Erratum to: Computational complexity of stochastic programming problems

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Corrigendum

Computational Complexity of Stochastic Programming Problems, *Mathematical Programming 106(3)*, 423-432, 2006.

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Hanasusanto et al. [HKW] pointed out a mistake in our paper. To be precise, the error occurs in the first formula on page 429. We gratefully acknowledge their careful reading of our paper. We repair the error here in this corrigendum. We have chosen to completely rewrite Section 3.2 of the paper. We have done this so that the correction can be read almost independently of the rest of the paper, and to take the opportunity to improve the exposition.

3.2 Continuous distributions

For two-stage stochastic programming problems with continuously distributed parameters, $\sharp$P-hardness of obtaining the optimal solution can be established under even the mildest conditions on the distributions. For the proof we use a reduction from the problem of computing the volume of a knapsack polytope, proved $\sharp$P-hard in [3].

Let $\mathbb{Z}_+$ denote the nonnegative integers, and $\mathbb{Q}_+$ the rational numbers.

Definition 3.2. Let $P = \{w \in [0,1]^n \mid \sum_{j=1}^n \alpha_j w_j \leq \beta\}$ be the *knapsack polytope*, where $\alpha_j \in \mathbb{Z}_+$ $(j = 1, \dots, n)$ and $\beta \in \mathbb{Q}_+$. We consider the following two computational problems.

(i) Counting knapsack solutions. Compute $N(P) = |\{0,1\}^n \cap P|$.

(ii) Volume of knapsack polytope. Compute $\mathrm{Vol}(P)$, the volume of P.

These problems are $\sharp$P-hard. In [3], Volume of knapsack polytope was proved $\sharp$P-hard by reduction from the known $\sharp$P-hard problem Counting knapsack solutions.

Claim 1. Counting knapsack solutions remains $\sharp$P-hard if $\alpha_n = \lfloor \beta \rfloor + 1$.

Proof. If $\alpha_n = \lfloor \beta \rfloor + 1$, let $P' = \{w^{n-1} \in [0,1]^{n-1} \mid \sum_{j=1}^{n-1} \alpha_j w_j \leq \lfloor \beta \rfloor\}|$. Then $N(P') = N(P)$, since $\alpha_j \in \mathbb{Z}_+$ $(j = 1, \dots, n-1)$, and $w \in \{0,1\}^n \cap P$ implies $w_n = 0$. Clearly, computing $N(P')$ is $\sharp$P-hard. Hence computing $N(P)$ is $\sharp$P-hard when $\alpha_n = \lfloor \beta \rfloor + 1$. $\square$

Claim 2. Volume of knapsack polytope remains $\sharp$P-hard if $\alpha_n > \beta$.

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Proof. For P as in Claim 1, let $P_i = \{w \in \{0,1\}^n \mid \sum_{j=1}^{n} \alpha_j w_j \leq \lfloor \beta \rfloor + i/n\}$, where $(0 \leq i < n)$. So $N(P_i) = N(P)$ for all $0 \leq i < n$, since $\alpha_j \in \mathbb{Z}_+$ $(j = 1, \ldots, n)$, and $\lfloor \beta \rfloor \leq \lfloor \beta \rfloor + i/n < \lfloor \beta \rfloor + 1$. Thus computing $N(P_i)$ is ♯P-hard.

The reduction on p. 970 of [3] implies that, if computing $N(P)$ is ♯P-hard, then computing $\mathrm{Vol}(P_k)$ is ♯P-hard, for some $0 \leq k < n$. It follows that computing $\mathrm{Vol}(P)$ is ♯P-hard if $\beta = \lfloor \beta \rfloor + k/n < \lfloor \beta \rfloor + 1 = \alpha_n$. □

Theorem 3.2. *Determining the optimal solution of a two-stage stochastic programming problem with continuously distributed parameters is* ♯P*-hard, even if all the stochastic parameters have the uniform* $[0, 1]$ *distribution.*

Proof. Define i.i.d. random variables $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-1}$, each uniformly distributed on $[0, 1]$, and write $\mathbf{q}^{n-1} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{n-1})$. Now, given an instance of computing the volume of a knapsack polytope as in Claim 2, consider the following two-stage stochastic programming problem with continuously distributed parameters:

$$\max\{-cx + Q(x) \mid 0 \leq x \leq 1\},$$

where

$$Q(x) = \mathbb{E}_{\mathbf{q}^{n-1}}\big[\max\{\beta y - \textstyle\sum_{i=1}^{n-1} \boldsymbol{q}_j y_j \mid 0 \leq y \leq x,\ y_j \geq \alpha_j y\ \ (j = 1, \ldots, n)\}\big].$$

For any realisation q^{n-1} of $\mathbf{q}^{n-1}$, the optimal solution of the second stage problem is $y_j = \alpha_j y \geq 0$, $(j = 1, \ldots, n-1)$, so

$$Q(x) = \mathbb{E}_{\mathbf{q}^{n-1}}\big[\max\{(\beta - \textstyle\sum_{i=1}^{n-1} \boldsymbol{q}_j \alpha_j) y \mid 0 \leq y \leq x\}\big].$$

Thus $y = x$ if $\sum_{j=1}^{n-1} q_j \alpha_j \leq \beta$, and $y = 0$ if $\sum_{j=1}^{n-1} q_j \alpha_j > \beta$.

Notice that, since $\alpha_n > \beta$, the constraint $w_n \leq 1$ is redundant in P in Claim 2. Define $\varphi(w^{n-1}) = (\beta - \sum_{j=1}^{n-1} \alpha_j w_j)/\beta$. Thus $\varphi(w^{n-1}) \leq 1$ in P' as defined in the proof of Claim 1. In P this implies that $\alpha_n w_n/\beta \leq \varphi(w^{n-1})$.

Now, the solution value of the two-stage problem can be written as

$$\max\big\{\big(\beta\Phi - c\big)x \mid 0 \leq x \leq 1\big\}, \text{ where } \Phi = \mathbb{E}_{\mathbf{q}^{n-1}}\big[\max(\varphi(\mathbf{q}^{n-1}), 0)\big].$$

Thus $x = 1$, with optimal objective value $\beta\Phi - c$, if $\Phi > c/\beta$, and $x = 0$, with optimal objective value 0, otherwise. Therefore, computing the optimal solution to the stochastic program requires determining whether or not $\Phi = \mathbb{E}_{\mathbf{q}^{n-1}}\big[\max(\varphi(\mathbf{q}^{n-1}), 0)\big] > c/\beta$.

Claim 3. Computing Φ is ♯P-hard.

Proof. Since, from Claim 2 above, computing $\mathrm{Vol}(P)$ is ♯P-hard, this claim follows from

$$\begin{aligned}
\Phi = \mathbb{E}_{\mathbf{q}^{n-1}}\big[\max(\varphi(\mathbf{q}^{n-1}), 0)\big] &= \int_{P'} \varphi(w^{n-1})\,\mathrm{d}w^{n-1}, \text{ since } 0 \leq \varphi(w^{n-1}) \leq 1 \text{ in } P', \\
&= \int_{P'} \int_0^{\varphi(w^{n-1})} 1\ \mathrm{d}(\alpha_n w_n/\beta)\,\mathrm{d}w^{n-1}, \\
&= (\alpha_n/\beta) \int_P 1\ \mathrm{d}w_n\,\mathrm{d}w^{n-1}, \text{ since } 0 \leq \alpha_n w_n/\beta \leq \varphi(w^{n-1}) \text{ in } P, \\
&= (\alpha_n/\beta)\mathrm{Vol}(P).
\end{aligned}$$

□

Now, from [3], it follows that Vol(P) is a rational number with known denominator, when P is a knapsack polytope. Hence we can compute Vol(P) exactly if we can compute it to a close enough approximation. Hence there must exist values of c for which deciding $\Phi > c/\beta$ is ♯P-hard. Otherwise we could use bisection to compute the ♯P-hard quantity Vol(P) exactly, in a polynomial number of iterations, a contradiction. □

Showing that this problem is in $\mathrm{P}^{\sharp\mathrm{P}}$ would require additional conditions on the input distributions. We note that a result of Lawrence [9] implies that exact computation may not even be in PSPACE.

References

[3] M. E. Dyer and A. M. Frieze. On the complexity of computing the volume of a polyhedron. *SIAM Journal on Computing*, 17:967–974, 1988.

[9] J. Lawrence. Polytope volume computation. *Math. Comput.*, 57:259–271, 1991.

[HKW] G. A. Hanasusanto, D. Kuhn and W. Wiesemann. A comment on "Computational complexity of stochastic programming problems", 2015. `http://www.optimization-online.org/DB_HTML/2015/03/4825.html`