# Tensor decomposition and homotopy continuation

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**Abstract**

A computationally challenging classical elimination theory problem is to compute polynomials which vanish on the set of tensors of a given rank. By moving away from computing polynomials via elimination theory to computing pseudowitness sets via numerical elimination theory, we develop computational methods for computing ranks and border ranks of tensors along with decompositions. More generally, we present our approach using joins of any collection of irreducible and nondegenerate projective varieties $X_1, \ldots, X_k \subset \mathbb{P}^N$ defined over $\mathbb{C}$. After computing ranks over $\mathbb{C}$, we also explore computing real ranks. Various examples are included to demonstrate this numerical algebraic geometric approach.

**Key words and phrases.** tensor rank, homotopy continuation, numerical elimination theory, witness sets, numerical algebraic geometry, joins, secant varieties.

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## Introduction

Suppose that $X \subset \mathbb{P}^N$ is an irreducible and nondegenerate projective variety defined over $\mathbb{C}$. A point P is a nonzero vector in $\mathbb{C}^{N+1}$ while $[P]$ defines the line in $\mathbb{C}^{N+1}$ passing through the origin and $P$, i.e., $[P] \in \mathbb{P}^N$ is the projectivization of $P \in \mathbb{C}^{N+1}$. The $X$-*rank* of $[P] \in \mathbb{P}^N$ (or of $P \in \mathbb{C}^{N+1}$), denoted $\mathrm{rk}_X(P)$, is the minimum $r \in \mathbb{N}$ such that $P$ can be written as a linear combination of $r$ elements of $\mathcal{C}(X)$:

$$P = \sum_{i=1}^{r} x_i, \quad x_i \in \mathcal{C}(X), \tag{1}$$

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where $\mathcal{C}(X)$ is the affine cone of the projective variety $X$.

Let $\sigma_r^0(X) \subset \mathbb{P}^N$ denote the set of elements with rank at most $r$ and, for $[x_i] \in \mathbb{P}^N$, let $\langle [x_1], \ldots, [x_r] \rangle$ denoted the linear space spanned by $x_1, \ldots, x_r$. The $r^{\text{th}}$ *secant variety* of $X$ is

$$\sigma_r(X) = \overline{\sigma_r^0(X)} = \overline{\bigcup_{[x_1],\ldots,[x_r] \in X} \langle [x_1], \ldots, [x_r] \rangle}.$$

If $[P] \in \sigma_r(X)$, then $[P]$ is the limit of a sequence of elements of $X$-rank at most $r$. The $X$-*border rank* of $[P]$, denoted $\mathrm{brk}_X(P)$, is the minimum $r \in \mathbb{N}$ such that $[P] \in \sigma_r(X)$. Obviously, $\mathrm{brk}_X(P) \leq \mathrm{rk}_X(P)$.

Secant varieties are just particular cases of join varieties. In particular, for irreducible and nondegenerate projective varieties $X_1, \ldots, X_k$, the constructible join and join variety of $X_1, \ldots, X_k$, respectively, are

$$J^0(X_1, \ldots, X_k) = \bigcup_{[x_1] \in X_1, \ldots, [x_k] \in X_k} \langle [x_1], \ldots, [x_k] \rangle \quad \text{and} \quad J(X_1, \ldots, X_k) = \overline{J^0(X_1, \ldots, X_k)}. \quad (2)$$

Clearly, $\sigma_r^0(X) = J^0(\underbrace{X, \ldots, X}_{r})$ and $\sigma_r(X) = J(\underbrace{X, \ldots, X}_{r})$.

In principle, one can test if an element belongs to a certain join variety (or if it has certain $X$-border rank) by computing defining equations for the join variety (or the secant variety, respectively). Unfortunately, finding defining equations for secant and join varieties is generally a very difficult elimination problem which is far from being well understood at this time.

The knowledge of the $X$-rank of a particular element is an open condition. In fact, $\sigma_r^0(X)$ is almost always an open subset of $\sigma_r(X)$, so membership tests for $\sigma_r^0(X)$ based on evaluating polynomials to determine the $X$-rank of a given element do not exist in general. Currently, there are few theoretical algorithms for specific cases, e.g., [31, 16, 25, 17, 56, 65, 8, 83].

Equations defining secant and join varieties are used to decide membership but do not yield information about decompositions. Computing such decompositions plays a fundamental role in numerous application areas including computational complexity, signal processing for telecommunications [33, 40], scientific data analysis [58, 75], electrical engineering [30], and statistics [63]. Some other applications include the complexity of matrix multiplication [82], the complexity problem of P versus NP [84], the study of entanglement in quantum physics [42], matchgates in computer science [84], the study of phylogenetic invariants [4], independent component analysis [32], blind identification in signal processing [74], branching structure in diffusion images [72], and other multilinear data analytic techniques in bioinformatics and spectroscopy [34].

Rather than focusing on computing defining equations, e.g., via classical elimination theory, this paper uses numerical algebraic geometry (e.g, see [12, 80] for a general overview) for performing membership tests and computing decompositions. In particular, we use *numerical elimination theory* to perform the computations based on the methods developed in [51, 52] (see also [12, Chap. 16]). These differ from several previous approaches of combining numerical algebraic geometry and elimination theory, e.g., [10, § 3.3-3.4] and [78, 79], in that these previous methods relied upon interpolation. Section 1 describes our approach for join varieties.

Once an element is known to be in $J(X_1, \ldots, X_k)$ or $\sigma_r(X)$, numerical elimination theory can also be used to decide if the element is in the corresponding constructible set $J^0(X_1, \ldots, X_k)$ or $\sigma_r^0(X)$. Moreover, an approach presented in [51] can be used to compute the codimension one component of $\overline{J(X_1, \ldots, X_k) \setminus J^0(X_1, \ldots, X_k)}$ and $\overline{\sigma_r(X) \setminus \sigma_r^0(X)}$. For example, this corresponds to the codimension one component of the closure of the set of points of $X$-border rank at most $r$ whose rank is larger than $r$. These computations are the focus of Section 2.

Another problem considered in this paper from the numerical point of view regards computing real decompositions of a real element. For example, after computing the $X$-rank $r$ of a real element $P$, we would like to know if it has a decomposition using $r$ real elements. That is, we determine if the real $X$-rank of $P$ is the same as its complex $X$-rank. This is discussed in Section 3 using the homotopy-based approach of [45]. Computing the real $X$-rank (not approximated) has recently been studied by various authors, especially in regards to the typical ranks of symmetric tensors, i.e., $r$ such that the symmetric tensors whose real rank is $r$ is an open real set. Theoretic results in this direction are provided in [6, 9, 19, 21, 23, 29, 35].

We emphasize that this approach works well on generic elements. For example, a numerical algebraic geometric based approach was presented in [50] for computing the total number of decompositions of a general element in the so-called perfect cases, i.e., when the general element has finitely many decompositions. In every case, one can track a single solution path defined by a Newton homotopy to compute the decomposition of a general element, as shown in Section 4. The simplicity of our method for generic elements suggests that our algorithm can be used to test the regularity or the defectivity of certain secant variety. A secant variety is in fact said to be defective if it has dimension smaller than the expected dimension. Terracini's lemma is often used to locate defective cases, with [14] considering a numerical version. One way to search for defective secant varieties is to consider the first one that should fill the ambient space. Thus, we believe that our approach can be used to investigate other possibly defective cases.

The last section, Section 5, considers several examples.

# 1 Membership tests

Let $X_1, \ldots, X_k \subset \mathbb{P}^N$ be irreducible and nondegenerate projective varieties with $J^0(X_1, \ldots, X_k)$ and $J(X_1, \ldots, X_k)$ defined by (2). This section focuses on the join variety $J(X_1, \ldots, X_k)$ while the next focuses on the construcible set $J^0(X_1, \ldots, X_k)$. Consider the smooth irreducible variety

$$\mathcal{J} = \left\{ ([P], x_1, \ldots, x_k) \;\middle|\; x_i \in \mathcal{C}(X_i), P = \sum_{i=1}^{k} x_i \right\} \subset \mathbb{P}^N \times \mathcal{C}(X_1) \times \cdots \times \mathcal{C}(X_k), \tag{3}$$

where $\mathcal{C}(X_i)$ is the affine cone on the projective variety $X_i$. The variety $\mathcal{J}$ is called the *abstract join variety*. For the projection $\pi([P], x_1, \ldots, x_k) = [P]$, it is clear that

$$\pi(\mathcal{J}) = J^0(X_1, \ldots, X_k) \quad \text{and} \quad \overline{\pi(\mathcal{J})} = J(X_1, \ldots, X_k). \tag{4}$$

The key to using the numerical elimination theory approaches of [51, 52] is to perform all computations on the abstract join variety $\mathcal{J}$. Moreover, one only needs a numerical algebraic geometric description, i.e., a witness set or a pseudowitness set, of the irreducible varieties $X_i$ to perform such computations on $\mathcal{J}$.

In Section 1.1, we define witness sets and pseudowitness sets. We will simplify the presentation and define these data structures using affine varieties. As [52, Remark 8] states, we can naturally extend from affine varieties to projective spaces by considering coordinates as sections of the hyperplane section bundle and accounting for the fact that coordinatewise projections have a *center*, i.e., a set of indeterminacy, that is contained in each fiber. Another option is to simply consider the affine cone. A third option, which is used in Section 5 to perform the necessary computations, is to restrict to a general affine coordinate patch and introduce scalars as illustrated in Section 1.3. Due to this implementation choice, there is the potential for ambiguity in Section 1.2, e.g., the dimension of the affine cone over a projective variety is one more than the dimension of the projective variety. In that section, the meaning of dimension is dependent on the implementation choice.

We explore a membership test for the join variety $J(X_1, \ldots, X_k)$ in Section 1.2. This test uses homotopy continuation without the need for computing defining equations, e.g., via interpolation or classical elimination, for $J(X_1, \ldots, X_k)$.

## 1.1 Witness and pseudowitness sets

The fundamental data structure in numerical algebraic geometry is a witness set with numerical elimination theory relying on pseudowitness sets first described in [52]. For simplicity, we provide a brief overview of both in the affine case with more details available in [12, Chap. 8 & 16].

Let $X \subset \mathbb{C}^N$ be an irreducible variety. A *witness set* for $X$ is a triple $\{f, \mathcal{L}, W\}$ where $f \subset \mathbb{C}[x_1, \ldots, x_N]$ such that $X$ is an irreducible component of $\mathcal{V}(f) = \{x \in \mathbb{C}^N \mid f(x) = 0\}$, $\mathcal{L}$ is a linear space in $\mathbb{C}^N$ with $\dim \mathcal{L} = \operatorname{codim} X$ which intersects $X$ transversally, and $W := X \cap \mathcal{L}$. In particular, $W$ is a collection of $\deg X$ points in $\mathbb{C}^N$, called a *witness point set*.

If the multiplicity of $X$ with respect to $f$ is greater than 1, we can use, for example, isosingular deflation [55] or a symbolic null space approach of [49], to replace $f$ with another polynomial system $f' \subset \mathbb{C}[x_1, \ldots, x_N]$ such that $X$ has multiplicity 1 with respect to $f'$. Therefore, without loss of generality, we will assume that $X$ has multiplicity 1 with respect to its *witness system* $f$. That is, $\dim X = \dim \operatorname{null} Jf(x^*)$ for general $x^* \in X$ where $Jf$ is the Jacobian matrix of $f$.

**Example 1** As an illustrative example, consider the irreducible variety $X := \mathcal{V}(f) \subset \mathbb{C}^3$ where $f = \{x_1^2 - x_2, x_1^3 + x_3\}$. The triple $\{f, \mathcal{L}, W\}$ forms a witness set for $X$ where $\mathcal{L} := \mathcal{V}(2x_1 - 3x_2 - 5x_3 + 1)$ and $W := X \cap \mathcal{L}$ which, to 3 decimal places, is the following set of three points:

$$\left\{ \ (-0.299, 0.089, 0.027), (0.450 \pm 0.683 \cdot \sqrt{-1}, -0.265 \pm 0.614 \cdot \sqrt{-1}, 0.539 \mp 0.095 \cdot \sqrt{-1}) \ \right\}$$

We note that $\mathcal{L}$ was defined using small integer coefficients for presentation purposes while, in practice, such coefficients are selected as random complex numbers.

A witness set for $X$ can be used to decide membership in $X$ [78]. Suppose that $p \in \mathbb{C}^N$ and $\mathcal{L}_p \subset \mathbb{C}^N$ is a sufficiently general linear space passing through $p$ with $\dim \mathcal{L}_p = \dim \mathcal{L} = \operatorname{codim} X$. Since $p \in X$ if and only if $p \in X \cap \mathcal{L}_p$, one simply needs to compute $X \cap \mathcal{L}_p$ by deforming from $X \cap \mathcal{L}$. That is, one computes the convergent endpoints, at $t = 0$, of the $\deg X$ paths starting, at $t = 1$, from the points in $W$ defined by the homotopy $X \cap (t \cdot \mathcal{L} + (1 - t) \cdot \mathcal{L}_p)$. With this setup, $p \in X$ if and only if $p$ arises as an endpoint in this deformation.

Suppose now that $\pi : \mathbb{C}^N \to \mathbb{C}^M$ is the projection defined by $\pi(x_1, \ldots, x_N) = (x_1, \ldots, x_M)$, $B = [I_M \ 0] \in \mathbb{C}^{M \times N}$ so that $\pi(x) = Bx$, and $Y := \overline{\pi(X)} \subset \mathbb{C}^M$. A *pseudowitness set* for $Y$ [52] is a quadruple $\{f, \pi, \mathcal{M}, U\}$ which is built from a witness set for $X$, namely $\{f, \mathcal{L}, W\}$, as follows. First, one computes the dimension of $Y$, for example, using [52, Lemma 3], namely

$$\dim Y = \dim X - \dim \operatorname{null} \begin{bmatrix} Jf(x^*) \\ B \end{bmatrix} \tag{5}$$

for general $x^* \in X$.

Suppose that $\mathcal{M}_1 \subset \mathbb{C}^M$ is a general linear space with $\dim \mathcal{M}_1 = \operatorname{codim} Y$ and $\mathcal{M}_2 \subset \mathbb{C}^N$ is a general linear space with $\dim \mathcal{M}_2 = \operatorname{codim} X - \operatorname{codim} Y =: \dim_{gf}(X, \pi)$, i.e., the dimension of a general fiber of $X$ with respect to $\pi$. Let $\mathcal{M} := (\mathcal{M}_1 \times \mathbb{C}^{N-M}) \cap \mathcal{M}_2$. We assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are chosen to be sufficiently general so that $\dim \mathcal{M} = \dim \mathcal{L} = \operatorname{codim} X$ and $U := X \cap \mathcal{M}$ consists of $\deg Y \cdot \deg_{gf}(X, \pi)$ points where $\deg_{gf}(X, \pi)$ is the degree of a general fiber of $X$ with respect to $\pi$. Thus, for the *pseudowitness point set* $U$, $\deg Y = |\pi(U)|$ and $\deg_{gf}(X, \pi) = |U|/|\pi(U)|$.

**Remark 2** Relation (5) provides an approach for determining if the join variety is *defective*, i.e., smaller than the expected dimension. In fact, since the abstract join $\mathcal{J}$ of (3) always has the expected dimension $\sum_{i=1}^{k} \dim X_i$, we may take $Y$ to be the join and $X = \mathcal{J}$ with $Y = \overline{\pi(\mathcal{J})}$.

**Example 3** Continuing with the setup from Ex. 1, consider the map $\pi(x_1, x_2, x_3) = (x_1, x_2)$. Clearly, $Y := \overline{\pi(X)}$ is the parabola defined by $x_2 = x_1^2$, but we will proceed from the witness set for $X$ to construct a pseudowitness set for $Y$.

We have

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \dim \operatorname{null} \begin{bmatrix} 2x_1^* & -1 & 0 \\ 0 & 3(x_1^*)^2 & x_3^* \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = 0 \text{ for general } (x_1^*, x_2^*, x_3^*) \in X.$$

Thus, $\dim Y = \dim X = 1$ and we can take $\mathcal{M} := \mathcal{V}(2x_1 - 3x_2 + 1) \times \mathbb{C} \subset \mathbb{C}^3$.

To compute the pseudowitness point set $U := X \cap \mathcal{M}$, we consider the homotopy defined by $X \cap (t \cdot \mathcal{L} + (1-t) \cdot \mathcal{M})$ with the three start points $W = X \cap \mathcal{L}$. For this homotopy, two paths converge and one diverges where the two convergent endpoints forming $U$ are

$$(1, 1, -1), \qquad (-1/3, 1/9, 1/27).$$

In particular, $\deg Y = |\pi(U)| = 2$ and $\deg_{gf}(X, \pi) = |U|/|\pi(U)| = 1$, meaning that $\pi$ is generically a one-to-one map from $X$ to $Y$.

**Example 4** As an example of computing a pseudowitness set for joins of varieties which are not rational, we consider curves $C_i \in \mathbb{P}^4$ which are defined by the intersection of 3 random quadric hypersurfaces. Hence, $C_i = \mathcal{V}(f_{i1}, f_{i2}, f_{i3})$ where $f_{ij}$ has degree 2 so that each $C_i$ is a complete intersection with $\deg C_i = 2^3 = 8$. Consider the abstract joins

$$\begin{aligned} \mathcal{J}_{12} &= \{([P], x_1, x_2) \mid P = x_1 + x_2, f_{ij}(x_i) = 0 \text{ for } i = 1, 2, j = 1, 2, 3\}, \\ \mathcal{J}_{11} &= \{([P], x_1, x_2) \mid P = x_1 + x_2, f_{1j}(x_i) = 0 \text{ for } i = 1, 2, j = 1, 2, 3\}. \end{aligned}$$

For $\pi([P], x_1, x_2) = [P]$, we have $\overline{\pi(\mathcal{J}_{12})} = J(C_1, C_2)$ and $\overline{\pi(\mathcal{J}_{11})} = J(C_1, C_1) = \sigma_2(C_1)$.

Witness sets for $\mathcal{J}_{12}$ and $\mathcal{J}_{11}$ shows that both abstract join varieties have degree 64. Then, by converting from a witness set to a pseudowitness set as described above, we find that $J(C_1, C_2)$ is a hypersurface of degree 64 while $\sigma_2(C_1)$ is a hypersurface of degree 16 with $\deg_{gf}(\mathcal{J}_{11}, \pi) = 2$.

In practice, we may compute a pseudowitness point set $U$ by starting with one sufficiently general point in the image and performing monodromy loops. Such an approach has been used in various applications, e.g., [39, 62], and will be used in many of the examples in Section 5. Since $Y = \overline{\pi(X)}$, we can compute a general point $y \in Y$ given a general point $x \in X$. Then, pick a general linear space $L$ passing through $y$ so that $y \in U = Y \cap L$. A random monodromy loop consists of two steps, each of which is performed using a homotopy. First, we move $L$ to some other general linear space $L'$. Next, we move back to $L$ via a randomly chosen path. During this loop, the path starting at $y \in L$ may arrive at some other point in $U$. We repeat this process until no new points are found for several loops. The completeness of the set is verified via a *trace test*. More information about this procedure can be found in, e.g., [39, 62] and [38, § 2.4.2].

Next, we discuss extending the membership test from witness sets to pseudowitness sets.

## 1.2 Membership test for images

As mentioned above, we can extend the notion of pseudowitness sets from the affine to the projective case. For the join variety $J := J(X_1, \ldots, X_k) = \overline{\pi(\mathcal{J})}$ where $\mathcal{J}$ is the abstract join variety, we will simply assume that we have a pseudowitness set $\{f, \pi, \mathcal{M}, U\}$ for $J$. This pseudowitness set for $J$ provides the required information needed to decide membership in the join variety [51]. As with the membership test using a witness set, testing membership in $J$ requires the tracking of at most $\deg J$ many paths, i.e., one only needs $U' \subset U$ with $\pi(U') = \pi(U)$ as discussed in [51, Remark 2].

Let $d := \dim J$ and suppose that $\mathcal{M}_1$ is the corresponding sufficiently general codimension $d$ linear space from the pseudowitness set with $\pi(U) = \pi(U') = J \cap \mathcal{M}_1$.

Given a point $[P] \in J$, suppose that $\mathcal{L}_P$ is a sufficiently general linear space of codimension $d$ passing through $[P]$. As with the membership test using a witness set, we want to compute $J \cap \mathcal{L}_p$ from $J \cap \mathcal{M}_1$. That is, we consider the $\deg J$ paths starting, at $t = 1$, from the points in $\pi(U') = \pi(U)$ defined by $J \cap (t \cdot \mathcal{M}_1 + (1 - t) \cdot \mathcal{L}_P)$. Since polynomials vanishing on $J$ are not accessible, we use the pseudowitness set for $J$ to lift these paths to the abstract join variety $\mathcal{J}$ which, by assumption, is an irreducible component of $\mathcal{V}(f)$. Thus, $f$ permits path tracking on the abstract join variety $\mathcal{J}$ and hence permits the tracking along the join variety $J$. Given $w \in U'$, let $Z_w(t)$ denote the path defined on $\mathcal{J}$ where $Z_w(1) = w$. In particular, we only need to consider $U' \subset U$ since, for any $w' \in U$ with $\pi(w) = \pi(w')$, $\pi(Z_w(t)) = \pi(Z_{w'}(t))$. With this setup we have the following membership test, which is an expanded version of [51, Lemma 1].

**Proposition 5** *For the setup described above with $J^0 := J^0(X_1, \ldots, X_k)$, the following hold.*

1. *$[P] \in J$ if and only if there exists $w \in U'$ such that $\lim_{t\to 0} \pi(Z_w(t)) = [P]$.*
2. *$[P] \in J^0$ if there exists $w \in U$ such that $\lim_{t\to 0} Z_w(t) \in \mathcal{J}$ and $\lim_{t\to 0} \pi(Z_w(t)) = [P]$.*
3. *If, for every $w \in U'$, $\lim_{t\to 0} Z_w(t) \in \mathcal{J}$, then $[P] \in J^0$ if and only if there exists $w \in U'$ such that $\lim_{t\to 0} \pi(Z_w(t)) = [P]$.*
4. *If $\lim_{t\to 0} Z_w(t)$ does not exist in $\mathcal{J}$ for every $w \in U$ such that $[P] = \lim_{t\to 0} \pi(Z_w(t))$, then either $[P] \in \mathcal{J} \setminus \mathcal{J}^0$ or $\dim\left(\pi^{-1}([P]) \cap \mathcal{J}\right) > \dim_{gf}(\mathcal{J}, \pi)$.*
5. *If $\dim J = 1$, then $[P] \in J^0$ if and only if there exists $w \in U$ such that $\lim_{t\to 0} Z_w(t) \in \mathcal{J}$ with $\lim_{t\to 0} \pi(Z_w(t)) = [P]$.*

**Proof.** With the setup above, we know that $J \cap \mathcal{L}_P$ consists of finitely many points. Thus, it follows from [64] that $[Q] \in J \cap \mathcal{L}_P$ if and only if there exists $w \in U'$ such that $[Q] = \lim_{t\to 0} \pi(Z_w(t))$. Item 1 follows since $[P] \in J$ if and only if $[P] \in J \cap \mathcal{L}_P$.

Item 2 follows from $\pi(\mathcal{J}) = J^0$. In fact, $L = \lim_{t\to 0} Z_w(t) \in \mathcal{J}$ with $[P] = \pi(L)$.

The assumption in Item 3 yields $J \cap \mathcal{L}_P = J^0 \cap \mathcal{L}_P$. Thus, this item follows immediately from Item 1 since $[P] \in J^0$ if and only $[P] \in J^0 \cap \mathcal{L}_P = J \cap \mathcal{L}_P$.

For Item 4, if $\dim\left(\pi^{-1}([P]) \cap \mathcal{J}\right) = \dim_{gf}(\mathcal{J}, \pi)$, then it follows from [64] that there must exist $w \in U$ such that $\lim_{t\to 0} Z_w(t) \in \mathcal{J}$ with $[P] = \lim_{t\to 0} \pi(Z_w(t))$. The statement follows since $\dim\left(\pi^{-1}([P]) \cap \mathcal{J}\right) < \dim_{gf}(\mathcal{J}, \pi)$ implies $\pi^{-1}([P]) \cap \mathcal{J} = \emptyset$, i.e., $[P] \notin \mathcal{J}^0$.

For Item 5, since $\mathcal{J}$ is irreducible with $\dim J = 1$, we know $\dim_{gf}(\mathcal{J}, \pi) = \dim \mathcal{J} - 1$. Hence, every fiber must be either empty or have dimension equal to $\dim_{gf}(\mathcal{J}, \pi)$. □

**Remark 6** In [47], the secant variety $X := \sigma_6(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3)$ was considered. The main theoretical result of [47] was constructing an exact polynomial vanishing on $X$ which was nonzero at $M_2$, the $2 \times 2$ matrix multiplication tensor, thereby showing that the border rank of $M_2$ was at least 7. However, before searching for such a polynomial, a version of the membership test described in Prop. 5 was used in [47, § 3.1] to show that $M_2 \notin X$ by tracking $\deg X = 15{,}456$ paths.

The following presents an illustrative example with Section 4 focusing on the special case in which $J(X_1, \ldots, X_k)$ fills the ambient space.

## 1.3 Illustrative example using membership test

To demonstrate the procedure, we consider computing the border rank of the cubic polynomial $x^2y$ in $\mathrm{Sym}^3\,\mathbb{C}^2$, thereby verifying the results of [61, Table 1], namely $\mathrm{brk}(x^2y) = 2$ (see also [83, 31, 16]). The computation also yields that either $\mathrm{rk}(x^2y) > 2$ or $\mathrm{rk}(x^2y) = 2$ with infinitely many decompositions. We will consider computing the rank in Sections 1.4 and 1.5. This subsection ends with a general discussion of Waring problems.

**Border rank 1 test using affine cones**

We start our computation with the affine cone of the abstract join variety built from a parameterization, namely

$$\mathcal{J} = \{(P, a) \mid P = v_3(a)\} \subset \mathbb{C}^4 \times \mathbb{C}^2 \quad \text{where} \quad v_3(a) = (a_1^3, 3a_1^2a_2, 3a_1a_2^2, a_2^3)$$

and $P = (P_1, \ldots, P_4)$. With $\pi(P, a) = P$, the affine cone of elements of $\mathrm{Sym}^3\,\mathbb{C}^2$ of border rank 1 is

$$J = \overline{\pi(\mathcal{J})} = \overline{\{v_3(a) \mid a \in \mathbb{C}^2\}} \subset \mathbb{C}^4.$$

To compute $\dim J$, where $J$ is considered an affine variety in $\mathbb{C}^4$, we use (5) with $\dim \mathcal{J} = 2$, $B = [I_4\ 0]$, and $f(P, a) = P - v_3(a)$. It is easy to verify that $\dim \mathrm{null} \begin{bmatrix} Jf(v_3(a^*), a^*) \\ B \end{bmatrix} = 0$ for general $a^* \in \mathbb{C}^2$ thereby showing $\dim J = \dim \mathcal{J} = 2$.

We construct a pseudowitness set for $J$, say $\{f, \pi, \mathcal{M}, U\}$ where $\mathcal{M}$ is defined by

$$\begin{bmatrix} P_1 - 3P_3 - 5P_4 - 2 \\ P_2 + 2P_3 + 4P_4 - 3 \end{bmatrix} = 0 \quad \text{and} \quad U = \left\{ \begin{array}{l} (-0.754, 3.29, -4.78, 2.32, -0.91\omega^k, 1.32\omega^k), \\ (-6.01, 7.99, -3.54, 0.523, -1.82\omega^k, 0.806\omega^k), \\ (3.39, 2.06, 0.416, 0.028, 1.50\omega^k, 0.304\omega^k) \end{array} \right\}$$

for $k = 0, 1, 2$ where $\omega$ is the third root of unity. Hence, $\deg J = 3$ meaning that we can test membership by tracking at most 3 paths, say starting at $U' \subset U$ obtained with $k = 0$.

Since $x^2y$ corresponds to the point $P = (0, 1, 0, 0)$, we consider the linear space $\mathcal{L}_P$ containing $P$ defined by the linear equations

$$\begin{bmatrix} P_1 + (2 + \sqrt{-1})P_3 - 3P_4 \\ (P_2 - 1) + 3P_3 - (4 + 2\sqrt{-1})P_4 \end{bmatrix} = 0.$$

The projection under $\pi$ of the endpoints of the three paths derived from deforming $\mathcal{M}$ to $\mathcal{L}_p$ are

$$\begin{array}{l} (0.181 + 0.284\sqrt{-1}, 0.120 + 0.313\sqrt{-1}, 0.054 + 0.330\sqrt{-1}, -0.014 + 0.332\sqrt{-1}), \\ (-0.732 + 0.256\sqrt{-1}, 0.269 + 0.309\sqrt{-1}, 0.099 - 0.193\sqrt{-1}, -0.114 - 0.010\sqrt{-1}), \\ (-0.400 - 0.149\sqrt{-1}, 0.138 - 0.233\sqrt{-1}, 0.129 + 0.112\sqrt{-1}, -0.084 + 0.068\sqrt{-1}). \end{array}$$

Since $P$ is not one of these three points, $1 < \mathrm{brk}(x^2y) \leq \mathrm{rk}(x^2y)$.

**Border rank 2 test using affine cones**

Every polynomial in $\mathrm{Sym}^3\,\mathbb{C}^2$ has border rank at most 2. We can verify this simply using the affine cone formulation

$$\mathcal{J} = \{(P,a,b) \mid P = v_3(a)+v_3(b)\} \subset \mathbb{C}^4 \times \mathbb{C}^2 \times \mathbb{C}^2 \tag{6}$$

together with (5). That is, $J = \overline{\pi(\mathcal{J})} = \mathbb{C}^4$. Since $\mathrm{brk}(x^2y) > 1$, we know $\mathrm{brk}(x^2y) = 2$ which we can verify by tracking one path defined by

$$P(t) = (1-t)(0,1,0,0) + t(-9\sqrt{-1}, 3-5\sqrt{-1}, -1-11\sqrt{-1}, -13) = v_3(a)+v_3(b). \tag{7}$$

At $t = 1$, we can start with $a = (1+\sqrt{-1}, 1-2\sqrt{-1})$ and $b = (2-\sqrt{-1}, 1-\sqrt{-1})$. One clearly has $\lim_{t\to 0} P(t) = (0,1,0,0)$, but the corresponding $(a,b)$ diverge to infinity at $t \to 0$. Since starting from any of the possible decompositions of $P(1)$, namely

$$P(1) = v_3(\omega^j a) + v_3(\omega^k b) = v_3(\omega^j b) + v_3(\omega^k a)$$

where $j,k = 0,1,2$ and $\omega$ is the third root of unity, yields a divergent path, Item 4 of Prop. 5 states that either $\mathrm{rk}(x^2y) > 2$ or $\mathrm{rk}(x^2y) = 2$ with infinitely many decompositions.

We will focus on distinguishing between these two cases in Sections 1.4 and 1.5.

**Remark 7** Once (5) has been used to verify that a certain secant or join variety fill the ambient space, this technique of tracking only one path can be used in general as discussed in Section 4. Remarkably, it turns out that defective secant varieties are almost always those that one was expecting to fill the ambient space.

**Border rank 2 test using affine coordinate patches**

We compare the behavior of using affine cones above with the use of an affine coordinate patch with scalars. The advantage here is that, assuming sufficiently general coordinate patches, all paths will converge with this setup. The paths for which $\lim_{t\to 0} Z_w(t)$ does not exist in $\mathcal{J}$ have the corresponding scalar, namely $\lambda_0$, equal to zero. Consider

$$Z = \left\{ (P,\lambda,a,b) \;\middle|\; \begin{array}{l} P\lambda_0 = v_3(a)\lambda_1 + v_3(b)\lambda_2 \\ r_1(P) = r_2(\lambda) = r_3(a) = r_4(b) = 0 \end{array} \right\} \subset \mathbb{C}^4 \times \mathbb{C}^3 \times \mathbb{C}^2 \times \mathbb{C}^2$$

where each $r_i$ is a general affine linear polynomial. The irreducible component of interest inside of $Z$ is $\overline{Z \setminus \mathcal{V}(\lambda_0)}$. Since this irreducible set plays a similar role of the abstract join variety, we will call it $\mathcal{J}$. With projection $\pi(P,\lambda,a,b) = P$, we have that $\overline{\pi(\mathcal{J})} = \mathcal{V}(r_1(P))$ verifying every element in $\mathrm{Sym}^3\,\mathbb{C}^2$ has border rank at most 2.

For concreteness and simplicity, we take $r_1(P) = P_2 - 1$, $r_2(\lambda) = 2\lambda_0 - \lambda_1 + 3\lambda_2 - 1$, $r_3(a) = 3a_1 - 2a_2 - 1$, and $r_4(b) = 2b_1 + 3b_2 - 1$. We consider tracking the corresponding path defined by $P(t) = (1-t)(0,1,0,0) + t(1+\sqrt{-1}, 1, 2-\sqrt{-1}, 1-2\sqrt{-1})$ starting at $t = 1$ with

$$\begin{aligned}
&\lambda = (0.00145 + 0.000914\sqrt{-1}, 0.0482 + 0.0524\sqrt{-1}, 0.348 + 0.0169\sqrt{-1}),\\
&a = (0.21 + 0.0532\sqrt{-1}, -0.184 + 0.0797\sqrt{-1}), \quad \text{and}\\
&b = (0.157 + 0.0666\sqrt{-1}, 0.229 - 0.0444\sqrt{-1}).
\end{aligned}$$

As mentioned above, the advantage is that this path is convergent, but the endpoint has $\lambda_0 = 0$ thereby showing that it would have diverged if we used an affine cone formulation.

**Waring problem**

This example of computing the rank of $x^2y$ in $\mathrm{Sym}^3\,\mathbb{C}^2$ is an example of the so-called Waring problem, namely writing a homogeneous polynomial as a sum of powers of linear forms. We leave it as an open challenge problem to derive a general formula for the degrees of the corresponding secant varieties in this case, which is the maximum number of paths that need to be tracked in order to decide membership. We highlight some of the known partial results.

The Veronese variety that parameterizes $d^{\text{th}}$ powers of linear forms in $n+1$ variables is classically known to have degree $d^n$.

In the binary case, the degree of $\sigma_2(X)$ where $X$ is the rational normal curve of degree $d$ parametrizing forms of type $(ax+by)^d$ is $\binom{d-1}{2}$ [2, 68]. More generally in [36] it is shown that the variety parameterizing forms of type $(a_0x_0+\cdots+a_nx_n)^d+(b_0x_0+\cdots+b_nx_n)^d$ has degree

$$\frac{1}{2}\left(d^{2n}-\sum_{j=0}^{n}(-1)^{n-j}d^j\binom{2n+1}{j}\binom{2n-j}{j}\right).$$

The same paper also computes the degree of $\sigma_2(X)$ where $X$ is any Segre-Veronese variety which parameterizes multihomogeneous polynomials of type $L_1^{d_1}\cdots L_k^{d_k}$ where $L_i$ is a linear form in the variables $x_{0,i},\ldots,x_{n_i,i}$ for $i=1,\ldots,k$. In this case, $\deg\sigma_2(X)$ is

$$\frac{1}{2}\left(((n_1,\ldots,n_k)!d_1^{n_1}\cdots d_k^{n_k})^2-\sum_{l=0}^{n}\binom{2n+1}{l}(-1)^{n-l}\sum_{\sum j_i=n-l}(n_1-j_1,\ldots,n_k-j_k)!\Pi_{i=1}^{k}\binom{n_i+j_i}{j_j}d_i^{n_i-j_i}\right).$$

The degree of some $k$-secants of ternary forms is known, e.g., [43, Thm. 1.4] and [56, Rem. 7.20].

## 1.4 Reduction to the curve case

Proposition 5 can determine membership in join varieties as well as provide some insight regarding membership in the constructible join. In particular, Item 5 of Prop. 5 shows that deciding membership in a join variety and the corresponding constructible join is equivalent when the join variety is a curve. The following describes one approach for deciding membership in the constructible join by reducing down to the curve case. Section 1.5 considers computing all decompositions of the form (1) and hence can also be used to decide membership in the constructible join by simply deciding if such a decomposition exists.

Suppose that $\mathcal{J}$ is the abstract join with corresponding join variety $J=\overline{\pi(\mathcal{J})}$. Since we want to reduce down to the curve case, we will assume that $d:=\dim J>1$. Let $C$ be a general curve section of $J$, that is, $C=J\cap\mathcal{L}$ where $\mathcal{L}$ is a general linear space with $\operatorname{codim}\mathcal{L}=d-1$. Since $J$ is irreducible and $\mathcal{L}$ is general, $C$ is also irreducible. Hence, $\mathcal{J}_C=\pi^{-1}(C)\cap\mathcal{J}$ is irreducible with $C=\overline{\pi(\mathcal{J}_C)}$. Therefore, one can use Item 5 of Prop. 5 to test membership in $C$ and $C^0=\pi(\mathcal{J}_C)$. However, testing membership in $C^0$ and $C$ is typically not the problem of interest.

Given $[P]$, we want to decide if $[P]$ is a member of $J^0$ or $J$. Thus, we could modify the description above to replace $\mathcal{L}$ with $\mathcal{L}_P$, a general linear slice of codimension $d-1$ passing through $[P]$. If $C_P=J\cap\mathcal{L}_P$, then $\mathcal{J}_{C_P}=\pi^{-1}(C_P)\cap\mathcal{J}$ need not be irreducible. In fact, since $\mathcal{L}_P$ is general through $[P]$, the closure of the images under $\pi$ of each irreducible component of $\mathcal{J}_{C_P}$ must either be the singleton $\{[P]\}$ or $C_P$. Therefore, one can apply Item 5 of Prop. 5 to each irreducible component of $\mathcal{J}_{C_P}$ whose image under $\pi$ is $C_P$.

The following illustrate this reduction to the curve case.

**Example 8** Consider $\mathcal{J}$ as in (6) with $J = \overline{\pi(\mathcal{J})} = \mathbb{C}^4$ and $d = 4$. Since a general curve section of $J$ is simply a general line in $\mathbb{C}^4$, we have $C = \mathcal{L}$ where $\mathcal{L}$ is a general line. It is easy to verify that $\mathcal{J}_C = \pi^{-1}(C) \cap \mathcal{J}$ is an irreducible curve of degree 30. We now consider the point $P$ corresponding to $x^2y$, namely $(0, 1, 0, 0)$. Hence, $C_P = \mathcal{L}_P$ where $\mathcal{L}_P$ is a general line through this point. In this case, $\mathcal{J}_{C_P} = \pi^{-1}(C_P) \cap \mathcal{J}$ is also an irreducible curve of degree 30. Hence, we can apply Item 5 of Prop. 5 to decide membership of $x^2y$ in $J^0 = \pi(\mathcal{J})$ by deciding membership in $C_P^0 = \pi(\mathcal{J}_{C_P})$. Since $C_P^0$ is a line, this is equivalent to tracking paths defined between a general point and $P$, as in (7). Since all paths diverge, Item 5 of Prop. 5 yields $\mathrm{rk}(x^2y) > 2$.

For comparison, suppose that we want to consider $C_Q = \mathcal{L}_Q$, which is a general line through the point $Q = (1, 1, 1, 1)$ corresponding to $x^3 + x^2y + xy^2 + y^3$ which has rank one. Then, the curve $\mathcal{J}_{C_Q} = \pi^{-1}(C_Q) \cap \mathcal{J}$ is the union of two irreducible curves, say $V_1$ and $V_2$ with $\pi(V_1) = \{Q\}$ and $\overline{\pi(V_2)} = C_Q$, both of which yield decompositions.

## 1.5 Computing all decompositions

A fundamental question related to rank is to describe the set of all of decompositions of a point $[P]$. In numerical algebraic geometry, this means compute a *numerical irreducible decomposition*, i.e., a witness set for each irreducible component, of the fiber over $[P]$, namely

$$\mathcal{F}_P := \pi^{-1}([P]) \cap \mathcal{J}(X_1, \ldots, X_k).$$

Computing $\mathcal{F}_P$ yields a membership test for $J^0(X_1, \ldots, X_k)$ since $[P] \in J^0(X_1, \ldots, X_k)$ if and only if $\mathcal{F}_P \neq \emptyset$. One approach is to directly compute a numerical irreducible decomposition using (1). Another approach is to perform a cascade [53, 76] starting with a witness set for $\mathcal{J}$. Since $\pi^{-1}([P])$ is defined by linear equations, computing $\mathcal{F}_P$ can be simply obtained by degenerating each general slicing hyperplane to a general hyperplane containing $\pi^{-1}([P])$. After each degeneration, the resulting points are either contained in $\pi^{-1}([P])$, forming *witness point supersets*, or not. The ones not contained in $\pi^{-1}([P])$ are used as the start points for the next degeneration. This process is described in detail in [54, § 2.2]. From the witness point supersets, standard methods in numerical algebraic geometry (e.g., see [12, Chap. 10]) are used to yield the numerical irreducible decomposition of $\mathcal{F}_P$.

After determining that $[P] \in J^0(X_1, \ldots, X_k)$ by showing $\mathcal{F}_P \neq \emptyset$, a numerical irreducible decomposition of $\mathcal{F}_P$ can then be used to perform additional computations. One such computation is deciding if $\mathcal{F}_P$ contains real points, i.e., determining if there is a real decomposition, which is described in Section 3. Another application is to determine if there exists "simpler" decompositions of $[P]$, e.g., deciding if $[P] \in J^0(X_1, \ldots, X_{k-1})$. In the secant variety case, this is equivalent to deciding if the rank of $[P]$ is strictly less than $k$.

**Example 9** With the setup from Section 1.3, consider computing all of the rank 3 decompositions of $x^2y$ using affine cones. That is, we consider

$$\mathcal{J} = \{(P, a, b, c) \mid P = v_3(a) + v_3(b) + v_3(c)\} \subset \mathbb{C}^4 \times \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{C}^{10} \tag{8}$$

which is irreducible of dimension 6 and degree 57. Thus, in a witness set for $\mathcal{J}$, we have a general linear space $\mathcal{L}$ of codimension 6 defined by linear polynomials $\ell_i(P, a, b, c) = 0$, $i = 1, \ldots, 6$ and a witness point set $W = \mathcal{J} \cap \mathcal{L}$ consisting of 57 points.

For $i = 1, \ldots, 4$, let $m_i(P)$ be a general linear polynomial which vanishes at $(0, 1, 0, 0)$. The cascade simply replaces the condition $\ell_i = 0$ with $m_i = 0$ sequentially for $i = 1, \ldots, 4$. In this case, for $i = 1, 2, 3$, $\mathcal{J} \cap \mathcal{V}(m_1, \ldots, m_i, \ell_{i+1}, \ldots, \ell_6)$ consists of 57 points, none of which are not contained in $\mathcal{F}_P$, while $\mathcal{J} \cap \mathcal{V}(m_1, \ldots, m_4, \ell_5, \ell_6)$ consists of 45 points, all of which are contained in $\mathcal{F}_P$. In fact, these 45 points form a witness point set for $\mathcal{F}_P$, which is an irreducible surface of degree 45. This, by itself, shows that $\text{rk}(x^2 y) \leq 3$ since for $\mathcal{F}_P \neq \emptyset$.

Even though Ex. 8 shows that $\text{rk}(x^2 y) > 2$, we verify this by computing $\mathcal{F}_P \cap \mathcal{V}(c)$ using the witness point set for $\mathcal{F}_P$ computed above. For $i = 1, 2$, letting $r_i(c)$ be a general linear polynomial vanishing at $c = 0$, we computed that $\mathcal{J} \cap \mathcal{V}(m_1, \ldots, m_4, r_1, \ell_6)$ consists of 36 points, none of which have $c = 0$. Since $\mathcal{F}_P \cap \mathcal{V}(c) = \mathcal{J} \cap \mathcal{V}(m_1, \ldots, m_4, r_1, r_2) = \emptyset$, we have shown that $\text{rk}(x^2 y) > 2$ and hence $\text{rk}(x^2 y) = 3$.

# 2 Boundary

By using the approaches described in Section 1, one is able to use numerical algebraic geometry to determine membership in $J^0(X_1, \ldots, X_k)$ and $J(X_1, \ldots, X_k)$. An interesting object is the *boundary* $\partial := \overline{J(X_1, \ldots, X_k) \setminus J^0(X_1, \ldots, X_k)}$ which is the closure of the elements which arose by closing the constructible set $J^0(X_1, \ldots, X_k)$. As a subset of $J(X_1, \ldots, X_k)$, the boundary $\partial$ may consist of irreducible components of various codimension. In the following, we describe an approach for computing the irreducible components of $\partial$ which have codimension one with respect to $J(X_1, \ldots, X_k)$, denoted $\partial_1$, derived from [51, § 3].

Before considering the computation of the codimension one components of the boundary for arbitrary joins, we first consider the case where $\mathcal{C} \subset \mathbb{P}^N \times \mathbb{C}^M$ is an irreducible curve and the projection $\pi([P], X) = [P]$ is generically finite-to-one on $\mathcal{C}$, i.e., $\dim \mathcal{C} = 1$ and $\dim_{gf}(\mathcal{C}, \pi) = 0$. The boundary of $C = \pi(\mathcal{C})$ consists of at most finitely many points $\partial_C = \overline{C} \setminus C$.

**Example 10** Consider the irreducible curve $\mathcal{C} = \{([a, b], x) \mid a \cdot x = b\} \subset \mathbb{P}^1 \times \mathbb{C}$ with projection $\pi([a, b], x) = [a, b]$. Generically, $\pi$ is a one-to-one map from $\mathcal{C}$ to $\mathbb{P}^1$. Since we have $x = b/a \in \mathbb{C}$ when $a \neq 0$, one can easily verify that the boundary of $C = \pi(\mathcal{C})$ is $\partial_C = \{[0, 1]\} \subset \mathbb{P}^1$.

In order to compute a finite superset of $\partial_C$, we consider the closure of $\mathcal{C} \subset \mathbb{P}^N \times \mathbb{C}^M$ in $\mathbb{P}^N \times \mathbb{P}^M$, say $\overline{\mathcal{C}}$. Then, a finite superset of $\partial_C$ is the set of points in $\overline{C}$ whose fiber intersects "infinity." That is, if we have coordinates $x \in \mathbb{C}^M$ and $[y] \in \mathbb{P}^M$ with the embedding given by

$$(x_1, \ldots, x_M) \in \mathbb{C}^M \mapsto [1, x_1, \ldots, x_M] \in \mathbb{P}^M,$$

then a finite superset of $\partial_C$ is $\pi(\overline{\mathcal{C}} \cap \mathcal{V}(y_0))$.

**Example 11** Continuing with the setup from Ex. 10, one can verify that

$$\overline{\mathcal{C}} = \{([a, b], [y_0, y_1]) \mid a \cdot y_1 = b \cdot y_0\} \subset \mathbb{P}^1 \times \mathbb{P}^1 \quad \text{and} \quad \pi(\overline{\mathcal{C}} \cap \mathcal{V}(y_0)) = \{[0, 1]\}.$$

In Ex. 11, it was the case that $\partial_C = \pi(\overline{\mathcal{C}} \cap \mathcal{V}(y_0))$. However, since $\partial_C \subset \pi(\overline{\mathcal{C}} \cap \mathcal{V}(y_0))$ in general, we must investigate each point in $\pi(\overline{\mathcal{C}} \cap \mathcal{V}(y_0))$ via Sections 1.4 and 1.5 to determine if it is contained in $\partial_C$.

We now turn the general case of $\mathcal{J}$ as in (3). Let $J$ and $J^0$ be the corresponding join variety and constructible join, i.e., $J = \overline{\pi(\mathcal{J})}$ and $J^0 = \pi(\mathcal{J})$, and $d = \dim J$. Since the case $d = 0$ is trivial, we will assume $d \geq 1$. Since we aim to compute witness points sets for the codimension 1 components, $\partial_1$, of the boundary $\partial = \overline{J \setminus J^0}$, i.e., $\partial_1$ has pure-dimension $d - 1$, we can restrict our attention to a general curve section of $J$, say $C$. This restriction cuts $\partial_1$ down to finitely many points, i.e., witness points, which we aim to compute.

Since $C$ is a general curve section, $\mathcal{M} = \pi^{-1}(C) \cap \mathcal{J}$ is irreducible. Finally, we take a general curve section of $\mathcal{M}$, say $\mathcal{C}$. Thus, $\mathcal{C}$ is an irreducible curve with $\overline{\pi(\mathcal{C})} = C$ and $\dim_{gf}(\mathcal{C}, \pi) = 0$. Applying the procedure described above yields a finite set of points containing $\partial_C$. Since the restriction from $\mathcal{M}$ to a general curve section $\mathcal{C}$ may have introduced new points in the boundary, we simply need to investigate each point with respect to $\mathcal{M}$ rather than $\mathcal{C}$ via Sections 1.4 and 1.5. In the end, we obtain the finitely many points forming a witness point set for $\partial_1$.

**Example 12** As with Section 1.3, we use a parameterization to compute the codimension one component of the boundary, $\partial$, in $\operatorname{Sym}^3 \mathbb{C}^2$ of border rank 2. Since every polynomial in $\operatorname{Sym}^3 \mathbb{C}^2$ has border rank 2, the codimension one component of $\partial$ is a hypersurface: it is the tangential variety of the rational normal cubic curve.

Since $J = \operatorname{Sym}^3 \mathbb{C}^2$, a general curve section $C$ of $J$ is simply a general line. Following an affine cone formulation as in (6), we take, for exposition, $C$ defined by the equations

$$P_1 + 3P_4 - 2 = P_2 - 4P_4 + 3 = P_3 - 2P_4 - 4 = 0.$$

Since $\dim_{gf}(\mathcal{J}, \pi) = 0$, $\mathcal{M} = \pi^{-1}(C) \cap \mathcal{J}$ is the curve, i.e., $\mathcal{C} = \mathcal{M}$ with $\overline{\pi(\mathcal{C})} = C$ where

$$\mathcal{C} = \{(P, a, b) \mid P = v_3(a) + v_3(b) \in C\} \subset \mathbb{C}^4 \times \mathbb{C}^2 \times \mathbb{C}^2.$$

Next, we compute the closure, $\overline{\mathcal{C}}$, of $\mathcal{C}$ in $\mathbb{C}^4 \times \mathbb{P}^4$ where $\mathbb{C}^2 \times \mathbb{C}^2 \hookrightarrow \mathbb{P}^4$ given by

$$(a, b) \in \mathbb{C}^2 \times \mathbb{C}^2 \mapsto [1, a_1, a_2, b_1, b_2] \in \mathbb{P}^4.$$

With coordinates $[y_0, \ldots, y_4] \in \mathbb{P}^4$, we find that $\pi(\overline{\mathcal{C}} \cap \mathcal{V}(y_0))$ consists of the following four points:

$$\begin{array}{ll} (2.308, -3.410, 3.794, -0.103), & (-35.743, 47.325, 29.163, 12.581), \\ (4.328, -6.103, 2.448, -0.776), & (0.018, -0.357, 5.321, 0.661). \end{array}$$

Finally, we verified that each of these points has rank larger than 2 meaning that the codimension one component of $\partial$ is a degree 4 hypersurface.

Although we can use numerical algebraic geometry to test membership in this hypersurface via Section 1, we can also easily recover the defining equation exactly in this case using [10]:

$$P_1^2 P_4^2 - 6P_1P_2P_3P_4 + 4P_1P_3^3 + 4P_2^3P_4 - 3P_2^2P_3^2 = 0.$$

Clearly, $(0, 1, 0, 0)$, corresponding to $x^2y$, lies on this hypersurface.

# 3 Real decompositions

For a real $[P]$, after computing the fiber $\mathcal{F}_P$ as in Section 1.5, a natural question is to determine if *real* decompositions exist. With a witness set for each irreducible component of $\mathcal{F}_P$, where all general choices involve selecting real numbers, the homotopy-based approach of [45] (see also [85]) can be used to determine if the component contains real points. The techniques described in [45, 85] rely upon computing critical points of the distance function as proposed by Seidenberg [73] (see also [5, 41, 70]). For secant varieties, this yields a method to determine the real rank of real elements.

Let $F$ be a system of $n$ polynomials in $N$ variables with real coefficients and $V \subset \mathcal{V}(F) \subset \mathbb{C}^N$ be an irreducible component. Fix a real point $y \in \mathbb{R}^N$ such that $y \notin \mathcal{V}(F)$. Following Seidenberg [73], we consider the optimization problem

$$\min\{\|x - y\|_2^2 \mid x \in V \cap \mathbb{R}^N\}. \tag{9}$$

Every connected component $C$ of $V \cap \mathbb{R}^N$ has a global minimizer of the distance functions from $y$ to $C$, i.e., there exists $x \in C$ such that $\|x - y\|_2^2 \leq \|z - y\|_2^2$ for every $z \in C$. Thus, there exists $\lambda \in \mathbb{P}^n$ with

$$G(x, \lambda) = \begin{bmatrix} F(x) \\ \lambda_0(x - y) + \sum_{i=1}^n \lambda_i \nabla F_i(x) \end{bmatrix} = 0 \tag{10}$$

where $\nabla F_i$ is the gradient of $F_i$. For the projection map $\pi(x, \lambda) = x$, the set $\pi(\mathcal{V}(G)) \subset \mathbb{C}^N$ is called the set of *critical points* of (9) and it intersects every connected component of $V \cap \mathbb{R}^N$. Hence, $V \cap \mathbb{R}^N = \emptyset$ if and only if there are no real critical points for (9).

This method allows one to decide if a real decomposition exists by deciding if there exists a real critical point of the distance function. As a by-product, one obtains the closest decomposition to the given point. When the set of critical points may be positive-dimensional, the approach presented in [45] uses a homotopy-based approach to reduce down to testing the reality of finitely many critical points. Therefore, the problem of deciding if a real decomposition exists can be answered by deciding the reality of finitely many points.

**Example 13** Consider deciding if the real rank of $x^2y$ in $\mathrm{Sym}^3\,\mathbb{C}^2$ is the same as the complex rank, namely 3. In Ex. 9, we computed $\mathcal{F}_P$, which is irreducible of dimension 2 and degree 45. In particular, we can take

$$F(a, b, c) = \begin{bmatrix} a_1^3 + b_1^3 + c_1^3 \\ 3(a_1^2 a_2 + b_1^2 b_2 + c_1^2 c_2) - 1 \\ 3(a_1 a_2^2 + b_1 b_2^2 + c_1 c_2^2) \\ a_2^3 + b_2^3 + c_2^3 \end{bmatrix}.$$

We aim to compute the critical points of the distance from

$$\alpha = (1, 2), \quad \beta = (-2, 1), \quad \gamma = (1, -1) \tag{11}$$

which arise from the solutions $(a, b, c) \in \mathbb{C}^6$ and $\delta \in \mathbb{P}^4$ of

$$G(a, b, c, \delta) = \begin{bmatrix} F(a, b, c) \\ \delta_0(a - \alpha, b - \beta, c - \gamma) + \sum_{i=1}^4 \delta_i \nabla F_i(a, b, c) \end{bmatrix}.$$

Solving $G = 0$ yields 234 critical points, of which 8 are real. Hence, the real rank of $x^2y$ is indeed 3 (cfr. [24]) where the one of minimal distance from $(\alpha, \beta, \gamma)$ is the decomposition (to three digits)

$$x^2y = (0.721x + 0.2849y)^3 + (-1.429x + 1.101y)^3 + (1.365x - 1.107y)^3. \tag{12}$$

The computation of *all* critical points provides a global approach for deciding if a real decomposition exists. Such a global approach may be computationally expensive when the number of critical points is large. Thus, we also describe a local approach based on *gradient descent homotopies* [44] with the aim of computing a real critical point. Although there is no guarantee, such a local approach can provide a quick affirmation that real decompositions exists.

With the setup as above, we consider the gradient descent homotopy

$$H(z, \delta, t) = \begin{bmatrix} F(z) - t \cdot F(y) \\ \delta_0(z - y) + \sum_{i=1}^n \delta_i \nabla F_i(z) \end{bmatrix} = 0.$$

Clearly, $H(x, \lambda, 0) = G(x, \lambda) = 0$ for $G$ as in (10). We consider the homotopy path $(z(t), \delta(t))$ where $z(1) = y \in \mathbb{R}^N$ and $\delta(1) = [1, 0, \ldots, 0] \in \mathbb{P}^n$. If this homotopy path is smooth for $0 < t \leq 1$

and converges as $t \to 0$, then $z(0)$ is a real critical point with respect to $F$. We note that $H$ is a so-called *Newton homotopy* since $\partial H/\partial t$ is independent of $z$ and $\delta$. Newton homotopies will also be used below in Section 4.

One can quickly try gradient descent homotopies for various $y \in \mathbb{R}^N$ with the goal of computing a real critical point, provided one exists.

**Example 14** With the setup from Ex. 13, we consider the gradient descent homotopy

$$H(a,b,c,\delta,t) = \left[ \begin{array}{c} F(a,b,c) - tF(\alpha,\beta,\gamma) \\ \delta_0(a-\alpha, b-\beta, c-\gamma) + \sum_{i=1}^4 \delta_i \cdot \nabla F_i(a,b,c) \end{array} \right].$$

The path, which starts at $t = 1$ with $(\alpha, \beta, \gamma, [1, 0, \ldots, 0])$ yields a smooth and convergent path with the endpoint corresponding the decomposition of minimal distance from $(\alpha, \beta, \gamma)$ in (12).

# 4 Generic cases

When the join variety $J(X_1, \ldots, X_k)$ fills the ambient projective space, the degree of the join variety is 1. In this section, we modify the approach presented in Prop. 5 to use a Newton homotopy which can compute a decomposition of a generic tensor by tracking one path. Such paths can even be tracked certifiably [46, 48].

Let $[P] \in J(X_1, \ldots, X_k)$ be generic. Thus, the dimension and degree of the fiber over $[P]$ is the same over a nonempty Zariski open subset of $J(X_1, \ldots, X_k)$, i.e.,

$$d := \dim_{gf}(\mathcal{J}, \pi) = \dim \pi^{-1}([P]) \cap \mathcal{J}$$

and $\deg_{gf}(J, \pi) = \deg \pi^{-1}([P]) \cap \mathcal{J}$. The first step for computing a decomposition of $[P]$ is to produce a starting point. This is performed by selecting generic $x_i^* \in \mathcal{C}(X_i)$ and computing $P^* = \sum_{i=1}^k x_i^*$. That is, $([P^*], x_1^*, \ldots, x_k^*) \in \mathcal{J}$ is sufficiently generic with respect to $\mathcal{J}$ and $[P]$.

With this setup, consider the homotopy that deforms the fiber as we move along the straight line from $[P^*]$ to $[P]$, namely $\pi^{-1}(t[P^*] + (1-t)[P]) \cap \mathcal{J}$. In order to reduce down to tracking along a path, i.e., curve, we simply select a generic linear space $\mathcal{L}$ of codimension $d$ passing through the point $([P^*], x_1^*, \ldots, x_k^*)$. This results in the Newton homotopy

$$\pi^{-1}(t[P^*] + (1-t)[P]) \cap \mathcal{J} \cap \mathcal{L}$$

where, at $t = 1$, we have start point $(x_1^*, \ldots, x_k^*)$. The endpoint of this path yields a decomposition of $[P]$ in the form (1).

## 4.1 Illustrative example

We demonstrate decomposing a general element via tracking one path on cubic forms in 3 variables. For a cubic form $C(x)$, we aim to write it as

$$C(x) = Q(x) \cdot L_1(x) + L_2(x)^3 \tag{13}$$

where $Q(x) = q_0 x_0^2 + q_1 x_0 x_1 + \cdots + q_5 x_2^2$ is a quadratic form and $L_i(x) = x_0 + a_{i1} x_1 + a_{i2} x_2$ is a linear form with $q_k, a_{ij} \in \mathbb{C}$. Geometrically, this means $C(x) \in J(\mathcal{O}^2(\nu_3(\mathbb{P}^2)), \nu_3(\mathbb{P}^2))$ where $\mathcal{O}^2(\nu_3(\mathbb{P}^2))$ is the second osculating variety to the Veronese surface $\nu_3(\mathbb{P}^2)$. Using linear algebra, it is easy to verify that a general cubic $C(x)$ must have finitely many decompositions of the form (13). We will apply the approach to compute decompositions of the cubics

$$C_1(x) = x_0^3 + x_1^3 + x_2^3 \quad \text{and} \quad C_2(x) = 4x_0^2x_2 + x_1^2x_2 - 8x_0^3.$$

The cubic $C_2$ defines a curve called the "witch of Agnesi." Starting with

$$\begin{array}{rcl} C^*(x) & = & \left(x_0^2 + (1+\sqrt{-1})x_0x_1 + 3x_0x_2 - 2x_1^2 + (3-\sqrt{-1})x_1x_2 + 2x_2^2\right)(x_0 + 2x_1 + 3x_2) \\ & & + \ \left(x_0 - (3+\sqrt{-1})x_1 + 5x_2\right)^3, \end{array}$$

the Newton homotopy deforming from $C^*$ to $C_i$ which is obtained by taking coefficients of

$$tC^*(x)+(1-t)C_i(x) = (q_0x_0^2+q_1x_0x_1+\cdots+q_5x_2^2)(x_0+a_{11}x_1+a_{12}x_2)+(x_0+a_{21}x_1+a_{22}x_2)^3$$

yields the following decompositions, which we have converted to exact representation using [10]:

$$\begin{array}{rcl} C_1(x) & = & (-3x_0x_1 + 3(1-\sqrt{-3})x_0x_2/2 - 3x_1^2 + 3(1-\sqrt{-3})x_1x_2/2)(x_0 - (1-\sqrt{-3})x_2/2) \\ & & + \ (x_0 + x_1 - (1-\sqrt{-3})x_2/2)^3, \\ C_2(x) & = & (-9x_0^2 - 9x_1^2/4)(x_0 - \sqrt{3}x_1/6 - 4x_2/9) + (x_0 - \sqrt{3}x_1/2)^3. \end{array}$$

# 5 Examples

The previous sections have described various approaches for computing information about join and secant varieties along with several illustrative examples. In this section, we present several examples which were computed using the methods described above with computations facilitated by `Bertini` [11].

## 5.1 Complex multiplication tensor

Complex multiplication can be treated as a bilinear map from $\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$, namely

$$(a, b) \cdot (c, d) \mapsto (ac - bd, ad + bc),$$

which, using the standard approach, involves 4 multiplications. Treating this as a bilinear map from $\mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$, we will use the above approaches to compute the rank and border rank (over $\mathbb{C}$) of this bilinear map, both of which are 2. We will then demonstrate how our method shows that the real rank of this bilinear map is 3. The decomposition by Gauss, namely

$$ac - bd = (a+b)\cdot c - b\cdot(c+d), \qquad ad + bc = (a+b)\cdot c + a\cdot(d-c), \tag{14}$$

shows that the real rank is at most 3 with multiplications $(a+b)\cdot c$, $b\cdot(c+d)$, and $a\cdot(d-c)$.

**Over the complex numbers**

Let $T : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^2$ denote the complex multiplication bilinear map. We first aim to compute $\operatorname{brk} T$ and $\operatorname{rk} T$. Observe that $T \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ and that computing its rank corresponds to computing one of its minimal decompositions with respect to the Segre variety

$$S := Seg(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) = \mathbb{P}(\mathbb{C}^8).$$

To accomplish this, we compute a pseudowitness set for $S = \sigma_1(S) \subset \mathbb{P}(\mathbb{C}^8)$. Using 17 random monodromy loops followed by the trace test, we find $\deg \sigma_1(S) = 6$. The membership test tracked 6 paths and found that each path converged to some finite endpoint which does not correspond to $T$. Therefore, $\operatorname{rk} T \geq \operatorname{brk} T > 1$.

Next, we turn our attention to $\sigma_2(S)$, which fills the whole space. Therefore, our method only requires tracking one solution path which converges. This shows that $[T] \in \sigma_2^0(S) \subset \sigma_2(S)$, meaning $\operatorname{rk} T = \operatorname{brk} T = 2$.

For example, if we look for decompositions of the form

$$\begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} (a + y_{12}b) \cdot (c + z_{12}d) \\ (y_{21}a + b) \cdot (z_{21}c + d) \end{bmatrix}, \tag{15}$$

then, using [10], our setup tracking one path yielded the decomposition

$$\begin{aligned} ac - bd &= \quad ((a - ib) \cdot (c - id) - (b - ia) \cdot (d - ic))/2 \\ ad + bc &= i\,((a - ib) \cdot (c - id) + (b - ia) \cdot (d - ic))/2 \end{aligned}$$

where $i = \sqrt{-1}$.

**Over the real numbers**

Since (14) shows that $\operatorname{rk}_{\mathbb{R}} T \leq 3$, we know that $\operatorname{rk}_{\mathbb{R}} T = 3$ if and only if $\operatorname{rk}_{\mathbb{R}} T > 2$. In (15), we used a specialized form to compute a decomposition over $\mathbb{C}$. In this case, there were only finitely many decompositions and all were not real. To show that $\operatorname{rk}_{\mathbb{R}} T > 2$, we will work with a fully general formulation:

$$\begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} (y_{11}a + y_{12}b) \cdot (z_{11}c + z_{12}d) \\ (y_{21}a + y_{22}b) \cdot (z_{21}c + z_{22}d) \end{bmatrix}$$

which defines $V \subset \mathbb{C}^{12}$, the union of two irreducible varieties $V_1$ and $V_2$, each having dimension 4 and degree 9. Using two different approaches, we show that $V \cap \mathbb{R}^{12} = \emptyset$.

First, using the setup from Section 3, we compute the critical points of the distance between $V$ and a random point in $[-1, 1]^{12}$. Since this yields 18 critical points, all of which are nonreal, we know $\operatorname{rk}_{\mathbb{R}} T > 2$.

Alternatively, in this case, we have additional structure since the two irreducible components of $V$ are complex conjugates of each other. Hence, $V \cap \mathbb{R}^{12}$ is contained in $V_1 \cap V_2$. Since $V_1 \cap V_2 = \emptyset$, we again see that $\operatorname{rk}_{\mathbb{R}} T > 2$.

## 5.2 Comparison with cactus rank

The following example shows that our method computes $X$-border rank and not the different notion of *cactus rank*, which was recently reintroduced in the literature (in [56] it was defined as "scheme length," and the first definition of cactus rank is in [20] after paper [26] where the cactus variety was first introduced). This example, which follows, was first published in [18] thanks to a suggestion from W. Buczyńska and J. Buczyński who proved it in [26] as a very peculiar but illustrative case where the $X$-border rank of a polynomial cannot be computed from a punctual scheme of the same length:

$$T = x_0^2 x_2 + 6x_1^2 x_3 - 3(x_0 + x_1)^2 x_4.$$

The $X$-border rank of $T$ with respect to $X = \nu_3(\mathbb{P}^4)$ is 5. In fact, one can explicitly write down a family $T_\epsilon$ having rank 5 with $T_\epsilon \to T$, namely

$$T_\epsilon = \frac{1}{3\epsilon}(x_0 + \epsilon x_2)^3 + 6(x_1 + \epsilon x_3)^3 - 3(x_0 + x_1 + \epsilon x_4)^3 + 3(x_0 + 2\,x_1)^3 - (x_0 + 3x_1)^3.$$

However, it is not possible to find a scheme of length 5 apolar to $T$ so that the cactus rank (and the rank) of $T$ is at least 6 [18, 26].

To verify that $\operatorname{brk} T > 4$, we compute a pseudowitness set for $\sigma_4(\nu_3(\mathbb{P}^4))$, which was accomplished using 77 random monodromy loops. The trace test verifies that $\deg \sigma_4(\nu_3(\mathbb{P}^4)) = 36{,}505$. Thus, upon tracking 36,505 solution paths, we found that all converged and none of the endpoints correspond to $T$ yielding $\operatorname{brk} T > 4$.

Next, we compute a pseudowitness set for $\sigma_5(\nu_3(\mathbb{P}^4))$, which took 102 monodromy loops with $\deg \sigma_5(\nu_3(\mathbb{P}^4)) = 24{,}047$. After tracking 24,047 paths, we find an endpoint limiting to $T$, but not converging in the fiber, yielding $\operatorname{brk} T = 5$.

As with other examples, the pseudowitness sets computed for this example can be stored and reused to test if other given cubic forms in 5 variables have border rank 4 and 5, respectively.

### 5.3 Generic elements

We turn to computing decompositions of generic elements by tracking one path as in Section 4.

The following example was posed to one of the authors by M. Mella a few years ago when the algorithm in [65] was not developed yet. M. Mella asked for a decomposition of a general polynomial of degree 5 in 3 variables, such as:

$$\begin{aligned} T \;=\;& 17051x_0^5 + 41500x_0^4x_1 + 720x_0^3x_1^2 + 11360x_0^2x_1^3 + 95010x_0x_1^4 + \\ & 19345x_1^5 - 18095x_0^4x_2 - 281420x_0^3x_1x_2 + 427290x_0^2x_1^2x_2 - \\ & 367940x_0x_1^3x_2 + 73860x_1^4x_2 + 243470x_0^3x_2^2 - 533370x_0^2x_1x_2^2 + \\ & 518670x_0x_1^2x_2^2 - 273140x_1^3x_2^2 + 156350x_0^2x_2^3 - 323300x_0x_1x_2^3 + \\ & 383760x_1^2x_2^3 + 80245x_0x_2^4 - 277060x_1x_2^4 + 84411x_2^5. \end{aligned}$$

In this case, the corresponding secant variety fills the image space and we can compute a decomposition by tracking one solution path. Using the setup

$$T = \sum_{j=1}^{7} \lambda_j (x_0 + a_{j1}x_1 + a_{j2}x_2)^5,$$

then endpoint of our path yielded the decomposition

$$\begin{aligned} T \;=\;& 243(x_0 + 8/3x_1 - 2/3x_2)^5 - 32768(x_0 - 3/4x_1 + 1/8x_2)^5 + \\ & 16807(x_0 - x_1 + x_2)^5 - 32(x_0 + 2x_1 - 4x_2)^5 + \\ & 32768(x_0 - 1/2x_2)^5 + 32(x_0 - 3/2x_1 + 5/2x_2)^5 + (x_0 - 5x_1 + 8x_2)^5. \end{aligned}$$

The algorithm in [65] can decompose general tensors in 3 variables up to degree 6. Our numerical homotopy algorithm can be used to decompose polynomials of higher degree, as the following example demonstrates. To illustrate, we start with the known decomposition

$$\begin{aligned} T \;=\;& 91(x_0 - 7/2x_1 + 9/2x_2)^7 + 58(x_0 - 3/2x_1 - 4/3x_2)^7 - 21(x_0 + 2x_1 - 9/2x_2)^7 \\ & + 33(x_0 + 3x_1 - x_2)^7 + 54(x_0 - 3x_1 - 5/3x_2)^7 + 88(x_0 - 3x_1 - 10/3x_2)^7 \\ & - 37(x_0 - 5x_1 + x_2)^7 + 93(x_0 - x_1 - 8x_2)^7 + 12(x_0 + 9/2x_1 + 10x_2)^7 \\ & - 89(x_0 - 5x_1 - 1/2x_2)^7 - 99(x_0 - x_1 - 3x_2)^7 - 22(x_0 - 1/3x_1 + 4x_2)^7. \end{aligned}$$

After expanding $T$ to extract the coefficients and tracking one path in 36 dimensions (after rescaling all coefficients to improve numerical conditioning), we compute the decomposition (where coefficients are rounded for readability and $i = \sqrt{-1}$):

$$
\begin{aligned}
T \;=\; & 90.5217(x_0 - 1.0016x_1 - 8.0256x_2)^7 + 133.8171(x_0 - 3.6909x_1 - 2.8478x_2)^7 \\
& - 97.4074(x_0 - 5.0606x_1 + 0.2459x_2)^7 + 89.4516(x_0 - 3.4857x_1 + 4.5217x_2)^7 \\
& - 20.6552(x_0 - 0.3125x_1 + 4.125x_2)^7 + 12.0133(x_0 + 4.4986x_1 + 9.9992x_2)^7 \\
& + 32.5455(x_0 + 3.0145x_1 - x_2)^7 + 83.1754(x_0 - 2.3582x_1 - 1.5306x_2)^7 \\
& - 19.4167(x_0 + 2.0658x_1 - 4.4909x_2)^7 - 83.0069(x_0 - 4.3651x_1 - 2.1818x_2)^7 \\
& - (30.0192 + 29.276i)(x_0 - (0.95833 + 0.2729i)x_1 - (3.9167 + 0.8299i)x_2)^7 \\
& - (30.0192 - 29.276i)(x_0 - (0.95833 - 0.2729i)x_1 - (3.9167 - 0.8299i)x_2)^7.
\end{aligned}
$$

We note that the original decomposition and this one are simply two points in the same fiber. Starting from this computed decomposition, we can use the approach of [50] to compute other points in the fiber. In this case, we obtain four other decompositions, one that we originally started with and the following three:

$$
\begin{aligned}
T \;=\; & -80.3535(x_0 - 0.96044x_1 - 3.2042x_2)^7 + 91.7624(x_0 - 3.4925x_1 + 4.4929x_2)^7 \\
& + 11.9529(x_0 + 4.5041x_1 + 10.005x_2)^7 - 42.331(x_0 - 5.1095x_1 - 1.1877x_2)^7 \\
& + 58.8757(x_0 - 1.9096x_1 - 0.70898x_2)^7 + 0.42442(x_0 - 6.5033x_1 - 3.8957x_2)^7 \\
& + 93.6035(x_0 - 1.0023x_1 - 7.9934x_2)^7 + 33.1366(x_0 + 2.9983x_1 - 1.0053x_2)^7 \\
& - 21.1233(x_0 + 2.0048x_1 - 4.4804x_2)^7 - 81.3951(x_0 - 4.9804x_1 + 0.50977x_2)^7 \\
& + 121.1404(x_0 - 3.0446x_1 - 3.0528x_2)^7 - 24.6931(x_0 - 0.46377x_1 + 3.855x_2)^7 \\
=\; & -19.5517(x_0 - 0.49254x_1 + 4.36x_2)^7 - 1.4462(x_0 + 3.3148x_1 + 5.9615x_2)^7 \\
& + 12.4957(x_0 + 4.48x_1 + 9.9434x_2)^7 - 64.3704(x_0 - 0.73438x_1 - 3.1471x_2)^7 \\
& + 94.5455(x_0 - 0.97674x_1 - 7.9818x_2)^7 - 18.506(x_0 + 1.7797x_1 - 4.9778x_2)^7 \\
& - 115.1045(x_0 - 5.0408x_1 + 0.0031746x_2)^7 + 30.4559(x_0 + 3.0345x_1 - 0.70492x_2)^7 \\
& + 126.7561(x_0 - 2.5128x_1 - 1.9074x_2)^7 + 89.4074(x_0 - 3.4483x_1 + 4.549x_2)^7 \\
& + (13.1591 + 9.58983i)(x_0 - (3.2017 - 1.1206i)x_1 - (4.0938 - 0.15711i)x_2)^7 \\
& + (13.1591 - 9.58983i)(x_0 - (3.2017 + 1.1206i)x_1 - (4.0938 + 0.15711i)x_2)^7 \\
=\; & -19.5946(x_0 - 0.21053x_1 + 4.1273x_2)^7 + 91.4966(x_0 + -0.99627x_1 + -8.0167x_2)^7 \\
& - 99.6415(x_0 - 5.1771x_1 - 0.12821x_2)^7 + 115.5185(x_0 - 2.7188x_1 - 3.4694x_2)^7 \\
& + 33.3191(x_0 + 2.9896x_1 - 0.99711x_2)^7 - 23.76(x_0 + 1.9074x_1 - 4.4706x_2)^7 \\
& + 88.0263(x_0 - 3.5054x_1 + 4.5294x_2)^7 + 12.0313(x_0 + 4.4976x_1 + 9.9967x_2)^7 \\
& - (3.5882 + 2.2523i)(x_0 - (5.4688 + 0.95833i)x_1 + (1.2571 - 0.93548i)x_2)^7 \\
& - (3.5882 - 2.2523i)(x_0 - (5.4688 - 0.95833i)x_1 + (1.2571 + 0.93548i)x_2)^7 \\
& - (14.6087 - 73.7949i)(x_0 - (1.6296 - 0.50355i)x_1 - (2.6154 + 1.0169i)x_2)^7 \\
& - (14.6087 + 73.7949i)(x_0 - (1.6296 + 0.50355i)x_1 - (2.6154 - 1.0169i)x_2)^7
\end{aligned}
$$

where $i = \sqrt{-1}$ and all numbers have been rounded for readability.

### 5.4 A degree 110 hypersurface

In [37], the authors consider the hypersurface $\mathcal{M} \subset \mathbb{P}^{15}$ defined by

$$
p_{ijkl} = \left( \sum_{s=0}^{1} a_{si} b_{sj} c_{sk} d_{sl} \right) \left( \sum_{r=0}^{1} e_{ri} f_{rj} g_{rk} h_{rl} \right) \text{ for all } \; (i, j, k, l) \in \{0, 1\}^4.
$$

The variety $\mathcal{M}$ is a Hadamard product, namely $\mathcal{M} = \sigma_2(S).\sigma_2(S)$ where $S$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^{15}$. The authors used this to show that $\deg \mathcal{M} = 110$ and the

Newton polytope of $\mathcal{M}$ has 17,214,912 vertices but did not compute an explicit equation. Since our approach is based on witness and pseudowitness sets, we do not need explicit equations to test for membership in $\mathcal{M}$.

We computed a pseudowitness set for $\mathcal{M}$ using 26 monodromy loops which also yielded $\deg \mathcal{M} = 110$, as expected. Let $\mathcal{M}^0$ denote the corresponding constructible set so that $\mathcal{M} = \overline{\mathcal{M}^0}$. To demonstrate our membership test, we consider the point

$$v = (2, 3, 0, -1, 4, 2, 0, 1, 1, -2, 2, 0, 1, 0, -4, 3) \in \mathbb{P}^{15}.$$

By tracking 110 paths, we find that $v \notin \mathcal{M}$. Next, we consider the point

$$\begin{aligned} w \;\; = \;\; & (2528064, -3079104, -2340576, 2038176, -1804032, 2398464, 1539648, -1524096, \\ & 1104000, 456086, -2403720, 284016, -511104, -502072, 1220472, -23424) \in \mathbb{P}^{15}. \end{aligned}$$

In this case, after tracking 110 paths, we find $w \in \mathcal{M}^0$. In particular, $w$ is the image of

$$\begin{array}{lll}
a_{00} = -6.5220 + 1.8885i & d_{00} = 18.1529 + 5.5948i & g_{00} = -1.4597 + 9.8573i \\
a_{01} = 17.1112 + 2.1887i & d_{01} = 11.1640 - 8.9931i & g_{01} = -3.9468 + 8.2471i \\
a_{10} = -0.0000 + 0.0000i & d_{10} = 0.0335 - 0.1311i & g_{10} = -1.6724 - 1.2813i \\
a_{11} = 4.7144 + 0.5813i & d_{11} = -0.0255 - 0.0286i & g_{11} = -2.9854 - 4.2021i \\
b_{00} = 1.0901 - 1.3154i & e_{00} = -16.9364 - 9.3010i & h_{00} = -0.3222 - 0.2848i \\
b_{01} = 0.2466 + 0.3406i & e_{01} = -2.8396 - 0.2652i & h_{01} = -0.1314 + 1.0045i \\
b_{10} = -22.1726 - 13.8102i & e_{10} = -2.7150 + 6.8142i & h_{10} = -0.8416 + 7.6337i \\
b_{11} = 0.7238 - 0.6901i & e_{11} = -0.1511 - 5.0503i & h_{11} = 3.0027 - 1.5781i \\
c_{00} = -12.5203 + 0.5994i & f_{00} = -3.0265 - 6.0562i & \\
c_{01} = 7.2261 - 2.6147i & f_{01} = 16.6817 - 14.5017i & \\
c_{10} = -10.9459 - 1.5479i & f_{10} = 2.9081 - 0.6907i & \\
c_{11} = -17.8454 + 2.8591i & f_{11} = 3.2373 + 6.1380i &
\end{array}$$

where $i = \sqrt{-1}$ and all decimals have been rounded for readability.

## 5.5 Joins for decomposable polynomials

Consider the closure of the spaces in $\mathrm{Sym}^4\mathbb{C}^4 \subset \mathbb{P}^{35}$ which can be written as the sum of $r$ squares of quadrics and the sum of $s$ fourth powers of linear forms:

$$f = \sum_{i=1}^{r} q_i^2 + \sum_{j=1}^{s} \ell_j^4,$$

i.e., $J(\sigma_r(\nu_2(\mathbb{P}(S^2\mathbb{C}^4))), \sigma_s(\nu_4(\mathbb{P}^3)))$. The following lists the expected dimension, which is the minimum of the dimension of the ambient space, namely 35, and $10r + 4s$, and the actual dimension for various $r$ and $s$. The ones in bold correspond to defective cases.

| $r$ | 0 | | | | | | | | | | 1 | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| Expected dim | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | 35 | 35 | 10 | 14 | 18 | 22 | 26 | 30 | 34 | 35 |
| Actual dim | 4 | 8 | 12 | 16 | 20 | 24 | 28 | 32 | **34** | 35 | 10 | 14 | 18 | 22 | 26 | 30 | 34 | 35 |

| $r$ | 2 | | | | | 3 | | | 4 | | 5 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $s$ | 0 | 1 | 2 | 3 | 4 | 0 | 1 | 2 | 0 | 1 | 0 |
| Expected dim | 20 | 24 | 28 | 32 | 35 | 30 | 34 | 35 | 35 | 35 | 35 |
| Actual dim | **19** | **23** | **27** | **31** | 35 | **27** | **31** | 35 | **34** | 35 | 35 |

There are two defective hypersurfaces, namely the case $(r, s) = (0, 9)$ and $(r, s) = (4, 0)$. We verified that the case $(r, s) = (0, 9)$ yields a degree 10 hypersurface [3] while the case $(r, s) = (4, 0)$ is a degree 38,475 hypersurface [22].

### 5.6 Best low rank approximation

Motivated by [66, Ex. 7 & 8], we consider $X = \sigma_2(\nu_4(\mathbb{P}^2)) \subset \mathbb{P}^{14}$. In our test, the computation of a pseudowitness set for $X$ required 14 monodromy loops which yielded $\deg X = 75$. With this, we can now test membership in $X$ by tracking at most 75 paths. For example, we consider the tensor from Thomas Schultz listed in [66, Ex. 7]:

$$\begin{aligned} T \;=\; & 0.1023x_0^4 + 0.0197x_1^4 + 0.1869x_2^4 + 0.0039x_0^2x_1^2 + 0.0407x_0^2x_2^2 - 0.00017418x_1^2x_2^2 \\ & - 0.002x_0^3x_1 + 0.0581x_0^3x_2 + 0.0107x_0x_1^3 + 0.0196x_0x_2^3 + 0.0029x_1^3x_2 - 0.0021x_1x_2^3 \\ & - 0.00032569x_0^2x_1x_2 - 0.0012x_0x_1^2x_2 - 0.0011x_0x_1x_2^2. \end{aligned}$$

Following the membership test from Section 1, since all 75 paths converged to points which did not correspond to $[T]$, we know that $[T] \notin X$.

Since it is expected that noise in the data moves one off of the variety, one often wants to compute the "best" low rank approximation. In this case, we want a real element in $X$ which minimizes the Euclidean distance of the coefficients, treated as a vector in $\mathbb{C}^{15}$. One approach is to compute all critical points which was performed in [66, Ex. 8]. This resulted in 195 points outside of the set of rank 1 elements, i.e., $\nu_4(\mathbb{P}^2)$, of which 9 are real. In particular, there are 2 are local minima and 7 saddle points, with the global minimum approximately being:

$$(0.0168x_0 - 0.00189x_1 + 0.657x_2)^4 + (0.56x_0 - 0.00254x_1 + 0.0988x_2)^4.$$

We additionally consider using a gradient descent homotopy as discussed in Section 3. Since $X$ is defined by 148 cubic polynomials, we utilized a random real combination of these polynomials. In our experiment, the path starting with $T$ produced the critical point corresponding to the same global minimizer as above.

### 5.7 Skew-symmetric tensors

We close by considering skew-symmetric tensors in $\bigwedge^3 \mathbb{C}^7 \subset \mathbb{C}^{35}$ with respect to the Grassmannian $G(3, 7)$. By dimension counting, one expects a general element to have rank 3, but one can easily verify using the methods described above that $\sigma_3(G(3, 7))$ is a hypersurface of degree 7. Hence, a general element has rank 4. The defectivity of this hypersurface has already been observed in [1, 15, 28] and it is conjectured that, together with $\sigma_3(G(4, 8))$, $\sigma_4(G(4, 8))$, $\sigma_4(G(3, 9))$ and their duals, they are the only defective secant varieties to a Grassmannian. To the best of our knowledge, the degree of this hypersurface had not been computed before and a degree 7 polynomial defining this hypersurface is an $SL(7)$-invariant polynomial whose cube is a determinant of certain contraction operator (this is described in [1, Theorem 5.1]). However, by tracking at most 7 paths, we are able to test membership in this hypersurface.

We now turn to $\sigma_2(G(3, 7)) \subset \mathbb{C}^{35}$ which is an irreducible variety of dimension 26 and degree 735. In particular, we aim to compute the codimension one components of its boundary as follows. To simplify the computation, we consider the maps $A_i : \mathbb{C}^5 \to \mathbb{C}^7$ defined by

$$A_1(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix},\ A_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \begin{bmatrix} 0 \\ \alpha_1 \\ 0 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix},\ A_3(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \begin{bmatrix} 0 \\ 0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix}.$$

For general affine linear polynomials $\ell_1, \ldots, \ell_{25}$ in 35 variables and $p$ in 27 variables, we consider

$$\mathcal{C} = \overline{\left\{ Z \cdot h^3 - \sum_{i=1}^{2} A_1(a_1^i, a_2^i, a_3^i, a_4^i, a_5^i) \wedge A_2(a_1^i, a_6^i, a_7^i, a_8^i, a_9^i) \wedge A_3(a_1^i, a_{10}^i, a_{11}^i, a_{12}^i, a_{13}^i) \;\middle|\; \ell_k(Z) = p(h, a_j^i) = 0 \right\}}$$

which is an irreducible curve. For a general $\beta \in \mathbb{C}$, we found that $\mathcal{C} \cap \mathcal{V}(h - \beta)$ consists of 48,930 points. By tracking the homotopy paths defined by $\mathcal{C} \cap \mathcal{V}(h - \beta \cdot t)$, 44,520 paths yielded points in $\mathcal{C} \cap \mathcal{V}(h)$. The corresponding points break down into two types. The first type, which consist of 3262 distinct $Z$ coordinates, each corresponding to the endpoint of 12 paths, either have $a_1^1 = 0$ or $a_1^2 = 0$. These points are in the boundary based on the choice of parameterization used in $\mathcal{C}$, but are not actually in the boundary of $\sigma_2(G(3,7))$. The second type, which consist of 1792 distinct $Z$ coordinates, each corresponding to the endpoint of 3 paths, are indeed in the boundary of $\sigma_2(G(3,7))$. In fact, these points form a witness point set for the following irreducible variety of dimension 25 and degree 1792:

$$\overline{\{v_1 \wedge v_2 \wedge w_1 + v_1 \wedge w_2 \wedge v_3 + w_3 \wedge v_2 \wedge v_3 \mid v_i, w_i \in \mathbb{C}^7\}} \subset \mathbb{C}^{35}$$

which is precisely the codimension one component of the boundary of $\sigma_2(G(3,7))$ and it is clearly the tangential variety to $G(3,7)$ (this is also well described in [1]). We note that $44{,}520 = 12 \cdot 3262 + 3 \cdot 1792$.


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