Suppose that there is a greedy $(n + 4)$-edge-colouring of G. Some edge is coloured $n + 4$. But such an edge has to be adjacent to at least $n + 3$ edges and thus to be one of the vu_i, say vu_n. The edge vu_n is adjacent to exactly $n + 3$ edges. So by Property (P), all edges adjacent to vu_n receive distinct colours in $\{1, \ldots, n + 3\}$.

Let us first prove by induction on $1 \leq j \leq n$ that the edge e_j incident to vu_n labelled $n + 5 - j$ is one of the vu_i, the result holding for $j = 1$. Suppose now that $j \geq 2$. The edge e_j must have degree at least $n + 5 - j$ since it is adjacent to vu_n and one edge of each colour in $\{1, \ldots, n + 4 - j\}$ by Property (P). Hence e_j must be incident to v since $u_n w_n$ is adjacent to four edges. Then e_j must have degree at least $n + 3$ since it is adjacent to the $j - 1$ edges e_l for $1 \leq l < j$ and one edge of each colour in $\{1, \ldots, n + 4 - j\}$. Hence e_j is one of the vu_i.

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Hence, without loss of generality, we may assume that $\phi(vu_i) = i + 4$ for all $1 \le i \le n$. The edge vu_i is adjacent to an edge coloured 4. This edge must be $u_i w_i$ since the edges av, bv and cv are adjacent to at most 2 edges coloured in $\{1, 2, 3\}$. Thus $\phi(u_i w_i) = 4$ for all $1 \le i \le n$.

Now every edge $u_i w_i$ is adjacent to three edges, one of each colour in $\{1, 2, 3\}$. Since $\phi(vu_i) \ge 5$, these three edges must be the three edges incident to w_i in H. Thus all the edges of H are coloured in $\{1, 2, 3\}$. Hence the restriction of ϕ to H is a 3-edge-colouring. $\square$

Remark 4. Observe that the graph G has chromatic index $\Delta(G)$. Indeed colour the edges adjacent to v with the colours $1, \dots, \Delta(G)$ and then extend greedily this colouring to the other edges. Since all the remaining edges are adjacent to at most four edges they will all get a colour less or equal to 5. Since $\Delta(G) \ge 5$, we obtain a $\Delta(G)$-edge colouring. Hence the above reduction shows that it is co-NP-complete to decide whether $\Gamma'(G) = \chi'(G)$.

Theorem 3 may be generalized as follows.

Theorem 5. *Let f be a function such that $k \le f(k) \le 2k - 2$ for all $k \in \mathbb{N}$. f-GREEDY EDGE-COLOURING is co-NP-Complete.*

Proof. f-GREEDY EDGE-COLOURING is clearly in co-NP, because a greedy edge-colouring of a graph G with more than $f(\Delta(G))$ colours is a certificate that $\Gamma'(G) > f(\Delta(G))$.

We now prove the co-NP-completeness by reduction from 3-EDGE-COLOU- RABILITY OF CUBIC GRAPHS.

Let H be a cubic graph on n vertices $w_1, \dots, w_n$ and let G be the graph defined as in the proof of Theorem 3. Set $p = f(n+3) - (n+3)$. Then $0 \le p \le n+1$. For $1 \le i \le p$, let T_i be the tree with vertex set $\{a_i, b_i, c_i, t_i\} \cup \{a_{i,j}, b_{i,j}, c_{i,j}, s_{i,j}, t_{i,j} \mid 1 \le j \le n-1\}$ and edge set $\{a_i t_i, b_i t_i, c_i t_i\} \cup \bigcup_{j=1}^{n-1} \{a_{i,j} t_{i,j}, b_{i,j} t_{i,j}, c_{i,j} t_{ij}, t_{i,j} s_{i,j}, s_{i,j} t_i\}$. Let G' be a graph obtained from the disjoint union of G and the T_i by adding the edge $u_n t_i$ for all $1 \le i \le p$. See Figure 2.
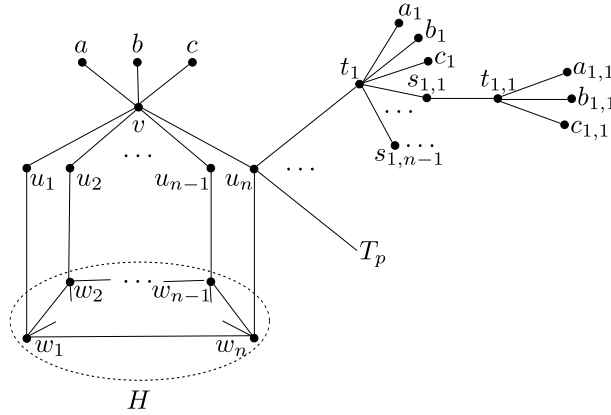

Figure 2: The graph G' obtained from a cubic graph H.

Observe that $\Delta(G') = n+3$ and the vertices of degree $n+3$ are v, $t_1, \dots, t_p$ and u_n when $p = n+1$. Moreover every edge is adjacent to at most $n + 3 + p$, so $\Gamma'(G) \le n + 3 + p + 1 = f(\Delta(G') + 1)$. The co-NP-completeness of f-GREEDY EDGE-COLOURING follows directly from the following claim.

Claim 5.1. $\chi'(H) = 3$ *if and only if* $\Gamma'(G') = f(\Delta(G')) + 1$.

($\Rightarrow$) Suppose that there exists a 3-edge-colouring ϕ of H. Let us extend ϕ into a greedy edge-colouring of G' with $f(\Delta(G')) + 1 = n + p + 4$ colours. We first extend it into a greedy $(n+4)$-colouring of G as we did in the proof of Theorem 3. In particular, we have $\phi(u_n w_n) = 4$ and $\phi(u_n v) = n + 4$. For all $1 \le i \le p$ and all $1 \le j \le n-1$, we set $\phi(t_i a_i) = 1$, $\phi(t_i b_i) = 2$, $\phi(t_i c_i) = 3$, $\phi(t_{i,j} a_{i,j}) = 1$, $\phi(t_{i,j} b_{i,j}) = 2$, $\phi(t_{i,j} c_{i,j}) = 3$, $\phi(t_{i,j} s_{i,j}) = j + 3$, and $\phi(t_i u_n) = n + 4 + i$. Then it is straightforward to check that ϕ is a greedy $(n + p + 4)$-edge-colouring of G'.

($\Leftarrow$) Suppose that G' admits a greedy $(n + p + 4)$-edge-colouring ϕ. For all $1 \le i \le p$, there is an edge e_i coloured $n + 4 + i$. This edge must has to be adjacent to at least $n + 3 + i$ edges by

Property (P). So all the e_i must be in $F = \{vu_n\} \cup \{u_n t_i \mid 1 \le i \le p\}$. Now the edge e_p is adjacent to an edge e_0 coloured $n + 4$. This edge is adjacent to at least $n + 4$ edges: one of each colour in $\{1, \ldots, n+3\}$ and e_p. Hence e_0 also has to be in F. Since $|F| = p+1$, all the edges in F are coloured with distinct labels in $\{n + 4, \ldots, n + p + 4\}$.

Now applying the same reasoning as in the proof of Theorem 3, we derive that the restriction to ϕ to H is a 3-edge-colouring. $\qquad\square$

3   Greedy edge-colouring of caterpillars

In this section, we show a polynomial-time algorithm solving GREEDY EDGE-COLOURING for caterpillars. A *caterpillar* is a tree such that the deletion of all leaves results in a path, called *backbone*.

We first show that the Grundy index of a caterpillar T is at mots $\Delta(T) + 1$, and so it it either $\Delta(T)$ or $\Delta(T) + 1$. some properties of a greedy edge-colouring of a caterpillar. We then give a polynomial-time algorithm that computes the Grundy index of a caterpillar.

3.1   Grundy index of a caterpillar

Lemma 6. *Let T be a caterpillar and v a vertex in its backbone. In every greedy edge-colouring of T, the colours $1, \ldots, d(v) - 2$ appear on the edges incident to v.*

Proof. By the contrapositive. Let c be an edge-colouring of T. Suppose that a colour $\alpha \in \{1, \ldots, d(v) - 2\}$ is not assigned to any edge incident to v. Then, since all the edges incident to v have different colours, at least three colours strictly greater than $d(v) - 2$ appear on three edges incidents to v. One of these colours, say β, must appear on an edge e incident with a leaf. But e is uniquely adjacent to edges incident to v. So e is adjacent to no edge coloured α. Since $\alpha \le d(v) - 2 < \beta$, the edge-colouring c is not greedy. $\qquad\square$

Lemma 7. *Let c be a greedy edge-colouring of a caterpillar T and v a vertex in the backbone of T. If two edges e_1 and e_2 incident to v receive colours greater than $d(v) - 1$, then e_1 and e_2 are two edges of the backbone and the edges incident to v and leaves are coloured $1, \ldots, d(v) - 2$.*

Proof. Suppose by way of contradiction that one of these two edges, say e_1, is incident to a leaf. Then e_1 is adjacent to $d(v) - 1$ other edges, and one of them, namely e_2, is assigned a colour greater than $d(v) - 1$. Thus e_1 is adjacent to at most $d(v) - 2$ edges whose colour is less or equal to $d(v) - 1$. So, there is a colour α in $\{1, \ldots, d(v) - 1\}$ such that no edge incident to e_1 is coloured α. This contradicts the fact that c is greedy. Hence e_1 and e_2 are edges of the backbone.

Now by Lemma 6, there must be edges incident to v of each colour in $\{1, \ldots, d(v) - 2\}$. So the $d(v) - 2$ edges distinct form e_1 and e_2, which are the edges linking v and leaves are coloured must be coloured in $\{1, \ldots, d(v) - 2\}$. $\qquad\square$

Theorem 8. *If T is a caterpillar, then $\Gamma'(T) \le \Delta(T) + 1$.*

Proof. Set $\Delta(T) = \Delta$. Suppose by way of contradiction that it is possible to greedily colour T with $\Delta + 2$ colours. Let e be an edge coloured $\Delta + 2$. It must be adjacent to at least $\Delta + 1$ edges, one of each colour $1, \ldots, \Delta + 1$. Thus, the edge e is in the backbone. According to Lemma 7, the edges e_1 and e_2 adjacent to e with colours Δ and $\Delta + 1$ are in the backbone. Furthermore all the edges adjacent to e which are neither e_1 nor e_2 are coloured in $\{1, \ldots, \Delta - 2\}$. Hence e is adjacent to no edge coloured $\Delta - 1$, a contradiction. $\qquad\square$

Theorem 8 is tight since there are caterpillars T whose Grundy index is greater than their maximum degree. For example, consider the caterpillar C_k with backbone (t, u, v, w) for which all the vertices t has degree $k - 1$, and v and w degree k. An edge-colouring in which the $k - 2$ edges incident to t and a leaf are coloured with $1, \ldots, k - 2$, the $k - 1$ edges incident to w and a leaf with $1, \ldots, k - 1$, the $k - 2$ edges incident to v and a leaf with $1, \ldots, k - 2$, the edge tu with $k - 1$, the edge vw with k and the edge uv with $k + 1$ is greedy. See Figure 3 for $k = 5$.
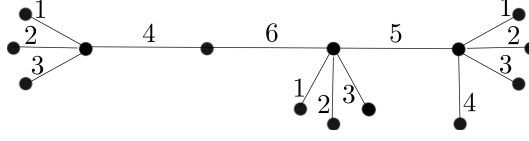
Figure 3: The caterpillar C_5 and a greedy edge-colouring with 6 colours.

3.2 Finding the Grundy index of a caterpillar

Theorem 8 implies that the Grundy index of a caterpillar T is either $\Delta(T)$ or $\Delta(T) + 1$. Hence determining the Grundy index of a caterpillar is equivalent to solve MINIMUM GREEDY EDGE-COLOURING for it. The aim of this subsection is to prove that it can be done in linear time.

Theorem 9. *Determining the Grundy index of a caterpillar T can be done in $O(|V(T)|)$.*

In order to prove these theorems we first give some definitions and prove some lemmas. Let T be a caterpillar with backbone $P = (v_1, v_2, \ldots, v_n)$. The *first edge* of P is $v_1 v_2$. For any edge $e = v_i v_{i+1} \in P$, removing e from T gives two caterpillars T_e^- and $T^+ e$, the first one containing v_i and the second one containing v_{i+1}. For convenience, the backbone of T_e^- is $P^-(e) = (v_i, v_{i-1}, \ldots, v_1)$ and the backbone of T_e^+ is $P_e^+ = (v_i, v_{i+1}, \ldots, v_n)$. Hence the first edge of T_e^- is (v_i, v_{i-1}) and the first edge of $T^+(e)$ is (v_{i+1}, v_{i+2}).

Lemma 10. *Let T be a caterpillar of maximum degree Δ with backbone $P = (v_1, \ldots, v_n)$. Then $\Gamma'(T) = \Delta + 1$ if and only if there is an edge $e \in E(P) \setminus \{v_1 v_2, v_{n-1} v_n\}$ such that*

(i) *one endvertex of e has degree Δ, and*

(ii) *one of the two caterpillars T_e^- and T_e^+ has a greedy edge-colouring such that the first edge of its backbone is coloured Δ and the other has a greedy edge-colouring such that its first edge of its backbone is coloured $\Delta - 1$.*

Proof. Assume that T has a greedy $(\Delta + 1)$-edge-colouring. Let e be an edge coloured $\Delta + 1$. By Lemma 7, e is in the backbone and incident to a vertex of degree Δ, proving (i). Moreover, the edge e is adjacent to an edge coloured Δ and another one labelled $\Delta - 1$. Again by Lemma 7 these two edges must also be in the backbone. In particular, e is not $v_1 v_2$ nor $v_{n-1} v_n$ because these two edges are adjacent to a unique edge of the backbone. Moreover the greedy edge-colourings induced on T_e^- and T_e^+ clearly satisfy (ii).

Conversely, assume that there is an edge $e \in E(P) \setminus \{v_1 v_2, v_{n-1} v_n\}$ satisfying (i) and (ii). Let ϕ^- and ϕ^+ be the greedy edge-colourings of T_e^- and T_e^+ respectively as in (ii). Let ϕ be the edge-colouring of T defined by $\phi(e) = \Delta + 1$, $\phi(f) = \phi^-(f)$ for all $f \in T_e^-$ and $\phi(f) = \phi^+(f)$ for all $f \in T_e^+$. We claim that ϕ is a greedy edge-colouring. Clearly, since ϕ^- and ϕ^+ are greedy, it suffices to prove that e is adjacent to an edge of every colour i in $\{1, \ldots, \Delta\}$. Since ϕ^+ and ϕ^- satisfy (ii), then e is adjacent to an edge labelled Δ and an edge labelled $\Delta - 1$, Now, e is incident to a vertex v of degree Δ. This vertex is incident to e and an edge f in the backbone. The edge f is the first edge a tree T_f in $\{T_e^+, T_e^-\}$. In the greedy edge-colouring of T_f, the edge f has a colour greater than $\Delta - 2$, so the $\Delta - 2$ edges incident to v which are not e nor f have all one colour in $1, \ldots, \Delta - 2$. Hence e is adjacent to an edge of every colour in $\{1, \ldots, \Delta\}$. □

Lemma 11. *Let T be a caterpillar with backbone P with first edge is $e = uv$. Then T has a greedy edge-colouring such that e is coloured k if and only if one of the following holds:*

(i) $d(u) \geq k$ *or* $d(v) \geq k$;

(ii) $d(u) = k - 1$ *and* T_e^+ *admits a greedy edge-colouring such that the first edge of P_e^+ is coloured $k - 1$.*

Proof. Let $e = uv$ with u the first vertex of P. Assume first that T has a greedy edge-colouring such that e is coloured k and that e is incident to no vertex of degree k. Then the edges incident to u must be coloured by $1, \ldots, d(u) - 1$ and the edges incident to u and a leaf are coloured by

$1, \ldots d(v) - 2$. Hence the edge incident to e and coloured $k - 1$ must be the first edge of P_e^+ is coloured $k - 1$ by Property (P). So the edge incident to e and coloured $k - 2$ must be incident to u, and thus $d(u) - 1 \geq k - 2$, that is $d(u) \geq k - 1$.

Assume now that (i) holds. Let x be a vertex in $\{u, v\}$ with degree at least k. One can colour all the edges incident to x with $1, \ldots, d(v)$ such that e is coloured k and then extend this edge-colouring greedily to obtain the desired greedy edge-colouring of T.

Finally assume that (ii) holds. Let ϕ be a greedy-edge colouring of T_e^+ such that the first edge of P_e^+ is coloured $k - 1$. One can extend it by assigning k to e, $1, \ldots, k - 2$ to the $k - 2$ edges incident to u and leaves and $1, \ldots, d(v) - 2$ to the edges incident to v. It is routine to check that this a a greedy-edge colouring of T. $\qquad\square$

Proof of Theorem 9. Theorem 8 and Lemma 10 imply that Algorithm 1 return the Grundy index of T provided that we have a subroutine FirstEdge(T,P,k) that returns 'yes' if a caterpillar T with backbone P admits a greedy-edge colouring such that the first edge of P is coloured k.

Algorithm 1: GrundyIndex(T)

Input: A caterpillar T.
Output: $\Gamma'(T)$.

1 Let $P = (v_1, v_2, \ldots, v_n)$ be the backbone of T. Compute $d(v_i)$ for all $1 \leq v_n$ and compute $\Delta = \Delta(T)$.

2 **for** $i = 2$ *to* $n - 2$ **do**

3      $e := v_i v_{i+1}$;

4   **if** $d(v_i) = \Delta$ *or* $d(v_{i+1}) = \Delta$ **then**

5   **if** *FirstEdge(T_e^+, P_e^+, Δ)=TRUE and FirstEdge(T_e^-, P_e^-, $\Delta - 1$)=TRUE* **then**

6   return $\Delta + 1$;

7   **if** *FirstEdge(T_e^+, P_e^+, $\Delta - 1$)=TRUE and FirstEdge(T_e^-, P_e^-, Δ)=TRUE* **then**

8   return $\Delta + 1$;

9 Return Δ;

Such a subroutine FirstEdge may be obtained by Algorithm 2 according to Lemma 11.

Algorithm 2: FirstEdge(T,P,k)

Input: A caterpillar T with backbone P and an integer k.
Output: $TRUE$ if there is a greedy k-edge-colouring of T with first edge of P coloured k, and $FALSE$ otherwise.

1 Let u be the first vertex of P and v its second. (So uv is the first edge.)

2 **if** $d(u) \geq k$ *or* $d(v) \geq k$ **then**

3   return $TRUE$;

4 **if** $d(u) \geq k - 1$ **then**

5   return FirstEdge($T - u$, $P - u$, $k - 1$);

6 return $FALSE$;

Let us now examine the complexity of Algorithm 1. Let us first observe that FirstEdge(T, P, k) makes a constant number of operations before calling FirstEdge($T - u$, $P - u$, $k - 1$). Hence an easy induction show that it makes $O(k)$ operations in total.

Algorithm 1 first computes (line 1) the degrees of all the v_i, which can be done in time $O(|V(T)|)$ and then takes the maximum of all these values which can also be done in time $O(|V(T)|)$.

In a second phase (line 2 to 8), for each edge $e \in P$ which is incident to a vertex of degree Δ, Algorithm 1 makes at most four calls of FirstEdge with last parameter $\Delta - 1$ or Δ. Hence for each $e \in P$ it makes $O(\Delta)$ operations. Let S be the set of vertices of degree Δ. The number of edges of P incident to a vertex of degree Δ is at most $2|S|$. But every vertex in S is adjacent to at least $\Delta - 2$ leaves. Hence $|V(T)| \geq |S| + (\Delta - 2)|S|$, so $|S| \leq |V(T)|/(\Delta - 1)$. Hence, in this second phase, the algorithm makes at most $O\left(2 \times \frac{|V(T)|}{\Delta-1}\Delta\right) = O(|V(T)|)$ operations.

Thus, in total, Algorithm 1 makes $O(|V(T)|)$ operations. □

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