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Gerber and Chopin combine SMC with RQMC to accelerate convergence. They apply RQMC as in the array-RQMC method discussed below, for which convergence rate theory remains thin despite impressive empirical performance. Their proof of $o(N^{-1/2})$ convergence rate is a remarkable contribution.

Array-RQMC simulates an array of N dependent realizations of a Markov chain, using RQMC so the distribution of the N states has low discrepancy D_t with respect to its theoretical distribution, at any step t. If $\mathbf{x}_t = \Gamma_t(\mathbf{x}_{t-1}, \mathbf{u}_t)$, where $\mathbf{x}_{t-1} \sim \mathcal{U}([0,1)^\ell)$ and $\mathbf{u}_t \sim \mathcal{U}([0,1)^d)$, array-RQMC approximates the $(\ell+d)$-dimensional integral $\mathbb{E}[D_t]$, using N RQMC points. It matches each state to a point whose first ℓ coordinates are near that point, and uses the next d coordinates for $\mathbf{u}_t$.

The *matching* step is crucial. If $\ell = 1$, just sort the states and points by their (increasing) first coordinate. If $\ell > 1$, one can map the states $\mathbf{x}_t \in [0,1)^\ell \to [0,1)$, then proceed as for $\ell = 1$. Gerber and Chopin do exactly this, using a Hilbert curve mapping. But the map $[0,1)^\ell \to [0,1)$ is not essential. E.g., if $\ell = 2$, one can sort the states and the points in $N^{-1/2}$ groups of size $N^{-1/2}$ by their first coordinate, then sort each group by the second coordinate. This extends to $\ell > 2$. With this multivariate sort, L’Ecuyer et al. (2009) observed an $\mathcal{O}(N^{-2})$ variance when pricing an Asian option with array-RQMC. I wonder what rate the Hilbert curve map can achieve for this example.

When $\mathcal{X} \subseteq \mathbb{R}^\ell$ instead of $[0,1)^\ell$, Gerber and Chopin map $\mathbf{x}_t \to [0,1)^\ell$ using a logistic transformation. The choice of transformation and its parameters may have a significant impact on the overall convergence. For the multivariate sort, no transformation is needed, it works directly in $\mathbb{R}^\ell$. The Hilbert sort could also be adapted to work directly in $\mathbb{R}^\ell$.

The SMC part uses importance sampling and particle resampling, a form of splitting (Kahn and Harris, 1949; L’Ecuyer et al., 2007) with fixed effort (N remains constant). Combinations of array-RQMC with several variants of splitting and with importance sampling were examined and tested by L’Ecuyer et al. (2007). Fixed-effort better fits array-RQMC than splitting variants that produce a random N at each step, and stratification improves the multinomial resampling process.

Array-RQMC was previously developed to estimate the expectation of a function of the state, not in the context of filtering for parameter estimation and for estimating posterior densities. But the same ideas apply. L’Ecuyer et al. (2008) proved an $\mathcal{O}(n^{-3/2})$ convergence rate for the variance for $\ell = 1$ and special types of RQMC points. Faster rates have been observed empirically in examples. Proving those rates in a general setting (under appropriate conditions) is challenging and deserves further research effort.

References

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