Harmonic-like Identification of Nonlinear Systems around an Equilibrium

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December 17, 2015

Abstract: In this work, we propose a method to compute the poles of the linear approximation to a nonlinear system, locally around a stable equilibrium point. The method uses periodic inputs, just like hermonic identification of linear systems. We show in passing that the class of Wiener-Hammerstein systems is but a small subset of the collection of all nonlinear control systems, locally around a hyperbolic equilibrium.For this we dwell on the exact local topological linearization of linear systems as developed in [6].

Keywords: Control, Identification.

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1 Introduction

Several classes of models have been proposed for black-box nonlinear system identification. They range from Volterra and multiple Fourier series [1, 2] to Hammerstein systems (*i.e.* a static nonlinear map followed by a linear dynamical system) or Wiener systems (*i.e.* a linear dynamical system followed by a static nonlinear map) and combinations thereof like the Wiener-Hammerstein model (a linear system sandwiched between two static nonlinearities) [3]. A variant of the Wiener model, where the dynamics is nilpotent, has become especially popular in recent years under the name of Nonlinear Autoregressive Moving Averages with Exogenous inputs (NARMAX) models [4]. Still, it is natural to ask how general these classes of models with respect to the wild variety of all nonlinear models.

In the present work, we approach a local, but otherwise fairly general version of nonlinear identification for finite-dimensional continuous time systems around an equilibrium point, provided that the latter is hyperbolic meaning that the linearized dynamics has no pure imaginary eigenvalue. We show in passing that Wiener-Hammerstein systems are very special, and highly non-generic in the class of all such systems, even locally. More importantly, we also show that stable nonlinear models lend themselve locally to a generalization of classical harmonic identification of linear systems, at least when the state is observed. To be specific, we will see that one can in principle determine the poles of the linearized approximation from the response to periodic inputs.

For this, we dwell on local linearization results from [6] saying that when a nonlinear system with observed state, around a hyperbolic equilibrium point, is feeded forward by another finite-dimensional nonlinear system, then it is locally conjugate to a linear system feeded forward in the same way. As will be established below, the local character in time of this result can be eliminated if the initial system is stable (however, the result remains local with respect to initial states). Now, in this construction, the conjugacy map generally depends on the feeding system that was used, and observable Wiener-Hammerstein systems are those for which such a dependency does not hold; this is why they are so special. The results we just mentioned appear in sections 2 and 3.

Pursuing this line of investigation, we set up in section 4 a framework for harmonic identification of stable nonlinear system with observed state around an equilibrium point, and we establish equations for identification to be solved in this context.

2 Local linearization of control systems

We consider a control system with observed state of the form:

$$\dot{x} = f(x, u)\ , \quad x \in \mathbb{R}^n\ , \quad u \in \mathbb{R}^m\ , \tag{1}$$

and we suppose that $f(0,0) = 0$, *i.e.* we work around an equilibrium point that we choose to be the origin without loss of generality. By a solution of (1) that remains in an open set $\Omega \subset \mathbb{R}^{n+m}$, we mean a mapping γ defined on a real interval I, say

$$\begin{array}{rccl} \gamma: & I & \to & \Omega \\ & t & \mapsto & \gamma(t) \;=\; (\,\gamma_{\mathrm{I}}(t)\,,\,\gamma_{\mathrm{II}}(t)\,) \end{array} \tag{2}$$

with $\gamma_{\mathrm{I}}(t) \in \mathbb{R}^n$ and $\gamma_{\mathrm{II}}(t) \in \mathbb{R}^m$, such that :

- γ is measurable, locally bounded, and γ_{I} is absolutely continuous,
- whenever $[T_1, T_2] \subset I$, we have :

$$\gamma_{\mathrm{I}}(T_2) - \gamma_{\mathrm{I}}(T_1) \;=\; \int_{T_1}^{T_2} f(\,\gamma_{\mathrm{I}}(t)\,,\,\gamma_{\mathrm{II}}(t)\,)\,dt\ .$$

Given another system

$$\dot{z} = g(z, v)\ , \quad z \in \mathbb{R}^n\ , \quad v \in \mathbb{R}^m\ , \tag{3}$$

we say that system (1) is conjugate to (3) if there is a local homeomorphism

$$\begin{array}{rccl} \chi : & \Omega & \rightarrow & \Omega' \\ & (x,u) & \mapsto & \chi(x,u) \ = \ (\, \chi_{\mathrm{I}}(x,u)\, ,\, \chi_{\mathrm{II}}(x,u)\,) \end{array} \tag{4}$$

between two open subsets of $\mathbb{R}^{n+m}$ such that, for any real interval I, a map $\gamma : I \to \Omega$ is a solution of (1) that remains in Ω if, and only if, $\chi \circ \gamma$ is a solution of (3) that remains in Ω'. In other words, trajectories are mapped to trajectories in a pointwise bijective, bi-continuous, and time-preserving manner. When the control variables u, v are not present, so that the component $\chi_{\mathrm{II}}(x,u)$ does not appear, we recover the definition of local conjugacy for ordinary differential equations (*i.e.* autonomous systems).

We say that (1) is locally linearizable if it is topologically conjugate to a linear control system at $(0,0)$.

Now, let us assume that f is continuously differentiable with respect to x and u. Subsequently, we single out the linear part of f by setting $A = \frac{\partial f}{\partial x}(0,0)$ and $B = \frac{\partial f}{\partial u}(0,0)$ so that (1) can be rewritten as

$$\begin{array}{rcl} \dot{x} & = & Ax \ + \ Bu \ + \ F(x,u) \\ \text{with} \quad F(0,0) & = & \dfrac{\partial F}{\partial x}(0,0) \ = \ \dfrac{\partial F}{\partial u}(0,0) \ = \ 0\ . \end{array} \tag{5}$$

If (1) is locally linearizable, then it is easy to see that it must be topologically conjugate to its linear approximation $\dot{z} = Az + Bu$. However, it is shown in [5] that this property is seldom satisfied, that is, it cannot hold unless some rather stringent conditions hold on f. In fact, it is proved in [5] that linearizability of system (1) implies that it is linearizable under feedback and that the linearizing feedback has quite a bit of smoothness; then, classical necessary conditions for smooth feddback local linearization [8] are to the effect that the derivatives of f must satisfy some integrability conditions which are highly non-generic.

As a next best thing, however, the following theorem was proved in [6]:

Theorem 1.[6, thm.3.1] *With f as in (5), consider the system*

$$\begin{array}{rcl} \dot{x} & = & Ax \ + \ Bh(\zeta) \ + \ F(x, h(\zeta)), \\ \dot{\zeta} & = & g(\zeta), \end{array} \tag{6}$$

where $g : \mathbb{R}^q \to \mathbb{R}^q$ is locally Lipschitz continuous, $h : \mathbb{R}^q \to \mathbb{R}^m$ is continuous with $h(0) = 0$, and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuously differentiable with $F(0,0) = \partial F/\partial x(0,0) = 0$, A being hyperbolic. Then, there exist neighborhoods V and W of 0 in $\mathbb{R}^n$ and $\mathbb{R}^q$ respectively, and a map $H : V \times W \to \mathbb{R}^n$ with $H(0,0) = 0$, such that

$$\begin{array}{rccc} H \times Id: & V \times W & \rightarrow & \mathbb{R}^n \times W \\ & (x,\zeta) & \mapsto & (H(x,\zeta), \zeta) \end{array}$$

is a homeomorphism from $V \times W$ onto its image that conjugates (6) to

$$\begin{array}{rcl} \dot{z} & = & Az \ + \ Bh(\zeta), \\ \dot{\zeta} & = & g(\zeta). \end{array} \tag{7}$$

Note that Theorem 1 tells us about conjugacy of ordinary differential equations rather than control systems. In this connection, the result comes close to the classical Grobman-Hartman theorem asserting that, around a hyperbolic equilibrium point, a differential equation is locally topologically conjugate to its linear approximation [7]. However, there is one important difference here, namely no hyperbolicity assumption is made on $\partial g/\partial\zeta(0)$, the derivative of g at zero, and it is not assumed that $g(0) = 0$, *i.e.* zero needs not be a fixed point of g. Accordingly, linearization in (7) is carried out with respect to the variable z only. The fact that $\partial g/\partial\zeta(0)$ needs not be hyperbolic in Theorem 1 will be crucial to us later on as we shall use periodic $\zeta(t)$ for harmonic identification. Let us mention that the linearizing homeomorphism is unique, up to composition with a linear invertible transformation (which induces a change of basis in the state space of the linearized system). We also mention that, even if , g, h are very smooth, the linearizing map $H(x,\zeta)$ needs not be differentiable at the origin: only Hölder-continuity will usually hold there.

3 Autonomous feed-forward of locally stable systems

In this section, we make Theorem 1 global in time for stable A and appropriate g.

Consider system (1) and let us restrict attention to inputs u that are themselves outputs of a (possibly nonlinear) control system:

$$\begin{array}{rcl} \dot{\zeta} & = & g(\zeta), \\ u & = & h(\zeta), \end{array} \tag{8}$$

where $\zeta(t) \in \mathbb{R}^q$, while $g : \mathbb{R}^q \to \mathbb{R}^q$ is locally Lipschitz continuous and $h : \mathbb{R}^q \to \mathbb{R}^m$ is continuous. In particular, $u(t)$ is entirely determined by the finite-dimensional data $\zeta(0)$. This operating mode is of course quite special and, from the control viewpoint, it may be described as open loop feed-forward of system (1) by system (8). Note that particular use of such a feed-forward occurs in classical harmonic identification of stable linear systems, where the system to be identified is subject to periodic inputs in order to estimate pointwise values of its transfer-function from the steady-state response. In fact, such a description of the steady-state response to a periodic signal at a given frequency as an operator which depends on the frequency is widely used in circuit theory, even in a nonlinear setting [9]. The discussion in section 4 will dwell on this framework.

Now, if we feed forward system (1) with system (8), we obtain system (6). Thus, from Theorem 1, we deduce the result below which gives additional information when system (1) is locally stable. As a piece of terminology, let us agree that the equilibrium $(0,0)$ of system (1) is said to be strictly stable if all eigenvalues of $A = \partial f/\partial x(0,0)$ have strictly negative real parts, that is, if the tangent linear control system at $(0,0)$ is strictly stable. Moreover, if system (8) happens to have an equilibrium at ζ_0, that is, if $g(\zeta_0) = 0$, then this equilibrium is said to be strongly asymtotically stable if initial conditions sufficiently close to ζ_0 at $t = 0$ generate trajectories $\zeta(t)$ that remain arbitrarily close to ζ_0 for all $t \geq 0$.

Theorem 2. *Consider system (1) where f is continuously differentiable with respect to both arguments in a neighborhood of $(0,0)$ and $f(0,0) = 0$. Set $A = \frac{\partial f}{\partial x}(0,0)$ and $B = \partial f/\partial u(0,0)$ so that (1) may be rewritten as (5). Assume that A is hyperbolic (i.e. has no purely imaginary eigenvalues). Consider also system (8) where g is Lipschitz continuous and h is continuous, say in the neighborhood of some $\zeta_0 \in \mathbb{R}^q$.*

(i) If we feed system (1) with initial state $x(0)$ by the output of system (8) with initial state $\zeta(0)$, and if we denote by $\hat{x}(t, x(0), \zeta(0))$ the solution at time t of the first equation in (6), then there exist two neighborhoods V and W of 0 and ζ_0 in $\mathbb{R}^n$ and $\mathbb{R}^q$ respectively, and a map $H : V \times W \to \mathbb{R}^n$ with $H(0,0) = 0$, of the form

$$\begin{array}{rccc} H \times Id: & V \times W & \to & \mathbb{R}^n \times W \\ & (x,\zeta) & \mapsto & (H(x,\zeta),\zeta) \end{array},$$

which is a homeomorphism from $V \times W$ onto its image such that

$$\begin{array}{rcl} H(\hat{x}(t, x(0), \zeta(0)), \zeta(t)) & = & e^{At} H(x(0), \zeta(0)) \\ & + & \int_0^t e^{A(t-\tau)} B h(\zeta(\tau))\, d\tau \end{array} \tag{9}$$

for all $x(0) \in V$, $\zeta(0) \in W$, and all t such that $\hat{x}(t, x(0), \zeta(0)) \in V$ and $\zeta(t) \in W$.

(ii) When the equilibrium of (1) is strictly stable, and if system (8) has an equilibrium at ζ_0 which is strongly asymtotically stable with $h(\zeta_0) = 0$, then (i) holds for all $t \geq 0$, possibly at the cost of shrinking V and W.

Proof: to deduce (i) from Theorem 1, first extend f, g,and h to the whole of $\mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^q$ respectively, keeping the same regularity, using a partition of unity. This step is purely formal, since Theorem 2 is local and therefore the result will not depend on the precise extension which is made. It is necessary, though, because Theorem 1 was stated for globally defined maps. We may also assume without loss of generality that $\zeta_0 = 0$. Then, we observe that the right hand side of (9) is a solution to

$$\dot{z} = Az + Bu, \qquad u(t) = h(\zeta(t)), \tag{10}$$

with initial condition $z(0) = H(x(0), \zeta(0))$, and we appeal to the local conjugacy of (6) to (7). As to (ii), note that if the equilibrium of (1) is strictly stable and the equilibrium of (8) is strongly asymtotically stable, and if moreover $h(\zeta(0)) = 0$, then $(\hat{x}(t, x(0), \zeta(0)), \zeta(t))$ will stay in $V \times W$ for all $t \geq 0$ provided that $x(0)$ and $\zeta(0)$ are small enough. Indeed, a strictly stable linear system subject to an input which is sufficiently small for all $t \geq 0$ (namely (10)) has a state-trajectory that remains arbitrarily close to the origin if the initial state is small enough. Thus, on a possibly smaller neighborhood $V \times W$ of the origin, the linearization in equation (9) will be valid for all $t \geq 0$.

Assumptions and notation being the same as in Theorem 2, let $\Omega \subset \mathbb{R}^n \times W$ be the image of the homeorphism $H \times Id$. Its inverse is clearly of the form $\Psi \times Id : \Omega \to V \times W$. Now, if we let $\hat{z}(t, z(0), u)$ indicate the state at time t of the linear control system $\dot{z} = Az + Bu$ with initial state $z(0)$ and control $u(t)$, then (9) expresses that

$$\hat{x}(t, x(0), \zeta(0)) = \Psi\Big(\hat{z}\big(t, H(x(0), \zeta(0)), h(\zeta)\big), \zeta(t)\Big). \tag{11}$$

Thus, for fixed input $u(t) = h(\zeta(t))$ resulting from a given $\zeta(0)$, the state trajectory $\hat{x}(t, x(0), \zeta(0))$ is obtained by applying the nonlinear map $\Psi(., \zeta(t))$ to the state of the linear control system $\dot{z} = Az + Bu$ initialized at $z(0) = H(x(0), \zeta(0))$.

In the special case where system (1) is of Hammerstein type, so that we can write

$$\dot{x} \quad = \quad A_1 x \ + \ B_1 \varphi(u) \tag{12}$$

where (A_1, B_1) is a controllable pair in $\mathbb{R}^{n\times(n+m_1)}$ with A strictly stable and φ a map from $\mathbb{R}^m$ into $\mathbb{R}^{m_1}$ assumed to be continuous, we see from (12), upon setting in (11) $m = m_1$, $q = m$, $A = A_1$, $B = B_1$ and $h = \varphi$, that $H = \psi = Id$ will satisfy the latter equation when $u = \zeta$, no matter what the function ζ is. In particular, a Hammerstein system fed with the state of an arbitrary nonlinear system is a cascade of the form (11) where Ψ (thus also H) is independent of its second argument. This indicates that such systems are extremely special and highly non-generic among all nonlinear systems, already locally around a strictly stable equilibrium point.

4 Harmonic-like identification

In this section, we assume throughout that f is continuously differentiable and that (1) has a strictly stable equilibrium at $(0, 0)$. For simplicity, we also assume that (1) has a single input. Let us pick system (8) to be a simple harmonic oscillator whose frequency is parametrized by the initial condition of the system's equations:

$$\begin{array}{rcl} \zeta &=& (\zeta_1, \zeta_2, \omega) \\ h(\zeta) &=& \zeta_1 \end{array} \qquad \begin{array}{rcl} \dot{\zeta}_1 &=& -\lambda\zeta_2, \\ \dot{\zeta}_2 &=& \lambda\,\zeta_1, \\ \dot{\lambda} &=& 0. \end{array} \tag{13}$$

Subject to initial conditions $\zeta_1(0) = a$ and $\zeta_2(0) = b$, and $\lambda(0) = \omega$, the solution to (13) is given by $\lambda(t) = \omega$ and

$$\begin{array}{rcl} \zeta_1(t) &=& a\cos(\omega t) - b\sin(\omega t), \\ \zeta_2(t) &=& a\sin(\omega t) + b\cos(\omega t), \end{array} \tag{14}$$

so that $\zeta_1(t) + i\zeta_2(t) = (a + ib)e^{i\omega t}$. Since $\|\zeta\|^2 = \zeta_1^2 + \zeta_2^2$ is independent of t, we certainly have that system (13) is strongly asymptotically stable in the neighborhood of an equilibrium $(0, 0, \omega_0)$. Therefore, by Theorem 2 point (ii), (9) is valid for all $t \geq 0$ if a, b are small enough and ω varies in the neighborhood of some reference frequency ω_0.

By (13), (14) and the commutativity of convolution, we may rewrite the right-hand side of (9) in the case at hand as

$$e^{At} H(x(0), \zeta(0)) + \int_0^t e^{A\tau} B\big(a\cos(\omega(t-\tau)) - b\sin(\omega(t-\tau)\big)\, d\tau.$$

Now, granted that A is strictly stable so that e^{At} goes to zero exponentially fast as $t \to +\infty$, the above quantity is equivalent for large positive t to

$$\begin{aligned}
&\int_0^\infty e^{A\tau} B(a\cos(\omega(t-\tau)) - b\sin(\omega(t-\tau)))\,d\tau \\
&= \int_0^\infty e^{A\tau} B\,\mathrm{Re}\Big((a+ib)e^{i\omega(t-\tau)}\Big)\,d\tau \\
&= \mathrm{Re}\left((a+ib)e^{i\omega t}\int_0^\infty e^{A\tau} B\, e^{-i\tau}\,d\tau\right) \\
&= \mathrm{Re}\left((a+ib)e^{i\omega t} F(i\omega)\right),
\end{aligned}$$

where $F(i\omega) = (i\omega Id - A)^{-1}B$ is the Fourier transform of $e^{A\tau}B$ evaluated at ω, that is to say F is the transfer function of the linear control system $\dot{z} = Az + Bu$.

From (9) and the previous calculation, we now see that for large positive t

$$H(\hat{x}(t, x(0), \zeta(0)), \zeta(t)) \sim \mathrm{Re}\left((a+ib)e^{i\omega t} F(i\omega)\right), \tag{15}$$

where the symbol "$\sim$" means that the difference of the two quantities involved goes to zero uniformly (in fact exponentially fast) as $t \to +\infty$. Recalling from (11) the definition of Ψ and using that it is Hölder continuous, we deduce from (15) that

$$\hat{x}(t, x(0), \zeta(0)) \sim \Psi\Big(\mathrm{Re}\left((a+ib)e^{i\omega t} F(i\omega)\right), \zeta(t)\Big), \tag{16}$$

as $t \to +\infty$. Because $\zeta(t)$ is $2\pi/\omega$-periodic by (14), this entails that $\hat{x}(t, x(0), \zeta(0))$ tends uniformly to a limit cycle given by the right-hand side of (16). In particular, this limit cycle is independent of $x(0)$. We will denote it by $\Gamma(t, \zeta(0))$.

Note that, from the identification point of view, $\Gamma(t, \zeta(0))$ can be estimated from the steady-state output of (1) when the state of the latter is observed, at least for large enough t. Hereafter, we assume that this is indeed the case and that $\Gamma(t, \zeta(0))$ is known for all t sufficiently large. The relation to be identified, which generalizes standard harmonic identification for linear systems, may then be written as:

$$\Psi\Big(\mathrm{Re}\left((a+ib)e^{i\omega t} F(i\omega)\right), \zeta(t)\Big) = \Gamma(t, \zeta(0)), \tag{17}$$

where $\zeta(0) = (a, b, \omega)$ is a vector of control variables which can be chosen arbitrarily in a neighborhhod of $(0, 0, \omega_0)$, while (A, B, Ψ) is the set of unknown quantities to be determined. Observe, due to the presence of Ψ, that this set of unknowns lies in no finite dimensional space.

To better understand the situation, let us consider first the simplest but already interesting case where $n = 1$ so that F is complex-valued. Let $\mathcal{V}_0$ be a ball around $(0, 0)$ in $\mathbb{R}^2$ and $\mathcal{W}_0$ be a real interval centered at some $\omega_0 \neq 0$, such that (16), thus also (17) are valid for all $(a, b) \in \mathcal{V}_0$ and $\omega \in \mathcal{W}_0$. Pick such a triple $(a, b, \omega) = \zeta(0)$ with $(a, b) \neq (0, 0)$, thereby giving rise to a limit cycle $\Gamma(t, \zeta(0))$ through (17). Choose $\omega' \in \mathcal{W}_0$, with $\omega' \neq \omega$, and define for each $\tau \in \mathbb{R}$ a pair $(a_\tau, b_\tau) \in \mathcal{V}_0$ by the formula

$$a_\tau + ib_\tau = (a+ib)e^{i(\omega-\omega')\tau}. \tag{18}$$

Let $\zeta_\tau(t)$ be the solution to (13) with initial conditions a_τ, b_τ, ω'. *We claim* that there are arbitrary large τ_0 for which

$$\Gamma(\tau_0, \zeta(0)) = \Gamma(\tau_0, \zeta_{\tau_0}(0)). \tag{19}$$

Indeed, since Ψ is injective and $\zeta(\tau) = \zeta_\tau(\tau)$ for each τ by (18), relation (19) is equivalent to the equality

$$\mathrm{Re}\left((a+ib)e^{i\omega\tau_0} F(i\omega)\right) = \mathrm{Re}\left((a_{\tau_0} + ib_{\tau_0})e^{i\omega\tau_0} F(i\omega')\right) \tag{20}$$

which amounts to

$$\operatorname{Re}\left((a+ib)e^{i\omega\tau_0}\big(F(i\omega)-F(i\omega')\big)\right)=0 \tag{21}$$

since $(a+ib)e^{i\omega\tau_0}=(a_{\tau_0}+ib_{\tau_0})e^{i\omega\tau_0}$. Clearly, relation (21) holds if and only if

$$\tau_0=-\frac{\arg\{(a+ib)(F(i\omega)-F(i\omega'))\}+k\pi}{\omega},\quad k\in\mathbb{N}, \tag{22}$$

which provides us with infinitely many equally spaced positive solutions to (19), *thereby proving the claim.*

By (22), finding τ_0 meeting (19) reduces to a 1-dimensional search for the zeros of

$$\tau\mapsto\Gamma(\tau,\zeta(0))-\Gamma(\tau,\zeta_\tau(0)) \tag{23}$$

over any real interval of length at least π/ω whose center is a sufficiently large real number. As both $\Gamma(\tau,\zeta(0))$ and $\Gamma(\tau,\zeta_\tau(0))$ can be evaluated for each τ large enough, τ_0 can in principle be found by dichtomy because of the continuity of the map in (23). Note in passing that the two estimates involved have different degree of complexity: $\Gamma(\tau,\zeta(0))$ can be obtained for all τ in one stroke by recording the output of system (1) with input $h(\zeta)$ where ζ is the solution to (13) initialized at (a,b,ω), whereas *each* value $\Gamma(\tau,\zeta_\tau(0))$ requires to feed system (1) with input $h(\zeta_\tau)$ where ζ_τ is the solution to (13) initialized at (a_τ,b_τ,ω'), and then to evaluate the corresponding output at time τ.

Once τ_0 is found, the pole of F is easily computed. For if we write $F(z)=c/(z+d)$ where c,d are nonzero real numbers and $d>0$ by the assumed stability of F, and if we set $(a+ib)=\rho e^{i\theta_0}$ where $\rho>0$, then we can rewrite (21) as

$$\operatorname{Im}\left(\frac{e^{i(\omega\tau_0+\theta_0)}}{(i\omega+d)(i\omega'+d)}\right)=0, \tag{24}$$

which is equivalent to d being the positive root of the equation

$$z^2-z\frac{\omega+\omega'}{\tan(\omega\tau_0+\theta_0)}-\omega\omega'=0. \tag{25}$$

in the unknown z. Note that $\tan(\omega\tau_0+\theta_0)\neq 0$, since $d>0$ hence $e^{i(\omega\tau_0+\theta_0)}$ cannot be real by (24). Solving (25), the coefficients of which are known, gives us d as announced. Altogether, we were able to recover F up to a multiplicative constant which is the best we can hope for since pre-multiplication by a constant can be incorporated to the definition of Ψ. Section 5 will illustrate this procedure numerically.

We now discuss the case where the dimension n of system (1) is arbitrary. Then $F(i\omega)$ is of the form

$$F(i\omega)=\frac{1}{q(i\omega)}\begin{pmatrix}p_1(i\omega)\\ p_2(i\omega)\\ \cdots\\ p_n(i\omega)\end{pmatrix} \tag{26}$$

where q is a monic polynomial of degree n with roots in the open left half-plane, since F is assumed to be stable, and $p_1,\cdots p_n$ are polynomials of degree at most $n-1$ with real coefficients which are jointly coprime with q.

We pick the dynamics of system (8) to be a concatenation of n harmonic oscillators of the form (13) with respective states $(\zeta_{1,j},\zeta_{2,j},\omega_j)$ for $1\le j\le n$. This results in a global state

$$\zeta=(\zeta_{1,1},\zeta_{2,1},\omega_1,\zeta_{1,2},\zeta_{2,2},\omega_2,\cdots,\zeta_{1,n},\zeta_{2,n},\omega_n), \tag{27}$$

with initial condition

$$\zeta(0)=(a_1,b_1,\omega_1,a_2,b_2,\omega_2,\cdots,a_n,b_n,\omega_n), \tag{28}$$

and we set $h(\zeta)=\Sigma_{j=1}^n\zeta_{1,j}$. This time, computations similar to those which led us to (16) yield

$$\hat{x}(t,x(0),\zeta(0))\sim\Psi\Big(\operatorname{Re}\left(\Sigma_{j=1}^n(a_j+ib_j)e^{i\omega_j t}F(i\omega_j)\right),\zeta(t)\Big), \tag{29}$$

as $t \to +\infty$ and, instead of (17), we get

$$\Psi\Big(\mathrm{Re}\left(\Sigma_{j=1}^{n}(a_j+ib_j)e^{i\omega_j t}F(i\omega_j)\right),\zeta(t)\Big)=\Gamma(t,\zeta(0)), \tag{30}$$

where $\Gamma(t,\zeta(0))$ indicates a limit cycle in $\mathbb{R}^n$ which we suppose again to be known, since it can in principle be estimated by observing the steady state of system (1) fed by system (8), provided that the $\zeta_{k,j}(0)$ are sufficiently small and that the ω_j are sufficiently close to reference frequencies $\omega_{0,j}$. Extending a previous notation, we let $\mathcal{V}_0$ be a ball around $(0,0)$ in $\mathbb{R}^2$ and $\mathcal{W}_{0,j}$ be a real interval centered at $\omega_{0,j}$ such that (29) and (30) are valid as soon as all $(a_j,b_j)\in\mathcal{V}_0$ and $\omega_{0,j}\in\mathcal{W}_{0,j}$ for all $j\in\{1,\cdots,n\}$.

Pick $(\omega_1,\cdots,\omega_n)\in\mathcal{W}_{0,1}\times\cdots\times\mathcal{W}_{0,n}$ so that they are all distinct, nonzero, and the diference between any two of them is a rational multiple of 2π: $\omega_i-\omega_j=\pi 2n_{i,j}/d_{i,j}$ for some relative integers $n_{i,j},d_{i,j}$. Choose also $(\omega_1',\cdots,\omega_n')\in\mathcal{W}_{0,1}\times\cdots\times\mathcal{W}_{0,n}$ with $\omega_j'\neq\omega_j$. The $n+1$ vectors

$$\left\{\mathrm{Re}(F(i\omega_j)-F(\omega_j')),\mathrm{Im}(F(i\omega_n)-F(\omega_n'))\right\}_{1\le j\le n}$$

are linearly dependent over $\mathbb{R}$, hence there are real numbers ρ_j, $1\le j\le n$, not all zero, and a real number θ_0 such that

$$\mathrm{Re}\Big(\Sigma_{j=1}^{n-1}\rho_j\left(F(i\omega_j)-F(i\omega_j')\right)$$

$$+\rho_n e^{i\theta_0}\left(F(i\omega_n)-F(i\omega_n')\right)\Big)=0.$$

Set $\tau_0=\theta_0/\omega_n$, $a_n=\rho_n$, $b_n=0$, and $(a_j+ib_j)=\rho_j e^{-i\omega_j\tau_0}$, $1\le j\le n-1$. Then, we have that

$$\mathrm{Re}\left(\Sigma_{j=1}^{n}(a_j+ib_j)e^{i\omega_j\tau_0}\left(F(i\omega_j)-F(i\omega_j')\right)\right)=0, \tag{31}$$

and in (31) we may replace τ_0 by any number of the form τ_0+kd where k is an interger and d a common multiple of the $d_{i,j}$. In addition, we may assume that $\rho_1^2+\cdots+\rho_n^2$ is fixed and small enough that $(a_j,b_j)\in\mathcal{V}_0$ for $j\in\{1,\cdots,n\}$. Next, paralleling (18), we define for $\tau\in\mathbb{R}$:

$$a_{j,\tau}+ib_{j,\tau}=(a_j+ib_j)e^{i(\omega_j-\omega_j')\tau},\quad 1\le j\le n. \tag{32}$$

Let $\zeta(t)$, $\zeta_\tau(t)$ be defined by (27) with initial condition

$$\zeta(0)=(a_1,b_1,\omega_1,a_2,b_2,\omega_2,\cdots,a_n,b_n,\omega_n)$$

and

$$\zeta_\tau(0)=(a_{1,\tau},b_{1,\tau},\omega_1',a_{2,\tau},b_{2,\tau},\omega_2',\cdots,a_{n\tau},b_{n,\tau},\omega_n')$$

respectively. From (32), (31) and (30), we deduce as before (*cf.* (19)) that there are $(a_j,b_j)\in\mathcal{V}_0$, $1\le j\le n$, and arbitrary large τ_0 for which

$$\Gamma(\tau_0,\zeta(0))=\Gamma(\tau_0,\zeta_{\tau_0}(0)). \tag{33}$$

Since both sides of (33) can be observed for τ_0 large enough by our very assumptions, it is in principle possible to find adequate values for a_j, b_j, and τ_0.

This time of course, determining the (a_j,b_j) and τ_0 no longer reduces to a one-dimensional search that can be solved by dichotomy: it amounts to find zeros of a function of $n+1$ variables on a rectangular domain (the numbers ρ_j and τ_0 as introduced before (31)). Granted the normalization $\rho_1^2+\cdots+\rho_n^2=\varepsilon$ small enough, this is indeed the correct number of unknowns to fulfill the n equations in (31)). The problem is difficult, of the same type as black-box optimization, and we shall not discuss here which algorithm or heuristics is adapted to solve it. We merely assume hereafter that the (a_j,b_j) and τ_0 can be found.

Subsequently, we obtain a family of equations of the form (31), parametrized by the ω_j' (we fix hereafter the ω_j). We explain below how these equations allow us to determine the dynamics of the linearized system, that is, the polynomial q in (26). For the sake of simplicity, we only discuss the generic case where the p_j in

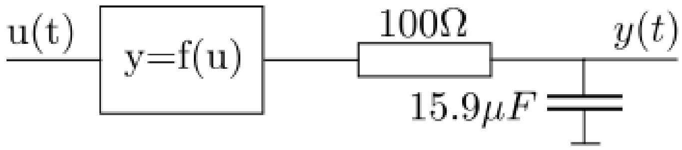

Figure 1: System

(26) are linearly independent over $\mathbb{R}$. Then, upon redefining Ψ in (29) and (30) by composing it to the right with a linear invertible transformation, we may assume that $p_j(z) = z^{j-1}$ so that (26) can be rewritten as

$$F(i\omega) = \frac{1}{q(i\omega)} \begin{pmatrix} 1 \\ i\omega \\ \cdots \\ (i\omega)^{n-1} \end{pmatrix}. \tag{34}$$

Then,

$$F(i\omega_j) - F(i\omega'_j) = \frac{q(-i\omega_j)q(-i\omega'_j)}{|q(i\omega_j)|^2|q(i\omega'_j)|^2} \tag{35}$$

$$\times \begin{pmatrix} q(i\omega'_j) - q(i\omega_j) \\ i\omega q(i\omega'_j) - i\omega'_j q(i\omega_j) \\ \cdots \\ (i\omega)^{n-1} q(i\omega'_j) - (i\omega'_j)^{n-1} q(i\omega_j) \end{pmatrix},$$

and substituting in (31) gives us, after chasing denominators, a system of n polynomial equations of degree $4n-1$ in the n unknows coefficients of q (the fact that when $n=1$ we found in (25) an equation for d of degree 2 rather than 3 is because the constant coefficient factors out in (35) when $n=1$). This system can be solved by elimination, using for instance a Gröbner basis algorithm. The fact that, for generic choices of $\omega'_1, \cdots, \omega'_j$) at least, the set of solution indeed consists of isolated n-tuples (*i.e.* has dimension 0) and that exactly one of them yields a stable polynomial requires an argument that is beyond the scope of the present paper. The reconstruction of the map Ψ in (17), which could in principle be carried out pointwise once q is known, is also left for future study.

In the previous analysis, we did not take into account uncertainty arising from measurement errors nor numerical sensitivity. In this connection, let us mention that the ω'_j are essentially (locally) free parameters of the procedure described above, that can be used to cross-correlate the results and compute averages. We shall not dwell on this issue.

5 Numerical experiments

In this section, we illustrate the procedure described in the previous section in the case of a first order system, *i.e.* when $n=1$. Although this is the simplest instance of our method, it is already interesting and it displays essential features of the algorithm without having to grapple with a heavy numerical search. The system under consideration is the simple circuit shown in figure 1, with input $u(t)$ and output $y(t)$ which is equal to the state $x(t)$. The nonlinear map f is given by $f(u) = u - 10u^2$, and the values of the resistor and capacitor are $R = 100\Omega$ and $C = 15.9\mu F$ respectively. The system has a fixed point at $(0,0)$ with strictly stable linear approximation, and in fact $A \sim -628.9$. Note that the system is of Hammerstein type although this played no role in our analysis. With the notation of (20), we pick $\omega = 100\pi$ and $\omega' = 150\pi$, corresponding to frequencies $50Hz$ and $75Hz$ respectively. The vector (a,b) is chosen to be $(0.001, 0)$ which corresponds to a small amplitude signal. Figure 2 shows the difference between both sides of (20) (*i.e.* the left hand side of (20)) when τ_0 gets replaced by a generic time instant τ, as a function of τ over a period. The seeked values for τ_0, that is, time instants at which the right hand side of (23) vanishes, are those τ for

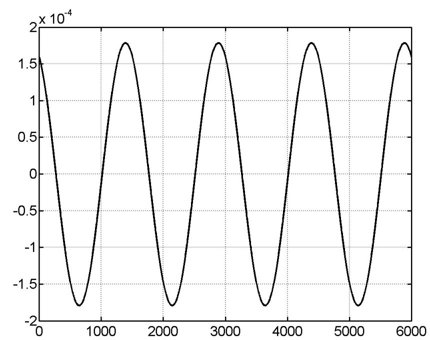

Figure 2: $\Gamma(\tau, \zeta(0)) - \Gamma(\tau, \zeta_\tau(0))$ as a function of τ.

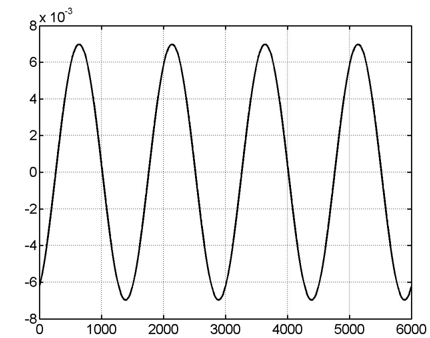

Figure 3: $\Gamma(\tau, \zeta(0)) - \Gamma(\tau, \zeta_\tau(0))$ as a function of τ.

which this difference is zero. It must be said that the use of harmonic balance techniques to simulate the output of the system gets us rid from the need to wait for the transient to vanish and the output to become in steady state. If we select for instance the value $\tau_0 = 0.5351$ and solve (25) (with $\theta_0 = 0$ and $\rho = 0.001$), we find that the positive root is 627.4 which is a rather good approximation of the theoretical d, namely 628.9.

We should mention that if we perform the same experiment except that $(a, b) = (0.1, 0)$, that is, if we use a much larger amplitude signal, the error on d becomes as bad as 1000 (to be compared with 628.9), which seems to indicate that we leave the domain where the exact linearization asserted in Theorem 2 is valid.

The system in figure 1 has infinitely many other equilibrium points than $(0, 0)$, associated with constant inputs. We performed experiments similar to the above for the value $u \equiv 2$. In figure 3, we show the diagram corresponding to figure 2 for this case. If we select for τ_0 from this diagram, whichever zero we choose, we find that the distance to the theoretical d when solving (25) is consistently of the order of 10^{-5} (to be compared with 628.9), which is excellent.

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