Numeric and Certified Isolation of the Singularities of the Projection of a Smooth Space Curve

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Abstract. Let $\mathcal{C}_{P\cap Q}$ be a smooth real analytic curve embedded in $\mathbb{R}^3$, defined as the solutions of real analytic equations of the form $P(x,y,z) = Q(x,y,z) = 0$ or $P(x,y,z) = \frac{\partial P}{\partial z} = 0$. Our main objective is to describe its projection $\mathcal{C}$ onto the (x,y)-plane. In general, the curve $\mathcal{C}$ is not a regular submanifold of $\mathbb{R}^2$ and describing it requires to isolate the points of its singularity locus Σ. After describing the types of singularities that can arise under some assumptions on P and Q, we present a new method to isolate the points of Σ. We experimented our method on pairs of independent random polynomials (P,Q) and on pairs of random polynomials of the form $(P, \frac{\partial P}{\partial z})$ and got promising results.

Keywords: Topology of Projected Real Curve, Topology of Apparent Contour, Singularities Isolation, Topology Computation, Numeric Certified Methods

1 Introduction

Consider a smooth analytic curve $\mathcal{C}_{P\cap Q} \subset \mathbb{R}^3$ defined by $P(x,y,z) = Q(x,y,z) = 0$ with P, Q analytic functions, and its projection $\mathcal{C} \subset \mathbb{R}^2$ on the (x,y)-plane. Computing the topology of $\mathcal{C}$, or computing a graph topologically equivalent to $\mathcal{C}$, requires computing the set Σ of its singularities (see 1.2 for a rigorous definition). In a second step, the study of the complement of Σ allows one to recover the topology of the curve. This fundamental problem arises in fields such as mechanical design, robotics and biology. A specific case of interest is when $Q = P_z$ (where P_z is the partial derivative $\frac{\partial P}{\partial z}$). In this case, the curve $\mathcal{C}$ is the apparent contour of the surface $P(x,y,z) = 0$. This case has been intensively studied and extended in the framework of the catastrophe theory (see [10] and references therein). Moreover, determining the topology of a projection of a space curve is an important step to compute its topology [7,11]. Similarly determining the topology of the apparent contour of a surface is an important step to compute its topology [1,5].

The goal of this paper is to take advantage of the specific structure of the singularities Σ and to propose a characterization allowing to isolate them efficiently. Since we do not restrict our work to the case $P = P_z = 0$, we also give a mathematical description of the types of singularities arising in the projection of curves defined by $P = Q = 0$ under some generic assumptions.

Our approach to isolating the singularities Σ is to construct a new system so-called *ball system*, the roots of which are in a one-to-one correspondence with the points of Σ. As shown with experimental results, this system suits numerical certified solvers such as subdivision methods or homotopy solvers in the polynomial case.

The rest of the paper is organized as follows. Section 2 classifies the singularities of $\mathcal{C}$ and relates them to the points where the projection Π_{xy} is not a diffeomorphism. The construction of the ball system and a proof of regularity of its solutions are exhibited in Section 3. Section 4 is dedicated to experiments. The rest of this section presents previous and related works, and gives explicitly the assumptions on P and Q for our method.

1.1 Previous Works

State-of-the-art symbolic methods that compute topology of real plane curves defined by polynomials are closely related to bivariate system solving. Many methods use resultant and sub-resultant theory to isolate critical points, see for instance the book chapter [24] and references within. There are some alternatives, using for instance Gröbner bases and rational univariate representations [28,6].

Numerical methods can be used together with interval arithmetic to compute and certify the topology of a non-singular curve when the interest area is a compact subset of the plane [20,16,27]. However they fail near any singular point of the curve. Isolating singularities of a plane curve $f(x, y) = 0$ with a numerical method is a challenge since it is described by the non-square system $f = f_x = f_y = 0$, and singularities are not necessarily regular solutions of this system. The latter system can be translated into a square system using combinations of its equations with first derivatives [8], and non-regular solutions can be handled through deflation systems (see for instance [12,26,18,19,13,3]), but the resulting systems are usually still overdetermined or contain spurious solutions.

Such systems are usually solved with symbolic bivariate solvers relying on Gröbner bases or rational univariate representations [28,6]. Well determined systems which solutions are regular can be handled by numerical approaches. Classical homotopy solvers [22] find all complex solutions of latter systems when their equations are polynomials. Subdivision methods [21,25,29] are numeric certified approaches to find all real solutions lying in an initial bounded domain of a system of analytic equations. When the latter are polynomial, these approaches can be extended to unbounded initial domains [29,25].

Starting with the work of Whitney [30], the catastrophe theory was developed to classify the singularities arising while deforming generic mappings (see [2,10] for example). From an algorithmic point of view, the authors of [9] use elements of the catastrophe theory to derive an algorithm isolating the singularities arising in mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$.

More specifically, the problem of isolating the singularities of the projection of a generic algebraic space curve was investigated in [15]. The authors use resultant and sub-resultant theory to represent Σ as the solutions of a regular bivariate system suited to a branch and bound solving approach. In Section 4 we compare this approach with the new approach proposed in this article.

1.2 Notations and Assumptions

In the following, $\mathcal{C}_{P\cap Q}$ denotes the curve defined as the zero set of the real analytic functions $P(x, y, z)$ and $Q(x, y, z)$ and B_0 is an open subset of $\mathbb{R}^2$. We will denote by Π_{xy} the projection from $\mathcal{C}_{P\cap Q}$ to the (x, y)-plane, and by $\mathcal{C}$ the projection $\Pi_{xy}(\mathcal{C}_{P\cap Q})$.

Regular points and $A_k^{\pm}$ singularities. A point p of the curve $\mathcal{C}$ is *regular* if there is a small neighborhood U of p in $\mathbb{R}^2$ such that $\mathcal{C} \cap U$ is a *regular submanifold* of $\mathbb{R}^2$. Otherwise it is *singular*. A singular point p of a curve $\mathcal{C}$ is of type $A_k^{\pm}$ if and only if $\mathcal{C}$ is equal to the solutions of the equation $x^2 \pm y^{k+1} = 0$ on a neighborhood U of p, up to a diffeomorphism from $U \subset \mathbb{R}^2$ to $V \subset \mathbb{R}^2$ ([2, §9.8]). Remark that those are not the only type of singularities that can appear on a plane curve. Notice that the types A_{2k}^{+} and A_{2k}^{-} are equivalent and simply denoted by A_{2k}. We will call *node* a singularity of type A_1^{-} or equivalently a transverse intersection of two real curve branches. We also call *cusp* a singularity of type A_{2k} and *ordinary cusp* the singularity A_2. With this notation, a point p of $\mathcal{C}$ is regular if and only if it is of type A_0.

In Section 2, we will describe the types of singularities of $\mathcal{C}$ assuming that :

- (A_1) The curve $\mathcal{C}_{P\cap Q}$ is smooth above B_0.
- (A_2) For any (α, β) in B_0, the system $P(\alpha, \beta, z) = Q(\alpha, \beta, z) = 0$ has at most 2 real roots counted with multiplicities.
- (A_3) There is at most a discrete set of points (α, β) in B_0 such that $P(\alpha, \beta, z) = Q(\alpha, \beta, z) = 0$ has 2 real roots counted with multiplicities.
- (A_4) Π_{xy} is a proper map from $\mathcal{C}_{P\cap Q} \cap (B_0 \times \mathbb{R})$ to its image, that is the inverse image of a compact subset is compact.

Then in Section 3, we will introduce the system of analytic equations that we will use to compute the singularities of $\mathcal{C}$. The solutions of this system will be regular under the following additional assumption:

- (A_5) The singularities of the curve $\mathcal{C}$ are either nodes or ordinary cusps.

Notice that Thom Transversality Theorem implies that $(A_1), (A_2), (A_3)$ and (A_5) hold for generic analytic maps P, Q defining $\mathcal{C}_{P\cap Q}$ (see [10, Th. 3.9.7 and §4.7]), and (A_4) holds at least for generic polynomial maps. If we assume only that the curve is smooth (assumption (A_1)), it would be interesting to prove that all the other assumptions hold after a generic linear change of coordinates.

If P, Q are polynomials, a semi-algorithm checking these conditions is given in [15, Semi-Algo. 1]. Otherwise when P, Q are analytic maps, the latter semi-algorithm can be adapted only when B_0 is bounded.

2 Description of the Singularity Locus Σ

The different types of singularities of a plane curve have been classified in [2] for example. We describe in this section the types of singularities that can arise on the curve $\mathcal{C}$ under the Assumptions $(A_1) - (A_4)$, and we relate those singularities with the projection mapping Π_{xy}. More precisely, using Arnold's notation recalled below, we show that under the Assumptions $(A_1) - (A_4)$, the singularities of $\mathcal{C}$ are of type $A_k^{\pm}$ (Lemma 2 and Corollary 1). Moreover, we show that a singular point of $\mathcal{C}$ is either a critical value of Π_{xy}, or the image of two distinct points of $\mathcal{C}_{P\cap Q}$ by Π_{xy}.

Singularities of $\mathcal{C}$ and critical points of Π_{xy}. The critical points of Π_{xy} are the points of $\mathcal{C}_{P\cap Q}$ where the tangent to the curve is vertical, i.e. aligned with the z-axis. Assuming that the conditions $(A_1)-(A_4)$ are satisfied by the curve $\mathcal{C}_{P\cap Q}$, we show that for p a point on the curve $\Pi_{xy}(\mathcal{C}_{P\cap Q})$:

1. if p is a critical point of Π_{xy}, then it is a cusp point of $\mathcal{C}$ (singularity of type $A_{2(k+1)}$);
2. if p is the image of two distinct points of $\mathcal{C}_{P\cap Q}$, then it is a singularity of type A_{2k+1}^{-};
3. otherwise, it is a regular point.

In particular, this implies that a point p is singular if and only if it is a critical value of Π_{xy} or it has two antecedents by Π_{xy}.

Lemma 1. *Let p be a point of $\mathcal{C}$. If p is not a critical value of Π_{xy} and $\Pi_{xy}^{-1}(p)$ has only one antecedent, then p is a regular point of $\mathcal{C}$.*

Proof. For U an open set of $\mathbb{R}^2$, we will denote by Π_{xy}^U the restriction of Π_{xy} to $\mathcal{C}_{P\cap Q}\cap\Pi_{xy}^{-1}(U)$. Since p is not a critical value of Π_{xy}, there exists a neighborhood U of p such that U does not contain any critical value of Π_{xy}, such that Π_{xy}^U is an immersion. Then, since p has a unique antecedent, (A_3) ensures that there is a neighborhood V of p such that Π_{xy}^V is a homeomorphism. Thus $\Pi_{xy}^{U\cap V}$ is an embedding and p is a regular point. □

Lemma 2. *Let p be a point of $\mathcal{C}$. If p has two antecedents by Π_{xy}, then p is a singularity of $\mathcal{C}$ of type A_{2k+1}^{-} with $k \geq 0$.*

Proof. If $\Pi_{xy}^{-1}(p)$ contains more than one antecedent of p, then (A_2) implies that p has exactly two antecedents q_u and q_v. Since Π_{xy} is proper by Assumption (A_4) and $\mathcal{C}_{P\cap Q}$ is smooth by Assumption (A_1), for a small enough neighborhood U of p, $\Pi_{xy}^{-1}(U)$ is bounded and is the union of two smooth connected branches of $\mathcal{C}_{P\cap Q}$. And (A_3) implies that in a small enough neighborhood of p, p is the only point with two antecedents. Let $u = (u_x, u_y, u_z)$ and $v = (v_x, v_y, v_z)$ be the two vectors tangent to $\mathcal{C}_{P\cap Q}$ at the antecedents q_u and q_v of p. Assumption (A_2) implies that neither u nor v are vertical, hence $\tilde{u} = (u_x, u_y)$ and $\tilde{v} = (v_x, v_y)$ are non-zero vectors of $\mathbb{R}^2$. We now distinguish two cases.

First, $\tilde{u}$ and $\tilde{v}$ are independent vectors. In this case, the mapping $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right)^{-1} \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is a diffeomorphic change of coordinates. Moreover $\left(\begin{smallmatrix} P_X(q_u) \\ Q_X(q_u) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} P_Y(q_u) \\ Q_Y(q_u) \end{smallmatrix}\right) \neq \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. Thus by the analytic implicit function theorem, there exists an analytic function $f : \mathbb{R} \mapsto \mathbb{R}$ such that $Y = f(X)$ and $f(0) = f'(0) = 0$ such that the projection of the branch at q_u has an equation of the form $Y = X^2 \tilde{f}(X)$. Symmetrically, the projection of the branch at q_v has an equation of the form $X = Y^2 \tilde{g}(Y)$. Thus, up to a diffeomorphism of $\mathbb{R}^2$, the curve $\mathcal{C}$ around p has an equation of the form $(Y - X^2 \tilde{f}(X))(X - Y^2 \tilde{g}(Y)) = 0$, or equivalently $(X + Y - X^2 \tilde{f}(X) - Y^2 \tilde{g}(Y))^2 - (X - Y - X^2 \tilde{f}(X) + Y^2 \tilde{G}(Y))^2 = 0$. That is, p is a singularity of type A_1^-, also called a node.

In the case where $\tilde{u}$ and $\tilde{v}$ are co-linear, we follow the same approach, using this time the diffeomorphic change of coordinate $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right) = \left(\begin{smallmatrix} u_x & u_y \\ -u_y & u_x \end{smallmatrix}\right)^{-1} \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Moreover $\left(\begin{smallmatrix} P_X(q_u) \\ Q_X(q_u) \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$. As in the previous case, we use the analytic implicit function theorem at q_u and q_v, and we conclude that there exist two analytic functions f and g such that on a neighborhood of p, the curve $\mathcal{C}$ is given by the equation $(Y - X^2 f(X)(Y - X^2 g(X)) = 0$. That can be rewritten as $(2Y - X^2(f(X) + g(X)))^2 - X^4(g(X) - f(X))^2 = 0$. Assumption (A_3) ensures that the projections of the 2 branches have only one common point, such that $g(X) - f(X)$ does not vanish identically. Then, denoting by k the valuation of $f(X) - g(X)$, p is a singularity of type A_{2k+3}^-. $\square$

Finally, if p is a critical value of Π_{xy} we use Arnold's classification of singularities and prove that p is a singular point of type $A_{2(k+1)}$ with $k \geq 0$.

Lemma 3. *Assume that the curve $\mathcal{C}_{P \cap Q}$ satisfies $(A_1) - (A_3)$. Let q be a critical point of Π_{xy}. Then, there exists a neighborhood U of q and an invertible 2×2 matrix M of real analytic functions such that:*

$$\left(\begin{smallmatrix} P \\ Q \end{smallmatrix}\right) = M \cdot \left(\begin{smallmatrix} X - Z^{3+2k} \\ Y - Z^2 \end{smallmatrix}\right) \circ \Phi(x, y, z) \tag{1}$$

where $\Phi : (x, y, z) \mapsto (\phi(x, y), \psi(z))$ is a diffeomorphism and k is a natural integer.

Corollary 1. *Let p be a point of $\mathcal{C}$. If p is a critical value of Π_{xy}, then p is a cusp of $\mathcal{C}$ of type $A_{2(k+1)}$ with $k \geq 0$.*

Proof (of the corollary). Let q be the critical point associated to p and denote π_{xy} the projection from $\mathbb{R}^3$ to $\mathbb{R}^2$. First we show that it is sufficient to study the behavior of $\mathcal{C}_{P \cap Q}$ in a neighborhood of q to describe the curve $\mathcal{C}$ in a neighborhood of p. Indeed, Assumptions (A_2) and (A_4) imply that above a small enough neighborhood of p, the curve $\mathcal{C}_{P \cap Q}$ has a unique connected branch. In particular for any neighborhood U of the critical point q there exists a neighborhood $V \subset U$ such that $\pi_{xy}(V) \cap \mathcal{C} \subset \Pi_{xy}(U \cap \mathcal{C}_{P \cap Q})$.

Then, Lemma 3 shows that there exists a neighborhood U of q and a diffeomorphism ϕ from $\pi_{xy}(U) \subset \mathbb{R}^2$ to $V \subset \mathbb{R}^2$ a neighborhood of $(0, 0)$ such that

$\phi(\Pi_{xy}(\mathcal{C}_{P\cap Q}\cap U)) = \{(X,Y)\in V \mid X^2 - Y^{3+2k}\}$. In particular, p is a singularity of type $A_{2(k+1)}$ with $k \geq 0$, that is a cusp. $\square$

Proof (of Lemma 3). This lemma is essentially a consequence of the analytic implicit function theorem, combined with our assumptions. First, q is a critical point thus $\mathcal{C}_{P\cap Q}$ has a vertical tangent at q, up to a translation, we assume $q = (0,0,0)$. Since $\mathcal{C}_{P\cap Q}$ is non-singular (Assumption (A_1)), the matrix $\left(\begin{smallmatrix} P_x(q) & P_y(q) \\ Q_x(q) & Q_y(q)\end{smallmatrix}\right)$ is invertible. Using the analytic implicit function theorem ([17] or [10, Corollary 2.7.3]), there exist two real analytic functions f, g from $\mathbb{R}$ to $\mathbb{R}$ such that $P(f(z), g(z), z) = Q(f(z), g(z), z) = 0$ on a small enough neighborhood of 0. In particular, letting $X := x - f(z)$ and $Y = y - g(z)$ we have $P = P(X+f(z), Y+g(z), z)$ and $Q = Q(X+f(z), Y+g(z), z)$. Using Hadamard's lemma ([10, Proposition 4.2.3]), there exist real analytic functions a, b, c, d such that $P = a\cdot X + b\cdot Y$ and $Q = c\cdot X + d\cdot Y$. Moreover, since $\left(\begin{smallmatrix} P_x(q) & P_y(q) \\ Q_x(q) & Q_y(q)\end{smallmatrix}\right)$ is invertible, the matrix $\left(\begin{smallmatrix} a(q) & b(q) \\ c(q) & d(q)\end{smallmatrix}\right)$ is also invertible. Let M_1 be the inverse of $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ on a small enough neighborhood of q. Then we have:

$$\left(\begin{smallmatrix} \tilde{P} \\ \tilde{Q}\end{smallmatrix}\right) := \left(\begin{smallmatrix} x-f(z) \\ y-g(z)\end{smallmatrix}\right) = M_1 \cdot \left(\begin{smallmatrix} P \\ Q\end{smallmatrix}\right) \quad . \tag{2}$$

Moreover, since the curve has a vertical tangent at q, we have $f_z(0) = g_z(0) = 0$. And according to Assumption (A_2), either $f_{zz}(0)$ or $g_{zz}(0)$ is not zero. Without restriction of generality, assume $\mu := g_{zz}(0) \neq 0$. Up to a scale of the variable z, we can assume that $\mu = 2$. Thus, there exist analytic functions u, v such that f and g are of the form:

$$f(z) = z^2 u(z) \tag{3}$$
$$g(z) = z^2(1 + zv(z)) \quad . \tag{4}$$

Letting $\psi : z \mapsto Z := z\sqrt{1+zv(z)}$, we have $\tilde{Q}(x, y, \psi^{-1}(Z)) = y - Z^2 = 0$. In particular, the function $\tilde{P} = x - z^2 u(z)$ can be rewritten as $\tilde{P}(x, y, \psi^{-1}(Z)) = x - Z^2(s(Z^2) + Zt(Z^2))$ with s and t two real analytic functions. Note that t cannot have all its derivatives vanishing at 0 since otherwise there would be a strictly positive dimensional set of points with two or more antecedents, contradicting Assumption (A_3). Let $k \in \mathbb{N}$ be the valuation of t, i.e. its first non vanishing derivative at 0. Then, there exists t' an analytic function such that $t(Z^2)$ is of the form $Z^{2k}(\eta + Z^2 t'(Z^2))$. The function $\tilde{P}(x, y, \psi^{-1}(Z))$ is of the form $x - Z^2(s(Z^2) + Z^{1+2k}(\eta + Z^2 t'(Z^2)))$. Using $\tilde{Q}$ to substitute $\psi(z)^2$ by y in $\tilde{P}$, there exists a matrix $M_2 := \left(\begin{smallmatrix} 1 & e \\ 0 & 1\end{smallmatrix}\right)$ where e is an analytic function, such that:

$$\left(\begin{smallmatrix} x - \frac{s(y)}{y} - \psi(z)^{3+2k}(\eta + yt(y)) \\ y - \psi(z)^2\end{smallmatrix}\right) = M_2 \cdot M_1 \cdot \left(\begin{smallmatrix} P \\ Q\end{smallmatrix}\right) \quad . \tag{5}$$

Finally, with:

$$\phi(x,y) = \left(\frac{x - \frac{s(y)}{y}}{\eta + y t(y)}, y \right) \tag{6}$$

$$\psi(z) = z\sqrt{1 + z v(z)} \tag{7}$$

$$M = M_1^{-1} \cdot M_2^{-1} \cdot \begin{pmatrix} \frac{1}{\eta + y t(y)} & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

we recover (1). □

3 Modeling System

Following the result of Section 2, a naive approach to represent the singularities Σ of $\mathcal{C}$ is to use the two following systems.

1. For $(x, y, z_1, z_2) \in B_0 \times \mathbb{R}^2$:

$$P(x,y,z_1) = P(x,y,z_2) = Q(x,y,z_1) = Q(x,y,z_2) = 0 \text{ and } z_1 \neq z_2 \ .$$

2. For $(x, y, z) \in B_0 \times \mathbb{R}$:

$$P(x,y,z) = Q(x,y,z) = P_z(x,y,z) = Q_z(x,y,z) = 0 \ .$$

However, the first system is numerically unstable near the set $z_1 = z_2$ and the second one is over-determined. Instead, we will introduce an unified system. First we define the operators that will be used to construct our system.

3.1 Ball System

Definition 1. *Let $A(x,y,z)$ be a real analytic function. We denote by $S.A$ and $D.A$ the functions:*

$$S.A(x,y,c,r_2) = \begin{cases} \frac{1}{2}(A(x,y,c+\sqrt{r_2}) + A(x,y,c-\sqrt{r_2})) & \text{if } r_2 > 0 \\ A(x,y,c) & \text{if } r_2 = 0 \\ \frac{1}{2}(A(x,y,c+i\sqrt{-r_2}) + A(x,y,c-i\sqrt{-r_2})) & \text{if } r_2 < 0 \end{cases} \tag{9}$$

$$D.A(x,y,c,r_2) = \begin{cases} \frac{1}{2\sqrt{r_2}}(A(x,y,c+\sqrt{r_2}) - A(x,y,c-\sqrt{r_2})) & \text{if } r_2 > 0 \\ A_z(x,y,c) & \text{if } r_2 = 0 \\ \frac{1}{2\sqrt{-r_2}}(A(x,y,c+i\sqrt{-r_2}) - A(x,y,c-i\sqrt{-r_2})) & \text{if } r_2 < 0 \end{cases} \ . \tag{10}$$

By abuse of notation, if M is a matrix of real analytic functions, $S.M$ and $D.M$ denote the matrices with the operator applied on each entry.

If A is a real analytic function, then $S.A$ and $D.A$ are also real analytic functions (see Lemma 6). This allows us to introduce the so-called *ball system* that we will use to compute Σ. In this system we map two solutions (x, y, z_1) and (x, y, z_2) of $P = Q = 0$ (or $P = P_z = 0$) to their center (x, y, c) and the square of their radius $r_2 = r^2$, with $r = |z_1 - c| = |z_2 - c|$. Figure 1 illustrates this mapping for singularities of the apparent contour of a torus. Its left part shows the surface $P = 0$, its set of z-critical points $\mathcal{C}_{P \cap P_z}$ and the apparent contour $\mathcal{C} = \Pi_{xy}(\mathcal{C}_{P \cap P_z})$. Its right part shows, for nodes and ordinary cusp singularities, their respective antecedents by Π_{xy}, centers c and radii r.

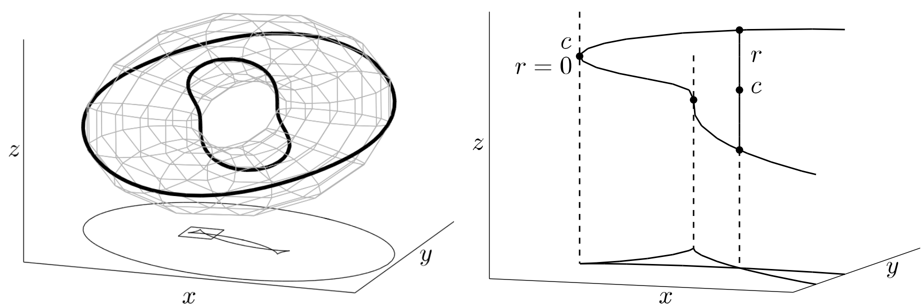

Fig. 1. Left: a torus, in bold line its set of z-critical points, its apparent contour, and the zoom zone corresponding to the right figure. Right: a detail, with antecedents, centers and radius corresponding to singularities.

Lemma 4. *Let $\mathcal{S}$ be the set of solutions of the so-called* ball system*:*

$$\begin{cases} S.P(x, y, c, r_2) = 0 \\ S.Q(x, y, c, r_2) = 0 \\ D.P(x, y, c, r_2) = 0 \\ D.Q(x, y, c, r_2) = 0 \end{cases} \tag{11}$$

in $B_0 \times \mathbb{R} \times \mathbb{R}^+$. Then $\Pi'_{xy}(\mathcal{S}) = \Sigma$, where Π'_{xy} is the projection from $\mathbb{R}^4$ to the (x, y)-plane.

Proof. According to Section 2, the singularity locus of $\mathcal{C}$ is exactly the union of the critical values of Π_{xy} and of the points that have several antecedents. They correspond respectively to the solutions of $\mathcal{S}$ such that $r = 0$ and such that $r > 0$. $\square$

One of the main advantage of this system is that its solutions are regular when the condition (A_5) is satisfied, and thus can be solved using certified numerical algorithms such as homotopy or subdivision methods (see Section 4).

Lemma 5. *Under the Assumptions* $(A_1)-(A_4)$*, all the solutions of the system* $S.P = S.Q = D.P = D.Q = 0$ *in* $B_0 \times \mathbb{R} \times \mathbb{R}^+$ *are regular if and only if* (A_5) *is satisfied.*

The next subsection is dedicated to the proof of this lemma.

3.2 Regularity Condition

Lemma 6. *If* A *is a real analytic function, then* $S.A$ *and* $D.A$ *are real analytic functions. Moreover, the derivatives of* $S.A$ *with respect to* x, y, c, r_2 *are respectively* $S.A_x, S.A_y, S.A_z, \frac{1}{2}D.A_z$*. The derivative of* $D.A$ *with respect to* x, y, c, r_2 *are respectively* $D.A_x, D.A_y, D.A_z$ *and* $\frac{S.A_z - D.A}{2r_2}$ *if* $r_2 > 0$ *and* $\frac{1}{6}A_{zzz}$ *if* $r_2 = 0$*.*

Proof. First, on a neighborhood of $r_2 > 0$, $S.A$ and $D.A$ are compositions of analytic functions, and thus are analytic. Likewise, for $r_2 < 0$, $S.A$ and $D.A$ are analytic functions, and all the coefficients of their series expansions are real, thus they are real valued analytic functions. Finally, on a neighborhood of $(x, y, c, 0)$, if $A(x, y, c + r) = \sum_{n=0}^{\infty} a_n(x, y, c) r^n$, the series expansions of $S.A$ and $D.A$ for $r_2 < 0$, $r_2 = 0$ and $r_2 > 0$ coincide as:

$$S.A(x, y, c, r_2) = \sum_{n=0}^{\infty} a_{2n}(x, y, c) r_2^n \tag{12}$$

$$D.A(x, y, c, r_2) = \sum_{n=0}^{\infty} a_{2n+1}(x, y, c) r_2^n \ . \tag{13}$$

Thus $S.A$ and $D.A$ are analytic functions. The expressions of their derivatives follow from the formulas. □

Lemma 7. *If* $\psi : U \subset \mathbb{R}^3 \mapsto V \subset \mathbb{R}^3$ *is an analytic diffeomorphism of the form* $\psi(x, y, z) = (\psi_1(x, y), \psi_2(x, y), \psi_3(x, y, z))$*, so-called* triangular*, then the mapping:*

$$SD.\psi : (x, y, c, r_2) \mapsto (\psi_1(x, y), \psi_2(x, y), S.\psi_3(x, y, c, r_2), r_2(D.\psi_3(x, y, c, r_2))^2)$$

is a real analytic diffeomorphism from $\{(x, y, c, r_2) \in \mathbb{R}^3 \times \mathbb{R}^+ \mid (x, y, c + \sqrt{r_2}) \in U\}$ *to* $\{(X, Y, C, R_2) \in \mathbb{R}^3 \times \mathbb{R}^+ \mid (X, Y, C + \sqrt{R_2}) \in V\}$*.*

Moreover, if $A : \mathbb{R}^3 \to \mathbb{R}$ *is an analytic map, we have:*

$$\begin{aligned} S.(A \circ \psi) &= (S.A) \circ (SD.\psi) \\ D.(A \circ \psi) &= (D.A) \circ (SD.\psi) \times D.\psi_3 \end{aligned}$$

Proof. According to the previous lemma, $SD.\psi$ is analytic. Moreover, since ψ^{-1} is analytic, $SD.(\psi^{-1})$ is also analytic. Assuming that the inequalities at the end of the lemma are correct, we can use them to check that $SD.(\psi^{-1}) \circ SD.\psi$ is the identity by developing the formula. Such that $SD.\psi$ is a diffeomorphism.

To prove the final identities of the lemma, let $(X, Y, C, R_2) = SD.\psi(x, y, c, r_2)$. We can observe that $\psi_3(x, y, c+\sqrt{r_2}) = C+\sqrt{R_2}$ and $\psi_3(x, y, c-\sqrt{r_2}) = C-\sqrt{R_2}$

by expanding $S.\psi_3 + \sqrt{r_2(D.\psi_3)^2}$ and $S.\psi_3 - \sqrt{r_2(D.\psi_3)^2}$. Using these formula, we can deduce the identities by expanding the right and left hand side of the equalities. ◻

Lemma 8. *Let P, Q be two analytic functions from $U \subset \mathbb{R}^3$ to $\mathbb{R}$ and assume that there exist two analytic functions $\tilde{P}, \tilde{Q}$, a 2×2 invertible matrix of analytic functions and a triangular diffeomorphism $\phi : U \to V \subset \mathbb{R}^3$ such that $\binom{P}{Q} = M \cdot \binom{\tilde{P}}{\tilde{Q}} \circ \phi$. Then we have:*

$$\begin{pmatrix} S.P \\ S.Q \\ D.P \\ D.Q \end{pmatrix} = \underbrace{\begin{pmatrix} S.M & r_2 D.\phi_3 D.M \\ & \\ D.M & D.\phi_3 S.M \end{pmatrix}}_{T} \begin{pmatrix} S.\tilde{P} \\ S.\tilde{Q} \\ D.\tilde{P} \\ D.\tilde{Q} \end{pmatrix} \circ SD.\phi$$

where the matrix T is invertible, of inverse $\tilde{T} := \begin{pmatrix} S.M^{-1} & r_2 D.M^{-1} \\ D.M^{-1}/D.\phi_3 & S.M^{-1}/D.\phi_3 \end{pmatrix}$.

Proof. First, using the identity $ab + cd = \frac{1}{2}(a+c)(b+d) + \frac{1}{2}(a-c)(b-d)$, we can deduce :

$$\begin{pmatrix} S.P \\ S.Q \\ D.P \\ D.Q \end{pmatrix} = \begin{pmatrix} S.M & r_2 D.M \\ & \\ D.M & S.M \end{pmatrix} \begin{pmatrix} S.(\tilde{P} \circ \phi) \\ S.(\tilde{Q} \circ \phi) \\ D.(\tilde{P} \circ \phi) \\ D.(\tilde{Q} \circ \phi) \end{pmatrix}$$

Finally, expanding the operators in the righthand side vector using the formula in Lemma 7, we prove the desired identity. Finally, since ϕ is a triangular diffeomorphism, we can use the formula of Lemma 7 with $A = \phi_3^{-1}$ to get $1 = D((\phi^{-1})_3 \circ \phi) = D.(\phi^{-1})_3 \circ SD.\phi \times D.\phi_3$. In particular, $D.\phi_3$ is never 0 and $\tilde{T}$ is well defined. Expanding $\tilde{T} \cdot T$, we get the identity, such that $\tilde{T}$ is the inverse of T.

Corollary 2. *A point p solution of the system $S.P = S.Q = D.P = D.Q = 0$ is regular if and only if the point $SD.\phi(p)$ is regular in the system $S.\tilde{P} = S.\tilde{Q} = D.\tilde{P} = D.\tilde{Q} = 0$.*

Proof. The claim of the lemma can be verified by developing the product vector. For the corollary, it is sufficient to observe that on a point p solution of the system, the Jacobian matrices satisfy the relation:

$$\mathrm{Jac}_p \begin{pmatrix} S.P \\ S.Q \\ D.P \\ D.Q \end{pmatrix} (p) = T \cdot \mathrm{Jac}_{SD.\phi(p)} \begin{pmatrix} S.\tilde{P} \\ S.\tilde{Q} \\ D.\tilde{P} \\ D.\tilde{Q} \end{pmatrix} \cdot \mathrm{Jac}_p(SD.\phi) \ .$$

◻

We have now all the tools necessary to prove Lemma 5.

Proof (of Lemma 5). First, let q be a solution of our system with $r_2 = 0$. Then, according to Lemma 3, there exists an invertible matrix M and a triangular diffeomorphism ϕ such that on a neighborhood of q we have:

$$\begin{pmatrix} P \\ Q \end{pmatrix} = M \cdot \begin{pmatrix} X - Z^{3+2k} \\ Y - Z^2 \end{pmatrix} \circ \Phi(x, y, z) \ .$$

Thus, the point q is regular in the ball system if and only if $(0, 0, 0)$ is regular in the ball system generated by $X - Z^{3+2k}$ and $Y - Z^2$ (Corollary 2). Computing the associated Jacobian matrix, we can check that q is regular if and only if $k = 0$, that is, if and only if its projection p is an ordinary cusp.

Now, let $q = (x, y, c, r_2)$ be a solution of the ball system with $r_2 > 0$. In this case q represents two points $q_1 = (x, y, c + \sqrt{r_2})$ and $q_2 = (x, y, c - \sqrt{r_2})$ of $\mathcal{C}_{P\cap Q}$ with the same projection. According to Lemma 6 the equation $\det Jac_{(x,y,c,r_2)}(S.P, S.Q, D.P, D.Q) = 0$ can be written

$$\begin{vmatrix} S.P_x & S.P_y & S.P_z & \frac{D.P_z}{2} \\ S.Q_x & S.Q_y & S.Q_z & \frac{D.Q_z}{2} \\ D.P_x & D.P_y & D.P_z & \frac{S.P_z - D.P}{2r_2} \\ D.Q_x & D.Q_y & D.Q_z & \frac{S.Q_z - D.Q}{2r_2} \end{vmatrix} = 0$$

This determinant simplifies using the facts that a) $D.P = D.Q = 0$ at the solutions, b) one can multiply lines 3 and 4 by $\sqrt{r_2}$ and column 4 by $2\sqrt{r_2}$, c) one can replace lines ℓ_1, ℓ_3 by $\ell_1 + \ell_3, \ell_1 - \ell_3$ and ℓ_2, ℓ_4 by $\ell_2 + \ell_4, \ell_2 - \ell_4$. The equation is then equivalent to

$$\begin{vmatrix} P_x(q_1) & P_y(q_1) & P_z(q_1) & P_z(q_1) \\ Q_x(q_1) & Q_y(q_1) & Q_z(q_1) & Q_z(q_1) \\ P_x(q_2) & P_y(q_2) & P_z(q_2) & -P_z(q_2) \\ Q_x(q_2) & Q_y(q_2) & Q_z(q_2) & -Q_z(q_2) \end{vmatrix} = 0$$

Expending this expression, one can check that it is equivalent to

$$\begin{vmatrix} P_y(q_1)Q_z(q_1) - P_z(q_1)Q_y(q_1) & P_y(q_2)Q_z(q_2) - P_z(q_2)Q_y(q_2) \\ P_z(q_1)Q_x(q_1) - P_x(q_1)Q_z(q_1) & P_z(q_2)Q_x(q_2) - P_x(q_2)Q_z(q_2) \end{vmatrix} = 0$$

The later expression is equivalent to the condition that projection on the (x, y) plane of the tangent vectors of the 3D curve $\mathcal{C}_{P\cap Q}$ at the points q_1 and q_2 are collinear. Thus in the case where $r_2 > 0$, a solution of the ball system is regular iff it projects to a node. $\square$

4 Experiments

We propose some quantitative results on the isolation of the singularities of the projection $\mathcal{C}$ of a space real curve $\mathcal{C}_{P\cap Q}$ (or $\mathcal{C}_{P\cap P_z}$ in the case of an apparent contour) by solving the ball system proposed in this paper. We consider here that P and Q are polynomials, hence the equations of the ball system are polynomials and $\mathcal{C}$ admits at most finitely many singularities in $\mathbb{R}^2$. Under our assumptions, the curve $\mathcal{C}'$ defined as the resultant of P and Q with respect to z ($Q = P_z$ in the case of an apparent contour) is the union of $\mathcal{C}$ and a finite set of isolated points. Its singularities can be characterized as real solutions of a bivariate system based on the sub-resultant chain of P and Q (or P_z) (see [15]). We compare the resolution with three state-of-the-art methods of the sub-resultant system, denoted by $\mathcal{S}_2$ in what follows, and the ball system $S.P = S.Q = D.P = D.Q = 0$ defined in Subsection 3.1, denoted by $\mathcal{S}_4$.

Experimental Data are random dense polynomials P, Q generated with degree d and integer coefficients chosen uniformly in $[\![-2^8, 2^8]\!]$. Unless explicitly stated, the given running times are averages over five instances for a given degree d.

Testing Environment is a Intel(R) Xeon(R) CPU L5640 @ 2.27GHz machine with Linux.

4.1 Resolution Methods

Gröbner Basis and Rational Univariate Representations allow one to find all real roots of a system of polynomials. The routine `Isolate` of the mathematical software `Maple` implements this approach.

Homotopy Continuation provides all the complex solutions of a system of polynomials and relies on a numerical path-tracking step. Among available open-source software implementing homotopy, we chose `Bertini`[1] notably because it handles both double precision (DP) and an Adaptive Multi-Precision (AMP) arithmetics [4]. This is necessary to prevent the loss of solutions in system $\mathcal{S}_2$ which coefficients are quotients of big integers (see Table 2).

Subdivision uses interval arithmetic (see [21,25,29] for an introduction) to compute for a given system all its regular solutions lying in an initial open box $B_0 \subset \mathbb{R}^n$. Here $n = 2$ for system $\mathcal{S}_2$ and $n = 4$ for system $\mathcal{S}_4$. When P, Q are polynomials, the initial box can be $\mathbb{R}^n$ (see [25, p.210] or [29, p.233]). Otherwise, B_0 is bounded, and the number of singularities is finite. Since we focus on singularities induced by projection of real parts of the curve $\mathcal{C}_{P\cap Q}$ or $\mathcal{C}_{P\cap P_z}$, we did only research solutions of the ball system having $r_2 \geq 0$. We implemented a subdivision solver in `c++`, using the `boost` or `mpfi` interval arithmetic library. The implementation is described with more details in [15].

4.2 Singularities Isolation: Comments on Tables 1, 2 and 3

Tables 1, 2 and 3 report the sequential running times (columns t) in seconds to compute the singularities of projection and apparent contour curves, using system $\mathcal{S}_2$ or system $\mathcal{S}_4$ to represent their singularities.

[1] https://bertini.nd.edu/

Table 1 shows that for `Isolate` running times are better when solving system $\mathcal{S}_2$, due to its lower number of variables.

Table 2 refers to resolution with `Bertini`, using DP and AMP arithmetics. In addition to running times, it reports the number of missed solutions (columns Mis. Sols.) when using DP arithmetic. The resolution by homotopy in DP of system $\mathcal{S}_2$ is not satisfactory due to the high number of missed solutions. The use of AMP arithmetic resolves this problem: for all systems we tested, all solutions were found. But it induces an important additional cost. System $\mathcal{S}_4$ seems better suited to homotopy resolution. In DP arithmetic, fewer solutions are missed and the cost of AMP arithmetic is more acceptable. Notice however that for three examples, a solution was missed both with DP and AMP arithmetic due to the truncation of a path considered as converging to a solution at infinity.

Table 3 reports results obtained with our implementation of subdivision. For a given degree, resolution times are subject to an important variance. For low degrees it is more efficient to solve system $\mathcal{S}_2$ than system $\mathcal{S}_4$ due to the higher dimension (i.e. 4 instead of 2) of the research space in the latter case. The difference of running times decreases when d increases, due to the size (in terms of degree, number of monomials and bit-size of coefficients) of the resultant and sub-resultant polynomials that have to be evaluated to solve system $\mathcal{S}_2$.

Table 1. Isolating singularities of projection and apparent contour curves with the routine `Isolate` of `Maple`. Input polynomials have degree d. The running times are in seconds.

	Projection		Apparent contour	
	system $\mathcal{S}_2$	system $\mathcal{S}_4$	system $\mathcal{S}_2$	system $\mathcal{S}_4$
d	t	t	t	t
4	1.321	4.293	0.206	0.1874
5	26.92	100.4	5.439	6.501
6	(a)	(a)	98.59	155.8
7	(a)	(a)	(a)	(a)

(a) Fails with error

5 Conclusion

Given an analytic curve $\mathcal{C}_{P\cap Q}$ satisfying some specific generic assumptions, we have described the different possible types of singularities Σ of its projection $\mathcal{C} = \Pi_{xy}(\mathcal{C}_{P\cap Q})$. Moreover we have shown that these singularities can be computed as the regular solutions of a new so-called *ball system*.

Even if our characterization increases the number of variables of the system to solve in order to compute Σ, we have shown with experimental results that

Table 2. Isolating singularities of projection and apparent contour curves with `Bertini` using DP and AMP arithmetic. Input polynomials have degree d. The running times are in seconds.

	`Bertini` with DP arithmetic											
	Projection						Apparent contour					
	system $\mathcal{S}_2$			system $\mathcal{S}_4$			system $\mathcal{S}_2$			system $\mathcal{S}_4$		
d	t	Mis. Sols.		t	Mis. Sols.		t	Mis. Sols.		t	Mis. Sols.	
4	0.864	0		1.376	1	(c)	0.174	0		0.46	1	
5	16.03	3		8.326	0		3.638	0		3.818	2	(c)
6	177.6	2		40.21	0		54.49	1		20.80	1	
7	1458	193		152.1	1	(c)	617.9	6		88.50	0	
8	$\geq$ 3000	599	(b)	508.5	3		2799	885		319.3	0	
9	$\geq$ 3000	1389	(b)	1429	7		$\geq$ 3000	1178	(b)	935.6	2	

(b) Has been run on a unique example
(c) Solution(s) is (are) missing due to infinite path(s) truncation

	`Bertini` with AMP arithmetic					
	Projection			Apparent contour		
	system $\mathcal{S}_2$	system $\mathcal{S}_4$		system $\mathcal{S}_2$	system $\mathcal{S}_4$	
d	t	t		t	t	
4	2.332	1.804	(c)	2.332	1.434	
5	147.8	13.888		147.852	15.01	(c)
6	$\geq$ 3000	123.41		1005	165.7	
7	$\geq$ 3000	1089	(c)	$\geq$ 3000	1147	
8	$\geq$ 3000	$\geq$ 3000		$\geq$ 3000	$\geq$ 3000	

Table 3. Isolating singularities of projection and apparent contour curves with subdivision. Input polynomials have degree d. The average running times t are given in seconds together with the standard deviation σ.

	Projection		Apparent contour	
	system $\mathcal{S}_2$	system $\mathcal{S}_4$	system $\mathcal{S}_2$	system $\mathcal{S}_4$
d	t $\pm$ σ	t $\pm$ σ	t $\pm$ σ	t $\pm$ σ
4	0.078 $\pm$ 0.03	0.759 $\pm$ 0.02	0.040 $\pm$ 0.02	1.509 $\pm$ 1.97
5	0.351 $\pm$ 0.13	1.973 $\pm$ 0.72	0.251 $\pm$ 0.23	25.34 $\pm$ 47.5
6	1.918 $\pm$ 0.55	6.442 $\pm$ 3.07	1.353 $\pm$ 0.57	11.38 $\pm$ 6.98
7	9.528 $\pm$ 3.92	22.43 $\pm$ 8.36	124.1 $\pm$ 142	54.21 $\pm$ 50.3
8	42.69 $\pm$ 16.8	57.00 $\pm$ 16.4	57.72 $\pm$ 63.7	99.22 $\pm$ 89.3
9	163.3 $\pm$ 111	137.5 $\pm$ 93	54.74 $\pm$ 33.3	95.11 $\pm$ 44.5

the ball system can be solved with numerical methods. With homotopy it is more often complete and faster to solve the latter system than the sub-resultant system. A certified resolution is provided by a subdivision solver. In term of computational cost, such solvers are known to suffer from the increase of the dimension of the research space. However for high degrees of input polynomials, the price to pay for solving the sub-resultant system seems higher than the one induced by the increasing of number of variables.

Finally, our characterization could be extended to higher dimensions, for instance for singularities of the projection in 2D of a curve in 4D or the projection in 3D of a surface in 4D.

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