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Abstract

We are interested in the increment stationarity property of L^2-indexed stochastic processes, which is a fairly general concern since many random fields can be interpreted as the restriction of a more generally defined L^2-indexed process. We first give a spectral representation theorem in the sense of Ito [9], and see potential applications on random fields, in particular on the L^2-indexed extension of the fractional Brownian motion. Then we prove that this latter process is characterized by its increment stationarity and self-similarity properties, as in the one-dimensional case.

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1 INTRODUCTION

It is known since the works of Ito [9] and Yaglom [20], that if a (multiparameter) stochastic process X is increment-stationary in the sense that for any s, s', t, t' and $h \in \mathbb{R}^d$:

$$\mathbb{E}\left((X_{t+h} - X_{s+h})(X_{t'+h} - X_{s'+h})\right) = \mathbb{E}\left((X_t - X_s)(X_{t'} - X_{s'})\right) ,$$

then X admits a spectral representation, which is understood as follows. There exist a complex-valued random measure M on $\mathbb{R}^d$, with control measure μ, and an $\mathbb{R}^d$-valued random vector Y, such that:

$$\forall t \in \mathbb{R}^d, \quad X_t = \int_{\mathbb{R}^d} \left(e^{i\langle t,x\rangle} - 1\right) M(\mathrm{d}x) + \langle t, Y\rangle .$$

Besides, Y and M are uncorrelated, in the sense that they have finite second moments and for any Borel set A, $\mathbb{E}\left(Y\overline{M(A)}\right) = 0$. Such representations have important applications in the study of sample path properties of stochastic processes (see [13, 19], to cite but a few). However, some processes that appear now frequently in the literature (for instance in the domain of stochastic partial differential equations [1, 5]) possess a different type of stationarity. This is the case of the Brownian sheet (the random field whose distributional derivative is the white noise on $\mathbb{R}^d$), and more generally of the fractional Brownian sheet (see Example 2.5). This led Basse-O'Connor et al. [2] to propose another spectral representation theorem for these processes, which permitted the construction of multiparameter stochastic integrals against these processes in the sense of Walsh.

Using a different technique, we obtain a similar result in Section 2, for a larger class of processes. Our Theorem 2.1 states that any random field $\{X(f),\ f \in L^2(T,m)\}$, where (T,m) is any measure space such that $L^2(T,m)$ is separable, which has second moments and satisfies the following increment-stationarity property: $\forall f, f', g, g', h \in L^2(T,m)$,

$$\mathbb{E}\left[\left(X(f+h) - X(g+h)\right)\left(X(f'+h) - X(g'+h)\right)\right] = \mathbb{E}\left[\left(X(f) - X(g)\right)\left(X(f') - X(g')\right)\right] ,$$

admits a spectral representation. We explain in paragraph 2.2 why this property covers many random fields, and how such random fields appear as the restriction of some $L^2(T,m)$-indexed process. In particular, all the known multiparameter extensions of the fractional Brownian motion are part of this class of processes. The counterpart for having such level of generality is that in some cases the resulting sprectral representation is either degenerate, or expressed in a too abstract setting for potential applications. However there are examples where the theorem permits to deduce sample path properties of multiparameter processes [16]. The prototypical example of a process to which our spectral representation theorem applies is the $L^2(T,m)$-indexed fractional Brownian motion (defined in [15] as an extension of the set-indexed fractional Brownian motion [7]).

Hence in Section 3 of this paper, we focus on the $L^2(T,m)$-indexed fBm. For any $H \in (0,1/2]$, this real-valued centred Gaussian process has a covariance given by:

$$\mathbb{E}\left(\boldsymbol{B}^H(f)\,\boldsymbol{B}^H(g)\right) = \frac{1}{2}\left(m\left(f^2\right)^{2H} + m\left(g^2\right)^{2H} - m\left((f-g)^2\right)^{2H}\right), \quad \forall f,g \in L^2(T,m) \tag{1.1}$$

where $m(\cdot)$ denotes the linear functional $\int_T \cdot \, \mathrm{d}m$ of $L^2(T,m)$. It encompasses most of the different known extensions of the fractional Brownian motion. We characterize the $L^2(T,m)$-indexed fractional Brownian motions in terms of self-similarity and increment-stationarity properties. Let us recall that the fractional Brownian motion of Hurst parameter $H \in (0,1)$ is the only (up to normalization of its variance) Gaussian process on $\mathbb{R}$ that has stationary increments and self-similarity of order H. In the multiparameter setting, there are several possible definitions of increment stationarity as well as self-similarity. For instance, the Lévy fractional Brownian motion of parameter H, whose covariance is given by $\mathbb{E}\left(\mathbb{X}_s^H \mathbb{X}_t^H\right) = \frac{1}{2}\left(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H}\right)$, is self-similar of order H and has a strong increment stationarity property on $\mathbb{R}^d$, i.e. against translations and rotations in $\mathbb{R}^d$:

$$\forall g \in \mathcal{G}(\mathbb{R}^d), \quad \{\mathbb{X}_{g(t)} - \mathbb{X}_{g(0)},\ t \in \mathbb{R}^d\} \overset{(d)}{=} \{\mathbb{X}_t,\ t \in \mathbb{R}^d\},$$

where $\mathcal{G}(\mathbb{R}^d)$ is the group of rigid motions of $\mathbb{R}^d$. Reciprocally, it is the only Gaussian process having these properties, up to normalization of its variance [17, p.393]. There is no such simple characterization for the fractional Brownian sheet (see the review [8]). We extend the notions of self-similarity and increment stationarity introduced in [7, 8], and give two characterizations of the L^2-fBm, depending on the definition of self-similarity and increment stationarity that are chosen for L^2-indexed processes.

2 Spectral representation of L^2-stationary processes

2.1 Preliminaries

A special structure on Hilbert spaces will appear frequently here, and will be referred to as *triple of Hilbert spaces*, or simply *triple* (this is a special case of Gel'fand triple). A triple consists of a separable Hilbert space $\mathcal{H}$ and a larger separable Hilbert space E such that $\mathcal{H}$ is densely and continuously embedded into E. We shall denote by E^* the topological dual of E, thus the inclusion $E^* \subset \mathcal{H}^*$ leads to write $E^* \subset \mathcal{H} \subset E$ by identifying $\mathcal{H}$ with $\mathcal{H}^*$. To continue with notations, we will use the duality bracket symbol $\langle \xi, x \rangle$, for any $\xi \in E^*$ and $x \in E$. Note that the previous properties imply that E^* is dense in $\mathcal{H}^*$ and that the canonical injection, that we shall denote by S, is also continuous (see for example [4, pp.136-137]). By a slight abuse of notations, we may write $S : E^* \to \mathcal{H}$ for this embedding.
For an extension of Bochner's theorem to be valid, we will need the embedding of these triples to be Hilbert-Schmidt. The following lemma gives the existence of such triples and is proved in [16] (actually with slightly stronger conclusions than written here).

Lemma 2.1. *Let $\mathcal{H}$ be a separable Hilbert space. There is a separable Hilbert space $(E, \|\cdot\|)$ such that $E^* \subset \mathcal{H} \subset E$ is a* triple *and the embedding $\mathcal{H} \subset E$ is Hilbert-Schmidt.*

Reproducing kernel Hilbert spaces (RKHS) will be particularly useful: let (T,d) be a separable metric space and C be a continuous covariance on $T \times T$. We denote by $\mathcal{H}(C)$ the associated RKHS (for a definition, see for instance [12, p.203]), which is separable [12, Theorem 5.3.1]. In particular, $\mathcal{H}(C)$ is spanned by the set of mappings $\{C(t,\cdot),\ t \in T\}$ with inner product given by $(C(t,\cdot), C(s,\cdot))_{\mathcal{H}(C)} = C(t,s)$ for any $t,s \in T$. Thus one can extract a basis of $\mathcal{H}(C)$ of the form $\{C(t_n,\cdot),\ t_n \in T\}$ by the separability

property of $\mathcal{H}(C)$.
In addition, we will always consider a Borel measure m on T and write (T,m) for the metric measure space (instead of (T,d,m)). $L^2(T,m)$ will be central in Section 2.3, and (T,m) is chosen such that $L^2(T,m)$ is separable (this the case for example when T is locally compact and separable).

Spectral representations involve random measures. We provide a formal definition of such objects.

Definition 2.2. *Let μ be a finite measure on the Borel sets of a topological space $\mathcal{X}$, which are denoted by $\mathcal{B}(\mathcal{X})$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. A complex-valued* random measure *on $\mathcal{B}(\mathcal{X})$ with control measure μ is a mapping $M:\mathcal{B}(\mathcal{X})\to L^2_{\mathbb{C}}(\Omega)$ satisfying:*

(i) zero mean: $\mathbb{E}(M(A))=0$ for any $A\in\mathcal{B}(\mathcal{X})$;

(ii) finite additivity: $M(A\cup B)=M(A)+M(B)$ a.s. for any disjoint $A,B\in\mathcal{B}(\mathcal{X})$;

(iii) covariance: $\mathbb{E}\left(M(A)\,\overline{M(B)}\right)=\mu(A\cap B)$ for any $A,B\in\mathcal{B}(\mathcal{X})$;

(iv) symmetry: $M(A)=\overline{M(-A)}$ a.s. for any $A\in\mathcal{B}(\mathcal{X})$.

Stochastic integrals with respect to a random measure can be defined for deterministic integrands. As usual, the first step is to define it for elementary functions via the relation $\int \mathbf{1}_A \mathrm{d}M = M(A)$, then extending it to simple functions. This establishes a linear isometry between the simple functions of $L^2(\mathcal{X},\mu)$ and $L^2_{\mathbb{C}}(\Omega)$ (in the sequel we drop the $\mathbb{C}$ indexing), which extends to the entire space $L^2(\mathcal{X},\mu)$.

2.2 Definitions of increment-stationarity and examples

In this paragraph, we precise the terminology related to stationarity. Note that our main result concerns L^2-indexed stochastic processes, and since most random fields of interest are neither indexed by an infinite-dimensional vector space, nor even a vector space, our goal is also to explain why this setting is interesting nonetheless.

For a given second-order T-indexed random field X with covariance C, we will consider the following objects: if there exist an $\mathcal{H}(C)$-valued mapping $\mathtt{f}: t\in T\mapsto \mathtt{f}_t\in\mathcal{H}(C)$ and an $\mathcal{H}(C)$-indexed process $\widehat{X}$ such that $X_t=\widehat{X}(\mathtt{f}_t)$ for any $t\in T$, then we say that X is *compatible with $\mathcal{H}$-indexing*. In case there exist a set-valued mapping $A: t\in T\mapsto A_t\in\mathcal{B}(T)$ and an isometry mapping $\mathtt{f}_t$ to $\mathbf{1}_{A_t}$ in some $L^2(T,m)$ space, we say that X is *compatible with set-indexing*.

Example 2.3 (Set-valued mappings). *1. The simplest example that comes to mind is the collection of rectangles of $\mathbb{R}^d$: $A_t=[0,t]$ and m is the Lebesgue measure.*

2. There is a mapping A and a measure m_d on $\mathbb{R}^d$ such that $m_d(A_t\triangle A_s)=\|t-s\|$ for any $s,t\in\mathbb{R}^d$, where $\|\cdot\|$ is the Euclidean norm and $\triangle$ is the symmetric difference of sets. Roughly, A_t is the set of all hyperplanes that separates 0 and t. This construction is fully described in [11, Chap. 4] or [17, p.401].

3. A similar construction due to Takenaka (see also [17, p.402-403]) gives the existence for $H\in(0,1/2]$ of a measure m_d^H and a set-valued mapping A such that $m_d^H(A_t\triangle A_s)=\|t-s\|^{2H}, \forall t,s\in\mathbb{R}^d$. Identically for a vector $\mathbf{H}=(H_1,\dots,H_d)\in(0,1/2]^d$, one can construct, by tensorization of one-dimensional measures, a new measure $m_d^{\mathbf{H}}$ and a set-valued mapping A such that $m_d^{\mathbf{H}}(A_t\triangle A_s)=\prod_{k=1}^d \|t_k-s_k\|^{2H_k}$.

Definition 2.4. *Let (T,m) be a measure space. We will say that a centred random field X indexed by T is* wide-sense increment-stationary *if the following set of assumptions holds:*

(i) X is compatible with L^2-indexing for the mapping $\mathtt{f}$ ($\widehat{X}(\mathtt{f}_t)=X_t$ for any $t\in T$) and dom $\widehat{X}$ is a subvector space of $\mathcal{H}(C)$;

(ii) $\widehat{X}$ is L^2-increment stationary, i.e. it has finite second moments at any point and it satisfies, for any f_1,f_2,g_1,g_2 and $h\in$ dom$\widehat{X}$ (dom$\widehat{X}$ is the domain of definition of $\widehat{X}$):

$$\mathbb{E}\left((\widehat{X}(f_1+h)-\widehat{X}(f_2+h))(\widehat{X}(g_1+h)-\widehat{X}(g_2+h))\right)=\mathbb{E}\left((\widehat{X}(f_1)-\widehat{X}(f_2))(\widehat{X}(g_1)-\widehat{X}(g_2))\right).$$

Let us remark that the existence of $\widehat{X}$ is close to the notion of "model" described in [11], although it is slightly less demanding. The choice of this type of stationarity for $\widehat{X}$ is motivated by the spectral representation theorem of the next section.

We present now a few wide-sense increment-stationary processes based on the examples of measure spaces given above.

Example 2.5. *1. For any fixed $H \in (0,1)$ ($H = 1/2$ corresponds to the Brownian case), there is a centred Gaussian process indexed by $\mathbb{R}^d$ which has the following increments:*

$$\mathbb{E}\left((\mathbb{X}^H_t - \mathbb{X}^H_s)^2\right) = \|t-s\|^{2H} .$$

This process is called Lévy (fractional) Brownian motion and has the simple increment stationarity *property: $\mathbb{E}\left((\mathbb{X}^H_{t+h} - \mathbb{X}^H_{s+h})(\mathbb{X}^H_{t'+h} - \mathbb{X}^H_{s'+h})\right) = \mathbb{E}\left((\mathbb{X}^H_t - \mathbb{X}^H_s)(\mathbb{X}^H_{t'} - \mathbb{X}^H_{s'})\right)$ for any $s, s', t, t', h \in \mathbb{R}^d$. Besides, the Euclidean space is compatible with set-indexing (see Example 2.3 point 2 for the definition of A_t and m_d) and the $L^2(\mathbb{R}^d, m_d)$-indexed Gaussian process defined by:*

$$\mathbb{E}\left(\widehat{\mathbb{X}}^H(f)\, \widehat{\mathbb{X}}^H(g)\right) = \frac{1}{2}\left(m_d(f^2)^{2H} + m_d(g^2)^{2H} - m_d((f-g)^2)^{2H}\right)$$

is well-defined for $H \leq 1/2$ (see [7]) and for any $t \in \mathbb{R}^d$, $\widehat{\mathbb{X}}^H(\mathbf{1}_{A_t}) = \mathbb{X}^H_t$.

2. The fractional Brownian sheet $\mathbb{W}^{\mathbf{H}}$ of Hurst parameter $\mathbf{H} = (H_1, \dots, H_d) \in (0,1)^d$ is the centred Gaussian process with covariance: let $\mathbf{t} = (t_1, \dots, t_d) \in \mathbb{R}^d_+$, $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}^d_+$,

$$\mathbb{E}\left(\mathbb{W}^{\mathbf{H}}_{\mathbf{t}}\, \mathbb{W}^{\mathbf{H}}_{\mathbf{s}}\right) = 2^{-d} \prod_{k=1}^{d} \left(|t_k|^{2H_k} + |s_k|^{2H_k} - |t_k - s_k|^{2H_k}\right) = R^{\otimes d}_{\mathbf{H}}(\mathbf{1}_{[0,\mathbf{t}]}, \mathbf{1}_{[0,\mathbf{s}]}) .$$

$R^{\otimes d}_{\mathbf{H}}$ is a notation that holds when $\mathbf{H} \in (0,1/2]^d$. Indeed, for $H_k \in (0,1/2]$ and $f,g \in L^2(\mathbb{R}_+, \lambda_1)$, $R_{H_k}(f,g) = 1/2(\|f\|^{4H_k} + \|g\|^{4H_k} - \|f-g\|^{4H_k})$ is a particular case of (1.1) (λ_d will denote the d-dimensional Lebesgue measure) and $R_{H_k}(\mathbf{1}_{[0,t_k]}, \mathbf{1}_{[0,s_k]})$ appears in the above product. The tensor product of such covariances yields a covariance on $\bigotimes_{k=1}^d L^2(\mathbb{R}_+, \lambda_1)$ which is isometric to $L^2(\mathbb{R}^d_+, \lambda_d)$, thus $R^{\otimes d}_{\mathbf{H}} = \bigotimes_{k=1}^d R_{H_k}$. Let $\widehat{\mathbb{W}}^{\mathbf{H}}$ be the $L^2(\mathbb{R}^d_+)$-indexed Gaussian process with covariance $R^{\otimes d}_{\mathbf{H}}$.

$\widehat{\mathbb{W}}^{\mathbf{H}}$ is L^2-increment stationary: this follows from the sheet increment stationarity *property of $\mathbb{W}^{\mathbf{H}}$. This property is the main object of study in [2] and is expressed as follows: for any $s \preccurlyeq t$, $s' \preccurlyeq t'$ and $u \in \mathbb{R}^d$,*

$$\mathbb{E}\left(\Delta\mathbb{W}^{\mathbf{H}}([s+u, t+u])\, \Delta\mathbb{W}^{\mathbf{H}}([s'+u, t'+u])\right) = \mathbb{E}\left(\Delta\mathbb{W}^{\mathbf{H}}([s,t])\, \Delta\mathbb{W}^{\mathbf{H}}([s',t'])\right)$$

where $\Delta\mathbb{W}^{\mathbf{H}}$ is the process obtained by the inclusion-exclusion formula. That is, for $\mathbf{s} \preccurlyeq \mathbf{t}$, $\Delta\mathbb{W}^{\mathbf{H}}([\mathbf{s},\mathbf{t}]) = \sum_{\epsilon \in \{0,1\}^d} (-1)^{\epsilon} \mathbb{W}^{\mathbf{H}}_{c_1(\epsilon_1),\dots,c_d(\epsilon_d)}$, where $\epsilon = |\boldsymbol{\epsilon}| = \epsilon_1 + \cdots + \epsilon_d$ and $c_k(\epsilon_k) = t_k$ if $\epsilon_k = 0$, s_k otherwise.

3. For any $H \in (0,1/2]$, the multiparameter fractional Brownian motion is the Gaussian process with covariance given by:

$$\mathbb{E}\left(\mathbb{B}^H_s \mathbb{B}^H_t\right) = \frac{1}{2}\left(\lambda_d([0,s])^{2H} + \lambda_d([0,t])^{2H} - \lambda_d([0,s] \triangle [0,t])^{2H}\right) , \; s,t \in \mathbb{R}^d_+ . \tag{2.1}$$

Its extension to an $L^2(\mathbb{R}^d_+, \lambda)$-indexed process which is L^2-increment stationary is straightforward from (1.1) and has been studied in [15]. Hence it is also increment stationary in the wide sense. When only observed as a multiparameter process, it satisfies: $\forall t \preccurlyeq t'$ and any $\tau \in \mathbb{R}^d_+$,

$$\lambda\left([0,t'] \setminus [0,t]\right) = \lambda([0,\tau]) \;\Rightarrow\; \mathbb{B}^H_{t'} - \mathbb{B}^H_t \overset{(d)}{=} \mathbb{B}^H_\tau . \tag{2.2}$$

This is in fact a weak form of the measure increment stationarity *presented in Section 3.*
When $H = \frac{1}{2}$ and $\mathbf{H} = (\frac{1}{2}, \dots, \frac{1}{2})$, $\mathbb{B}^H$ and $\mathbb{W}^{\mathbf{H}}$ above are the same process, known as Brownian sheet.

One of our initial motivations for this work was to obtain a spectral representation theorem for processes having the measure increment stationarity, and a fractal characterization of the multiparameter fBm based on this property, but this sole property seems in fact too weak for these purposes.
We only presented Gaussian examples but stable process could also be exhibited ([17]). These were examples of processes that are compatible with set-indexing and that extend naturally to a function space indexing. If no such extension is available, one can always resort to the following result.

Proposition 2.1. *Let (T,m) be a measure space such that $L^2(T,m)$ is separable. Any second order T-indexed process with covariance C extends to a linear $\mathcal{H}(C)$-indexed process and thus is wide-sense increment-stationary.*

Proof. Let C be the covariance of X and let $\{t_n \in T, n \in \mathbb{N}\}$ such that $\{C(t_n,\cdot),\ n\in\mathbb{N}\}$ is a basis of $\mathcal{H}(C)$ (recall that $\mathcal{H}(C)$ is separable, see the beginning of this section). Then define $\widehat{X}(C(t_n,\cdot)) = X_{t_n}$ for any n and extend $\widehat{X}$ to $\mathrm{Span}\{C(t_n,\cdot),\ n\in\mathbb{N}\}$ by linearity. $\widehat{X}$ is now a linear isometry from $\mathrm{Span}\{C(t_n,\cdot),\ n\in\mathbb{N}\}$ to $L^2(\Omega)$. As such it can be extended to a process from $\mathcal{H}(C)$ to $L^2(\Omega)$ by density. The assertion follows. $\square$

This result is only here to emphasize how general our definition of increment stationarity is. In fact, having in hands a *linear* $\mathcal{H}(C)$-indexed process might not be very useful (at least for the applications we have in mind), since it yields a somehow degenerate spectral decomposition, as we will see in the next section. However this linear process can be considered as a stochastic integral against X, whose space of (deterministic) integrands coincides with the RKHS of X.

2.3 Spectral representation theorem for L^2-increment stationary processes

In this section, no particular property of $L^2(T,m)$ is used except that it is a separable Hilbert space. Hence, the stochastic processes that appear here are indexed by a separable Hilbert space $\mathcal{H}$.
In the sequel, $E^* \subset \mathcal{H} \subset E$ is a triple as in Lemma 2.1, and S denotes the canonical injection from E^* to $\mathcal{H}$ (and S^* is its dual). Note that the norm of E is denoted by $\|\cdot\|$ as it will be the most frequently used. Any other norm will be written with a subscript, for instance $\|\cdot\|_{L^2(\mu)}$ or $\|\cdot\|_{\mathcal{H}}$. With these notations, the Hilbert-Schmidt property of the embedding reads: S, resp. S^*, is a Hilbert-Schmidt operator of $(E^*,\|\cdot\|_{E^*}) \to (\mathcal{H},\|\cdot\|_{\mathcal{H}})$, resp. of $(\mathcal{H},\|\cdot\|_{\mathcal{H}}) \to (E,\|\cdot\|)$.

Proposition 2.2. *Let $C : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a covariance of the form $C(\kappa,\kappa') = \frac{1}{2}(\Phi(\kappa)+\Phi(\kappa')-\Phi(\kappa-\kappa'))$ for some symmetric continuous function Φ. Then there exist a non-negative symmetric operator $R : E^* \to E$, and a finite Borel measure μ on E such that:*

$$\forall \xi, \quad \Phi(S\xi) = \langle \xi, R\xi\rangle + 2\int_E \frac{1-\cos\langle \xi, x\rangle}{1\wedge \|x\|^2}\, \mu(\mathrm{d}x)\ . \tag{2.3}$$

Besides, $R\circ i_E$ is a trace-class operator on E (where i_E is the Riesz isomorphism of $E \to E^*$), and $\mu(\{0\}) = 0$. (Note that the norm appearing in the above integral is the norm of E, and we do not precise it in the sequel unless the context is unclear.)

Proof. Due to the form of C, the application $\xi \in E^* \mapsto \Phi(S\xi)$ is continuous and negative definite (see Definition 4.3 and Proposition 4.4 in [18]). Thus, according to Schoenberg's theorem, $\xi \mapsto \exp(-t\Phi(S\xi))$ is positive definite for any $t \in \mathbb{R}_+^*$. The existence of b, R, μ_0 in the next paragraph is explained in [3], but we give the main ingredients for the sake of completeness.
It follows from Lemma 2.1 and Sazonov's theorem (see [21, Theorem 3.2]), according to which a Hilbert-Schmidt map is radonifying, that since $\kappa \mapsto \exp(-\frac{t}{2}\Phi(\kappa))$ is continuous on $\mathcal{H}$ for each $t>0$, it is the Fourier transform of a measure ν_t on E. By Lévy's continuity theorem in Hilbert spaces ([3]), $\{\nu_t,\ t>0\}$ weakly converges as $t \to 0$ to the Dirac mass δ_0. Hence $\xi \in E^* \mapsto \exp\left(-\frac{1}{2}\Phi(S\xi)\right)$ is the characteristic function of the infinitely divisible distribution ν_1. So by the Lévy-Khintchine theorem [14, Theorem VI.4.10]:

$$\forall \xi \in E^*, \quad \Phi(S\xi) = 2i\langle \xi, b\rangle + \langle \xi, R\xi\rangle - 2\int_E \left(e^{i\langle \xi, x\rangle} - 1 - \frac{i\langle \xi, x\rangle}{1+\|x\|^2}\right)\mu_0(\mathrm{d}x)\ ,$$

where $b \in E$, R satisfies the hypotheses stated in the proposition, and μ_0 is a Lévy measure, in the sense that $\mu_0(\{0\}) = 0$ and $\int_E \left(1 \wedge \|x\|^2\right) \mu_0(\mathrm{d}x) < \infty$.
Using the equality $\Phi(-\xi) = \Phi(\xi)$, we obtain that for any $\xi \in E^*$:

$$\langle \xi, b \rangle = \int_E \left(\frac{\langle \xi, x \rangle}{1 + \|x\|^2} - \sin\langle \xi, x \rangle \right) \mu_0(\mathrm{d}x) \ .$$

The linearity of the left hand side term implies that for any $n \in \mathbb{N}$, $\int_E \sin\langle n\xi, x\rangle \, \mu_0(\mathrm{d}x) = n \int_E \sin\langle \xi, x\rangle \, \mu_0(\mathrm{d}x)$. Hence $\int_E \sin\langle \xi, x\rangle \, \mu_0(\mathrm{d}x) = 0$ for any $\xi \in E^*$ and it follows that μ_0 is a symmetric measure. Thus $b = \int_E \frac{x}{1+\|x\|^2} \, \mu_0(\mathrm{d}x) = 0$ also. The result follows by defining $\mu(\mathrm{d}x) = (1 \wedge \|x\|^2) \, \mu_0(\mathrm{d}x)$. □

Similarly to Definition 2.4 for $L^2(T,m)$-indexed processes, we define L^2-increment stationarity for $\mathcal{H}$-indexed processes.

Definition 2.6. *A real-valued $\mathcal{H}$-indexed centred*[1] *random field Y is L^2-increment stationary if it has finite second moments at any point and if it satisfies, for any $\kappa_1, \kappa_2, \kappa_1', \kappa_2'$ and $h \in \mathcal{H}$:*

$$\mathbb{E}\left((Y(\kappa_1 + h) - Y(\kappa_2 + h))(Y(\kappa_1' + h) - Y(\kappa_2' + h))\right) = \mathbb{E}\left((Y(\kappa_1) - Y(\kappa_2))(Y(\kappa_1') - Y(\kappa_2'))\right) \ .$$

Theorem 2.1. *Let Y be a real-valued $\mathcal{H}$-indexed L^2-increment stationary process with continuous covariance, and let $E^* \subset \mathcal{H} \subset E$ be a triple with Hilbert-Schmidt embedding. Then, there exist a random measure M on E, and an E-valued random variable Z such that:*

$$\forall \xi \in E^*, \quad Y_\xi = Y_0 + \int_E \frac{e^{i\langle \xi, x\rangle} - 1}{1 \wedge \|x\|} \, M(\mathrm{d}x) + \langle \xi, Z \rangle \ ,$$

and M and Z have the following properties, for μ and R as in Proposition 2.2:

- *M has control measure μ and Z has finite second moments with covariance operator $R : E^* \to E$, i.e. $\mathbb{E}\left(\langle \eta, Z\rangle \langle \xi, Z\rangle\right) = \langle \eta, R\xi\rangle < \infty$ for any $\eta, \xi \in E^*$;*
- *M and Z are uncorrelated, in the sense that for any $A \in \mathcal{B}(E)$ and any $\xi \in E^*$, $\mathbb{E}\left(\langle \xi, Z\rangle \overline{M(A)}\right) = 0$.*

The previous decomposition extends to $\mathcal{H}$ in the following manner: there exists a linear mapping $\mathcal{Z} : \mathcal{H} \to L^2(\Omega)$ which is uncorrelated with M, such that $\mathbb{E}\left(\mathcal{Z}(\kappa)^2\right) = (\kappa, \tilde{R}\kappa)_{\mathcal{H}}$, where $\tilde{R}$ is a symmetric non-negative operator on $\mathcal{H}$ and

$$\forall \kappa \in \mathcal{H}, \quad Y(\kappa) = Y(0) + \int_E \gamma(\kappa, x) \, M(\mathrm{d}x) + \mathcal{Z}(\kappa) \ ,$$

where γ is the unique uniformly continuous extension of $\xi \in E^* \mapsto \frac{1 - e^{i\langle \xi, \cdot\rangle}}{1 \wedge \|\cdot\|} \in L^2(\mu)$ to a mapping from $\mathcal{H} \to L^2(\mu)$. Conversely, any $\mathcal{H}$-indexed process with this representation is L^2-increment stationary.

Before proving the theorem, let us state the following useful lemma, which can be proved with the tools of [6, Chap. 39].

Lemma 2.7. *Let E be a separable Banach space and μ a finite Borel measure on E. Then the space of trigonometric polynomials $\mathcal{T} = Span\left\{e^{i\langle \xi, \cdot\rangle}, \ \xi \in E^*\right\}$ is dense in $L^2(E, \mu)$.*

Proof of theorem 2.1. This proof is carried out in two steps. In the first one, we prove the decomposition on E^*, while in the second step, we extend it to $\mathcal{H}$. Without restriction, we may assume that $Y(0) = 0$, since otherwise we can define $\widetilde{Y}(\kappa) = Y(\kappa) - Y(0)$.

First Step. The L^2-increment stationarity implies that the covariance of Y is of the form given in Proposition 2.2 (with a continuous function Φ), thus we let μ and R be defined according to the result

[1]Theorem 2.1 still holds true if instead of Y centred, one assumes that $\mathbb{E}\left(Y(\kappa_1) - Y(\kappa_2)\right) = \mathbb{E}\left(Y(\kappa_1 - \kappa_2) - Y(0)\right)$, $\forall \kappa_1, \kappa_2 \in \mathcal{H}$.

of this proposition. For some non-zero $\xi_0 \in E^*$, let X be defined by $X_\xi = Y_{\xi+\xi_0} - Y_\xi$. Then X is L^2-stationary, defined in a similar sense to L^2-increment stationarity, i.e. it has finite second moments and for any κ_1, κ_2 and $h \in \mathcal{H}$: $\mathbb{E}\left(X(\kappa_1+h)X(\kappa_2+h)\right) = \mathbb{E}\left(X(\kappa_1)X(\kappa_2)\right)$. Its covariance satisfies:

$$\begin{aligned}\mathbb{E}\left(X_\xi \, X_\eta\right) &= \mathbb{E}\left(\left(Y_{\xi+\xi_0} - Y_\xi\right)\left(Y_{\eta+\xi_0} - Y_\eta\right)\right) \\ &= \frac{1}{2}\left(\Phi(\xi-\eta+\xi_0) + \Phi(\xi-\eta-\xi_0) - 2\Phi(\xi-\eta)\right)\end{aligned}$$

and one can check that this quantity can be written $\Psi(\xi-\eta)$ (we omit the dependence in ξ_0 in this notation), where Ψ reads:

$$\forall \xi \in E^*, \quad \Psi(\xi) = \langle \xi_0, R\xi_0\rangle + 2\int_E e^{i\langle \xi, x\rangle} \frac{1-\cos\langle \xi_0, x\rangle}{1\wedge \|x\|^2}\, \mu(\mathrm{d}x)\ .$$

Let us define a new finite measure on the Borel sets of E by:

$$\tilde{\mu}_{\xi_0}(\mathrm{d}x) = 2\frac{1-\cos\langle \xi_0, x\rangle}{1\wedge\|x\|^2}\mathbf{1}_{\{x\neq 0\}}\mu(\mathrm{d}x) + \mathbf{1}_{\{x=0\}}\langle \xi_0, R\xi_0\rangle\ ,$$

so that Ψ can be written $\Psi(\xi) = \int_E e^{i\langle \xi, x\rangle}\, \tilde{\mu}_{\xi_0}(\mathrm{d}x)$.
We shall now define a process T_{ξ_0} on the vector space $\mathrm{Span}\{e^{i\langle \xi, \cdot\rangle},\ \xi \in E^*\}$ satisfying the following linearity properties: for any $\lambda \in \mathbb{R},\ \xi, \eta \in E^*$,

$$\begin{aligned}T_{\xi_0}\left(\lambda e^{i\langle \xi, \cdot\rangle}\right) &= \lambda X_\xi \\ T_{\xi_0}\left(e^{i\langle \xi, \cdot\rangle} + e^{i\langle \eta, \cdot\rangle}\right) &= X_\xi + X_\eta\ .\end{aligned}$$

We claim that this process is well-defined, as there does not exist either couples $(\lambda, \xi) \neq (\lambda', \xi') \in (\mathbb{R}\setminus\{0\}) \times E^*$ such that $\lambda e^{i\langle \xi, \cdot\rangle} = \lambda' e^{i\langle \xi', \cdot\rangle}$, nor does there exist couples $(\xi, \eta) \neq (\xi', \eta') \in E^* \times E^*$ such that $e^{i\langle \xi, \cdot\rangle} + e^{i\langle \eta, \cdot\rangle} = e^{i\langle \xi', \cdot\rangle} + e^{i\langle \eta', \cdot\rangle}$. Note that T_{ξ_0} is a linear isometry of $\mathcal{T} \to L^2(\Omega)$ (recall that $\mathcal{T}$ is the space of trigonometric polynomials).
Since $\mathcal{T}$ is dense in $L^2(\tilde{\mu}_{\xi_0})$ (see Lemma 2.7), T_{ξ_0} extends into a linear isometry of $L^2(\tilde{\mu}_{\xi_0}) \to L^2(\Omega)$, and we are able to define the following random measure:

$$\tilde{M}_{\xi_0}(A) = T_{\xi_0}(\mathbf{1}_A)\ , \quad \forall A \in \mathcal{B}(E)\ ,$$

so that $\tilde{M}_{\xi_0}$ has control measure $\tilde{\mu}_{\xi_0}$: $\mathbb{E}\left(\tilde{M}_{\xi_0}(A)\right) = 0$ and $\mathbb{E}\left(\tilde{M}_{\xi_0}(A)\, \overline{\tilde{M}_{\xi_0}(B)}\right) = \tilde{\mu}_{\xi_0}(A\cap B)$, for all $A, B \in \mathcal{B}(E)$. One can now construct a stochastic integral against $\tilde{M}_{\xi_0}$ which satisfies, for any $f \in L^2(\tilde{\mu}_{\xi_0})$:

$$\int_E f(x)\, \tilde{M}_{\xi_0}(\mathrm{d}x) = T_{\xi_0}(f)\ .$$

In particular, for $f = e^{i\langle \xi, \cdot\rangle}$, we recover $X_\xi^{(\xi_0)} = \int_E e^{i\langle \xi, x\rangle}\, \tilde{M}_{\xi_0}(\mathrm{d}x)$. Note that we shall use the notation $X_\xi^{(\xi_0)}$ for X_ξ in the rest of this proof. By the same density argument as above, there is a random variable Z_{ξ_0} in the $L^2(\Omega)$-closure of $\mathrm{Span}\{X_\xi,\ \xi \in E^*\}$ such that:

$$Z_{\xi_0} = \tilde{M}_{\xi_0}(\{0\})\ .$$

At the end of this proof, we give more details on Z_{ξ_0}. But first, let us define the random measure $\underline{M}_{\xi_0}$ and the process $\underline{X}^{(\xi_0)}$ by:

$$\begin{aligned}\forall A \in \mathcal{B}(E), &\quad \underline{M}_{\xi_0}(A) = \tilde{M}_{\xi_0}(A) - \mathbf{1}_{\{A\cap\{0\}\neq\emptyset\}}\tilde{M}_{\xi_0}(\{0\}) \\ \forall \xi \in E^*, &\quad \underline{X}_\xi^{(\xi_0)} = X_\xi^{(\xi_0)} - Z_{\xi_0} = \int_E e^{i\langle \xi, x\rangle}\underline{M}_{\xi_0}(\mathrm{d}x)\ .\end{aligned}$$

A few facts can be easily deduced from the previous definitions: firstly, the control measure of $\underline{M}_{\xi_0}$ is $\underline{\mu}_{\xi_0} = 2\frac{1-\cos\langle\xi_0,x\rangle}{1\wedge\|x\|^2}\mathbf{1}_{\{x\neq 0\}}\mu(\mathrm{d}x)$; secondly, $\underline{X}^{(\xi_0)}$ is still a stationary process; and finally, for any $\xi\in E^*$, Z_{ξ_0} and $\underline{X}^{(\xi_0)}_\xi$ are uncorrelated.

Let us come back to X and let $\xi_0'\in E^*$: observe that for any $\xi\in E^*$,

$$X^{(\xi_0+\xi_0')}_\xi = X^{(\xi_0')}_{\xi+\xi_0} + X^{(\xi_0)}_\xi \ .$$

Thus for any $\xi\in E^*$, $\int_E e^{i\langle\xi,x\rangle}\tilde{M}_{\xi_0+\xi_0'}(\mathrm{d}x) = \int_E e^{i\langle\xi+\xi_0,x\rangle}\tilde{M}_{\xi_0'}(\mathrm{d}x) + \int_E e^{i\langle\xi,x\rangle}\tilde{M}_{\xi_0}(\mathrm{d}x)$. By symmetry, this implies:

$$\forall\xi\in E^*,\quad \int_E e^{i\langle\xi,x\rangle}\left(e^{i\langle\xi_0',x\rangle}-1\right)\tilde{M}_{\xi_0}(\mathrm{d}x) = \int_E e^{i\langle\xi,x\rangle}\left(e^{i\langle\xi_0,x\rangle}-1\right)\tilde{M}_{\xi_0'}(\mathrm{d}x)\ ,$$

which can be transposed to $\underline{M}$, since the previous integrals cannot charge $\{0\}$:

$$\forall\xi\in E^*,\quad \int_E e^{i\langle\xi,x\rangle}\left(e^{i\langle\xi_0',x\rangle}-1\right)\underline{M}_{\xi_0}(\mathrm{d}x) = \int_E e^{i\langle\xi,x\rangle}\left(e^{i\langle\xi_0,x\rangle}-1\right)\underline{M}_{\xi_0'}(\mathrm{d}x)\ . \tag{2.4}$$

Let us define the following finite Borel measure $\underline{\mu}_{\xi_0,\xi_0'}(\mathrm{d}x) := 2(1-\cos\langle\xi_0',x\rangle)\,\underline{\mu}_{\xi_0}(\mathrm{d}x)$, and define also for any $A\in\mathcal{B}(E)$ the mapping $\varphi_{\xi_0,\xi_0',A} : x\in E\mapsto \mathbf{1}_A(x)(1\wedge\|x\|)\left((e^{i\langle\xi_0,x\rangle}-1)(e^{i\langle\xi_0',x\rangle}-1)\right)^{-1}$. $\varphi_{\xi_0,\xi_0',A}$ belongs to $L^2(\underline{\mu}_{\xi_0,\xi_0'})$ since

$$\int_E \left|\varphi_{\xi_0,\xi_0',A}(x)\right|^2\underline{\mu}_{\xi_0,\xi_0'}(\mathrm{d}x) = \int_E\left|\varphi_{\xi_0,\xi_0',A}(x)\left(e^{i\langle\xi_0',x\rangle}-1\right)\right|^2\underline{\mu}_{\xi_0}(\mathrm{d}x) = \int_A\mu(\mathrm{d}x)$$

is finite. Recall that Lemma 2.7 states that $\varphi_{\xi_0,\xi_0',A}$ can be approximated by elements in $\mathrm{Span}\{e^{i\langle\xi,\cdot\rangle},\ \xi\in E^*\}$. Hence Equation (2.4) yields that for any $A\in\mathcal{B}(E)$ such that $A\cap\{0\}=\emptyset$, $\int_A(1\wedge\|x\|)\left(e^{i\langle\xi_0,x\rangle}-1\right)^{-1}\underline{M}_{\xi_0}(\mathrm{d}x)$ is independent of ξ_0 (and by definition, $\underline{M}_{\xi_0}(\{0\})=0$). Thus we call this quantity $M(A)$, and one can verify that M is a random measure whose control measure is precisely μ. From the equality:

$$\int_E\frac{e^{i\langle\xi,x\rangle}-1}{1\wedge\|x\|}\,M(\mathrm{d}x) = \int_E\frac{e^{i\langle\xi,x\rangle}-1}{1\wedge\|x\|}\,\frac{1\wedge\|x\|}{e^{i\langle\xi,x\rangle}-1}\,\underline{M}_\xi(\mathrm{d}x) = \underline{M}_\xi(E)\ ,\quad\forall\xi\in E^*\ ,$$

and due to $\underline{X}^{(\xi)}_0 = \underline{M}_\xi(E)$, it is now clear that Y admits the following representation:

$$\forall\xi\in E^*,\quad Y_\xi = Z_\xi + \int_E\frac{e^{i\langle\xi,x\rangle}-1}{1\wedge\|x\|}\,M(\mathrm{d}x)\ . \tag{2.5}$$

To conclude this part of the proof, we need to show that there exists a random variable Z with values in E such that $Z_\xi=\langle\xi,Z\rangle$ and whose covariance operator is R. One can easily check that R is the covariance operator of $\{Z_\xi,\ \xi\in E^*\}$, so let us prove that for any $\xi,\eta\in E^*$, $Z_\xi+Z_\eta = Z_{\xi+\eta}$ a.s. Using Equality (2.5),

$$\begin{aligned}\mathbb{E}\left((Z_{\xi+\eta}-Z_\xi-Z_\eta)^2\right) &= \mathbb{E}\left(\left|Y_{\xi+\eta}-Y_\xi-Y_\eta+\int_E\frac{e^{i\langle\xi,x\rangle}+e^{i\langle\eta,x\rangle}-e^{i\langle\xi+\eta,x\rangle}-1}{1\wedge\|x\|}\,M(\mathrm{d}x)\right|^2\right)\\ &= \mathbb{E}\left(\left|Y_{\xi+\eta}-Y_\xi-Y_\eta\right|^2\right)+\mathbb{E}\left(\left|\int_E\frac{e^{i\langle\xi,x\rangle}+e^{i\langle\eta,x\rangle}-e^{i\langle\xi+\eta,x\rangle}-1}{1\wedge\|x\|}\,M(\mathrm{d}x)\right|^2\right)\\ &\quad+2\mathbb{E}\left((Y_{\xi+\eta}-Y_\xi-Y_\eta)\overline{\int_E\frac{e^{i\langle\xi,x\rangle}+e^{i\langle\eta,x\rangle}-e^{i\langle\xi+\eta,x\rangle}-1}{1\wedge\|x\|}\,M(\mathrm{d}x)}\right)\ .\end{aligned} \tag{2.6}$$

We analyze the three summands of the last line separately, and recall that the covariance of Y is given by $C(\xi,\eta)=\frac{1}{2}(\Phi(\xi)+\Phi(\eta)-\Phi(\xi-\eta))$, so that $\mathbb{E}\left(\left(Y_{\xi+\eta}-Y_\xi-Y_\eta\right)^2\right)=2\Phi(\xi)+2\Phi(\eta)-\Phi(\xi+\eta)-\Phi(\xi-\eta)$. The decomposition of Φ given in (2.3) implies that:

$$\mathbb{E}\left(\left(Y_{\xi+\eta}-Y_\xi-Y_\eta\right)^2\right)=4\int_E\frac{1-\cos\langle\xi,x\rangle}{1\wedge\|x\|^2}\,\mu(\mathrm{d}x)+4\int_E\frac{1-\cos\langle\eta,x\rangle}{1\wedge\|x\|^2}\,\mu(\mathrm{d}x)$$
$$-2\int_E\frac{1-\cos\langle\xi+\eta,x\rangle}{1\wedge\|x\|^2}\,\mu(\mathrm{d}x)-2\int_E\frac{1-\cos\langle\xi-\eta,x\rangle}{1\wedge\|x\|^2}\,\mu(\mathrm{d}x)\,, \qquad (2.7)$$

because the quadratic terms cancel one another.
Next, we remark that $Y_{\xi+\eta}-Y_\xi-Y_\eta=-\int_E\left(e^{i\langle\xi,x\rangle}+e^{i\langle\eta,x\rangle}-e^{i\langle\xi+\eta,x\rangle}-1\right)\,M(\mathrm{d}x)+\rho(\xi,\eta)$, where $\rho(\xi,\eta)=\tilde{M}_{\xi+\eta}(\{0\})-\tilde{M}_\xi(\{0\})-\tilde{M}_\eta(\{0\})$, and also that $\mathbb{E}\left(\tilde{M}_\xi(\{0\})\,\overline{M(A)}\right)=0$ for any $\xi\in E^*$ and $A\in\mathcal{B}(E)$. Hence $\rho(\xi,\eta)$ is uncorrelated with M, so the sum of the second and third summand in Equation (2.6) is in fact equal to:

$$-\mathbb{E}\left(\left|\int_E\frac{e^{i\langle\xi,x\rangle}+e^{i\langle\eta,x\rangle}-e^{i\langle\xi+\eta,x\rangle}-1}{1\wedge\|x\|}\,M(\mathrm{d}x)\right|^2\right)=-\int_E\frac{\left|e^{i\langle\xi,x\rangle}+e^{i\langle\eta,x\rangle}-e^{i\langle\xi+\eta,x\rangle}-1\right|^2}{1\wedge\|x\|^2}\,\mu(\mathrm{d}x)\,.$$

The sum between this term and (2.7) is precisely 0. Thus $\mathbb{E}\left((Z_{\xi+\eta}-Z_\xi-Z_\eta)^2\right)=0$. We prove similarly that for any $\lambda\in\mathbb{R}$, $Z_{\lambda\xi}=\lambda Z_\xi$ a.s. Hence Z_ξ is linear.
To find an E-valued random variable Z, let $\{\xi_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis of E^* (with scalar product $(\cdot,\cdot)_{E^*}$) and $\{e_n\}_{n\in\mathbb{N}}$ be the dual basis. For any $N\in\mathbb{N}$, let us define $Z_N=\sum_{n=1}^N Z_{\xi_n}e_n$. For $\xi\in E^*$, we write $\xi=\sum_{n=1}^\infty(\xi,\xi_n)_{E^*}\xi_n$ the decomposition of ξ in the previous basis, and $\xi^N=\xi-\sum_{n=1}^N(\xi,\xi_n)_{E^*}\xi_n$. Then, by linearity of Z_ξ,

$$\mathbb{E}\left(|Z_\xi-\langle\xi,Z_N\rangle|\right)=\mathbb{E}\left(|Z_{\xi^N}|\right)$$
$$\leq\sqrt{\mathbb{E}\left(|Z_{\xi^N}|^2\right)}=\sqrt{\langle\xi^N,R\xi^N\rangle}\,.$$

Hence for each $\xi\in E^*$, $\langle\xi,Z_N\rangle\to Z_\xi$ a.s. as $N\to\infty$. To prove that Z_N has a limit in E, observe that for any integers $N\geq P$, $\mathbb{E}(\|Z_N-Z_P\|)\leq\sqrt{\mathbb{E}(\sum_{n=P+1}^N Z_{\xi_n}^2)}$. But $\mathbb{E}(\sum_{n=P+1}^N Z_{\xi_n}^2)=\sum_{n=P+1}^N\langle\xi_n,R\xi_n\rangle\leq\sum_{n=1}^\infty\langle\xi_n,R\xi_n\rangle$ which is finite since R is trace-class, thus there exists an E-valued random vector such that Z_ξ takes the announced form.

Second Step. Let $\Xi(S\xi)=2\int_E\frac{1-\cos\langle\xi,x\rangle}{1\wedge\|x\|^2}\,\mu(\mathrm{d}x)$ be the second part of the covariance Φ. Then Ξ extends to a function on $\mathcal{H}$. Indeed, the mapping:

$$\gamma:\ S(E^*)\to L^2(\mu)$$
$$S\xi\mapsto\frac{1-e^{i\langle\xi,\cdot\rangle}}{1\wedge\|\cdot\|}$$

satisfies $\|\gamma(S\xi)-\gamma(S\eta)\|_{L^2(\mu)}=\|\gamma(S(\xi-\eta))\|_{L^2(\mu)}\leq\Phi(S(\xi-\eta))^{1/2}$ for any $\xi,\eta\in E^*$, where the inequality holds since the difference between both terms is precisely $\langle\xi-\eta,R(\xi-\eta)\rangle\geq 0$. Note that γ is well-defined (since S is an injection) and that $\Phi^{1/2}$ is only a seminorm on $\mathcal{H}$ (it might not separate points). Hence we consider the quotient space $S(E^*)/\Phi$ endowed with the proper norm $\Phi^{1/2}$, where the equivalence relation is given by $\xi\sim\eta\iff\Phi(S(\xi-\eta))=0$. We still denote by γ the previous mapping. Thus γ is uniformly continuous as a mapping from $S(E^*)/\Phi$ to $L^2(\mu)$. Hence by a classical analysis result, it extends to a uniformly continuous mapping (still denoted by γ) on the completion of $S(E^*)/\Phi$ with respect to the $\Phi^{1/2}$ norm. Since Φ is continuous in $\mathcal{H}$, the closure of $S(E^*)/\Phi$ includes $\mathcal{H}/\Phi$. So γ can be finally considered as a mapping on the space $\mathcal{H}/\Phi$. Now define $\tilde{R}$ as follows:

$$\forall\bar{\kappa}\in\mathcal{H}/\Phi,\quad\tilde{R}(\bar{\kappa},\bar{\kappa})=\Phi(\bar{\kappa})-\|\gamma(\bar{\kappa})\|^2_{L^2(\mu)}\,,$$

and then $\tilde{R}(\bar{\kappa},\bar{\kappa}')$ by polarization. This is a nonnegative definite symmetric bilinear operator, as the limit of R on E^*/Φ. In fact, $\tilde{R}$ and γ are well-defined on $\mathcal{H}$ by $\gamma(\kappa)=\gamma(\bar{\kappa})$ and $\tilde{R}(\kappa,\kappa')=\tilde{R}(\bar{\kappa},\bar{\kappa'})$ for any $\kappa,\kappa'\in\mathcal{H}$ ($\bar{\kappa}$ denotes the equivalence class of κ). Indeed if κ_1,κ_2 are two elements in the same equivalence class, $\|\gamma(\kappa_1)-\gamma(\kappa_2)\|_{L^2(\mu)}\leq\Phi(\kappa_1-\kappa_2)^{1/2}=0$, and:

$$\begin{aligned}\tilde{R}(\bar{\kappa},\kappa_1)-\tilde{R}(\bar{\kappa},\kappa_2)=\tilde{R}(\bar{\kappa},\bar{0})&=\frac{1}{2}\left(\Phi(\bar{\kappa})-\|\gamma(\bar{\kappa})\|^2_{L^2(\mu)}+\Phi(\bar{0})-\|\gamma(\bar{0})\|^2_{L^2(\mu)}-\Phi(\bar{\kappa}+\bar{0})+\|\gamma(\bar{\kappa}+\bar{0})\|^2_{L^2(\mu)}\right)\\&=0\ .\end{aligned}$$

As for the processes, we proceed as follows: define $\{\mathcal{M}(\kappa)=\int_E\gamma(\kappa)(x)\ M(\mathrm{d}x),\ \kappa\in\mathcal{H}\}$. This process is well-defined due to the preceding construction of γ, and it coincides with the process $\int_E\frac{1-e^{i\langle\cdot,x\rangle}}{1\wedge\|x\|}M(\mathrm{d}x)$ on E^*. Then define $\mathcal{Z}(\kappa)=Y(\kappa)-\mathcal{M}(\kappa)$, which coincides with Z if $\kappa\in E^*$. This concludes the proof. □

2.4 Discussion

Given a T-indexed random field X with covariance C, the linear $\mathcal{H}(C)$-indexed process $\widehat{X}$ constructed in Proposition 2.1 has the following spectral representation: $\forall f\in\mathcal{H}(C)$, $\widehat{X}(f)=\mathcal{Z}(f)$ where $\mathcal{Z}:\mathcal{H}(C)\to L^2(\Omega)$. Hence $\widehat{X}$ has no spectral measure and our theorem does not carry much information in that case. However as we will see in the next example, this does not mean that there is not another process whose restriction is X and which has a spectral measure.

Our second remark is related to the spectral representation of some fractional processes. We recall that the covariance of the multiparameter fractional Brownian motion is given in (2.1). In [16], a spectral representation was obtained as a special case of our theorem, due to special results available for stable measures on Hilbert spaces. Hence the present work yields a more generic and complete (although more lengthy) way to prove that:

$$\forall t\in\mathbb{R}^d_+,\quad \mathbb{B}^H_t=\int_E\gamma\left(\mathbf{1}_{[0,t]},x\right)\ M^H(\mathrm{d}x)\ ,$$

where E is some Hilbert space in which $L^2(\mathbb{R}^d_+)$ is (Hilbert-Schmidt) embedded, γ is defined as in Theorem 2.1, and M^H has control measure Δ^H, where Δ^H is the Lévy measure of a stable measure on E. In particular, this representation helps studying the sample path regularity of the multiparameter fBm, since B^H_t can now be written as a sum of independent processes if E is sliced into disjoint subsets [16]. It is also interesting to notice that Δ^H has a similar form to the control measure of the usual fractional Brownian motion. Indeed, we recall the spectral representation of the fractional Brownian motion:

$$B^H_t=c_H\int_{\mathbb{R}}\frac{e^{itx}-1}{|x|^{H+\frac{1}{2}}}\mathbb{W}(\mathrm{d}x)\ ,$$

where c_H is a normalizing constant and $\mathbb{W}$ is a complex Gaussian white noise. Hence in that case the control measure is simply $\frac{\lambda(\mathrm{d}x)}{|x|^{1+2H}}$ while from [10], we know that $\Delta^H(B)=\int_0^\infty\frac{\mathrm{d}r}{r^{1+2H}}\int_S\mathbf{1}_B(ry)\ \sigma^H(\mathrm{d}y)$, where σ^H is a finite, rotationally invariant measure on the unit sphere S of E.

3 Stationarity and self-similarity characterization

The $L^2(T,m)$-fractional Brownian motion is the centred Gaussian process with covariance (1.1). In this section, $L^2(T,m)$ becomes simply L^2, and $\|\cdot\|$ refers to the $L^2(T,m)$ norm. We give two characterizations of the L^2-fBm: the first one is similar to the characterization of the Lévy fBm, while the second one uses a notion of stationarity similar to the one defined for set-indexed processes in [7, 8].

We start with some definitions. Consider the set $\mathcal{G}$, which is the restriction of the general linear group of L^2 to bounded linear mappings $\varphi:L^2\to L^2$ such that:

$$\forall f,g\in L^2,\quad \|f\|=\|g\|\Rightarrow\|\varphi(f)\|=\|\varphi(g)\|\ .$$

Let $\varrho : \mathcal{G} \to \mathbb{R}_+$ be the application that maps φ to the square of its operator norm. Note that for any $\varphi \in \mathcal{G}$ and any $f \in L^2$, $\|\varphi(f)\| = \sqrt{\varrho(\varphi)}\,\|f\|$, and that ϱ is a group morphism.

We will say that an L^2-indexed stochastic process X is:

- H–self-similar, if:

$$\forall a > 0, \quad \{a^{-H} X_{af},\ f \in L^2\} \overset{(d)}{=} \{X_f,\ f \in L^2\}\ ; \tag{SS1}$$

- strongly H–self-similar, if:

$$\forall \varphi \in \mathcal{G}, \quad \{X_{\varphi(f)},\ f \in L^2\} \overset{(d)}{=} \{\varrho(\varphi)^H X_f,\ f \in L^2\}\ ; \tag{SS2}$$

- strongly L^2-increment stationary, if for any translation or orthogonal transformation ψ of L^2:

$$\{X_{\psi(f)} - X_{\psi(0)},\ f \in L^2\} \overset{(d)}{=} \{X_f - X_0,\ f \in L^2\}\ ; \tag{SI1}$$

- weakly L^2-increment stationary, if for any $f_1, \dots, f_n \in L^2$, $g_1, \dots, g_n$ and $h \in L^2$:

$$\left(X_{f_1+h} - X_{g_1+h}, \dots, X_{f_n+h} - X_{g_n+h}\right) \overset{(d)}{=} \left(X_{f_1} - X_{g_1}, \dots, X_{f_n} - X_{g_n}\right)\ . \tag{SI2}$$

The L^2-fBm satisfies all the above properties. (SS1) and (SI1) are direct analogues of the multiparameter properties presented in the introduction. They give a similar characterization:

Proposition 3.1. *Let X be an L^2-indexed Gaussian process. X is an L^2-fBm if and only if it is H–self-similar and increment-stationary in the strong sense (i.e. it satisfies (SS1) and (SI1)), up to normalization of its variance.*

The proof is similar to the characterization of the Lévy fractional Brownian motion ([17, p.393]).

Before stating our second characterization theorem, note that property (SI2) is equivalent to L^2-increment stationarity defined in Section 2 if X is a Gaussian process. We briefly discuss (SI2) and (SS2) for T-indexed processes which are compatible with set-indexing. So let X be such process, $\widehat{X}$ be its $L^2(T,m)$-indexed extension and A be the associated set-valued mapping. The definition of *measure increment stationarity* (presented in a weak form in (2.2)) is made precise here, in a form suited to non-Gaussian processes: for any $n \in \mathbb{N}$, any $t_0, t_1, \dots, t_n \in T$, and any $\tau_1, \dots, \tau_n \in T$,

$$\forall i,j,\ m\Big((A_{t_i} \triangle A_{t_0}) \cap (A_{t_j} \triangle A_{t_0})\Big) = m\left(A_{\tau_i} \cap A_{\tau_j}\right) \Rightarrow \left(X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_0}\right) \overset{(d)}{=} \left(X_{\tau_1}, \dots, X_{\tau_n}\right)\ .$$

If X is a process such that $\widehat{X}$ satisfies properties (SI2) and (SS2), then X has the measure increment stationarity. Note that the property (SS2) is a generalization of the self-similarity proposed in [7], initially introduced for set-indexed processes. It is well-suited for multiparameter processes, as in (SS2), for the special choice of mappings φ_a, $a \in \mathbb{R}_+^*$ defined by $\varphi_a\left(\mu_1 \mathbf{1}_{[0,t_1]} + \mu_2 \mathbf{1}_{[0,t_2]}\right) = \mu_1 \mathbf{1}_{[0,at_1]} + \mu_2 \mathbf{1}_{[0,at_2]}$, we can say that a multiparameter process is H–self-similar if $X_{at} = \widehat{X}(\mathbf{1}_{[0,at]}) \overset{(d)}{=} \varrho(\varphi_a)^H X_t$. Note that here, $\varrho(\varphi_a) = a^d$.

Proposition 3.2. *Let X be an L^2-indexed Gaussian process. X is an L^2-fractional Brownian motion of parameter $H \in (0,1)$ if and only if X satisfies (SI2) and (SS2) of order H, up to normalization of its variance.*

Proof. We first prove that X is centred. Let $f_0 \in L^2$ be a unit vector, and for any $f, g \in L^2$ we have:

$$\mathbb{E}\left(X_{f+g} - X_g\right) = \mathbb{E}\left(\varrho(\varphi_1)^H X_{f_0} - \varrho(\varphi_2)^H X_{f_0}\right)$$

where $\varphi_1, \varphi_2 \in \mathcal{G}$ are such that $f + g = \varphi_1(f_0)$ and $g = \varphi_2(f_0)$. We also have, by (SI2), that:

$$\mathbb{E}\left(X_{f+g} - X_g\right) = \mathbb{E}\left(X_f\right) = \varrho(\varphi_3)^H \mathbb{E}(X_{f_0})$$

where $\varphi_3 \in \mathcal{G}$ is such that $f = \varphi_3(f_0)$. We know by definition of ϱ that $\varrho(\varphi_1) = \|f+g\|^2$, $\varrho(\varphi_2) = \|g\|^2$ and $\varrho(\varphi_3) = \|f\|^2$. Hence, the equality between the last two equations implies that:

$$\left(\|f+g\|^{2H} - \|g\|^{2H}\right) \mathbb{E}\left(X_{f_0}\right) = \|f\|^{2H}\, \mathbb{E}\left(X_{f_0}\right) .$$

Since this is true for any $f, g \in L^2$, we must have $\mathbb{E}(X_{f_0}) = 0$, and so $\mathbb{E}(X_f) = 0,\ \forall f \in L^2$. To obtain the covariance, just notice that by using (SI2) and (SS2) in the same fashion:

$$\mathbb{E}\left((X_f - X_g)^2\right) = \|f-g\|^{2H}\, \frac{\mathbb{E}\left(X_{f_0}^2\right)}{\|f_0\|^{2H}} = \|f-g\|^{2H}\, \mathbb{E}\left(X_{f_0}^2\right) .$$

Therefore, $\mathbb{E}\left(X_f\, X_g\right) = \frac{1}{2}\mathbb{E}\left(X_{f_0}^2\right)\left(\|f\|^{2H} + \|g\|^{2H} - \|f-g\|^{2H}\right)$. Finally, stationarity implies that $\mathbb{E}\left(X_{f_0}^2\right) = \mathbb{E}\left(X_{g_0}^2\right)$ for any g_0 of norm 1. □

As a final remark, let us observe that we could not prove any such fractal characterization for the multiparameter fractional Brownian motion (defined in (2.1)). Despite that $\mathbb{B}^H$ is a process compatible with set-indexing (with $A_t = [0,t]$), that it is measure increment stationary and H–self-similar, we do not know if a centred Gaussian process X with these three properties is a multiparameter fractional Brownian motion. If one was willing to use Proposition 3.2 to prove this, the main difficulty would be to construct an L^2-indexed process extending the definition of X, which we leave as an open problem.

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References

[1] R. Balan, M. Jolis, and L. Quer-Sardanyons. SPDEs with fractional noise in space with index $1/4 < H < 1/2$. *Electron. J. Probab.*, pages 1–36, 2015.

[2] A. Basse-O'Connor, S.-E. Graversen, and J. Pedersen. Multiparameter processes with stationary increments: spectral representation and integration. *Electron. J. Probab.*, 17, 2012.

[3] L. Beznea, A. Cornea, and M. Röckner. Potential theory of infinite dimensional Lévy processes. *J. Funct. Anal.*, 261(10):2845–2876, 2011.

[4] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer, 2011.

[5] R. Dalang, D. Khoshnevisan, C. Mueller, D. Nualart, and Y. Xiao. *A minicourse on stochastic partial differential equations*. Number 1962. Springer, 2009.

[6] B. Driver. *Probability tools with examples*. Lectures notes available on B. Driver's page.

[7] E. Herbin and E. Merzbach. A set-indexed fractional Brownian motion. *J. Theoret. Probab.*, 19(2): 337–364, 2006.

[8] E. Herbin and E. Merzbach. The multiparameter fractional Brownian motion. In *Math Everywhere*, pages 93–101. Springer, 2007.

[9] K. Ito. Stationary random distributions. *Mem. College Sci. Univ. Kyoto Ser. A Math.*, 28(3):209–223, 1954.

[10] J. Kuelbs. A representation theorem for symmetric stable processes and stable measures on H. *Z. Wahrsch. Verw. Gebiete*, 26(4):259–271, 1973.

[11] M. A. Lifshits. *Gaussian random functions*, volume 322. Springer Science & Business Media, 1995.

[12] M. B. Marcus and J. Rosen. *Markov processes, Gaussian processes, and local times*. Cambridge University Press, Cambridge, 2006.

[13] D. Monrad and H. Rootzén. Small values of Gaussian processes and functional laws of the iterated logarithm. *Probab. Theory Related Fields*, 101(2):173–192, 1995.

[14] K. R. Parthasarathy. *Probability measures on metric spaces*. Academic Press, 1967.

[15] A. Richard. A fractional Brownian field indexed by L^2 and a varying Hurst parameter. *Stochastic Process. Appl.*, 125:1394–1425, 2015.

[16] A. Richard. Some singular sample path properties of a multiparameter fractional Brownian motion. *Submitted*, 2015. ArXiv:1410.4430.

[17] G. Samorodnitsky and M. S. Taqqu. *Stable non-Gaussian random processes*. 1994.

[18] R.L. Schilling, R. Song, and Z. Vondracek. *Bernstein functions: theory and applications*, volume 37. Walter de Gruyter, 2012.

[19] M. Talagrand. Hausdorff measure of trajectories of multiparameter fractional Brownian Motion. *Ann. Probab.*, 23(2):767–775, 1995.

[20] A. M. Yaglom. Some classes of random fields in n-dimensional space, related to stationary random processes. *Theory Probab. Appl.*, 2(3):273–320, January 1957.

[21] J. A. Yan. Generalizations of Gross' and Minlos' theorems. In *Séminaire de probabilités XXIII*, pages 395–404, 1989.