



# Realizability games for arithmetical formulæ

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# REALIZABILITY GAMES FOR ARITHMETICAL FORMULÆ

ÉTIENNE MIQUEY

## TALK PROPOSAL

Game semantics is a useful tool for reasoning on proof of arithmetical formulæ. Given a  $\Sigma_{2k}^0$ -formula

$$\Phi = \exists x_1 \forall y_1 \dots \exists x_k \forall y_k f(\vec{x}, \vec{y}) = 0$$

Coquand defines a natural intuitionistic game between two players named Eloise (defender) and Abelard (opponent) [1]. Basically, Eloise has to give a value  $m_1$  for the first existential quantifier, then Abelard replies with a value  $n_1$  for the universal quantifier that follows. Next Eloise gives another value for the second existential quantifier, and so on until all the quantifiers are instantiated. We check then whether  $\mathbb{N} \models f(\vec{m}, \vec{n}) = 0$  : if so, we say Eloise wins, otherwise Abelard does. This game is easily turned into a classical game by giving Eloise the ability of changing her mind and starting again from a former position. Such a game admits winning strategies if and only if  $\mathbb{N} \models \Phi$

*Krivine's classical realizability* [6] is a reformulation of Kleene's realizability allowing to reason with classical principle. It uses as language of terms the  $\lambda_c$ -calculus, which is Church's  $\lambda$ -calculus enriched with (at least) the control operator `call/cc`. To each formula  $A$  of Peano Arithmetic is associated a truth value  $|A|$ , the set of *realizers*, which is defined by orthogonality to a falsity value  $\|A\|$ . The elements of  $\|A\|$  are stacks, and realizers of  $A$  are the terms that are somehow able to pass successfully the bunch of tests in  $\|A\|$ . In practice, a realizability model is given by a pole, which is a set of processes  $t \star \pi$ . Intuitively, we can think of realizers as defenders and stacks as opponents, the referee being the pole.

In this talk, we will explain how to combine the underlying intuition of a game in classical realizability with Coquand's game, in the case of arithmetical formulæ. Our framework is a reformulation of games as defined by Krivine [5] or Guillermo [2], using an inductive presentation that is more suitable for proofs. Given a formula  $\exists x \forall y f(x, y) = 0$ , a winning strategy for Eloise is a term  $t$  that will interact with its opponent, putting on the stack the integers it plays, possibly backtracking to a former state, and eventually choosing a winning position. After explaining more in details the principle of the game, we will show that the winning strategies are exactly the realizers.

This characterization is broken as soon as we add some non-substitutive instructions to the language of terms (e.g. `quote`), but we will show that we can recover it, changing slightly the rules of the game. Finally, we will explain how this characterization allows us to tackle two important problems.

- First, it gives a precise specification of the realizers of arithmetical formulæ: Whereas in intuitionistic realizability the computational content of a realizer does not contain more information than the formula itself, in classical realizability, the problem is much more subtle, due to the presence of control operator.
- Second, it proves the absoluteness of  $\Sigma_n^0$ -formulæ for classical realizability. We know that classical realizability can simulate forcing, which is known to be limited to  $\Sigma_2^1/\Pi_2^1$ -formulæ by Shoenfield's barrier. But knowing whether realizability might go below this barrier or even lower was unknown.

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