-Continuous Kleene ω-Algebras

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Abstract. We define and study basic properties of *-continuous Kleene ω-algebras that involve a *-continuous Kleene algebra with a *-continuous action on a semimodule and an infinite product operation that is also *-continuous. We show that *-continuous Kleene ω-algebras give rise to iteration semiring-semimodule pairs, and that for Büchi automata over *-continuous Kleene ω-algebras, one can compute the associated infinitary power series.

1 Introduction

A continuous (or complete) Kleene algebra is a Kleene algebra in which all suprema exist and are preserved by products. These have nice algebraic properties, but not all Kleene algebras are continuous, for example the semiring of regular languages over some alphabet. Hence a theory of *-continuous Kleene algebras has been developed to cover this and other interesting cases.

For infinite behaviors, complete semiring-semimodule pairs involving an infinite product operation have been developed. Motivated by some examples of structures which are not complete in this sense, cf. the energy functions of [5], we generalize here the notion of *-continuous Kleene algebra to one of *-continuous Kleene ω-algebra. These are idempotent semiring-semimodule pairs which are not necessarily complete, but have enough suprema in order to develop a fixed-point theory and solve weighted Büchi automata (*i.e.,* to compute infinitary power series).

We will define both a finitary and a non-finitary version of *-continuous Kleene ω-algebras. We then establish several properties of *-continuous Kleene ω-algebras, including the existence of the suprema of certain subsets related to regular ω-languages. Then we will use these results in our characterization of the free finitary *-continuous Kleene ω-algebras. We also show that each *-continuous Kleene ω-algebra gives rise to an iteration semiring-semimodule pair and that Büchi automata over *-continuous Kleene ω-algebras can be solved algebraically. For proofs of the results in this paper, and also for further motivation and results related to energy functions, we refer to [4].

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A Kleene algebra [12] is an idempotent semiring $S = (S, \vee, \cdot, \bot, 1)$ equipped with a star operation $^* : S \to S$ such that for all $x, y \in S$, yx^* is the least solution of the fixed point equation $z = zx \vee y$ and x^*y is the least solution of the fixed point equation $z = xz \vee y$ with respect to the natural order.

Examples of Kleene algebras include the language semiring $P(A^*)$ over an alphabet A, whose elements are the subsets of the set A^* of all finite words over A, and whose operations are set union and concatenation, with the languages $\emptyset$ and $\{\varepsilon\}$ serving as $\bot$ and 1. Here, ε denotes the empty word. The star operation is the usual Kleene star: $X^* = \bigcup_{n \geq 0} X^n = \{u_1 \ldots u_n : u_1, \ldots, u_n \in X,\ n \geq 0\}$.

Another example is the Kleene algebra $P(A \times A)$ of binary relations over any set A, whose operations are union, relational composition (written in diagrammatic order), and where the empty relation $\emptyset$ and the identity relation id serve as the constants $\bot$ and 1. The star operation is the formation of the reflexive-transitive closure, so that $R^* = \bigcup_{n \geq 0} R^n$ for all $R \in P(A \times A)$.

The above examples are in fact *continuous Kleene algebras*, i.e., idempotent semirings S such that equipped with the natural order, they are all complete lattices (hence all suprema exist), and the product operation preserves arbitrary suprema in either argument:

$$y(\bigvee X) = \bigvee yX \quad \text{and} \quad (\bigvee X)y = \bigvee Xy$$

for all $X \subseteq S$ and $y \in S$. The star operation is given by $x^* = \bigvee_{n \geq 0} x^n$, so that x^* is the supremum of the set $\{x^n : n \geq 0\}$ of all powers of x. It is well-known that the language semirings $P(A^*)$ may be identified as the *free* continuous Kleene algebras (in a suitable category of continuous Kleene algebras).

A larger class of models is given by the *-*continuous Kleene algebras* [12]. By the definition of a *-continuous Kleene algebra $S = (S, \vee, \cdot, \bot, 1)$, only suprema of sets of the form $\{x^n : n \geq 0\}$ need to exist, where x is any element of S, and x^* is given by this supremum. Moreover, product preserves such suprema in both of their arguments:

$$y(\bigvee_{n \geq 0} x^n) = \bigvee_{n \geq 0} yx^n \quad \text{and} \quad (\bigvee_{n \geq 0} x^n)y = \bigvee_{n \geq 0} x^n y.$$

For any alphabet A, the collection $R(A^*)$ of all regular languages over A is an example of a *-continuous Kleene algebra which is not a continuous Kleene algebra. The Kleene algebra $R(A^*)$ may be identified as the free *-continuous Kleene algebra on A. It is also the free Kleene algebra on A, cf. [11]. There are several other characterizations of $R(A^*)$, the most general of which identifies $R(A^*)$ as the free iteration semiring on A satisfying the identity $1^* = 1$, cf. [1,13].

For non-idempotent extensions of the notions of continuous Kleene algebras, *-continuous Kleene algebras and Kleene algebras, we refer to [6, 7].

When A is an alphabet, let A^ω denote the set of all ω-words (infinite sequences) over A. An ω-*language* over A is a subset of A^ω. It is natural to consider the set $P(A^\omega)$ of all languages of ω-words over A, equipped with the operation of set union as $\vee$ and the empty language $\emptyset$ as $\bot$, and the left action of $P(A^*)$

on $P(A^\omega)$ defined by $XV = \{xv : x \in X,\ v \in V\}$ for all $X \subseteq A^*$ and $V \subseteq A^\omega$. Then that $(P(A^\omega), \vee, \perp)$ is a $P(A^*)$-semimodule and thus $(P(A^*), P(A^\omega))$ is a semiring-semimodule pair. We may also equip $(P(A^*), P(A^\omega))$ with an infinite product operation mapping an ω-sequence $(X_0, X_1, \ldots)$ over $P(A^*)$ to the ω-language $\prod_{n\geq 0} X_n = \{x_0 x_1 \ldots \in A^\omega : x_n \in X_n\}$. (Thus, an infinite number of the x_n must be different from ε. Note that $1^\omega = \perp$ holds.) The semiring-semimodule pair so obtained is an example of a continuous Kleene ω-algebra.

More generally, we call a semiring-semimodule pair (S, V) a *continuous Kleene ω-algebra* if S is a continuous Kleene algebra (hence S and V are idempotent), V is a complete lattice with the natural order, and the action preserves all suprema in either argument. Moreover, there is an infinite product operation which is compatible with the action and associative in the sense that the following hold:

1. For all $x_0, x_1, \ldots \in S$, $\prod_{n\geq 0} x_n = x_0 \prod_{n\geq 0} x_{n+1}$.
2. Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \leq n_1 \cdots$ be a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\prod_{n\geq 0} x_n = \prod_{k\geq 0} y_k$.

Moreover, the infinite product operation preserves all suprema:

3. $\prod_{n\geq 0}(\bigvee X_n) = \bigvee\{\prod_{n\geq 0} x_n : x_n \in X_n,\ n \geq 0\}$, for all $X_0, X_1, \ldots \subseteq S$.

The above notion of continuous Kleene ω-algebra may be seen as a special case of the not necessarily idempotent complete semiring-semimodule pairs of [9].

Our aim in this paper is to provide an extension of the notion of continuous Kleene ω-algebras to **-continuous Kleene ω-algebras*, which are semiring-semimodule pairs (S, V) consisting of a *-continuous Kleene algebra S acting on a necessarily idempotent semimodule V, such that the action preserves certain suprema in its first argument, and which are equipped with an infinite product operation satisfying the above compatibility and associativity conditions and some weaker forms of the last axiom.

2 Free continuous Kleene ω-algebras

We have defined continuous Kleene ω-algebras in the introduction as idempotent semiring-semimodule pairs (S, V) such that $S = (S, \vee, \cdot, \perp, 1)$ is a continuous Kleene algebra and $V = (V, \vee, \perp)$ is a continuous S-semimodule. Thus, equipped with the natural order $\leq$ given by $x \leq y$ iff $x \vee y = y$, S and V are complete lattices and the product and the action preserve all suprema in either argument. Moreover, there is an infinite product operation, satisfying the compatibility and associativity conditions, which preserves all suprema.

In this section, we offer descriptions of the free continuous Kleene ω-algebras and the free continuous Kleene ω-algebras satisfying the identity $1^\omega = \perp$.

A homomorphism between continuous Kleene algebras preserves all operations. A homomorphism is continuous if it preserves all suprema. We recall the following folklore result.

Theorem 1. *For each set A, the language semiring $(P(A^*), \vee, \cdot, \perp, 1)$ is the free continuous Kleene algebra on A.*

Equivalently, if S is a continuous Kleene algebra and $h : A \to S$ is any function, then there is a unique continuous homomorphism $h^\sharp : P(A^*) \to S$ extending h.

In view of Theorem 1, it is not surprising that the free continuous Kleene ω-algebras can be described using languages of finite and ω-words. Suppose that A is a set. Let A^ω denote the set of all ω-words over A and $A^\infty = A^* \cup A^\omega$. Let $P(A^*)$ denote the language semiring over A and $P(A^\infty)$ the semimodule of all subsets of A^∞ equipped with the action of $P(A^*)$ defined by $XY = \{xy : x \in X, y \in Y\}$ for all $X \subseteq A^*$ and $Y \subseteq A^\infty$. We also define an infinite product by $\prod_{n\geq 0} X_n = \{u_0u_1 \ldots : u_n \in X_n\}$.

Homomorphisms between continuous Kleene ω-algebras (S, V) and (S', V') consist of two functions $h_S : S \to S'$, $h_V : V \to V'$ which preserve all operations. A homomorphism (h_S, h_V) is continuous if h_S and h_V preserve all suprema.

Theorem 2. *$(P(A^*), P(A^\infty))$ is the free continuous Kleene ω-algebra on A.*

Consider now $(P(A^*), P(A^\omega))$ with infinite product defined, as a restriction of the above infinite product, by $\prod_{n\geq 0} X_n = \{u_0u_1 \ldots \in A^\omega : u_n \in X_n,\ n \geq 0\}$. It is also a continuous Kleene ω-algebra. Moreover, it satisfies $1^\omega = \bot$.

Theorem 3. *For each set A, $(P(A^*), P(A^\omega))$ is the free continuous Kleene ω-algebra satisfying $1^\omega = \bot$ on A.*

3 *-Continuous Kleene ω-algebras

In this section, we define **-continuous Kleene ω-algebras* and *finitary *-continuous Kleene ω-algebras* as an extension of the *-continuous Kleene algebras of [11]. We establish several basic properties of these structures, including the existence of the suprema of certain subsets corresponding to regular ω-languages.

We define a *-continuous Kleene ω-algebra (S, V) as a *-continuous Kleene algebra $(S, \vee, \cdot, \bot, 1, ^*)$ acting on a (necessarily idempotent) semimodule $V = (V, \vee, \bot)$ subject to the usual laws of unitary action as well as the following axiom

- Ax0: For all $x, y \in S$ and $v \in V$, $xy^*v = \bigvee_{n\geq 0} xy^n v$.

Moreover, there is an infinite product operation mapping an ω-word $x_0x_1 \ldots$ over S to an element $\prod_{n\geq 0} x_n$ of V. Thus, infinite product is a function $S^\omega \to V$, where S^ω denotes the set of all ω-words over S.

The infinite product is subject to the following axioms relating it to the other operations of Kleene ω-algebras and operations on ω-words. The first two axioms are the same as for continuous Kleene ω-algebras. The last two are weaker forms of the complete preservation of suprema of continuous Kleene ω-algebras.

Ax1: For all $x_0, x_1, \ldots \in S$, $\prod_{n\geq 0} x_n = x_0 \prod_{n\geq 0} x_{n+1}$.

Ax2: Let $x_0, x_1, \ldots \in S$ and $0 = n_0 \leq n_1 \cdots$ be a sequence which increases without a bound. Let $y_k = x_{n_k} \cdots x_{n_{k+1}-1}$ for all $k \geq 0$. Then $\prod_{n\geq 0} x_n = \prod_{k\geq 0} y_k$.

Ax3: For all $x_0, x_1, \dots$ and y, z in S, $\prod_{n\geq 0}(x_n(y \vee z)) = \bigvee_{x'_n \in \{y,z\}} \prod_{n\geq 0} x_n x'_n$.
Ax4: For all $x, y_0, y_1, \dots \in S$, $\prod_{n\geq 0} x^* y_n = \bigvee_{k_n \geq 0} \prod_{n \geq 0} x^{k_n} y_n$.

It is clear that every continuous Kleene ω-algebra is *-continuous.

Some of our results will also hold for weaker structures. We define a *finitary* *-continuous Kleene ω-algebra as a structure (S, V) as above, equipped with a star operation and an infinite product $\prod_{n\geq 0} x_n$ restricted to *finitary ω-words* over S, i.e., to sequences $x_0, x_1, \dots$ such that there is a finite subset F of S such that each x_n is a finite product of elements of F. (Note that F is not fixed and may depend on the sequence $x_0, x_1, \dots$) It is required that the axioms hold whenever the involved ω-words are finitary.

Finally, a *generalized *-continuous Kleene algebra* (S, V) is defined as a *-continuous Kleene ω-algebra, but without the infinite product (and without Ax1–Ax4). However, it is assumed that Ax0 holds.

The above axioms have a number of consequences. For example, if $x_0, x_1, \dots \in S$ and $x_i = \bot$ for some i, then $\prod_{n\geq 0} x_n = \bot$. Indeed, if $x_i = \bot$, then $\prod_{n\geq 0} x_n = x_0 \cdots x_i \prod_{n \geq i+1} x_n = \bot \prod_{n\geq i+1} x_n = \bot$. By Ax1 and Ax2, each *-continuous Kleene ω-algebra is an ω-semigroup.

Suppose that (S, V) is a *-continuous Kleene ω-algebra. For each word $w \in S^*$ there is a corresponding element $\overline{w}$ of S which is the product of the letters of w in the semiring S. Similarly, when $w \in S^* V$, there is an element $\overline{w}$ of V corresponding to w, and when $X \subseteq S^*$ or $X \subseteq S^* V$, then we can associate with X the set $\overline{X} = \{\overline{w} : w \in X\}$, which is a subset of S or V. Below we will denote $\overline{w}$ and $\overline{X}$ by just w and X, respectively, The following two lemmas are well-known (and follow from the fact that the semirings of regular languages are the free *-continuous Kleene algebras [11] and the free Kleene algebras [12]).

Lemma 4. *Suppose that S is a *-continuous Kleene algebra. If $R \subseteq S^*$ is regular, then $\bigvee R$ exists. Moreover, for all $x, y \in S$, $x(\bigvee R)y = \bigvee xRy$.*

Lemma 5. *Let S be a *-continuous Kleene algebra. Suppose that R, R_1 and R_2 are regular subsets of S^*. Then*

$$\bigvee(R_1 \cup R_2) = \bigvee R_1 \vee \bigvee R_2$$
$$\bigvee(R_1 R_2) = (\bigvee R_1)(\bigvee R_2)$$
$$\bigvee(R^*) = (\bigvee R)^*.$$

In a similar way, we can prove:

Lemma 6. *Let (S, V) be a generalized *-continuous Kleene algebra. If $R \subseteq S^*$ is regular, $x \in S$ and $v \in V$, then $x(\bigvee R)v = \bigvee xRv$.*

Lemma 7. *Let (S, V) be a *-continuous Kleene ω-algebra. Suppose that the languages $R_0, R_1, \dots \subseteq S^*$ are regular and that the set $\{R_0, R_1, \dots\}$ is finite. Moreover, let $x_0, x_1, \dots \in S$. Then*

$$\prod_{n\geq 0} x_n (\bigvee R_n) = \bigvee \prod_{n \geq 0} x_n R_n.$$

Lemma 8. *Let (S, V) be a finitary *-continuous Kleene ω-algebra. Suppose that the languages $R_0, R_1, \ldots \subseteq S^*$ are regular and that the set $\{R_0, R_1, \ldots\}$ is finite. Moreover, let $x_0, x_1, \ldots$ be a finitary sequence of elements of S. Then*

$$\prod_{n\geq 0} x_n(\bigvee R_n) = \bigvee \prod_{n\geq 0} x_n R_n.$$

Note that each sequence $x_0, y_0, x_1, y_1, \ldots$ with $y_n \in R_n$ is finitary.

Corollary 9. *Let (S, V) be a finitary *-continuous Kleene ω-algebra. Suppose that $R_0, R_1, \ldots \subseteq S^*$ are regular and that the set $\{R_0, R_1, \ldots\}$ is finite. Then $\bigvee \prod_{n\geq 0} R_n$ exists and is equal to $\prod_{n\geq 0} \bigvee R_n$.*

When $v = x_0 x_1 \ldots \in S^\omega$ is an ω-word over S, it naturally determines the element $\prod_{n\geq 0} x_n$ of V. Thus, any subset X of S^ω determines a subset of V. Using this convention, Lemma 7 may be rephrased as follows. For any *-continuous Kleene ω-algebra (S, V), $x_0, x_1, \ldots \in S$ and regular sets $R_0, R_1, \ldots \subseteq S^*$ for which the set $\{R_0, R_1, \ldots\}$ is finite, it holds that $\prod_{n\geq 0} x_n(\bigvee R_n) = \bigvee X$ where $X \subseteq S^\omega$ is the set of all ω-words $x_0 y_0 x_1 y_1 \ldots$ with $y_i \in R_i$ for all $i \geq 0$, i.e., $X = x_0 R_0 x_1 R_1 \ldots$ Similarly, Corollary 9 asserts that if a subset of V corresponds to an infinite product over a finite collection of ordinary regular languages in S^*, then the supremum of this set exists.

In any (finitary or non-finitary) *-continuous Kleene ω-algebra (S, V), we define an *ω-power operation* $S \to V$ by $x^\omega = \prod_{n\geq 0} x$ for all $x \in S$. From the axioms we immediately have:

Corollary 10. *Suppose that (S, V) is a *-continuous Kleene ω-algebra or a finitary *-continuous Kleene ω-algebra. Then the following hold for all $x, y \in S$:*

$$\begin{aligned} x^\omega &= x x^\omega \\ (xy)^\omega &= x(yx)^\omega \\ x^\omega &= (x^n)^\omega, \quad n \geq 2. \end{aligned}$$

Thus, each *-continuous Kleene ω-algebra gives rise to a Wilke algebra [14].

Lemma 11. *Let (S, V) be a (finitary or non-finitary) *-continuous Kleene ω-algebra. Suppose that $R \subseteq S^\omega$ is ω-regular. Then $\bigvee R$ exists in V.*

Lemma 12. *Let (S, V) be a (finitary or non-finitary) *-continuous Kleene ω-algebra. For all ω-regular sets $R_1, R_2 \subseteq S^\omega$ and regular sets $R \subseteq S^*$ it holds that*

$$\begin{aligned} \bigvee (R_1 \cup R_2) &= \bigvee R_1 \vee \bigvee R_2 \\ \bigvee (R R_1) &= (\bigvee R)(\bigvee R_1). \end{aligned}$$

And if R does not contain the empty word, then

$$\bigvee R^\omega = (\bigvee R)^\omega.$$

4 Free finitary *-continuous Kleene ω-algebras

Recall that for a set A, $R(A^*)$ denotes the collection of all regular languages in A^*. It is well-known that $R(A^*)$, equipped with the usual operations, is a *-continuous Kleene algebra on A. Actually, $R(A^*)$ is characterized up to isomorphism by the following universal property.

Call a function $f : S \to S'$ between *-continuous Kleene algebras a *-continuous homomorphism if it preserves all operations including star, so that it preserves the suprema of subsets of S of the form $\{x^n : n \geq 0\}$, where $x \in S$.

Theorem 13 ([12]). *For each set A, $R(A^*)$ is the free *-continuous Kleene algebra on A.*

Thus, if S is any *-continuous Kleene algebra and $h : A \to S$ is any mapping from any set A into S, then h has a unique extension to a *-continuous Kleene algebra homomorphism $h^\sharp : R(A^*) \to S$.

Now let $R'(A^\infty)$ denote the collection of all subsets of A^∞ which are finite unions of finitary infinite products of regular languages, that is, finite unions of sets of the form $\prod_{n\geq 0} R_n$, where each $R_n \subseteq A^*$ is regular, and the set $\{R_0, R_1, \ldots\}$ is finite. Note that $R'(A^\infty)$ contains the empty set and is closed under finite unions. Moreover, when $Y \in R'(A^\infty)$ and $u = a_0a_1 \ldots \in Y \cap A^\omega$, then the alphabet of u is finite, i.e., the set $\{a_n : n \geq 0\}$ is finite. Also, $R'(A^\infty)$ is closed under the action of $R(A^*)$ inherited from $(P(A^*), P(A^\infty))$. The infinite product of a sequence of regular languages in $R(A^*)$ is not necessarily contained in $R'(A^\infty)$, but by definition $R'(A^\infty)$ contains all infinite products of finitary sequences over $R(A^*)$.

Example 14. Let $A = \{a, b\}$ and consider the set $X = \{aba^2b \ldots a^nb \ldots\} \in P(A^\infty)$ containing a single ω-word. X can be written as an infinite product of subsets of A^*, but it cannot be written as an infinite product $R_0R_1 \ldots$ of regular languages in A^* such that the set $\{R_0, R_1, \ldots\}$ is finite. Hence $X \notin R'(A^\infty)$.

Theorem 15. *For each set A, $(R(A^*), R'(A^\infty))$ is the free finitary *-continuous Kleene ω-algebra on A.*

Consider now $(R(A^*), R'(A^\omega))$ equipped with the infinite product operation $\prod_{n\geq 0} X_n = \{u_0u_1 \in A^\omega : u_n \in X_n,\ n \geq 0\}$, defined on finitary sequences $X_0, X_1, \ldots$ of languages in $R(A^*)$.

Theorem 16. *For each set A, $(R(A^*), R'(A^\omega))$ is the free finitary *-continuous Kleene ω-algebra satisfying $1^\omega = \bot$ on A.*

5 *-continuous Kleene ω-algebras and iteration semiring-semimodule pairs

In this section, we will show that every (finitary or non-finitary) *-continuous Kleene ω-algebra is an iteration semiring-semimodule pair.

Some definitions are in order. Suppose that $S = (S, +, \cdot, 0, 1)$ is a semiring. Following [1], we call S a *Conway semiring* if S is equipped with a star operation ${}^* : S \to S$ satisfying, for all $x, y \in S$,

$$\begin{aligned}(x+y)^* &= (x^*y)^*x^* \\ (xy)^* &= 1 + x(yx)^*y.\end{aligned}$$

It is known [1] that if S is a Conway semiring, then for each $n \geq 1$, so is the semiring $S^{n\times n}$ of all $n \times n$-matrices over S with the usual sum and product operations and the star operation defined by induction on n so that if $n > 1$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension $< n$, then

$$M^* = \begin{pmatrix} (a+bd^*c)^* & (a+bd^*c)^*bd^* \\ (d+ca^*b)^*ca^* & (d+ca^*b)^* \end{pmatrix}.$$

The above definition does not depend on how M is split into submatrices.

Suppose that S is a Conway semiring and $G = \{g_1, \dots, g_n\}$ is a finite group of order n. For each $x_{g_1}, \dots, x_{g_n} \in S$, consider the $n \times n$ matrix $M_G = M_G(x_{g_1}, \dots, x_{g_n})$ whose ith row is $(x_{g_i^{-1}g_1}, \dots, x_{g_i^{-1}g_n})$, for $i = 1, \dots, n$, so that each row (and column) is a permutation of the first row. We say that the group identity [2] associated with G holds in S if for each $x_{g_1}, \dots, x_{g_n}$, the first (and then any) row sum of M_G^* is $(x_{g_1} + \dots + x_{g_n})^*$. Finally, we call S an *iteration semiring* [1,3] if the group identities hold in S for all finite groups of order n.

Classes of examples of (idempotent) iteration semirings are given by the continuous and the *-continuous Kleene algebras defined in the introduction. As mentioned above, the language semirings $P(A^*)$ and the semirings $P(A \times A)$ of binary relations are continuous and hence also *-continuous Kleene algebras, and the semirings $R(A^*)$ of regular languages are *-continuous Kleene algebras.

When S is a *-continuous Kleene algebra and n is a nonnegative integer, then the matrix semiring $S^{n\times n}$ is also a *-continuous Kleene algebra and hence an iteration semiring, cf. [11]. The star operation is defined by

$$M^*_{i,j} = \bigvee_{m\geq 0,\ 1\leq k_1,\dots,k_m\leq n} M_{i,k_1}M_{k_1,k_2}\cdots M_{k_m,j},$$

for all $M \in S^{n\times n}$ and $1 \leq i, j \leq n$. It is not trivial to prove that the above supremum exists. The fact that M^* is well-defined can be established by induction on n together with the well-known matrix star formula mentioned above.

A semiring-semimodule pair (S, V) is a *Conway semiring-semimodule pair* if it is equipped with a star operation ${}^* : S \to S$ and an omega operation ${}^\omega : S \to V$ such that S is a Conway semiring acting on the semimodule $V = (V, +, 0)$ and the following hold for all $x, y \in S$:

$$\begin{aligned}(x+y)^\omega &= (x^*y)^*x^\omega + (x^*y)^\omega \\ (xy)^\omega &= x(yx)^\omega.\end{aligned}$$

It is known [1] that when (S,V) is a Conway semiring-semimodule pair, then so is $(S^{n\times n}, V^n)$ for each n, where V^n denotes the $S^{n\times n}$-semimodule of all n-dimensional (column) vectors over V with the action of $S^{n\times n}$ defined similarly to matrix-vector product, and where the omega operation is defined by induction so that when $n > 1$ and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension $< n$, then

$$M^\omega = \begin{pmatrix} (a+bd^*c)^\omega \ + \ (a+bd^*c)^*bd^\omega \\ (d+ca^*b)^\omega \ + \ (d+ca^*b)^*ca^\omega \end{pmatrix}.$$

We also define *iteration semiring-semimodule pairs* [1, 9] as those Conway semiring-semimodule pairs such that S is an iteration semiring and the omega operation satisfies the following condition: let $M_G = M_G(x_{g_1}, \dots, x_{g_n})$ like above, with $x_{g_1}, \dots, x_{g_n} \in S$ for a finite group $G = \{g_1, \dots, g_n\}$ of order n, then the first (and hence any) entry of M_G^ω is equal to $(x_{g_1} + \cdots + x_{g_n})^\omega$.

Examples of (idempotent) iteration semiring-semimodule pairs include the semiring-semimodule pairs $(P(A^*), P(A^\omega))$ of languages and ω-languages over an alphabet A, mentioned in the introduction. The omega operation is defined by $X^\omega = \prod_{n\ge 0} X$. More generally, it is known that every continuous Kleene ω-algebra gives rise to an iteration semiring-semimodule pair. The omega operation is defined as for languages: $x^\omega = \prod_{n\ge 0} x_n$ with $x_n = x$ for all $n \ge 0$.

Other not necessarily idempotent examples include the *complete* and the *(symmetric) bi-inductive semiring-semimodule pairs* of [8, 9].

Suppose now that (S,V) is a *-continuous Kleene ω-algebra. Then for each $n \ge 1$, $(S^{n\times n}, V^n)$ is a semiring-semimodule pair. The action of $S^{n\times n}$ on V^n is defined similarly to matrix-vector product (viewing the elements of V^n as column vectors). It is easy to see that $(S^{n\times n}, V^n)$ is a generalized *-continuous Kleene algebra for each $n \ge 1$.

Suppose that $n \ge 2$. We would like to define an infinite product operation $(S^{n\times n})^\omega \to V^n$ on matrices in $S^{n\times n}$ by

$$(\prod_{m\ge 0} M_m)_i = \bigvee_{1\le i_1, i_2, \ldots \le n} (M_0)_{i,i_1}(M_1)_{i_1,i_2}\cdots$$

for all $1 \le i \le n$. However, unlike in the case of complete semiring-semimodule pairs [9], the supremum on the right-hand side may not exist. Nevertheless it is possible to define an omega operation $S^{n\times n} \to V^n$ and to turn $(S^{n\times n}, V^n)$ into an iteration semiring-semimodule pair.

Lemma 17. *Let (S,V) be a (finitary or non-finitary) *-continuous Kleene ω-algebra. Suppose that $M \in S^{n\times n}$, where $n \ge 2$. Then for every $1 \le i \le n$,*

$$(\prod_{m\ge 0} M)_i = \bigvee_{1\le i_1, i_2, \ldots \le n} M_{i,i_1}M_{i_1,i_2}\cdots$$

exists, so that we define M^ω *by the above equality. Moreover, when $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where a and d are square matrices of dimension $< n$, then*

$$M^\omega = \begin{pmatrix} (a \vee bd^*c)^\omega \ \vee \ (a \vee bd^*c)^*bd^\omega \\ (d \vee ca^*b)^\omega \ \vee \ (d \vee ca^*b)^*ca^\omega \end{pmatrix}. \tag{1}$$

Theorem 18. *Every (finitary or non-finitary) *-continuous Kleene ω-algebra is an iteration semiring-semimodule pair.*

Relation to bi-inductive semiring-semimodule pairs. Recall that when P is a partially ordered set and f is a function $P \to P$, then a *pre-fixed point* of f is an element x of P with $f(x) \leq x$. Similarly, $x \in P$ is a *post-fixed point* of f if $x \leq f(x)$. Suppose that f is monotone and has x as its least pre-fixed point. Then x is a *fixed point*, i.e., $f(x) = x$, and thus the least fixed point of f. Similarly, when f is monotone, then the greatest post-fixed point of f, whenever it exists, is the greatest fixed point of f.

When S is a *-continuous Kleene algebra, then S is a Kleene algebra [11]. Thus, for all $x, y \in S$, x^*y is the least pre-fixed point (and thus the least fixed point) of the function $S \to S$ defined by $z \mapsto xz \vee y$ for all $z \in S$. Moreover, yx^* is the least pre-fixed point and the least fixed point of the function $S \to S$ defined by $z \mapsto zx \vee y$, for all $z \in S$. Similarly, when (S, V) is a generalized *-continuous Kleene algebra, then for all $x \in S$ and $v \in V$, x^*v is the least pre-fixed point and the least fixed point of the function $V \to V$ defined by $z \mapsto xz \vee v$.

A *bi-inductive semiring-semimodule pair* is defined as a semiring-semimodule pair (S, V) for which both S and V are partially ordered by the natural order relation $\leq$ such that the semiring and semimodule operations and the action are monotone, and which is equipped with a star operation $^* : S \to S$ and an omega operation $^\omega : S \to V$ such that the following hold for all $x, y \in S$ and $v \in V$:

- x^*y is the least pre-fixed point of the function $S \to S$ defined by $z \mapsto xz + y$,
- x^*v is the least pre-fixed point of the function $V \to V$ defined by $z \mapsto xz + v$,
- $x^\omega + x^*v$ is the greatest post-fixed point of the function $V \to V$ defined by $z \mapsto xz + v$.

A bi-inductive semiring-semimodule pair is said to be symmetric if for all $x, y \in S$, yx^* is the least pre-fixed point of the functions $S \to S$ defined by $z \mapsto zx + y$. It is known that every bi-inductive semiring-semimodule pair is an iteration semiring-semimodule pair, see [9]. By the above remarks we have:

Proposition 19. *Suppose that (S, V) is a finitary *-continuous Kleene ω-algebra. When for all $x \in S$ and $v \in V$, $x^\omega \vee x^*v$ is the greatest post-fixed point of the function $V \to V$ defined by $z \mapsto xz \vee v$, then (S, V) is a symmetric bi-inductive semiring-semimodule pair.*

6 Büchi automata in *-continuous Kleene ω-algebras

A generic definition of Büchi automata in Conway semiring-semimodule pairs was given in [1, 8]. In this section, we recall this general definition and apply it to *-continuous Kleene ω-algebras. We give two different definitions of the behavior of a Büchi automaton, an algebraic and a combinatorial, and show that these two definitions are equivalent.

Suppose that (S, V) is a Conway semiring-semimodule pair, S_0 is a subsemiring of S closed under star, and A is a subset of S. We write $S_0\langle A\rangle$ for the set of all finite sums $s_1a_1 + \cdots + s_m a_m$ with $s_i \in S_0$ and $a_i \in A$, for each $i = 1, \ldots, m$.

We define a (weighted) Büchi automaton over (S_0, A) of dimension $n \geq 1$ in (S, V) as a system $\mathbf{A} = (\alpha, M, k)$ where $\alpha \in S_0^{1\times n}$ is the initial vector, $M \in S_0\langle A\rangle^{n\times n}$ is the transition matrix, and k is an integer $0 \leq k \leq n$. In order to define the behavior $|\mathbf{A}|$ of $\mathbf{A}$, let us split M into 4 parts as above, $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, with $a \in S_0\langle A\rangle^{k\times k}$ the top-left k-by-k submatrix. Then we define

$$|\mathbf{A}| = \alpha \begin{pmatrix} (a + bd^*c)^\omega \\ d^*c(a + bd^*c)^\omega \end{pmatrix}.$$

We give another more combinatorial definition. Let (S, V) be a *-continuous Kleene ω-algebra. A Büchi automaton $\mathbf{A} = (\alpha, M, k)$ over (S_0, A) of dimension n may be represented as a transition system whose set of states is $\{1, \ldots, n\}$. For any pair of states i, j, the transitions from i to j are determined by the entry $M_{i,j}$ of the transition matrix. Let $M_{i,j} = s_1a_1 \vee \cdots \vee s_m a_m$, say, then there are m transitions from i to j, respectively labeled $s_1a_1, \ldots, s_n a_n$. An accepting run of the Büchi automaton from state i is an infinite path starting in state i which infinitely often visits one of the first k states, and the weight of such a run is the infinite product of the path labels. The behavior of the automaton in state i is the supremum of the weights of all accepting runs starting in state i. Finally, the behavior of the automaton is $\alpha_1 w_1 \vee \cdots \vee \alpha_n w_n$, where for each i, α_i is the ith component of α and w_i is the behavior in state i. Let $|\mathbf{A}|'$ denote the behavior of $\mathbf{A}$ according to this second definition.

Theorem 20. *For every Büchi automaton $\mathbf{A}$ over (S_0, A) in a (finitary or non-finitary) *-continuous Kleene ω-algebra, it holds that $|\mathbf{A}| = |\mathbf{A}|'$.*

For completeness we also mention a Kleene theorem for the Büchi automata introduced above, which is a direct consequence of the Kleene theorem for Conway semiring-semimodule pairs, cf. [8, 10].

Theorem 21. *An element of V is the behavior of a Büchi automaton over (S_0, A) iff it is regular (or rational) over (S_0, A), i.e., when it can be generated from the elements of $S_0 \cup A$ by the semiring and semimodule operations, the action, and the star and omega operations.*

It is a routine matter to show that an element of V is rational over (S_0, A) iff it can be written as $\bigvee_{i=1}^n x_i y_i^\omega$, where each x_i and y_i can be generated from $S_0 \cup A$ by $\vee$, $\cdot$ and *.

7 Conclusion

We have introduced continuous and (finitary and non-finitary) *-continuous Kleene ω-algebras and exposed some of their basic properties. Continuous Kleene

ω-algebras are idempotent complete semiring-semimodule pairs, and conceptually, *-continuous Kleene ω-algebras are a generalization of continuous Kleene ω-algebras in much the same way as *-continuous Kleene algebras are of continuous Kleene algebras: In *-continuous Kleene algebras, suprema of finite sets and of sets of powers are required to exist and to be preserved by the product; in *-continuous Kleene ω-algebras these suprema are also required to be preserved by the infinite product.

It is known that in a Kleene algebra, *-continuity is precisely what is required to be able to compute the reachability value of a weighted automaton (or its power series) using the matrix star operation. Similarly, we have shown that the Büchi values of automata over *-continuous ω-algebras can be computed using the matrix omega operation.

We have seen that the sets of finite and infinite languages over an alphabet are the free continuous Kleene ω-algebras, and that the free finitary *-continuous Kleene ω-algebras are given by the sets of regular languages and of finite unions of finitary infinite products of regular languages. A characterization of the free (non-finitary) *-continuous Kleene ω-algebras (and whether they even exist) is left open.

We have seen that every *-continuous Kleene ω-algebra is an iteration semiring-semimodule pair, which permits to compute the behavior of Büchi automata with weights in a *-continuous Kleene ω-algebra using ω-powers of matrices.

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Appendix: Extra Lemmas and Proofs

Proof (of Theorem 2). Suppose that (S, V) is any continuous Kleene ω-algebra an let $h : A \to S$ be a mapping. We want to show that there is a unique extension of h to a continuous homomorphism $(h^\sharp_S, h^\sharp_V)$ from $(P^\omega(A), P(A^\infty))$ to (S, V).

For each $u = a_0 \ldots a_{n-1}$ in A^*, define $h_S(u) = h(a_0) \cdots h(a_{n-1})$ and $h_V(u) = h(a_0) \cdots h(a_{n-1})1^\omega = \prod_{k \geq 0} b_k$, where $b_k = a_k$ for all $k < n$ and $b_k = 1$ for all $k \geq n$. When $u = a_0 a_1 \ldots \in A^\omega$, define $h_V(u) = \prod_{k \geq 0} h(a_k)$. Note that we have $h_S(uv) = h_S(u)h_S(v)$ for all $u, v \in A^*$ and $h_S(\epsilon) = 1$. Also, $h_V(uv) = h_S(u)h_V(v)$ for all $u \in A^*$ and $v \in A^\infty$. Thus, $h_V(XY) = h_S(X)h_V(Y)$ for all $X \subseteq A^*$ and $Y \subseteq A^\infty$. Moreover, for all $u_0, u_1, \ldots$ in A^*, if $u_i \neq \epsilon$ for infinitely many i, then $h_V(u_0 u_1 \ldots) = \prod_{k \geq 0} h_S(u_k)$. If on the other hand, $u_k = \epsilon$ for all $k \geq n$, then $h_V(u_0 u_1 \ldots) = h_S(u_0) \cdots h_S(u_{n-1})1^\omega$. In either case, if $X_0, X_1, \ldots \subseteq A^*$, then $h_V(\prod_{n \geq 0} X_n) = \prod_{n \geq 0} h_S(X_n)$.

Suppose now that $X \subseteq A^*$ and $Y \subseteq A^\infty$. We define $h^\sharp_S(X) = \bigvee h_S(X)$ and $h^\sharp_V(Y) = \bigvee h_V(Y)$. It is well-known that $h^\sharp_S$ is a continuous semiring morphism $P(A^*) \to S$. Also, $h^\sharp_V$ preserves arbitrary suprema, since $h^\sharp_V(\bigcup_{i \in I} Y_i) = \bigvee h_V(\bigcup_{i \in I} Y_i) = \bigvee \bigcup_{i \in I} h_V(Y_i) = \bigvee_{i \in I} \bigvee h_V(Y_i) = \bigvee_{i \in I} h^\sharp_V(Y_i)$.

We prove that the action is preserved. Let $X \subseteq A^*$ and $Y \subseteq A^\infty$. Then $h^\sharp_V(XY) = \bigvee h_V(XY) = \bigvee h_S(X)h_V(Y) = \bigvee h_S(X) \bigvee h_V(Y) = h^\sharp_S(X)h^\sharp_V(Y)$.

Finally, we prove that the infinite product is preserved. Let $X_0, X_1, \ldots \subseteq A^*$. Then $h^\sharp_V(\prod_{n \geq 0} X_n) = \bigvee h_V(\prod_{n \geq 0} X_n) = \bigvee \prod_{n \geq 0} h_S(X_n) = \prod_{n \geq 0} \bigvee h_S(X_n) = \prod_{n \geq 0} h^\sharp_S(X_n)$.

It is clear that h_S extends h, and that (h_S, h_V) is unique. $\square$

Lemma 22. *$(P(A^*), P(A^\omega))$ is a quotient of $(P(A^*), P(A^\infty))$ under the continuous homomorphism (φ_S, φ_V) such that φ_S is the identity on $P(A^*)$ and φ_V maps $Y \subseteq A^\infty$ to $Y \cap A^\omega$.*

Proof (of Lemma 22). Suppose that $Y_i \subseteq A^\infty$ for all $i \in I$. It holds that $\varphi_V(\bigcup_{i \in I} Y_i) = A^\omega \cap \bigcup_{i \in I} Y_i = \bigcup_{i \in I}(A^\omega \cap Y_i) = \bigcup_{i \in I} \varphi_V(Y_i)$.

Let $X \subseteq A^*$ and $Y \subseteq A^\infty$. Then $h_V(XY) = XY \cap A^\omega = X(Y \cap A^\omega) = \varphi_S(X)\varphi_V(Y)$.

Finally, let $X_0, X_1, \ldots \subseteq A^*$. Then $h_V(\prod_{n \geq 0} X_n) = \{u_0 u_1 \ldots \in A^\omega : u_n \in X_n\} = \prod_{n \geq 0} h_S(X_n)$. $\square$

Lemma 23. *Suppose that (S, V) is a continuous Kleene ω-algebra satisfying $1^\omega = \bot$. Let (h_S, h_V) be a continuous homomorphism $(P(A^*), P(A^\infty)) \to (S, V)$. Then (h_S, h_V) factors through (φ_S, φ_V).*

Proof (of Lemma 23). Define $h'_S = h_S$ and $h'_V : P(A^\omega) \to V$ by $h'_V(Y) = h_V(Y)$, for all $Y \subseteq A^\omega$. Then clearly $h_S = \varphi_S h'_S$. Moreover, $h_V = \varphi_V h'_V$, since for all $Y \subseteq A^\infty$, $Y\varphi_V h'_V = (Y \cap A^\omega)h_V = (Y \cap A^\omega)h_V \vee (Y \cap A^*)h_S 1^\omega = (Y \cap A^\omega)h_V \vee ((Y \cap A^*)1^\omega)h_V = ((Y \cap A^\omega) \cup (Y \cap A^*)1^\omega)h_V = Y h_V$.

Since (φ_S, φ_V) and (h_S, h_V) are homomorphisms. so is (h'_S, h'_V). It is clear that h'_V preserves all suprema. $\square$

Proof (of Theorem 3). Suppose that (S,V) is a continuous Kleene ω-algebra satisfying $1^\omega = \bot$. Let $h : A \to S$. By Theorem 2, there is a unique continuous homomorphism $(h_S, h_V) : (P(A^*), P(A^\infty)) \to (S, V)$ extending h. By Lemma 23, h_S and h_V factor as $h_S = \varphi_S h'_S$ and $h_V = \varphi_V h'_V$, where (h'_S, h'_V) is a continuous homomorphism $(P(A^*), P(A^\omega)) \to (S,V)$. This homomorphism (h'_S, h'_V) is the required extension of h to a continuous homomorphism $(P(A^*), P(A^\omega)) \to (S,V)$. Since the factorization is unique, so is this extension. □

Proof (of Lemma 6). Suppose that $R = \emptyset$. Then $x(\bigvee R)v = \bot = \bigvee xRv$. If R is a singleton set $\{y\}$, then $x(\bigvee R)v = xyv = \bigvee xRv$. Suppose now that $R = R_1 \cup R_2$ or $R = R_1R_2$, where R_1, R_2 are regular, and suppose that our claim holds for R_1 and R_2. Then, if $R = R_1 \cup R_2$,

$$\begin{aligned}
x(\bigvee R)v &= x(\bigvee R_1 \vee \bigvee R_2)v \quad \text{(by Lemma 5)}\\
&= x(\bigvee R_1)v \vee x(\bigvee R_2)v\\
&= \bigvee xR_1v \vee \bigvee xR_2v\\
&= \bigvee x(R_1 \cup R_2)v\\
&= \bigvee xRv,
\end{aligned}$$

where the third equality uses the induction hypothesis. If $R = R_1R_2$, then

$$\begin{aligned}
x(\bigvee R)v &= x(\bigvee R_1)(\bigvee R_2)v \quad \text{(by Lemma 5)}\\
&= \bigvee (xR_1(\bigvee R_2)v)\\
&= \bigvee \{y(\bigvee R_2)v : y \in xR_1\}\\
&= \bigvee \{\bigvee yR_2v : y \in xR_1\}\\
&= \bigvee xR_1R_2v\\
&= \bigvee xRv,
\end{aligned}$$

where the second equality uses the induction hypothesis for R_1 and the fourth the one for R_2. Suppose last that $R = R_0^*$, where R_0 is regular and our claim holds for R_0. Then, using the previous case, it follows by induction that

$$x(\bigvee R_0^n)v = \bigvee xR_0^nv \tag{2}$$

for all $n \geq 0$. Using this and $\mathsf{Ax0}$, it follows now that

$$
\begin{aligned}
x(\bigvee R)v &= x(\bigvee R_0^*)y \\
&= x(\bigvee_{n\geq 0} \bigvee R_0^n)v \\
&= x(\bigvee_{n\geq 0} (\bigvee R_0)^n)v \quad \text{(by Lemma 5)} \\
&= \bigvee_{n\geq 0} x(\bigvee R_0)^n v \quad (\text{by}\mathsf{Ax0}) \\
&= \bigvee_{n\geq 0} x(\bigvee R_0^n)v \quad \text{(by Lemma 5)} \\
&= \bigvee_{n\geq 0} \bigvee x R_0^n v \quad \text{(by (2))} \\
&= \bigvee x R_0^* v \\
&= \bigvee x R v.
\end{aligned}
$$

The proof is complete. $\square$

Proof (of Lemma 7). If one of the R_i is empty, our claim is clear since both sides are equal to $\perp$, so we suppose they are all nonempty.

Below we will suppose that each regular language comes with a fixed decomposition having a minimal number of operations needed to obtain the language from the empty set and singleton sets. For a regular language R, let $|R|$ denote the minimum number of operations needed to construct it. When $\mathcal{R}$ is a finite set of regular languages, let $\mathcal{R}'$ denote the set of non-singleton languages in it. Let $|\mathcal{R}| = \sum_{R\in\mathcal{R}'} 2^{|R|}$. Our definition ensures that if $\mathcal{R} = \{R, R_1, \ldots, R_n\}$ and $R = R' \cup R''$ or $R = R'R''$ according to the fixed minimal decomposition of R, and if $\mathcal{R}' = \{R', R'', R_1, \ldots, R_n\}$, then $|\mathcal{R}'| < |\mathcal{R}|$. Similarly, if $R = R_0^*$ by the fixed minimal decomposition and $\mathcal{R}' = \{R_0, R_1, \ldots, R_n\}$, then $|\mathcal{R}'| < |\mathcal{R}|$.

We will argue by induction on $|\mathcal{R}|$.

When $|\mathcal{R}| = 0$, then $\mathcal{R}$ consists of singleton languages and our claim follows from $\mathsf{Ax3}$. Suppose that $|\mathcal{R}| > 0$. Let R be a non-singleton language appearing in $\mathcal{R}$. If R appears only a finite number of times among the R_n, then there is some m such that R_n is different from R for all $n \geq m$. Then,

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{i<m} x_i(\bigvee R_i) \prod_{n\geq m} x_n(\bigvee R_n) \quad \text{(by } \mathsf{Ax1}\text{)}\\
&= (\bigvee x_0R_0 \cdots x_{n-1}R_{n-1}) \prod_{n\geq m} x_n(\bigvee R_n) \quad \text{(by Lemma 5)}\\
&= \bigvee (x_0R_0 \cdots x_{n-1}R_{n-1} \prod_{n\geq m} x_n(\bigvee R_n)) \quad \text{(by Lemma 6)}\\
&= \bigvee \{y \prod_{n\geq m} x_n(\bigvee R_n) : y \in x_0R_0 \cdots x_{n-1}R_{n-1}\}\\
&= \bigvee \{\bigvee y \prod_{n\geq m} x_nR_n : y \in x_0R_0 \cdots x_{n-1}R_{n-1}\}\\
&= \bigvee \prod_{n\geq 0} x_nR_n,
\end{aligned}$$

where the passage from the 4th line to the 5th uses induction hypothesis and $\mathsf{Ax1}$.

Suppose now that R appears an infinite number of times among the R_n. Let $R_{i_1}, R_{i_2}, \ldots$ be all the occurrences of R among the R_n. Define

$$\begin{aligned}
y_0 &= x_0(\bigvee R_0)\cdots(\bigvee R_{i_1-1})x_{i_1}\\
y_j &= x_{i_j+1}(\bigvee R_{i_j+1})\cdots(\bigvee R_{i_{j+1}-1})x_{i_{j+1}},
\end{aligned}$$

for $j \geq 1$. Similarly, define

$$\begin{aligned}
Y_0 &= x_0R_0 \cdots R_{i_1-1}x_{i_1}\\
Y_j &= x_{i_j+1}R_{i_j+1} \cdots R_{i_{j+1}-1}x_{i_{j+1}},
\end{aligned}$$

for all $j \geq 1$. It follows from Lemma 5 that

$$y_j = \bigvee Y_j$$

for all $j \geq 0$. Then

$$\prod_{n\geq 0} x_n(\bigvee R_n) = \prod_{j\geq 0} y_j(\bigvee R), \tag{3}$$

by $\mathsf{Ax2}$, and

$$\prod_{n\geq 0} x_nR_n = \prod_{j\geq 0} Y_jR.$$

If $R = R' \cup R''$, then:

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{j\geq 0} y_j(\bigvee(R' \cup R'')) \quad \text{(by (3))}\\
&= \prod_{j\geq 0} y_j(\bigvee R' \vee \bigvee R'') \quad \text{(by Lemma 5)}\\
&= \bigvee_{z_j \in \{\bigvee R', \bigvee R''\}} \prod_{j\geq 0} y_j z_j \quad \text{(by } \mathsf{Ax3}\text{)}\\
&= \bigvee_{z_j \in \{\bigvee R', \bigvee R''\}} \bigvee \prod_{j\geq 0} Y_j z_j\\
&= \bigvee_{Z_j \in \{R', R''\}} \bigvee \prod_{j\geq 0} Y_j Z_j\\
&= \bigvee \prod_{n\geq 0} x_n(R' \cup R'')\\
&= \bigvee \prod_{n\geq 0} x_n R,
\end{aligned}$$

where the 4th and 5th equalities hold by the induction hypothesis and $\mathsf{Ax2}$.

Suppose now that $R = R'R''$. Then applying the induction hypothesis almost directly we have

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{j\geq 0} y_j(\bigvee R'R'')\\
&= \prod_{j\geq 0} y_j(\bigvee R')(\bigvee R'') \quad \text{(by Lemma 5)}\\
&= \bigvee \prod_{j\geq 0} Y_j(\bigvee R')(\bigvee R'')\\
&= \bigvee \prod_{j\geq 0} Y_j R'R''\\
&= \bigvee \prod_{n\geq 0} x_n R'R''\\
&= \bigvee \prod_{n\geq 0} x_n R,
\end{aligned}$$

where the third and fourth equalities come from the induction hypothesis and $\mathsf{Ax2}$.

The last case to consider is when $R = T^*$, where T is regular. We argue as follows:

$$\begin{aligned}
\prod_{n\geq 0} x_n(\bigvee R_n) &= \prod_{j\geq 0} y_j(\bigvee T^*)\\
&= \prod_{j\geq 0} y_j(\bigvee T)^* \quad \text{(by Lemma 5)}\\
&= \bigvee_{k_0,k_1,\dots} \prod_{j\geq 0} y_j(\bigvee T)^{k_j} \quad \text{(by }\mathsf{Ax1}\text{ and }\mathsf{Ax4}\text{)}\\
&= \bigvee_{k_0,k_1,\dots} \bigvee \prod_{j\geq 0} Y_j(\bigvee T)^{k_j}\\
&= \bigvee_{k_0,k_1,\dots} \bigvee \prod_{j\geq 0} Y_j T^{k_j}\\
&= \bigvee_{j\geq 0} Y_j T^*\\
&= \bigvee_{j\geq 0} Y_j R_j\\
&= \bigvee_{n\geq 0} x_n R_n,
\end{aligned}$$

where the 4th and 5th equalities follow from the induction hypothesis and Ax2. The proof is complete. □

Proof (of Lemma 11). It is well-known that R can be written as a finite union of sets of the form $R_0(R_1)^\omega$ where $R_0, R_1 \subseteq S^*$ are regular, moreover, R_1 does not contain the empty word. It suffices to show that $\bigvee R_0(R_1)^\omega$ exists. But this holds by Corollary 9. □

Proof (of Lemma 12). The first claim is clear. The second follows from Lemma 6. For the last, see the proof of Lemma 11. □

Proof (of Theorem 15). Our proof is modeled after the proof of Theorem 2. First, is is clear from the fact that $(P(A^*), P(A^\infty))$ is a continuous Kleene ω-algebra, and that $R(A^*)$ is a *-continuous semiring, that $(R(A^*), R'(A^\infty))$ is indeed a finitary *-continuous Kleene ω-algebra.

Suppose that (S, V) is any finitary *-continuous Kleene ω-algebra and let $h : A \to S$ be a mapping. For each $u = a_0 \dots a_{n-1}$ in A^*, define $h_S(u) = h(a_0) \cdots h(a_{n-1})$ and $h_V(u) = h(a_0) \cdots h(a_{n-1}) 1^\omega = \prod_{k\geq 0} b_k$, where $b_k = a_k$ for all $k < n$ and $b_k = 1$ for all $k \geq n$. When $u = a_0 a_1 \ldots \in A^\omega$ whose alphabet is finite, define $h_V(u) = \prod_{k\geq 0} h(a_k)$. This infinite product exists in $R'(A^\infty)$.

Note that we have $h_S(uv) = h_S(u) h_S(v)$ for all $u, v \in A^*$, and $h_S(\epsilon) = 1$. And if $u \in A^*$ and $v \in A^\infty$ such that the alphabet of v is finite, then $h_V(uv) =$

$h_S(u)h_V(v)$. Also, $h_V(XY) = h_S(X)h_V(Y)$ for all $X \subseteq A^*$ in $R(A^*)$ and $Y \subseteq A^\infty$ in $R'(A^\infty)$.

Moreover, for all $u_0, u_1, \ldots$ in A^*, if $u_i \neq \epsilon$ for infinitely many i, such that the alphabet of $u_0u_1\ldots$ is finite, then $h_V(u_0u_1\ldots) = \prod_{k\geq 0} h_S(u_k)$. If on the other hand, $u_k = \epsilon$ for all $k \geq n$, then $h_V(u_0u_1\ldots) = h_S(u_0)\cdots h_S(u_{n-1})1^\omega$. In either case, if $X_0, X_1, \ldots \subseteq A^*$ are regular and form a finitary sequence, then the sequence $h_S(X_0), h_S(X_1), \ldots$ is also finitary as is each infinite word in $\prod_{n\geq 0} X_n$, and $h_V(\prod_{n\geq 0} X_n) = \prod_{n\geq 0} h_S(X_n)$.

Suppose now that $X \subseteq A^*$ is regular and $Y \subseteq A^\infty$ is in $R'(A^\infty)$. We define $h_S^\sharp(X) = \bigvee h_S(X)$ and $h_V^\sharp(Y) = \bigvee h_V(Y)$. It is well-known that $h_S^\sharp$ is a *-continuous semiring morphism $R(A^*) \to S$. Also, $h_V^\sharp$ preserves finite suprema, since when I is finite, $h_V^\sharp(\bigcup_{i\in I} Y_i) = \bigvee h_V(\bigcup_{i\in I} Y_i) = \bigvee \bigcup_{i\in I} h_V(Y_i) = \bigvee_{i\in I} \bigvee h_V(Y_i) = \bigvee_{i\in I} h_V^\sharp(Y_i)$.

We prove that the action is preserved. Let $X \in R(A^*)$ and $Y \in R'(A^\infty)$. Then $h_V^\sharp(XY) = \bigvee h_V(XY) = \bigvee h_S(X)h_V(Y) = \bigvee h_S(X) \bigvee h_V(Y) = h_S^\sharp(X)h_V^\sharp(Y)$.

Finally, we prove that infinite product of finitary sequences is preserved. Let $X_0, X_1, \ldots$ be a finitary sequence of regular languages in $R(A^*)$. Then, using Corollary 9, $h_V^\sharp(\prod_{n\geq 0} X_n) = \bigvee h_V(\prod_{n\geq 0} X_n) = \bigvee \prod_{n\geq 0} h_S(X_n) = \prod_{n\geq 0} \bigvee h_S(X_n) = \prod_{n\geq 0} h_S^\sharp(X_n)$.

It is clear that h_S extends h, and that (h_S, h_V) is unique. □

Lemma 24. *Suppose that (S, V) is a finitary *-continuous Kleene ω-algebra satisfying $1^\omega = \bot$. Let (h_S, h_V) be a *-continuous homomorphism $(R(A^*), R'(A^\infty)) \to (S, V)$. Then (h_S, h_V) factors through (φ_S, φ_V).*

Proof (of Theorem 16). This follows from Theorem 15 using Lemma 24. □

Proof (of Lemma 17). Suppose that $n = 2$. Then by Corollary 9, $(a \vee bd^*c)^\omega$ is the supremum of the set of all infinite products $A_{1,i_1}A_{i_1,i_2}\cdots$ containing a or c infinitely often, and $(a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{1,i_1}A_{i_1,i_2}\cdots$ containing a and c only finitely often. Thus, $(a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{1,i_1}A_{i_1,i_2}\cdots$. Similarly, $(d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega$ is the supremum of the set of all infinite products $A_{2,i_1}A_{i_1,i_2}\cdots$.

The proof of the induction step is similar. Suppose that $n > 2$, and let a be $k \times k$. Then by induction hypothesis, for every i with $1 \leq i \leq k$, the ith component of $(a \vee bd^*c)^\omega$ is the supremum of the set of all infinite products $A_{i,i_1}A_{i_1,i_2}\cdots$ containing an entry of a or c infinitely often, whereas the ith component of $(a \vee bd^*c)^*bd^\omega$ is the supremum of all infinite products $A_{i,i_1}A_{i_1,i_2}\cdots$ containing entries of a and c only finitely often. Thus, the ith component of $(a \vee bd^*c)^\omega \vee (a \vee bd^*c)^*bd^\omega$ is the supremum of the set of all infinite products $A_{i,i_1}A_{i_1,i_2}\cdots$. A similar fact holds for $(d \vee ca^*b)^\omega \vee (d \vee ca^*b)^*ca^\omega$. The proof is complete. □

Proof (of Theorem 18). Suppose that (S, V) is a finitary *-continuous Kleene ω-algebra. Then

$$(x \vee y)^\omega = (x^* y)^\omega \vee (x^* y)^* x^\omega,$$

since by Lemma 5 and Lemma 12, $(x^* y)^\omega$ is the supremum of the set of all infinite products over $\{x, y\}$ containing y infinitely often, and $(x^* y)^* x^\omega$ is the supremum of the set of infinite products over $\{x, y\}$ containing y finitely often. Thus, $(x^* y)^\omega \vee (x^* y)^* x^\omega$ is equal to $(x \vee y)^\omega$, which by Ax3 is the supremum of all infinite products over $\{x, y\}$. As noted above, also

$$(xy)^\omega = x(yx)^\omega$$

for all $x, y \in S$. Thus, (S, V) is a Conway semiring-semimodule pair and hence so is each $(S^{n \times n}, V^n)$.

To complete the proof of the fact that (S, V) is an iteration semiring-semimodule pair, suppose that $x_1, \ldots, x_n \in S$, and let $x = x_1 \vee \cdots \vee x_n$. Let A be an $n \times n$ matrix whose rows are permutations of the $x_1, \ldots, x_n$. We need to prove that each component of A^ω is x^ω. We use Lemma 17 and Ax3 to show that both are equal to the supremum of the set of all infinite products over the set $X = \{x_1, \ldots, x_n\}$.

By Lemma 17, for each $i_0 = 1, \ldots, n$, the i_0th row of A^ω is $\bigvee_{i_1, i_2, \ldots} a_{i_0, i_1} a_{i_1, i_2} \cdots$. It is clear that each infinite product $a_{i_0, i_1} a_{i_1, i_2} \cdots$ is an infinite product over X. Suppose now that $x_{j_0} x_{j_1} \cdots$ is an infinite product over X. We define by induction on $k \geq 0$ an index i_{k+1} such that $a_{i_k, i_{k+1}} = x_{j_k}$. Suppose that $k = 0$. Then let i_1 be such that $a_{i_0, i_1} = x_{j_0}$. Since x_{j_0} appears in the i_0th row, there is such an i_1. Suppose that $k > 0$ and that i_k has already been defined. Since x_{j_k} appears in the i_kth row, there is some i_{k+1} with $a_{i_k, i_{k+1}} = x_{j_k}$. We have completed the proof of the fact that the i_0th entry of A^ω is the supremum of the set of all infinite products over the set $X = \{x_1, \ldots, x_n\}$.

Consider now $x^\omega = xx \cdots$. We use induction on n to prove that x^ω is also the supremum of the set of all infinite products over the set $X = \{x_1, \ldots, x_n\}$. When $n = 1$ this is clear. Suppose now that $n > 0$ and that the claim is true for $n - 1$. Let $y = x_1 \vee \cdots \vee x_{n-1}$ so that $x = y \vee x_n$. We have:

$$\begin{aligned} x^\omega &= (y \vee x_n)^\omega \\ &= (x_n^* y)^* x_n^\omega \vee (x_n^* y)^\omega \\ &= (x_n^* y)^* x_n^\omega \vee (x_n^* x_1 \vee \cdots \vee x_n^* x_{n-1})^\omega. \end{aligned}$$

Now

$$(x_n^* y)^* x_n^\omega = \bigvee_{k, m_1, \ldots, m_k \geq 0} x_n^{m_1} y \cdots x_n^{m_k} y x_n^\omega,$$

by Lemma 6, which is the supremum of all infinite products over X containing $x_1, \ldots, x_{n-1}$ only a finite number of times.

Also, using the induction hypothesis and Ax4,

$$\begin{aligned} (x_n^* x_1 \vee \cdots \vee x_n^* x_{n-1})^\omega &= \bigvee_{1 \leq i_1, i_2, \ldots \leq n-1} x_n^* x_{i_1} x_n^* x_{i_2} \cdots \\ &= \bigvee_{1 \leq i_1, i_2, \ldots \leq n-1} \bigvee_{k_0, k_1, \ldots} x_n^{k_0} x_{i_1} x_n^{k_1} x_{i_2} \cdots \end{aligned}$$

which is the supremum of all infinite products over X containing one of $x_1, \ldots, x_{n-1}$ an infinite number of times. Thus, x^ω is the supremum of all infinite products over X as claimed. The proof is complete. □

Proof (of Theorem 20). This holds by Lemma 17 and the fact that any matrix semiring over a *-continuous Kleene algebra is itself a *-continuous Kleene algebra. □