Integral Equations for Electromagnetic Scattering at Multi-Screens

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Abstract. In [X. Claeys and R. Hiptmair, *Integral equations on multi-screens.* Integral Equations and Operator Theory, 77(2):167–197, 2013] we developed a framework for the analysis of boundary integral equations for acoustic scattering at so-called multi-screens, which are arbitrary arrangements of thin panels made of impenetrable material. In this article we extend these considerations to boundary integral equations for electromagnetic scattering.

We view tangential multi-traces of vector fields from the perspective of quotient spaces and introduce the notion of single-traces and spaces of jumps. We also derive representation formulas and establish key properties of the involved potentials and related boundary operators. Their coercivity will be proved using a splitting of jump fields. Another new aspect emerges in the form of surface differential operators linking various trace spaces.

Mathematics Subject Classification (2010). Primary 45A05; Secondary 65R20.

Keywords. screen, integral equation, scattering, wave propagation, Helmholtz, junction points.

1. Introduction

In this article we examine first-kind boundary integral equations (BIEs) related to the homogeneous Maxwell equations in frequency domain

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2\mathbf{u} = \mathbf{0}\ , \tag{1.1}$$

with wave number $\kappa \in \mathbb{C} \setminus (-\infty, 0]$, in the exterior of rather general two-dimensional surfaces (with boundary) that we have dubbed *complex screens* or *multi-screens*, see Section 2. This generalises the well established theory for Lipschitz screens as presented by Buffa and Christiansen in [4]. Interest in this generalisation is motivated by the ubiquity of complex screen geometries

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in engineering applications and by the widespread use of boundary integral equation techniques for numerical simulation, see [22, 26, 25, 8, 11] among others.

This article can be viewed as a companion to [9], where we focused on BIEs associated with the scalar Helmholtz equation and developed their theory on complex screens. The definition of suitable trace spaces and, in particular, of spaces of jumps turned out to be a major mathematical challenge in that work. We mastered it by consistently resorting to Green's formulas in the volume, following the modern paradigm for the analysis of BIEs [12]. Our current bid to generalise the theory of [4] to (1.1) and the underlying function spaces encounters further mathematical challenges related to the peculiarities of the Maxwell equations compared to the Helmholtz equation. In particular, to prove stability of boundary integral operators, we have to deal with Hodge type decompositions of vector fields in non-Lipschitz domains, which is outside the scope of traditional results for this kind of decomposition.

Let us briefly review the plan of the paper. In Section 2 we recapitulate the definition of multi-screens already introduced in [9]. Section 3 recalls the scalar trace spaces defined in [9, Sections 5 & 6]. In the core Section 4 we follow the reasonings of [9] to develop a clear idea of tangential traces of vector fields with **curl** in L^2. Next, Section 5 examines surface differential operators linking scalar and vector multi-trace, single-trace, and jump spaces. In Section 6, we examine the well-posedness of the electromagnetic scattering problem with perfectly conducting boundary conditions at a multi-screen. To guarantee existence and uniqueness of the solution, we introduce an additional geometrical assumption so as to guarantee that this boundary value problem enters the standard Riesz-Schauder theory. Aiming for boundary integral operators, Section 7 provides a representation formula for solutions of (1.1). Applying the trace operators introduced earlier to the representation formula yields boundary integral equations. In Section 8 we give an alternative definition of our new tangential jump trace spaces, which establishes a link to existing theory. In the final and crucial Section 9 we prove coercivity of the Maxwell single layer boundary integral operator on a multi-screen, see Theorem 9.7. To accomplish this, we have to resort to a novel variant of the usual splitting technique based on a Hodge-type decomposition of the jump traces.

Notations

Γ	Multi-screen with boundary $\partial\Gamma$
Ω_j	Finite collection of Lipschitz domains adjacent to Γ, see Definition 2.3
$\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma})$	Sobolev space of functions $\mathbb{R}^3\setminus\overline{\Gamma}\to\mathbb{C}$, see (3.1)
$\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma})$	Sobolev space of vector fields $\mathbb{R}^3\setminus\overline{\Gamma}\to\mathbb{C}^3$ with square integrable divergence
$\mathbb{H}^{\pm\frac{1}{2}}(\Gamma)$	Scalar values multi-trace spaces, see Definition 3.1
π_{D}	Dirichlet trace (point trace) $\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma})\to\mathbb{H}^{1/2}(\Gamma)$

π_{N}	Normal component trace $\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma})\to\mathbb{H}^{-1/2}(\Gamma)$
$\langle\!\langle\cdot,\cdot\rangle\!\rangle$	Bilinear duality pairing for scalar functions on Γ, see (3.3)
$\mathrm{H}^{\pm\frac{1}{2}}([\Gamma])$	Scalar-valued single traces spaces, see Definition 3.2
$\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})$	Sobolev space of vector fields $\mathbb{R}^3\setminus\overline{\Gamma}\to\mathbb{C}^3$ with square integrable **curl**
γ_{T}	Standard tangential trace operator, see (4.2), (7.1)
$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$	Tangential multi-trace space, see Definition 4.4
π_{T}	tangential multi-trace operator; canonical projection onto $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$, see (4.5)
$\langle\!\langle\cdot,\cdot\rangle\!\rangle_\times$	Skew-symmetric duality pairing in $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$, see (4.6), (4.8)
$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma])$	Tangential single-trace space, see Definition 4.12
$\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])$	Tangential jump space, see Definition 4.13
$[\ ]$	Jump operator, see (3.5) and Definitions 4.8
∇_Γ	surface gradient, see (5.2)
curl_Γ	surface rotation, see (5.4)
γ_{R}	tangential trace of **curl**, see (7.1)
$\mathcal{G}_\kappa$	Helmholtz fundamental solution with wave number κ
$\mathrm{DL}_\kappa,\mathrm{SL}_\kappa$	Vector single and double layer potentials, see (7.6)
$\imath$	imaginary unit
$\mathbf{H}_\times^{\frac{1}{2}}(\Gamma)$	tangential trace space for $(\mathrm{H}^1(\mathbb{R}^3))^3$, see (8.5)
$\mathcal{E}_{\mathrm{T}}$	Dirichlet harmonic vector fields, see (9.1)

2. Geometrical Setting: Definition of Multi-Screens

Since the treatment of particular geometries is the main focus of the present contribution, we start with a precise description of the geometries we consider, closely following [9, Section 2]. To begin with, we recall what is an orientable Lipschitz screen, a notion that was introduced by Buffa and Christiansen [4]. Here and in the sequel, we will only consider three dimensional situations, since we are interested in the study of Maxwell's equations.

Definition 2.1 (Lipschitz screen).
A *Lipschitz screen* (in the sense of Buffa-Christiansen) is a subset $\Gamma\subset\mathbb{R}^3$ that satisfies the following properties:

- the set $\overline{\Gamma}$ is a compact Lipschitz two-dimensional sub-manifold with boundary,
- denoting $\partial\Gamma$ the boundary of $\overline{\Gamma}$, we have $\Gamma=\overline{\Gamma}\setminus\partial\Gamma$,
- there exists a finite covering of $\overline{\Gamma}$ with cubes such that, for each such cube C, denoting by h the length of its sides, we have

* if C contains a point of $\partial\Gamma$, there exists an orthonormal basis of $\mathbb{R}^3$ in which C can be identified with $(0,h)^3$ and there are Lipschitz continuous functions $\psi : \mathbb{R} \to \mathbb{R}$ and $\phi : \mathbb{R}^2 \to \mathbb{R}$ with values in $(0,h)$ such that

$$\begin{aligned} \Gamma \cap C &= \{\ (x,y,z) \in C \ \mid\ y < \psi(x),\ z = \phi(x,y)\ \}\ , \\ \partial\Gamma \cap C &= \{\ (x,y,z) \in C \ \mid\ y = \psi(x),\ z = \phi(x,y)\ \}\ , \end{aligned} \tag{2.1}$$

* if C contains no boundary point, there exists a Lipschitz open set $\Omega \subset \mathbb{R}^3$ such that we have $\Gamma \cap C = \partial\Omega \cap C$.

In the sequel, we will refer to orientable screens as "screens in the sense of Buffa and Christiansen". Multi-screens are generalisations of such objects that allow the presence of several "panels" or "fins".

Definition 2.2 (Lipschitz partition).
A *Lipschitz partition* of $\mathbb{R}^3$ is a finite collection of Lipschitz open sets $(\Omega_j)_{j=0\dots n}$ such that $\mathbb{R}^3 = \cup_{j=0}^{n}\overline{\Omega}_j$ and $\Omega_j \cap \Omega_k = \emptyset$, if $j \neq k$.

Definition 2.3 (Multi-screen).
A *multi-screen* is a subset $\Gamma \subset \mathbb{R}^3$ such that there exists a Lipschitz partition of $\mathbb{R}^3$ denoted $(\Omega_j)_{j=0\dots n}$ satisfying $\Gamma \subset \cup_{j=0}^{n}\partial\Omega_j$ and such that, for each $j = 0 \dots n$, we have $\overline{\Gamma} \cap \partial\Omega_j = \overline{\Gamma}_j$ where $\Gamma_j \subset \partial\Omega_j$ is some Lipschitz screen (in the sense of Buffa-Christiansen).

Remark 2.4. Since the definition above allows the presence of several branches, multi-screens are not globally orientable a priori, although they are locally orientable away from junction points, i.e. points where several branches meet. Although such surfaces commonly occur in applications, beside our article [9], we could not find any literature on integral equations considering such objects, especially in the context of electromagnetics.

Remark 2.5. Concerning variational formulations of Maxwell's equations, however, there already exist references dealing with possibly non-Lipschitz geometries. In this direction, we would like to point out [21] that considers a geometrical setting (see in particular Theorem 3.6) that covers the situations considered in the present article.

3. Scalar Valued Function Spaces on Multi-Screens

To prepare the ground for treating traces of vector fields, we give a brief review of the functional framework that was developed in [9] for analysing scalar scattering by multi-screen objects. We shall provide no proofs of the results contained in this section, and refer the reader to [9, Sections 5 & 6].

3.1. Domain based function spaces

The trace spaces adapted to multi-screens that we introduced in [9] are built upon two domain based functional spaces. The first one, denoted $\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma})$, is defined as the space of functions $u \in \mathrm{L}^2(\mathbb{R}^3)$ such that there exists $\mathbf{p} \in \mathrm{L}^2(\mathbb{R}^3)$ satisfying*

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus\overline{\Gamma}} u \,\mathrm{div}(\mathbf{q})\, d\boldsymbol{x} = -\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{p}\cdot\mathbf{q}\, d\boldsymbol{x} \qquad \forall \mathbf{q} \in (\mathcal{D}(\mathbb{R}^3\setminus\overline{\Gamma}))^3 \\ &\text{and we set} \quad \|u\|^2_{\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma})} \;:=\; \|u\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \|\mathbf{p}\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\,. \end{aligned} \tag{3.1}$$

Naturally, this norm is well defined since, if such a $\mathbf{p}$ as above exists, it is unique. The Sobolev space $\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma})$ equipped with the norm defined in (3.1) is a Hilbert space. We also define $\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^3)$ the closure of $\mathcal{D}(\mathbb{R}^3\setminus\overline{\Gamma})$ in $\mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma})$ with respect to this norm. The second domain based space that we introduced in [9, Section 4], denoted by $\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma})$, is the space of fields $\mathbf{p} \in \mathrm{L}^2(\mathbb{R}^3)^3$ such that there exists $u \in \mathrm{L}^2(\mathbb{R}^3)$ satisfying

$$\begin{aligned} &\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{p}\cdot\nabla v\, d\boldsymbol{x} = -\int_{\mathbb{R}^3\setminus\overline{\Gamma}} u\, v\, d\boldsymbol{x} \qquad \forall v \in \mathcal{D}(\mathbb{R}^3\setminus\overline{\Gamma})\,, \\ &\text{and we set} \quad \|\mathbf{p}\|^2_{\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma})} \;:=\; \|u\|^2_{\mathrm{L}^2(\mathbb{R}^3)} + \|\mathbf{p}\|^2_{\mathrm{L}^2(\mathbb{R}^3)}\,. \end{aligned} \tag{3.2}$$

Once again, if such a u as above exists, it is unique, so that the norm $\|\;\|_{\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma})}$ is well defined. The space $\mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma})$ equipped with this norm is a Hilbert space. We also define $\mathbf{H}_{0,\Gamma}(\mathrm{div},\mathbb{R}^3)$ as the closure of $\mathcal{D}(\mathbb{R}^3\setminus\overline{\Gamma})^3$ with respect to this norm.

3.2. Multi-trace spaces

These trace spaces are defined in an abstract manner as factor spaces, see [9, Section 5].

Definition 3.1 (Scalar valued multi-trace spaces).
Scalar valued Dirichlet and Neumann multi-trace spaces, respectively, are defined as

$$\begin{aligned} \mathbb{H}^{+\frac{1}{2}}(\Gamma) \;&:=\; \mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma})/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^3) \\ \mathbb{H}^{-\frac{1}{2}}(\Gamma) \;&:=\; \mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma})/\mathbf{H}_{0,\Gamma}(\mathrm{div},\mathbb{R}^3). \end{aligned}$$

These spaces are equipped with their respective canonical quotient norms $\|\;\|_{\mathbb{H}^{\pm1/2}(\Gamma)}$.

We also consider trace operators $\pi_{\mathrm{D}} : \mathrm{H}^1(\mathbb{R}^3\setminus\overline{\Gamma}) \to \mathbb{H}^{1/2}(\Gamma)$ and $\pi_{\mathrm{N}} : \mathbf{H}(\mathrm{div},\mathbb{R}^3\setminus\overline{\Gamma}) \to \mathbb{H}^{-1/2}(\Gamma)$ simply as the canonical projections for these

*Given any open subset $\omega \subset \mathbb{R}^3$, $\mathcal{D}(\omega)$ denotes the set of elements of $\mathcal{C}^\infty(\mathbb{R}^3)$ that vanish in $\mathbb{R}^3\setminus\overline{\omega}$, and $\mathcal{D}'(\omega)$ designates its dual i.e., the space of distributions in ω.

quotient spaces. The multi-trace spaces $\mathbb{H}^{\pm 1/2}(\Gamma)$ are dual to each other via the bilinear pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ defined by the formula

$$\langle\!\langle \pi_{\mathrm{D}}(u), \pi_{\mathrm{N}}(\mathbf{p}) \rangle\!\rangle := \int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{p}\cdot\nabla u + \mathrm{div}(\mathbf{p})\, u \, d\boldsymbol{x} \;, \tag{3.3}$$

for all $u \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma})$ and all $\mathbf{p} \in \mathbf{H}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Gamma})$. Identity (3.3) should be understood as a generalised Green formula where Γ plays the role of the "boundary" of $\mathbb{R}^3 \setminus \overline{\Gamma}$.

3.3. Single-trace spaces and jumps

The elements of $\mathbb{H}^{\pm 1/2}(\Gamma)$ may be regarded as double-valued functions defined on Γ (each value being associated to a different face of Γ). We also consider subspaces of the multi-trace spaces that correspond to single valued functions.

Definition 3.2 (Scalar-valued single trace spaces).
Scalar valued single traces spaces for Dirichlet and Neumann data, respectively, are defined as

$$\begin{aligned} \mathrm{H}^{+\frac{1}{2}}([\Gamma]) &:= \mathrm{H}^1(\mathbb{R}^3)/\mathrm{H}^1_{0,\Gamma}(\mathbb{R}^3) \;=\; \pi_{\mathrm{D}}\big(\,\mathrm{H}^1(\mathbb{R}^3)\,\big) \\ \mathrm{H}^{-\frac{1}{2}}([\Gamma]) &:= \mathbf{H}(\mathrm{div},\mathbb{R}^3)/\mathbf{H}^1_{0,\Gamma}(\mathrm{div},\mathbb{R}^3) \;=\; \pi_{\mathrm{N}}\big(\,\mathbf{H}(\mathrm{div},\mathbb{R}^3)\,\big) \end{aligned}$$

These are closed subspaces of $\mathbb{H}^{\pm 1/2}(\Gamma)$ and, as such, inherit the norms $\|\;\|_{\mathbb{H}^{\pm 1/2}(\Gamma)}$.

The single trace spaces $\mathrm{H}^{\pm 1/2}([\Gamma])$ are polar to each other under the duality pairing (3.3). In particular we have $\langle\!\langle \dot{u}, \dot{p} \rangle\!\rangle = 0$ for every $\dot{u} \in \mathrm{H}^{1/2}([\Gamma])$, $\dot{p} \in \mathrm{H}^{-1/2}([\Gamma])$. We also define jump spaces as duals of the single trace spaces

$$\widetilde{\mathrm{H}}^{+\frac{1}{2}}([\Gamma]) \;=\; \mathrm{H}^{-\frac{1}{2}}([\Gamma])' \qquad \text{and} \qquad \widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma]) \;=\; \mathrm{H}^{+\frac{1}{2}}([\Gamma])' \tag{3.4}$$

We equip the jump spaces (3.4) with the dual norms. Note that any element of $\mathbb{H}^{\pm 1/2}(\Gamma)$ naturally induces an element of $\widetilde{\mathrm{H}}^{\pm 1/2}([\Gamma])$ via the pairing $\langle\!\langle \cdot,\cdot \rangle\!\rangle$. This allows to consider a continuous and surjective "jump" operator $[\,\cdot\,] : \mathbb{H}^{+1/2}(\Gamma) \to \widetilde{\mathrm{H}}^{+1/2}([\Gamma])$ defined by the formula

$$\langle [\dot{u}], \dot{q} \rangle \;:=\; \langle\!\langle \dot{u}, \dot{q} \rangle\!\rangle \qquad \forall \dot{q} \in \mathrm{H}^{-\frac{1}{2}}([\Gamma]) \tag{3.5}$$

where this holds for any $\dot{u} \in \mathbb{H}^{+1/2}(\Gamma)$. In a completely analogous manner we can define a jump operator $[\,\cdot\,] : \mathbb{H}^{-1/2}(\Gamma) \to \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$ that is continuous and surjective as well.

4. Tangential Traces on Multi-Screens

Now we study tangential traces of **curl**-conforming vector fields featuring jumps across the multi-screen Γ. Our considerations run parallel to those [9] for scalar Dirichlet and Neumann multi-traces. Let us also point out that our treatment of traces is in the spirit of [2, 17].

4.1. Function spaces for vector fields in the volume

First of all, we define spaces of vector fields on the unbounded domain $\mathbb{R}^3\setminus\overline{\Gamma}$. As usual, the space $\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})$ will designate the set of $\mathbf{u}\in\mathrm{L}^2(\mathbb{R}^3)^3$ such that there exists $\mathbf{p}\in\mathrm{L}^2(\mathbb{R}^3)^3$ satisfying

$$\int_{\mathbb{R}^3\setminus\overline{\Gamma}}\mathbf{u}\cdot\mathbf{curl}(\mathbf{v})\,d\boldsymbol{x} \;=\; \int_{\mathbb{R}^3\setminus\overline{\Gamma}}\mathbf{p}\cdot\mathbf{v}\,d\boldsymbol{x}\qquad\forall\mathbf{v}\in\mathcal{D}(\mathbb{R}^3\setminus\overline{\Gamma})^3\;. \tag{4.1}$$

Of course, according to this definition, we have $\mathbf{p}=\mathbf{curl}(\mathbf{u})|_{\mathbb{R}^3\setminus\overline{\Gamma}}$ in the sense of distribution in $\mathbb{R}^3\setminus\overline{\Gamma}$. However, in general $\mathbf{p}\neq\mathbf{curl}(\mathbf{u})$ in the sense of distributions in $\mathbb{R}^3$, as there may be tangential jumps of $\mathbf{u}$ across Γ. We equip the space $\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})$ with the scalar product

$$(\mathbf{u},\mathbf{v})_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})}:=\int_{\mathbb{R}^3\setminus\overline{\Gamma}}\mathbf{u}\overline{\mathbf{v}}\,d\boldsymbol{x}+\int_{\mathbb{R}^3\setminus\overline{\Gamma}}(\mathbf{curl}\;\mathbf{u}|_{\mathbb{R}^3\setminus\overline{\Gamma}})\cdot(\mathbf{curl}\;\overline{\mathbf{v}}|_{\mathbb{R}^3\setminus\overline{\Gamma}})\,d\boldsymbol{x}\;.$$

It is well known that $\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})$ is a Hilbert space when equipped with this scalar product. We denote by $\|\;\;\|_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})}$ the induced norm. We also define $\mathbf{H}_{0,\Gamma}(\mathbf{curl},\mathbb{R}^3)$ to be the closure of $\mathcal{D}(\mathbb{R}^3\setminus\overline{\Gamma})^3$ with respect to the norm $\|\;\;\|_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})}$. It is clear that both $\mathbf{H}_{0,\Gamma}(\mathbf{curl},\mathbb{R}^3)$ and $\mathbf{H}(\mathbf{curl},\mathbb{R}^3)$ are closed subspaces of $\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})$.

4.2. Tangential trace spaces on boundaries of Lipschitz domains

Of course, the treatment of multi-screens is founded on established results concerning traces of vector fields on the boundary of non-smooth domains. All results presented in this section are covered in [6]; see also [7, 3] for surveys.

In this section, we consider a generic Lipschitz domain $\Omega\subset\mathbb{R}^3$. According to Rademacher's theorem, the normal vector field $\mathbf{n}$ at $\Gamma:=\partial\Omega$ is a well defined function of $(\mathrm{L}^\infty(\Gamma))^3$. Let $\mathbf{H}(\mathbf{curl},\Omega)$ denote the space of vector fields $\mathbf{u}\in(\mathrm{L}^2(\Omega))^3$ such that $\mathbf{curl}(\mathbf{u})\in(\mathrm{L}^2(\Omega))^3$. We define the *tangential trace* γ_{T} as the operator that satisfies

$$\gamma_{\mathrm{T}}(\mathbf{u})=\mathbf{n}\times(\mathbf{u}\times\mathbf{n})\qquad\forall\mathbf{u}\in(\mathcal{D}(\mathbb{R}^3))^3\;. \tag{4.2}$$

This operator induces a surjective continuous trace operator $\gamma_{\mathrm{T}}:\mathbf{H}(\mathbf{curl},\Omega)\to\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$, see [6, Thm.4.1]. In addition, the following is a straightforward consequence of [6, Thm.5.1],

$$\mathbf{H}_0(\mathbf{curl},\Omega)=\ker(\,\gamma_{\mathrm{T}}\,)\;,$$

where $\mathbf{H}_0(\mathbf{curl},\Omega)$ is the completion of $(\mathcal{D}(\Omega))^3$ in $\mathbf{H}(\mathbf{curl},\Omega)$. Hence, there is an isomorphism between $\mathbf{H}(\mathbf{curl},\Omega)/\mathbf{H}_0(\mathbf{curl},\Omega)$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ which makes possible the identification

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)\;=\;\mathbf{H}(\mathbf{curl},\Omega)/\mathbf{H}_0(\mathbf{curl},\Omega)\;. \tag{4.3}$$

The trace γ_{T} can thus be read as canonical projection onto a factor space.

4.3. Tangential vector multi-traces

Now we introduce spaces obtained as tangential traces of vector fields belonging to $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$. The geometry of the multi-screen may be very complex, and this makes this trace space difficult to define. To overcome this geometrical difficulty, we take the cue from (4.3) and, in analogy to Definition 3.1 and [9, Section 5], use an abstract definition based on quotient spaces:

Definition 4.1 (Tangential multi-trace space).
The tangential multi-trace space on the multi-screen Γ is defined as

$$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \;:=\; \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})/\mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)\;. \tag{4.4}$$

Of course, the chosen notation contains "$-\frac{1}{2}$" as a superscript, as well as "curl_Γ" in order to suggest as explicitly as possible that this new space is a generalisation of the space $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ for $\Gamma = \partial\Omega$. In other words, Definition (4.4) is consistent with (4.3).

The space (4.4) will be equipped with the quotient norm, see the Appendix in [9] for example, and the trace operator is given by the canonical projection

$$\pi_{\mathrm{T}} : \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) \to \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)\;. \tag{4.5}$$

Now observe that, using elementary density arguments, for $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ we have $\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{curl}(\mathbf{u})\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{curl}(\mathbf{v})\, d\boldsymbol{x} = 0$ whenever $\mathbf{u} \in \mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)$ or $\mathbf{v} \in \mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)$. As a consequence, for any $\dot{\mathbf{u}}, \dot{\mathbf{v}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, we can define

$$\langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle\!\rangle_\times := \int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{curl}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}(\mathbf{v})\; d\boldsymbol{x}\;, \tag{4.6}$$

where $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ are such that $\pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}$ and $\pi_{\mathrm{T}}(\mathbf{v}) = \dot{\mathbf{v}}$. The bilinear form $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\times$ is clearly skew-symmetric and continuous on $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) \times \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$. It actually puts $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ into duality with itself, *cf.* [9, Section 5.1].

Proposition 4.2 (Self-duality of $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$, cf. [9, Proposition 4.1]**,** [7, Theorem 2]**).** *For any continuous linear form $\varphi : \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{C}$, there exists a unique $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that $\varphi(\dot{\mathbf{v}}) = \langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle\!\rangle_\times$ for all $\dot{\mathbf{v}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and $\|\varphi\|_{(\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma))'} = \|\dot{\mathbf{u}}\|_{\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)}$.*

Proof: For any $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, consider the unique minimal norm representative $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, that is $\pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}$. We have $\|\mathbf{u}\|_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})} = \|\dot{\mathbf{u}}\|_{\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)}$ by definition of the quotient norm and $\mathbf{u}$ satisfies the orthogonality condition

$$\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{curl}(\mathbf{u}) \cdot \mathbf{curl}(\mathbf{v}) + \mathbf{u} \cdot \mathbf{v}\, d\boldsymbol{x} = 0 \qquad \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)\;. \tag{4.7}$$

Set $\mathbf{p} = \mathbf{curl}(\mathbf{u}) \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3\setminus\overline{\Gamma})$, so that $\mathbf{curl}(\mathbf{p}) = -\mathbf{u}$ since $\mathbf{curl}(\mathbf{curl}(\mathbf{u})) + \mathbf{u} = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$, which is a direct consequence of (4.7). Since both $\mathbf{p}$ and

$\mathbf{u}$ linearly and continuously depend on $\dot{\mathbf{u}}$, we can set $\Phi(\dot{\mathbf{u}}) = \pi_{\mathrm{T}}(\mathbf{p})$, where $\Phi : \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ is a linear isometry. Then we have

$$\langle\!\langle\, \overline{\dot{\mathbf{u}}}, \Phi(\dot{\mathbf{u}})\, \rangle\!\rangle_\times \;=\; \int_{\mathbb{R}^3\setminus\overline{\Gamma}} |\,\mathbf{curl}(\mathbf{u})|^2 + |\mathbf{u}|^2\, d\boldsymbol{x} \;=\; \|\dot{\mathbf{u}}\|^2_{\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)}\ .$$

This clearly shows that the bilinear form $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\times$ induces an isometric isomorphism and concludes the proof. □

A consequence of the duality proved in Proposition 4.2 is the following characterisation of the space $\mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)$ as kernel of the bilinear form $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\times$.

Lemma 4.3 (Characterization of $\mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)$, cf. [9, Corollary 5.2]**).** *For any $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, we have $\mathbf{u} \in \mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)$, if and only if*

$$\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{curl}(\mathbf{u})\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{curl}(\mathbf{v})\, d\boldsymbol{x} = 0 \quad \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})\ .$$

4.4. Tangential multi-trace spaces in particular situations

In this subsection we will examine two particular situations where it is easy to give explicit descriptions of $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ in terms of more standard trace spaces. For scalar multi-trace spaces these considerations have been elaborated in [9, Section 5.2].

4.4.1. Skeleton of a Lipschitz partition. We first consider the case where $\Gamma = \cup_{j=0}^n \partial\Omega_j$, where $(\Omega_j)_{j=0\dots n}$ is a Lipschitz partition of $\mathbb{R}^3$, see Definition 2.2. Denote $\Gamma_j := \partial\Omega_j$. In this situation, the operator $\mathrm{Loc}(\mathbf{u}) = (\mathbf{u}|_{\Omega_j})_{j=0}^n$ provides an isometric isomorphism

$$\mathrm{Loc}\ : \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) \to \mathbf{H}(\mathbf{curl}, \Omega_0) \times \cdots \times \mathbf{H}(\mathbf{curl}, \Omega_n)\ .$$

For each subdomain Ω_j, let $\mathrm{Ext}_j : \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \partial\Omega_j) \to \mathbf{H}(\mathbf{curl}, \Omega_j)$ be a right inverse of the tangential trace operator local to Ω_j. As a consequence $\pi_{\mathrm{T}} \cdot \mathrm{Loc}^{-1} \cdot (\mathrm{Ext}_0 \times \cdot \times \mathrm{Ext}_n)$ is an isometric isomorphism, so that we can identify

$$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \cong \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_0) \times \cdots \times \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma_n)\ .$$

Let $\mathbf{n}_j$ stand for the exterior unit normal vector to $\partial\Omega_j$. Let $\dot{\mathbf{u}}, \dot{\mathbf{v}}$ be two elements of $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ that we identify with $(\dot{\mathbf{u}}_j)_{j=0}^n, (\dot{\mathbf{v}}_j)_{j=0}^n$ according to the isomorphism exhibited above. Standard Green's formula applied in each Ω_j, along with (4.6) and (4.7) yield

$$\langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle\!\rangle_\times = \sum_{j=0}^n \int_{\Gamma_j} \mathbf{n}_j \times \dot{\mathbf{u}}_j \cdot \dot{\mathbf{v}}_j d\sigma\ , \tag{4.8}$$

which agrees with the skew-symmetric duality pairing defined in [7, Formula (10)]. Formula (4.8) provides further motivation for the notation "$\langle\!\langle\cdot,\cdot\rangle\!\rangle_\times$". In the general case where Γ is not necessarily the skeleton of a Lipschitz partition, the above discussion shows that (i) $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ can at least be embedded into $\Pi_{j=0}^n \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma_j}, \Gamma_j)$, although this embedding is not an isomorphism anymore, and that (ii) Expression (4.8) still holds for smooth $\dot{\mathbf{u}}, \dot{\mathbf{v}}$.

4.4.2. Standard Lipschitz screens. Next, we consider the situation of a Lipschitz partition with two domains, $\mathbb{R}^3 = \overline{\Omega}_0 \cup \overline{\Omega}_1$, where Ω_1 is a bounded Lipschitz domain, and $\Gamma \subset \partial\Omega_0 \cap \partial\Omega_1$. Once again, let us denote $\Gamma_j := \Gamma \cap \partial\Omega_j$ for $j = 0, 1$. The injection $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) \subset \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \partial\Omega_0)$ induces a natural embedding

$$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})/\mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3) \hookrightarrow \\ \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \partial\Omega_0)/\mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)$$

From the isometric isomorphism $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \partial\Omega_0) \cong \mathbf{H}(\mathbf{curl}, \Omega_0) \times \mathbf{H}(\mathbf{curl}, \Omega_1)$ and the definition of trace spaces as quotient spaces in (4.3), we conclude that there is a natural embedding,

$$\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \hookrightarrow \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \times \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma). \tag{4.9}$$

Unless $\Gamma = \partial\Omega_0 = \partial\Omega_1$, this is a strict embedding (and not an isomorphism). Among all pairs $(\mathbf{u}, \mathbf{v}) \in (\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma))^2$, let us describe those belonging $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$. Consider an element of $\dot{\mathbf{u}} \in \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ and let $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ satisfy $\pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}$ and denote $\mathbf{u}_j := \mathbf{u}|_{\Omega_j}$. In accordance with the discussion above, we make the identification $\dot{\mathbf{u}} = (\pi_{\mathrm{T}}^0(\mathbf{u}_0), \pi_{\mathrm{T}}^1(\mathbf{u}_1))$ where $\pi_{\mathrm{T}}^j(\mathbf{u}_j) := (\mathbf{n}_j \times \mathbf{u}_j|_\Gamma) \times \mathbf{n}_j$ with $\mathbf{u}_j = \mathbf{u}|_{\Omega_j}$. Since tangential traces of $\mathbf{u}_0, \mathbf{u}_1$ coincide on $\partial\Omega_0 \setminus \Gamma$ i.e. $(\mathbf{n}_0 \times \mathbf{u}_0) \times \mathbf{n}_0 = (\mathbf{n}_1 \times \mathbf{u}_1) \times \mathbf{n}_1$ on $\partial\Omega_0 \setminus \Gamma$, we have

$$\pi_{\mathrm{T}}^1(\mathbf{u}_1) - \pi_{\mathrm{T}}^0(\mathbf{u}_0) \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \ := \\ \{\dot{\mathbf{v}} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \partial\Omega_0) \mid \dot{\mathbf{v}} = 0 \ \text{ on } \ \partial\Omega_0 \setminus \Gamma\} \ .$$

Using appropriate liftings of traces local to each subdomain, one shows that the condition above actually yields a characterisation of $\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$,

$$\begin{aligned} &\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \ = \\ &\{(\dot{\mathbf{v}}_1, \dot{\mathbf{v}}_2) \in \mathbf{H}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \mid \dot{\mathbf{v}}_1 - \dot{\mathbf{v}}_2 \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathbf{curl}_\Gamma, \Gamma) \ \} \ . \end{aligned} \tag{4.10}$$

Next, let us provide explicit formula for the duality pairing $\langle\langle \cdot, \cdot \rangle\rangle_\times$. Take two traces $\dot{\mathbf{u}}, \dot{\mathbf{v}} \in \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ and assume that $\mathbf{u}, \mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ satisfy $\pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}$, $\pi_{\mathrm{T}}(\mathbf{v}) = \dot{\mathbf{v}}$. Let us identify $\dot{\mathbf{u}} = (\dot{\mathbf{u}}_0, \dot{\mathbf{u}}_1)$ and $\dot{\mathbf{v}} = (\dot{\mathbf{v}}_0, \dot{\mathbf{v}}_1)$ in accordance with the discussion above, and denote $\mathbf{u}_j = \mathbf{u}|_{\Omega_j}$, $\mathbf{v}_j = \mathbf{v}|_{\Omega_j}$. Then we have

$$\begin{aligned} \langle\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle\rangle_\times &= \int_{\mathbb{R}^3 \setminus \overline{\Gamma}} \mathbf{curl}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{curl}(\mathbf{v}) \cdot \mathbf{u} \, d\boldsymbol{x} \\ &= \sum_{j=0,1} \int_{\Omega_j} \mathbf{curl}(\mathbf{u}_j) \cdot \mathbf{v}_j - \mathbf{curl}(\mathbf{v}_j) \cdot \mathbf{u}_j \, d\boldsymbol{x} \\ &= \sum_{j=0,1} \int_{\partial\Omega_j} (\mathbf{n}_j \times \dot{\mathbf{u}}_j) \cdot \dot{\mathbf{v}}_j \, d\sigma \\ &= \int_\Gamma (\mathbf{n}_0 \times \dot{\mathbf{u}}_0) \cdot \dot{\mathbf{v}}_0 + (\mathbf{n}_1 \times \dot{\mathbf{u}}_1) \cdot \dot{\mathbf{v}}_1 \, d\sigma \ . \end{aligned} \tag{4.11}$$

4.5. Single-trace spaces and jump spaces

Now we introduce a vector counterpart of single-trace spaces that correspond to tangential traces matching on both side of each panel of multi-screens. This space, and its dual, will play a pivotal role in the theoretical study of integral equations posed on Γ.

Definition 4.4 (Tangential single-trace space, cf. [9, Definition 6.1]**).** The *tangential single-trace space* is defined as the quotient space

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \;:=\; \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)/\mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)\;. \tag{4.12}$$

Note that this definition differs from (4.4) in that $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ is considered instead of $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, which induces transmission conditions across the panels of Γ. Obviously, we have $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \subset \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$ and the quotient norm on $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ agrees with the norm inherited from the multi-trace space $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$.

The single-trace space $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma])$ is actually polar (and not dual!) to itself with respect to the pairing $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\times$, which yields a variational characterisation.

Proposition 4.5 (Characterization of tangential single-trace space, cf. [9, Proposition 6.3]**).** *For* $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$*, we have*

$$\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \quad \Longleftrightarrow \quad \langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle\!\rangle_\times = 0 \quad \forall \dot{\mathbf{v}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])\;.$$

Proof: Take an arbitrary $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ and consider any $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ that satisfies $\pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}$. Assume first that $\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ so that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$. Thus, according to the very definition of $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\times$ given by (4.6) and Green's formula, for any $\dot{\mathbf{v}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ there exists $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ such that $\pi_{\mathrm{T}}(\mathbf{v}) = \dot{\mathbf{v}}$ which implies

$$\begin{aligned}
\langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle\!\rangle_\times &= \int_{\mathbb{R}^3 \setminus \overline{\Gamma}} \mathbf{curl}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}(\mathbf{v})\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \mathbf{curl}(\mathbf{u}) \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{curl}(\mathbf{v})\, d\boldsymbol{x} = 0\;.
\end{aligned}$$

This proves the "only if" part of the proposition. Now assume that $\langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}} \rangle\!\rangle_\times = 0$ for all $\dot{\mathbf{v}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, and let us show that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$. Set $\mathbf{p} = \mathbf{curl}(\mathbf{u})|_{\mathbb{R}^3 \setminus \overline{\Gamma}}$. For any $\mathbf{v} \in \mathcal{D}(\mathbb{R}^3)^3$, we have $\pi_{\mathrm{T}}(\mathbf{v}) = \dot{\mathbf{v}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, which implies

$$\begin{aligned}
\int_{\mathbb{R}^3} \mathbf{u} \cdot \mathbf{curl}(\mathbf{v})\, d\boldsymbol{x} &= \int_{\mathbb{R}^3 \setminus \overline{\Gamma}} \mathbf{u} \cdot \mathbf{curl}(\mathbf{v})\, d\boldsymbol{x} = \langle\!\langle \dot{\mathbf{v}}, \dot{\mathbf{u}} \rangle\!\rangle_\times + \int_{\mathbb{R}^3} \mathbf{v} \cdot \mathbf{p}\, d\boldsymbol{x} \\
&= \int_{\mathbb{R}^3} \mathbf{v} \cdot \mathbf{p}\, d\boldsymbol{x}, \quad \forall \mathbf{v} \in (\mathcal{D}(\mathbb{R}^3))^3\;.
\end{aligned}$$

This shows that $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ with $\mathbf{curl}(\mathbf{u}) = \mathbf{p}$ in $\mathbb{R}^3$, not just in $\mathbb{R}^3 \setminus \overline{\Gamma}$. Hence $\dot{\mathbf{u}} = \pi_{\mathrm{T}}(\mathbf{u}) \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. $\square$

As we pointed out above, the space $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ is not dual to itself.

As in [9, Section 6.2], this observation motivates the introduction to another type of trace spaces.

Definition 4.6 (Tangential jump space, cf. [9, Definition 6.4]**).** The *tangential jump space* on the multi-screen Γ is defined as the dual space

$$\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \;:=\; \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])' \,. \tag{4.13}$$

We equip the space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ with the dual norm

$$\|\dot{\mathbf{u}}\|_{\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma])} \;:=\; \sup_{\dot{\mathbf{v}}\in\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma])} \frac{|\langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}}\rangle\!\rangle_\times|}{\|\dot{\mathbf{v}}\|_{\mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)}} \;. \tag{4.14}$$

Since $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ is a closed subspace of $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, a direct application of the Hahn-Banach Theorem (see [23, Thm.3.6]) shows that for any $\varphi \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, there exists $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that $\langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}}\rangle\!\rangle_\times = \varphi(\dot{\mathbf{v}})$. This is a motivation for adopting $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\times$ as notation for the (self-)duality pairing between $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ and $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. Now, combining Proposition 4.2 and Proposition 4.5, we easily arrive at the following conclusion.

Lemma 4.7 (Quotient space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$). *The tangential jump space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ is isometrically isomorphic to the quotient space $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)/\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$.*

Clearly, an element of $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ induces an element of this space via the duality pairing $\langle\!\langle\cdot,\cdot\rangle\!\rangle_\times$.

Definition 4.8 (Jump operator, cf. [9, Definition 6.5]**).**
We define the *jump operator* $[\;\;] : \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ through

$$\langle\!\langle [\dot{\mathbf{u}}], \dot{\mathbf{v}}\rangle\!\rangle_\times \;:=\; \langle\!\langle \dot{\mathbf{u}}, \dot{\mathbf{v}}\rangle\!\rangle_\times \;, \qquad \forall \dot{\mathbf{v}} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \,.$$

It was shown above that the jump operator is surjective. It can also can be used to characterise single trace spaces. The following lemma is a direct consequence of Proposition 4.5.

Lemma 4.9 (Single-trace space as kernel of jump operator, cf. [9, Corollary 6.6]**).** *A trace* $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *belongs* $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$*, if and only if* $[\dot{\mathbf{u}}] = 0$.

4.6. Single-trace spaces and jump spaces in special situations

Again, we wish to comment on simple situations where it is possible to give rather explicit description of the single trace space $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, and the jump trace space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$.

4.6.1. Skeleton of a Lipschitz partition. As in Section 4.4.1, in this situation the screen $\Gamma = \cup_{j=0}^{n}\partial\Omega_j$ is the union of the boundaries of a Lipschitz partition $\mathbb{R}^3 = \cup_{j=0}^{n}\overline{\Omega}_j$. Write $\Gamma_j = \partial\Omega_j$. Take any $\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ and consider $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ that satisfies $\pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}$. Let $\mathbf{u}_j = \mathbf{u}|_{\Omega_j}$ and $\dot{\mathbf{u}}_j = (\mathbf{n}_j \times \mathbf{u}_j|_{\Gamma_j}) \times \mathbf{n}_j$. Following the arguments presented in §4.4, we can identify $\dot{\mathbf{u}}$ with $(\dot{\mathbf{u}}_j)_{j=0}^{n}$. The condition $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ amounts to $\mathbf{n}_j \times \mathbf{u}_j + \mathbf{n}_k \times \mathbf{u}_k = 0$ on $\partial\Omega_j \cap \partial\Omega_k$. In other words,

$$\begin{aligned}\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \cong \Big\{ \, (\dot{\mathbf{u}}_j)_{j=0}^{n} \in \Pi_{j=0}^{n}\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_{\Gamma_j}, \Gamma_j) \;|& \\ \mathbf{n}_j \times \dot{\mathbf{u}}_j + \mathbf{n}_k \times \dot{\mathbf{u}}_k = 0 \;\text{ on }\; \partial\Omega_j \cap \partial\Omega_k \;\forall j,k \, \Big\}&\end{aligned}$$

Unfortunately, a similarly explicit description of the space of jumps in the case of a Lipschitz skeleton remains elusive.

4.6.2. Standard Lipschitz screen. As in Section 4.4.2, we now examine the case where $\Gamma \subset \partial\Omega_1$ for some bounded Lipschitz open set Ω_1. For the complement we write $\Omega_0 = \mathbb{R}^3 \setminus \overline{\Omega}_1$. Take any element $\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, and consider $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$ such that $\pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}$. Denoting $\mathbf{u}_j = \mathbf{u}|_{\Omega_j}$ and $\dot{\mathbf{u}}_j := (\mathbf{n}_j \times \mathbf{u}_j|_\Gamma) \times \mathbf{n}_j$, according to the discussion of Section 4.4, we can identify $\dot{\mathbf{u}}$ with $(\dot{\mathbf{u}}_0, \dot{\mathbf{u}}_1)$. Since $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$, the tangential traces of $\mathbf{u}_0$ and $\mathbf{u}_1$ must coincide on $\partial\Omega_0$, and in particular on Γ. We conclude that $\dot{\mathbf{u}}_0 = \dot{\mathbf{u}}_1$ on Γ, and this turns out to be a characterisation,

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \;=\; \{\, (\dot{\mathbf{v}}_0, \dot{\mathbf{v}}_0) \mid \dot{\mathbf{v}}_0 \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \,\} \subset \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)\,.$$

Now take any element $\dot{\mathbf{u}} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma])$. There exists $\dot{\mathbf{p}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that $\langle\!\langle \dot{\mathbf{v}}, \dot{\mathbf{u}} \rangle\!\rangle_\times = \langle\!\langle \dot{\mathbf{v}}, \dot{\mathbf{p}} \rangle\!\rangle_\times$ for all $\dot{\mathbf{v}} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma])$. The trace $\dot{\mathbf{p}}$ can be identified with a pair $(\dot{\mathbf{p}}_0, \dot{\mathbf{p}}_1) \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that $\dot{\mathbf{p}}_0 - \dot{\mathbf{p}}_1 \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, see Section 4.4. Now for any $\dot{\mathbf{v}} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma])$ that we identify with $(\dot{\mathbf{v}}_0, \dot{\mathbf{v}}_0), \dot{\mathbf{v}}_0 \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$, according to (4.11), we have

$$\begin{aligned}\langle\!\langle \dot{\mathbf{v}}, \dot{\mathbf{u}} \rangle\!\rangle_\times = \langle\!\langle \dot{\mathbf{v}}, \dot{\mathbf{p}} \rangle\!\rangle_\times \quad &= \int_\Gamma \mathbf{n}_0 \times \dot{\mathbf{v}}_0 \cdot \dot{\mathbf{p}}_0 + \mathbf{n}_1 \times \dot{\mathbf{v}}_0 \cdot \dot{\mathbf{p}}_1 \, d\sigma \\ &= \int_\Gamma \mathbf{n}_0 \times \dot{\mathbf{v}}_0 \cdot (\dot{\mathbf{p}}_0 - \dot{\mathbf{p}}_1) \, d\sigma \\ &= \int_\Gamma \mathbf{n}_0 \times \dot{\mathbf{v}}_0 \cdot (\frac{\dot{\mathbf{p}}_0 - \dot{\mathbf{p}}_1}{2}) - \mathbf{n}_1 \times \dot{\mathbf{v}}_0 \cdot (\frac{\dot{\mathbf{p}}_0 - \dot{\mathbf{p}}_1}{2}) \, d\sigma\end{aligned}$$

Set $\dot{\mathbf{q}} := (\dot{\mathbf{p}}_0 - \dot{\mathbf{p}}_1)/2 \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$. The calculus above shows that the pair $(\dot{\mathbf{q}}, -\dot{\mathbf{q}})$ can also be chosen as representative of $\dot{\mathbf{u}}$. A careful inspection of the previous calculus actually shows that for any $\dot{\mathbf{u}} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma])$ there exists one and only one $\dot{\mathbf{q}} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that $(\dot{\mathbf{q}}, -\dot{\mathbf{q}})$ represents $\dot{\mathbf{u}}$. This proves that we can make the following identification,

$$\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \;=\; \{\, (\dot{\mathbf{q}}, -\dot{\mathbf{q}}) \mid \dot{\mathbf{q}} \in \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) \,\}\,.$$

5. Surface Differential Operators

Compared to the developments in [9] for scalar valued functions a completely new aspect for vector valued functions is the definition of the classical surface differential operators such as surface gradient, curl, and divergence. These operators will give rise to a De Rham diagram relating the scalar and tangential trace spaces. We also show that these operators map single trace spaces into single trace spaces, and jump traces to jump traces.

5.1. Surface gradient

Consider any function $p \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma})$. We clearly have $\mathbf{curl}(\nabla p) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$ so that $\nabla p \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$. Thus, the tangential trace $\pi_{\mathrm{T}}(\nabla p)$ is well defined and, according to (4.6), for all $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ and $p \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma})$ we have

$$\int_{\mathbb{R}^3 \setminus \overline{\Gamma}} \mathbf{curl}(\mathbf{u}) \cdot \nabla p \, d\boldsymbol{x} \;=\; \langle\!\langle \pi_{\mathrm{T}}(\mathbf{u}), \pi_{\mathrm{T}}(\nabla p) \rangle\!\rangle_{\times} \;. \tag{5.1}$$

Since the left-hand side above does not change when replacing p by $p + q$ where $q \in \mathrm{H}^1_{0,\Gamma}(\mathbb{R}^3)$, this formula allows to define the *surface gradient* $\nabla_\Gamma : \mathbb{H}^{1/2}(\Gamma) \to \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ according to the formula

$$\nabla_\Gamma(\, \pi_{\mathrm{D}}(p) \,) \;:=\; \pi_{\mathrm{T}}(\, \nabla p \,) \qquad \forall p \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma}) \;. \tag{5.2}$$

From this definition of the surface gradient we conclude that, if $p \in \mathrm{H}^1(\mathbb{R}^3)$, we have $\nabla p \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$, so that $\nabla_\Gamma \dot{p} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ whenever $\dot{p} \in \mathrm{H}^{1/2}([\Gamma])$. In other words, the surface gradient maps single traces to single traces.

5.2. Surface curl operator

For any $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, we clearly have $\mathrm{div}(\mathbf{curl}(\mathbf{u})) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$, so that $\mathbf{curl}(\mathbf{u}) \in \mathrm{H}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Gamma})$. As a consequence, by definition of the pairing (3.3) between $\mathbb{H}^{+1/2}(\Gamma)$ and $\mathbb{H}^{-1/2}(\Gamma)$, for all $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ and $p \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma})$ we have

$$\int_{\mathbb{R}^3 \setminus \overline{\Gamma}} \mathbf{curl}(\mathbf{u}) \cdot \nabla p \, d\boldsymbol{x} \;=\; \langle\!\langle\, \pi_{\mathrm{D}}(p), \pi_{\mathrm{N}}(\mathbf{curl}(\mathbf{u}))\, \rangle\!\rangle \;. \tag{5.3}$$

Examining the left-hand side of this identity, it is clear that $\pi_{\mathrm{N}}(\mathbf{curl}(\mathbf{u}))$ only depends on $\pi_{\mathrm{T}}(\mathbf{u})$ (the equivalence class modulo an element of $\mathbf{H}_{0,\Gamma}(\mathbf{curl}, \mathbb{R}^3)$), so that it actually defines a continuous mapping $\mathrm{curl}_\Gamma : \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbb{H}^{-1/2}(\Gamma)$, the *surface curl*, by the formula

$$\mathrm{curl}_\Gamma(\, \pi_{\mathrm{T}}(\mathbf{u}) \,) \;:=\; \pi_{\mathrm{N}}(\, \mathbf{curl}(\mathbf{u}) \,) \qquad \forall \mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) \;. \tag{5.4}$$

In addition observe that, if $\pi_{\mathrm{T}}(\mathbf{u}) = \nabla_\Gamma(\dot{p})$ for some $\dot{p} = \pi_{\mathrm{D}}(p) \in \mathbb{H}^{1/2}(\Gamma)$, then we have $\mathrm{curl}_\Gamma(\nabla_\Gamma \dot{p}) = \mathrm{curl}_\Gamma(\pi_{\mathrm{T}}(\nabla p)) = \pi_{\mathrm{N}}(\, \mathbf{curl}(\nabla p)\,) = 0$. In other words $\mathrm{curl}_\Gamma \cdot \nabla_\Gamma = 0$, which is a well known property of classical surface curl and grad operators on the boundary of a Lipschitz open set.

Moreover, if $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$, we clearly have $\mathbf{curl}(\mathbf{u}) \in \mathbb{H}(\mathrm{div}, \mathbb{R}^3)$, so that $\mathrm{curl}_\Gamma(\dot{\mathbf{u}}) \in \mathrm{H}^{-1/2}([\Gamma])$ whenever $\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. In other words, the surface curl operator maps single traces to single traces.

5.3. Surface Green's formula

Recall that the trace operators $\pi_{\mathrm{D}} : \mathrm{H}^1(\mathbb{R}^3 \backslash \overline{\Gamma}) \to \mathbb{H}^{1/2}(\Gamma)$, and $\pi_{\mathrm{T}} : \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \backslash \overline{\Gamma}) \to \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ are onto by construction. An immediate consequence of (5.1) and (5.3) is the following formula:

$$\langle\!\langle \dot{p}, \mathrm{curl}_\Gamma(\dot{\mathbf{v}}) \rangle\!\rangle \;=\; \langle\!\langle \dot{\mathbf{v}}, \nabla_\Gamma(\dot{p}) \rangle\!\rangle_\times \quad \forall \dot{p} \in \mathbb{H}^{\frac{1}{2}}(\Gamma)\;, \quad \forall \dot{\mathbf{v}} \in \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)\;. \tag{5.5}$$

This formula allows to extend the definition of surface differential operators to jump trace spaces easily. Indeed we define $\nabla_\Gamma : \widetilde{\mathrm{H}}^{1/2}(\Gamma) \to \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ as adjoint to curl_Γ by the formula

$$\langle\!\langle \dot{\mathbf{v}}, \nabla_\Gamma \dot{u} \rangle\!\rangle_\times \;:=\; \langle\!\langle \dot{u}, \mathrm{curl}_\Gamma \dot{\mathbf{v}} \rangle\!\rangle \quad \forall \dot{\mathbf{v}} \in \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma])\;, \tag{5.6}$$

for all $\dot{u} \in \widetilde{\mathrm{H}}^{1/2}(\Gamma)$. This definition is valid because surface gradient maps single traces to single traces, as proved above. In a similar manner we define a continuous operator $\mathrm{curl}_\Gamma : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \to \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$.

5.4. Summary: Commuting diagrams for trace spaces

The previous definitions and results allow to do vector calculus on the surface of multi-screens in a way very similar to standard calculus on the surface of 2D manifolds. In particular, the definitions and relationships of various trace spaces and trace operators may be summarised by means of commutative diagrams.

Lemma 5.1 (Commuting diagram for volume and surface differential operators: multi-trace case). *The volume and surface differential operators commute with the traces in the sense of the following commuting diagram*

$$\begin{array}{ccccc}
\mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma}) & \xrightarrow{\nabla} & \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) & \xrightarrow{\mathbf{curl}} & \mathrm{H}(\mathrm{div}, \mathbb{R}^3 \setminus \overline{\Gamma}) \\
\big\downarrow \pi_{\mathrm{D}} & & \big\downarrow \pi_{\mathrm{T}} & & \big\downarrow \pi_{\mathrm{N}} \\
\mathbb{H}^{+\frac{1}{2}}(\Gamma) & \xrightarrow{\nabla_\Gamma} & \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) & \xrightarrow{\mathrm{curl}_\Gamma} & \mathbb{H}^{-\frac{1}{2}}(\Gamma)
\end{array}$$

Obviously, if $\dot{p} \in \mathrm{H}^{1/2}([\Gamma])$, there exists $p \in \mathrm{H}^1(\mathbb{R}^3)$ such that $\pi_{\mathrm{D}}(p) = \dot{p}$. Then we have $\nabla p \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$, so that $\pi_{\mathrm{T}}(\nabla p) = \nabla_\Gamma(\dot{p}) \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. We prove in a completely similar manner that, if $\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, then $\mathrm{curl}_\Gamma(\dot{\mathbf{u}}) \in \mathrm{H}^{-1/2}([\Gamma])$. The following result summarises these two properties

Lemma 5.2 (Commuting diagram for volume and surface differential operators: single-trace case). *The following diagram connecting traces and differential operators commutes:*

$$\begin{array}{ccccc}
\mathrm{H}^1(\mathbb{R}^3) & \xrightarrow{\nabla} & \mathbf{H}(\mathbf{curl},\mathbb{R}^3) & \xrightarrow{\mathbf{curl}} & \mathrm{H}(\mathrm{div},\mathbb{R}^3) \\
\big\downarrow \pi_{\mathrm{D}} & & \big\downarrow \pi_{\mathrm{T}} & & \big\downarrow \pi_{\mathrm{N}} \\
\mathrm{H}^{+\frac{1}{2}}([\Gamma]) & \xrightarrow{\nabla_\Gamma} & \mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]) & \xrightarrow{\mathrm{curl}_\Gamma} & \mathrm{H}^{-\frac{1}{2}}([\Gamma])
\end{array}$$

In addition, note that the jump operators introduced in (3.5) and (4.8) commute with the surface differential operators defined above for jump trace spaces. This, along with Lemma 4.9 and [9, Corollary 6.6] proves the following lemma.

Lemma 5.3 (Commuting diagram for surface differential operators and jump operators). *Let ι denote canonical injections. Then the following diagram commutes, and the vertical sequences are exact.*

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathrm{H}^{+\frac{1}{2}}([\Gamma]) & \xrightarrow{\nabla_\Gamma} & \mathrm{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]) & \xrightarrow{\mathrm{curl}_\Gamma} & \mathrm{H}^{-\frac{1}{2}}([\Gamma]) \\
\big\downarrow \iota & & \big\downarrow \iota & & \big\downarrow \iota \\
\mathbb{H}^{+\frac{1}{2}}(\Gamma) & \xrightarrow{\nabla_\Gamma} & \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma) & \xrightarrow{\mathrm{curl}_\Gamma} & \mathbb{H}^{-\frac{1}{2}}(\Gamma) \\
\big\downarrow [\cdot] & & \big\downarrow [\cdot] & & \big\downarrow [\cdot] \\
\widetilde{\mathrm{H}}^{+\frac{1}{2}}([\Gamma]) & \xrightarrow{\nabla_\Gamma} & \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma]) & \xrightarrow{\mathrm{curl}_\Gamma} & \widetilde{\mathrm{H}}^{-\frac{1}{2}}([\Gamma]) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
0 & & 0 & & 0
\end{array}$$

6. Boundary Value Problem

Now, we consider a classical electromagnetic scattering problem in $\mathbb{R}^3 \setminus \overline{\Gamma}$ namely the homogeneous Maxwell's equations (1.1) with perfect conductor (PEC) boundary conditions, which amounts to an exterior Dirichlet problem: Given some tangential multi-trace $\mathbf{g} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$, we wish to find a vector field $\mathbf{u} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl},\mathbb{R}^3 \setminus \overline{\Gamma})$ that satisfies

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2\mathbf{u} = \mathbf{0} \ \text{ in } \ \mathbb{R}^3 \setminus \overline{\Gamma}, \qquad \pi_{\mathrm{T}}(\mathbf{u}) = \mathbf{g} \ \text{ on } \ \Gamma\,, \tag{6.1}$$

and the Silver-Müller radiation conditions

$$\lim_{r\to\infty}\int_{\partial \mathrm{B}_r} |\,\mathbf{curl}(\mathbf{u})\times\mathbf{n}_r - i\kappa\mathbf{u}|^2 d\sigma_r = 0\ . \tag{6.2}$$

Although this is a fairly standard problem, existence and uniqueness of its solution is not trivial due to the possibly highly irregular (non-Lipschitz) geometry under consideration here. Fredholm's alternative and Riesz-Schauder theory, that are key tools in the analysis of such boundary value problems (see e.g. [17]), heavily rely on compact embedding results.

6.1. Restriction on the geometry

Unfortunately generic multi-screens can be very rugged and may thwart key technical arguments linked with compact embeddings. Hence, we need to introduce slight restrictions on the geometries under consideration. In order to formulate this properly, we need an intermediate definition borrowed from [21, Def.3.3].

Definition 6.1 (Cone with a tame base). An open subset D of the unit sphere $\mathbb{S}^2$ is said to be *tame*, if for every $\mathbf{s}\in D$ there exists an open neighbourhood $U\subset\mathbb{S}^2$ of $\mathbf{s}$ such that $U\cap D$ has only finitely many connected components, which are simply connected and enjoy the Rellich compactness property (that is, the compact embedding of H^1 in L^2). In this case, the set $\mathcal{C}(D) := \{\tau\mathbf{s}\mid \tau\in(0,1) \text{ and } \mathbf{s}\in D\}$ is called a *cone with a tame base* D.

Guided by Theorem 3.6 of [21] that provides sufficient geometric conditions for the so-called Maxwell compactness property, see Theorem 6.5 below, we shall make the following hypothesis in the remainder of this section.

Assumption 6.2. *For every $\boldsymbol{x}\in\overline{\Gamma}$ there exists an open neighbourhood $U_{\boldsymbol{x}}$ centred at $\boldsymbol{x}$ such that $U_{\boldsymbol{x}}\setminus\overline{\Gamma}$ has only finitely many connected components that are Lipschitz diffeomorphic to some cone $\mathcal{C}(D)$ with a tame base D.*

This assumption begs for explanation. Let us show why it is mild and introduces only a slight restriction in the present geometric setting. The interior of a multi-screen Γ, denoted $\mathrm{int}(\Gamma)$, consists of those points $\boldsymbol{x}\in\Gamma$ such that $\mathrm{B}_{\boldsymbol{x}}\cap(\mathbb{R}^3\setminus\Gamma)$ has several connected components for any ball $\mathrm{B}_{\boldsymbol{x}}$ with sufficiently small radius centered at $\boldsymbol{x}$. Resorting to the Lipschitz partition $\mathbb{R}^3=\cup_{j=0}^n\overline{\Omega}_j$ from Definition 2.3, each connected component of $\mathrm{B}_{\boldsymbol{x}}\cap(\mathbb{R}^3\setminus\Gamma)$ is composed of intersections $\mathrm{B}_{\boldsymbol{x}}\cap\Omega_j$. Since each $\mathrm{B}_{\boldsymbol{x}}\cap\Omega_j$ is Lipschitz diffeomorphic to the intersection of $\mathrm{B}_{\boldsymbol{x}}$ with a straight half-space, Assumption 6.2 is clear for interior points of a multi-screen.

Next, denote by $\partial_{\boldsymbol{r}}\Gamma$ the set of regular points of the boundary defined as those points $\boldsymbol{x}\in\partial\Gamma:=\overline{\Gamma}\setminus\mathrm{int}(\Gamma)$ such that $\mathrm{B}_{\boldsymbol{x}}\cap\overline{\Gamma}=\mathrm{B}_{\boldsymbol{x}}\cap\overline{\Sigma}$ for some ball $\mathrm{B}_{\boldsymbol{x}}$ centred at $\boldsymbol{x}$ and some standard Lipschitz screen Σ as defined in Definition 2.1. For such points Assumption 6.2 holds, as is detailed in Example 4.1 in [21].

From the previous discussion, we conclude that Assumption 6.2 induces restrictions on the geometry only in the neighbourhood of non-regular points of the boundary, i.e., at points in $\partial_{\boldsymbol{s}}\Gamma := \partial\Gamma \setminus \partial_{\boldsymbol{r}}\Gamma$. In most relevant geometries, $\partial_{\boldsymbol{s}}\Gamma$ merely is a finite set of points. To give examples, we have marked these points in the geometries of Figure 1.

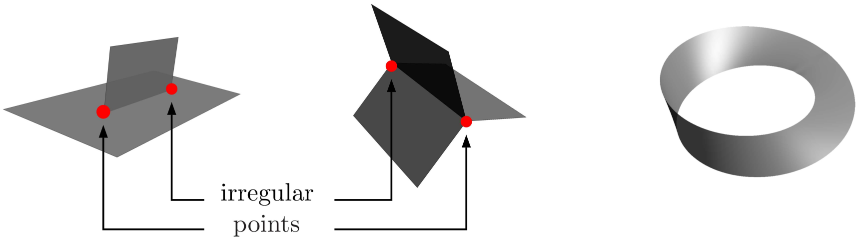

FIGURE 1. Irregular points of several multi-screens. For the Möbius strip represented on the right, the boundary does not have any irregular point so that Assumption 6.2 is satisfied. Each of the other two geometries in the left have two irregular points.

Fortunately, a rigorous justification of Assumption 6.2 is possible for an important class of multi-screens.

Definition 6.3 (Piecewise smooth multi-screen). We call a multi-screen *piecewise smooth*, if the adjacent Lipschitz domains Ω_j, $j = 0, \ldots, n$, stipulated by Definition 2.3, are curved Lipschitz polyhedra and $\overline{\Gamma} \cap \partial\Omega_j$ is the union of smooth faces of Ω_j.

Usually, only piecewise smooth multi-screens are faced in engineering applications, where computer aided design is used to create geometries. Examples are smooth sheets with piecewise smooth boundaries glued together at some edges, see Figure 1.

Recall that boundaries of curved Lipschitz polyhedra can locally be mapped to boundaries of polyhedra by means of C^∞-diffeomorphisms, see [13, p. 244]. Hence, if Γ is a piecewise smooth multi-screen, then for each $\boldsymbol{x} \in \Gamma$ there is a ball $B_{\boldsymbol{x}}$ centred at $\boldsymbol{x}$ and a Lipschitz diffeomorphism $\Phi : B_{\boldsymbol{x}} \to \Phi(B_{\boldsymbol{x}}) \subset \mathbb{R}^3$ such that $\Phi(\overline{\Gamma} \cap B_{\boldsymbol{x}})$ is the union of finitely many (cut) hyperplanes intersected with $\Phi(B_{\boldsymbol{x}})$: piecewise smooth multi-screens allow local flattening. The next result then shows that piecewise smooth multi-screens accommodate Assumption 6.2 above.

Lemma 6.4. *If* Γ *is a piecewise smooth multi-screen, then it satisfies Assumption* 6.2.

Proof: Take a $\boldsymbol{x} \in \Gamma$, and write $B_{\boldsymbol{x}}$ for a small ball around $\boldsymbol{x}$ and $\mathrm{B}_{j,\boldsymbol{x}} := \mathrm{B}_{\boldsymbol{x}} \cap \Omega_j$, where the domains Ω_j, $j = 0, \dots, n$, are those Lipschitz domains occurring in the Definition 2.3 of a multi-screen. Thanks to the possibility of local flattening discussed above, we can take for granted without loss of generality that every Ω_j is a genuine polyhedron. Thus, $\mathrm{B}_{j,\boldsymbol{x}}$ is a cone: $\mathrm{B}_{j,\boldsymbol{x}} = \{s \cdot \tau, \text{where } s \in [0, r), \tau \in \mathcal{U}\}$ for some small radius $r > 0$ and some subset $\mathcal{U} \subset \mathbb{S}^2$. Now we have to show that $\mathcal{U}$ is tame. Assume that $B_{\boldsymbol{x}}$ has been chosen small enough so that, in a suitable local coordinate system with origin $\boldsymbol{x}$, we have

$$\begin{aligned} r\mathcal{U} &= \partial \mathrm{B}_{\boldsymbol{x}} \cap \Omega_j \\ &= \{\boldsymbol{y} = (y_1, y_2, y_3)^T \mid (y_1, y_2)^T \in U^2_{\boldsymbol{x}},\ y_3 < f(y_1, y_2)\ , \|\boldsymbol{y}\| = r\}\ , \end{aligned}$$

where f is a uniformly Lipschitz continuous function $f : U^2_{\boldsymbol{x}} \to \mathbb{R}$, and $U^2_{\boldsymbol{x}} \subset \mathbb{R}^2$ a suitable neighbourhood of $\mathbf{0}$. This representation demonstrates that $\mathcal{U}$ is a Lipschitz subdomain of $\mathbb{S}^2$. For it the Rellich compactness property as introduced in Definition 6.1 is satisfied. $\Box$

6.2. Well posedness of the scattering problem

Now we recall and use a sophisticated compact embedding result by Picard, Weck and Witsch [21, Thm. 3.6] adapted to possibly highly irregular geometries. Its statement relies on the two Hilbert spaces

$$\begin{aligned} \mathbf{E}_{\mathrm{T}}(\Omega) &:= (\,\text{closure of } \mathcal{D}(\Omega)^3 \text{ in } \mathbf{H}(\mathbf{curl}, \Omega)\,) \cap\ \mathbf{H}(\mathrm{div}, \Omega)\ , \\ \mathbf{E}_{\mathrm{N}}(\Omega) &:=\ \mathbf{H}(\mathbf{curl}, \Omega) \cap (\,\text{closure of } \mathcal{D}(\Omega)^3 \text{ in } \mathbf{H}(\mathrm{div}, \Omega)\,)\ . \end{aligned} \tag{6.3}$$

for a bounded open set $\Omega \subset \mathbb{R}^3$. Both spaces above are closed in $\mathbf{H}(\mathbf{curl}, \Omega) \cap \mathbf{H}(\mathrm{div}, \Omega)$ equipped with the norm $\|\cdot\|_{\mathbf{H}(\mathbf{curl},\Omega)} + \|\cdot\|_{\mathbf{H}(\mathrm{div},\Omega)}$.

Theorem 6.5 (Maxwell compactness property). *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. If Assumption* 6.2 *holds, then both $\mathbf{E}_{\mathrm{N}}(\Omega)$ and $\mathbf{E}_{\mathrm{T}}(\Omega)$ are* compactly *embedded in $\mathrm{L}^2(\Omega)^3$.*

Now we take Assumption 6.2 for granted, so that the theorem above is applicable with $\Omega = \mathrm{B} \setminus \overline{\Gamma}$ where B is any ball with sufficiently large radius. As a direct consequence we have the well posedness of the scattering problem (6.1)-(6.2).

Proposition 6.6 (Existence and uniqueness of solutions of the exterior Dirichlet problem). *Assume that $\mathbb{R}^3 \setminus \overline{\Gamma}$ is connected and that Assumption* 6.2 *holds. For any tangential multi-trace $\mathbf{g} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ there exists a unique vector field $\mathbf{u} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ that satisfies*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2 \mathbf{u} = \mathbf{0} \ \text{ in } \ \mathbb{R}^3 \setminus \overline{\Gamma}, \qquad \pi_{\mathrm{T}}(\mathbf{u}) = \mathbf{g} \ \text{ on } \ \Gamma\ . \tag{6.4}$$

and Silver-Müller outgoing radiation condition. Moreover, $\mathbf{u}$ depends continuously on $\mathbf{g}$.

Proof: Using a lifting function provided by Lemma 7.1, the problem above is equivalent to finding $\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ such that $\pi_{\mathrm{T}}(\mathbf{v}) = 0$ on Γ, $\mathbf{v}$ satisfies the Silver-Müller radiation condition, and $\mathbf{curl}^2\,\mathbf{v} - \kappa^2\mathbf{v} = \mathbf{f}$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$ for some suitable compactly supported $\mathbf{f} \in \mathrm{L}^2(\mathbb{R}^3)^3$. According to [21, Theorem 2.10], uniqueness of the solution also implies existence and continuous dependency. Hence, the proposition will be established, if we can prove that there is uniqueness of the solution.

Assume that $\mathbf{u}$ is solution of Problem (6.4) with $\mathbf{g} = \mathbf{0}$. To prove that $\mathbf{u} = \mathbf{0}$ we simply reproduce a very classical argument of scattering theory [10, Thm. 6.11]. For sufficiently large $r > 0$, let B_r refer to the open ball of radius r, and $\Omega_r = \mathrm{B}_r \setminus \overline{\Gamma}$. Applying Green's formula (4.6) and using that $\mathbf{curl}^2\,\mathbf{u} - \kappa^2\mathbf{u} = \mathbf{0}$ in Ω_r, we obtain

$$
\begin{aligned}
&\int_{\Omega_r} |\,\mathbf{curl}(\mathbf{u})|^2 - \kappa^2|\mathbf{u}|^2\,d\boldsymbol{x}\\
&\qquad = \int_{\partial\mathrm{B}_r} \overline{\mathbf{u}} \times \mathbf{curl}(\mathbf{u}) \cdot \mathbf{n}_r\,d\sigma_r + \int_{\Omega_r} \overline{\mathbf{u}} \cdot \big(\mathbf{curl}^2\,\mathbf{u} - \kappa^2\mathbf{u}\big)\,d\boldsymbol{x}\\
&\qquad = \int_{\partial\mathrm{B}_r} \mathbf{curl}(\mathbf{u}) \times \mathbf{n}_r \cdot \overline{\mathbf{u}}\,d\sigma_r\\
&\qquad \Longrightarrow \Im m\Big\{ \int_{\partial\mathrm{B}_r} \mathbf{curl}(\mathbf{u}) \times \mathbf{n}_r \cdot \overline{\mathbf{u}}\,d\sigma_r \Big\} = 0
\end{aligned}
\tag{6.5}
$$

In the calculus above $\mathbf{n}_r$ and σ_r respectively are the outgoing normal to B_r, and the surface Lebesgue measure of $\partial\mathrm{B}_r$. Besides we have

$$
\begin{aligned}
&\int_{\partial\mathrm{B}_r} |\,\mathbf{curl}(\mathbf{u}) \times \mathbf{n}_r - i\kappa\mathbf{u}|^2 d\sigma_r =\\
&\qquad\qquad \int_{\partial\mathrm{B}_r} |\,\mathbf{curl}(\mathbf{u}) \times \mathbf{n}_r|^2 + \kappa^2|\mathbf{u}|^2 d\sigma_r\\
&\qquad\qquad +2\kappa\,\Im m\Big\{ \int_{\partial\mathrm{B}_r} \mathbf{curl}(\mathbf{u}) \times \mathbf{n}_r \cdot \overline{\mathbf{u}}\,d\sigma_r \Big\}
\end{aligned}
\tag{6.6}
$$

Plugging the results of (6.5) into (6.6), and using Silver-Müller's radiation condition, we deduce that $\lim_{r\to\infty} \int_{\partial\mathrm{B}_r} |\mathbf{u}|^2 d\sigma_r = 0$. This in turn implies that $\mathbf{u} = \mathbf{0}$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$ according to Rellich's Lemma [10, Lemma 2.12]. □

Remark 6.7. We emphasise that Theorem 6.5, and hence Assumption 6.2, is not required in Section 7 and 8 below that establish various results on layer potentials (such as representation formula, jump formula) associated to Maxwell's equations. It will be essential in Section 9, though.

7. Boundary Integral Equation

In the remaining of this article we will focus on boundary integral formulations of the scattering problem (6.1)-(6.2). Since representation formulas are a key stepping stone in the derivation of boundary integral equations, we start by deriving them for solutions of the homogeneous Maxwell equations

(1.1) in the exterior of the multi-screen Γ. Our approach employs distributional calculus and takes the cue from [7, Section 4] and [4, Section 3.2]. A key ingredient are traces associated with the 2nd-order Maxwell operator, corresponding to Dirichlet and Neumann traces for scalar 2nd-order operators: for sufficiently smooth vector fields they are defined as

$$\gamma_{\mathrm{T}}(\mathbf{u}) := \pi_{\mathrm{T}}(\mathbf{u}) \quad \text{and} \quad \gamma_{\mathrm{R}}(\mathbf{u}) := \pi_{\mathrm{T}}(\,\mathbf{curl}(\mathbf{u})\,)\,. \tag{7.1}$$

Next, we write $\mathbf{curl}^2 := \mathbf{curl}\ \mathbf{curl}$, and introduce the Hilbert space $\mathbf{H}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma}) := \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma}) \mid \mathbf{curl}^2(\mathbf{v}) \in \mathrm{L}^2(\Gamma)^3\}$, equipped with the natural norm $\sum_{j=0}^{2} \|\,\mathbf{curl}^j\,\mathbf{v}\|_{\mathrm{L}^2(\mathbb{R}^3\setminus\overline{\Gamma})}$. Then $\gamma_{\mathrm{T}}, \gamma_{\mathrm{R}} : \mathbf{H}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma}) \to \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ are clearly continuous. Moreover, they are surjective:

Lemma 7.1.
The operators $\gamma_{\mathrm{T}}, \gamma_{\mathrm{R}} : \mathbf{H}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma}) \to \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ *both admit a continuous right-inverse.*

Proof: We provide the proof for γ_{R}, since the proof for γ_{T} may follow the sames lines and is actually slightly simpler. For any given $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$, let $\Psi(\dot{\mathbf{u}})$ refer to the minimal-norm representative of $\dot{\mathbf{u}}$ (see the proof of Proposition 4.2). Then $\dot{\mathbf{u}} \mapsto -\,\mathbf{curl}\cdot\Psi(\dot{\mathbf{u}})$ is the right inverse we are looking for. $\square$

7.1. Representation formulas

Next, we establish a representation formula for radiating solutions of the homogeneous Maxwell equations (1.1) in the exterior of a multi-screen Γ. By "radiating" we mean that the Silver-Müller radiation conditions at ∞ are satisfied [10, Definition 6.6]. Pick any radiating function $\mathbf{u} \in \mathbf{H}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma})$ that satisfies $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2\mathbf{u} = \mathbf{0}$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$. In this case, for any $\mathbf{v} \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma})$, we find the following Green's formula

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\overline{\Gamma}} & \mathbf{curl}(\mathbf{curl}(\mathbf{u}))\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{curl}(\mathbf{curl}(\mathbf{v}))\,d\boldsymbol{x} \\ & = \langle\!\langle \gamma_{\mathrm{R}}(\mathbf{u}), \gamma_{\mathrm{T}}(\mathbf{v})\,\rangle\!\rangle_\times - \langle\!\langle \gamma_{\mathrm{R}}(\mathbf{v}), \gamma_{\mathrm{T}}(\mathbf{u})\,\rangle\!\rangle_\times\end{aligned} \tag{7.2}$$

On the other hand, since $\mathbf{curl}\,\mathbf{curl}\,\mathbf{u} - \kappa^2\mathbf{u} = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$, we deduce that $\mathrm{div}(\mathbf{u}) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$. As a consequence, for any $v \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma})$, we have

$$\begin{aligned}\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathbf{u}\cdot\nabla v\,d\boldsymbol{x} &= \langle\!\langle \pi_{\mathrm{D}}(v), \pi_{\mathrm{N}}(\mathbf{u})\rangle\!\rangle = \frac{1}{\kappa^2}\,\langle\!\langle \pi_{\mathrm{D}}(v), \pi_{\mathrm{N}}(\mathbf{curl}^2\ \mathbf{u})\rangle\!\rangle \\ &= \frac{1}{\kappa^2}\,\langle\!\langle \gamma_{\mathrm{D}}(v), \mathrm{curl}_\Gamma(\gamma_{\mathrm{R}}(\mathbf{u}))\rangle\!\rangle\,.\end{aligned} \tag{7.3}$$

Now recall that the vector Laplace operator is given by $-\Delta = \mathbf{curl}(\mathbf{curl}) - \nabla(\mathrm{div})$. Using this formula as well as (7.2)-(7.3), we conclude that for any $\mathbf{u} \in \mathbf{H}(\mathbf{curl}^2, \mathbb{R}^3\setminus\overline{\Gamma})$ satisfying $\mathbf{curl}(\mathbf{curl}\ \mathbf{u}) - \kappa^2\mathbf{u} = \mathbf{0}$ in $\mathbb{R}^3\setminus\overline{\Gamma}$, the following holds in the sense of distributions in $\mathbb{R}^3$ (and not just $\mathbb{R}^3 \setminus \overline{\Gamma}$!)

$$-\Delta\mathbf{u} - \kappa^2\mathbf{u} = -\gamma_{\mathrm{R}}'(\gamma_{\mathrm{T}}(\mathbf{u})) - \gamma_{\mathrm{T}}'(\gamma_{\mathrm{R}}(\mathbf{u})) + \kappa^{-2}\nabla(\,\gamma_{\mathrm{D}}'\cdot\mathrm{curl}_\Gamma(\gamma_{\mathrm{R}}(\mathbf{u}))\,)\,. \tag{7.4}$$

In the equation above the operators $\gamma_{\mathrm{D}}' : \mathbb{H}^{-1/2}(\Gamma) \to \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\Gamma})'$, and $\gamma_{\mathrm{T}}' : \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})'$ and $\gamma_{\mathrm{R}}' : \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma})'$ are defined as formal adjoints of the trace operators $\gamma_{\mathrm{D}}, \gamma_{\mathrm{T}}$ and γ_{R}:

$$\begin{aligned} \langle \gamma_{\mathrm{D}}' \dot{p}, \varphi \rangle &= \langle\!\langle \dot{p}, \gamma_{\mathrm{D}} \varphi \rangle\!\rangle && \forall \varphi \in \mathcal{D}(\mathbb{R}^3)\ , \\ \langle \gamma_*' \dot{\mathbf{u}}, \boldsymbol{\varphi} \rangle &= \langle\!\langle \dot{\mathbf{u}}, \gamma_* \boldsymbol{\varphi} \rangle\!\rangle_\times && \forall \boldsymbol{\varphi} \in (\mathcal{D}(\mathbb{R}^3))^3\ , * = \mathrm{T}, \mathrm{R}\ . \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ designates the $\mathrm{L}^2(\mathbb{R}^3)$ duality pairing between $\mathcal{D}(\mathbb{R}^3)$ and $\mathcal{D}'(\mathbb{R}^3)$ (or their vector-valued counterparts). Note that the spaces $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\Gamma})'$, $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})'$ and $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma})'$ are naturally embedded in the spaces of distributions $\mathcal{D}'(\mathbb{R}^3)$ and $(\mathcal{D}'(\mathbb{R}^3))^3$, respectively. Thus, the right hand-side of Expression (7.4), and in particular the ∇ operator, can be understood in the sense of distributions $\mathcal{D}'(\mathbb{R}^3)$. As the multi-screen Γ is assumed to be bounded, this right-hand side is clearly a distribution with bounded support.

Remark 7.2. Let us comment on how to read $\gamma_{\mathrm{T}}'(\gamma_{\mathrm{R}}(\mathbf{u}))$. Similar comments will also apply to $\gamma_{\mathrm{R}}'(\gamma_{\mathrm{T}}(\mathbf{u}))$. Consider an arbitrary element $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. For any $\varphi \in \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma})$ and any constant vector $\mathbf{v} \in \mathbb{C}^3$, we have $\varphi\,\mathbf{v} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$, so that

$$(\varphi, \mathbf{v}) \mapsto \langle\!\langle \dot{\mathbf{u}}, \gamma_{\mathrm{T}}(\varphi\,\mathbf{v}) \rangle\!\rangle_\times \tag{7.5}$$

is a continuous bilinear form over $\mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma}) \times \mathbb{C}^3$. In addition, we obviously have $\langle\!\langle \dot{\mathbf{u}}, \gamma_{\mathrm{T}}(\varphi\,\mathbf{v}) \rangle\!\rangle_\times = 0$ whenever $\varphi \in \mathrm{H}^1_{0,\Gamma}(\mathbb{R}^3)$, so that (7.5) actually induces a bilinear form defined over $\mathbb{H}^{1/2}(\Gamma) \times \mathbb{C}^3$, since by Definition 3.1 $\mathbb{H}^{1/2}(\Gamma) := \mathrm{H}^1(\mathbb{R}^3 \setminus \overline{\Gamma}) / \mathrm{H}^1_{0,\Gamma}(\mathbb{R}^3)$. From this discussion, we conclude that $\gamma_{\mathrm{T}}'(\gamma_{\mathrm{R}}(\mathbf{u}))$ is a distribution with values in $\mathbb{C}^3$ i.e. an element of $(\mathcal{D}'(\mathbb{R}^3))^3$. As such, it can legitimately be considered as a (distributional) vector field.

Let $\mathcal{G}_\kappa(\boldsymbol{x})$ denote the radiating fundamental solution for the Helmholtz equation with wave number κ, i.e. the unique distribution over $\mathbb{R}^3$ satisfying $-\Delta \mathcal{G}_\kappa - \kappa^2 \mathcal{G}_\kappa = \delta_0$ and Sommerfeld radiation conditions at ∞ (δ_0 is the Dirac distribution centred at $\mathbf{0}$). Since the right-hand side in (7.4) could be identified as a distributional vector field supported on Γ, the convolution of $\mathcal{G}_\kappa$ with each term of this right-hand side makes sense, see [23, Def. 6.36], and this leads to an explicit expression for $\mathbf{u}$, the multi-screen version of the Stratton-Chu representation formula [10, Theorem 6.2], [7, Formula (24)].

Proposition 7.3 (Representation formula, cf. [9, Proposition 8.2]**).** *Assume that* $\mathbf{u} \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})$ *is a radiating vector field satisfying* $\mathbf{curl}^2\ \mathbf{u} - \kappa^2 \mathbf{u} = 0$ *in* $\mathbb{R}^3 \setminus \overline{\Gamma}$. *Then it can be represented as*

$$\mathbf{u}(\boldsymbol{x}) = \mathrm{DL}_\kappa(\,\gamma_{\mathrm{T}}(\mathbf{u})\,)(\boldsymbol{x}) + \mathrm{SL}_\kappa(\,\gamma_{\mathrm{R}}(\mathbf{u})\,)(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Gamma}$$

$$\textit{with} \quad \mathrm{DL}_\kappa(\dot{\mathbf{u}}) := -\mathcal{G}_\kappa * \gamma_{\mathrm{R}}'(\dot{\mathbf{u}}) \tag{7.6a}$$

$$\mathrm{SL}_\kappa(\dot{\mathbf{u}}) := -\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) + \kappa^{-2} \nabla(\mathcal{G}_\kappa * \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})\,)\ . \tag{7.6b}$$

In the sequel, the two potential operators DL_κ and SL_κ will be called *double and single layer potentials*, respectively [7, Formulas (27) & (28)].

7.2. Continuity and jump formula for layer potentials

In this paragraph we will establish continuity properties for layer potentials, and study their behaviour across the multi-screen.

Lemma 7.4 (Continuity of the single layer potential, cf. [9, Proposition 8.3]**).** *The single layer potential* SL_κ *maps continuously the space* $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ *into* $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3)$.

Proof: According to the discussion above the distributions $\gamma_{\mathrm{T}}'(\dot{\mathbf{u}})$ and $\gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})$, respectively, belong to $(\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)')^3$ and $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)'$. Besides $\mathcal{G}_\kappa *$ is pseudo-differential operator of order -2 on $\mathbb{R}^3$ mapping continuously $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)' \to \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ [24, Theorem 3.1.2]. Thus we conclude that $\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) \in (\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3))^3 \subset \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3)$. We also deduce that $\mathcal{G}_\kappa * \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}}) \in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$ so that $\nabla(\mathcal{G}_\kappa * \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})\,) \in \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}, \mathbb{R}^3)$, which concludes the proof. □

An immediate consequence of the previous result is that $[\gamma_{\mathrm{T}}] \cdot \mathrm{SL}_\kappa(\dot{\mathbf{u}}) = 0$ for any $\dot{\mathbf{u}}$. Now let us establish a technical lemma that will help handle differential and trace operators.

Lemma 7.5. *For any* $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ *we have* $\mathrm{div}(\,\gamma_{\mathrm{T}}'(\dot{\mathbf{u}})) = -\gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})$*, where the operator* div *should be understood in the sense of distributions on* $\mathbb{R}^3$.

Proof: Take any element $\varphi \in \mathcal{D}(\mathbb{R}^3)$. The very definition of the divergence operator in the distributional sense shows that $\langle \varphi, \mathrm{div}(\,\gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\,)\rangle := -\langle \nabla\varphi, \gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\rangle$, where the pairing $\langle\cdot,\cdot\rangle$ refers to the duality between $\mathcal{D}(\mathbb{R}^3)$ and $\mathcal{D}'(\mathbb{R}^3)$. Now, according to the definition of $\nabla_\Gamma = \gamma_{\mathrm{T}} \cdot \nabla$, and the surface Green's formula (5.5), we have

$$\begin{aligned}
\langle \varphi, \mathrm{div}(\,\gamma_{\mathrm{T}}'(\,\dot{\mathbf{u}}\,)\,)\rangle \quad &= -\,\langle\!\langle \dot{\mathbf{u}}, \gamma_{\mathrm{T}}(\nabla\varphi)\rangle\!\rangle_\times \;=\; -\,\langle\!\langle \dot{\mathbf{u}}, \nabla_\Gamma\gamma_{\mathrm{D}}(\varphi)\rangle\!\rangle_\times \\
&= -\langle\!\langle\, \gamma_{\mathrm{D}}(\varphi), \mathrm{curl}_\Gamma(\dot{\mathbf{u}})\,\rangle\!\rangle \\
&= \;-\langle\,\varphi, \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})\,\rangle \quad \forall\varphi \in \mathcal{D}(\mathbb{R}^3)\;.
\end{aligned}$$

□

The next lemma exhibits useful identities linking both potential operators SL_κ and DL_κ from (7.6).

Lemma 7.6. *For any* $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$*, we have* $\mathrm{div}\,\mathrm{DL}_\kappa(\dot{\mathbf{u}}) = 0$ *and* $\mathrm{div}\,\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = -\kappa^{-2}\gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})$*, as well as* $\mathbf{curl}\;\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = \mathrm{DL}_\kappa(\dot{\mathbf{u}})$ *and* $\mathbf{curl}\;\mathrm{DL}_\kappa(\dot{\mathbf{u}}) = -\gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) + \kappa^2\mathrm{SL}_\kappa(\dot{\mathbf{u}})$*, where the operators* div *and* **curl** *are to be understood in the sense of distributions on* $\mathbb{R}^3$.

Proof: For any $\boldsymbol{\varphi} \in (\mathcal{D}(\mathbb{R}^3))^3$, and any $\dot{\mathbf{u}} = \gamma_{\mathrm{T}}(\mathbf{u}) \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, we have $\langle \boldsymbol{\varphi}, \gamma_{\mathrm{R}}'(\gamma_{\mathrm{T}}(\mathbf{u}))\rangle = \langle\!\langle \gamma_{\mathrm{T}}(\mathbf{u}), \gamma_{\mathrm{R}}(\boldsymbol{\varphi})\rangle\!\rangle_\times = \langle\!\langle \gamma_{\mathrm{T}}(\mathbf{u}), \gamma_{\mathrm{T}}(\mathbf{curl}\,\boldsymbol{\varphi})\rangle\!\rangle_\times$, which shows that

$$\gamma_{\mathrm{R}}'(\dot{\mathbf{u}}) = \mathbf{curl}\;\gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) \tag{7.7}$$

in the sense of $\mathcal{D}'(\mathbb{R}^3)$. From this, and standard properties of convolution (see [23, Thm. 6.37] for example), we obtain

$$\mathrm{DL}_\kappa(\dot{\mathbf{u}}) = -\,\mathbf{curl}\,\big(\,\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\,\big) \tag{7.8}$$

in the sense of distributions on $\mathbb{R}^3$. Further, (7.7) directly implies $\mathrm{div}\,\gamma_{\mathrm{R}}'(\gamma_{\mathrm{T}}(\mathbf{u})) = 0$, and also $\mathbf{curl}\,\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = \mathrm{DL}_\kappa(\dot{\mathbf{u}})$. Making use of Lemma 7.5, and standard properties of convolution, we obtain

$$\begin{aligned}
\mathrm{div}\,\mathrm{SL}_\kappa(\dot{\mathbf{u}}) &= -\mathcal{G}_\kappa * \big(\,\mathrm{div}\,\gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\,\big) + \kappa^{-2}\Delta\big(\,\mathcal{G}_\kappa * \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})\,\big)\\
&= \mathcal{G}_\kappa * \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}}) + \kappa^{-2}\Delta\big(\,\mathcal{G}_\kappa * \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})\,\big)\\
&= \kappa^{-2}(\Delta+\kappa^2)\big(\,\mathcal{G}_\kappa * \gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})\,\big) = -\kappa^{-2}\gamma_{\mathrm{D}}' \cdot \mathrm{curl}_\Gamma(\dot{\mathbf{u}})
\end{aligned}$$

To prove the last identity, we take the **curl** of Identity (7.8), standard properties of convolution and Lemma 7.5, which yields

$$\begin{aligned}
\mathbf{curl}\,\mathrm{DL}_\kappa(\dot{\mathbf{u}}) &= -\,\mathbf{curl}\,\mathbf{curl}\,\big(\,\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\,\big) = (\Delta - \nabla\mathrm{div})\big(\,\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\,\big)\\
&= -\gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) - \kappa^2\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) - \nabla\big(\,\mathcal{G}_\kappa * \mathrm{div}\,\gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\,\big)\\
&= -\gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) + \kappa^2\mathrm{SL}_\kappa(\dot{\mathbf{u}})\;.
\end{aligned}$$

□

These results show in particular that the potential operators naturally yield radiating solutions of the homogeneous Maxwell equations (1.1) in $\mathbb{R}^3\setminus\overline{\Gamma}$, *cf.* [7, Formula (29)].

Corollary 7.7 (Potentials solve homogeneous Maxwell's equations). *The double layer potential* DL_κ *maps continuously* $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])$ *into* $\mathbf{H}_{\mathrm{loc}}(\mathbf{curl},\mathbb{R}^3\setminus\overline{\Gamma})$. *Moreover we have* $\mathbf{curl}^2\;\mathrm{SL}_\kappa(\dot{\mathbf{u}}) - \kappa^2\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = -\gamma_{\mathrm{T}}'(\dot{\mathbf{u}})$ *and* $\mathbf{curl}^2\;\mathrm{DL}_\kappa(\dot{\mathbf{u}}) - \kappa^2\mathrm{DL}_\kappa(\dot{\mathbf{u}}) = -\,\mathbf{curl}(\,\gamma_{\mathrm{T}}'(\dot{\mathbf{u}})\,)$ *in the sense of distributions in* $\mathbb{R}^3$. *In addition* $\mathrm{DL}_\kappa(\dot{\mathbf{u}}), \mathrm{SL}_\kappa(\dot{\mathbf{u}})$ *both satisfy the Silver-Müller radiation conditions at* ∞ *for any* $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$.

From this we learn that $\mathrm{DL}_\kappa(\dot{\mathbf{u}})$ and $\mathrm{SL}_\kappa(\dot{\mathbf{u}})$ are continuous mappings from $\mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma,\Gamma)$ into $\mathbf{H}(\mathbf{curl}^2,\mathbb{R}^3\setminus\overline{\Gamma})$. Now we will derive jump formulas for potential operators across Γ. *cf.* [7, Theorem 7]. Lemma 7.4 already establishes that $\mathrm{SL}_\kappa(\dot{\mathbf{u}})$ does not admit any tangential jump. The next proposition provides sharper results.

Proposition 7.8 (Jump relations, cf. [9, Proposition 8.4]**).** *For all tangential traces* $\dot{\mathbf{u}} \in \mathbb{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$ *we have*

$$\begin{aligned}
[\gamma_{\mathrm{T}}]\cdot\mathrm{DL}_\kappa(\dot{\mathbf{u}}) &= [\dot{\mathbf{u}}]\quad, & [\gamma_{\mathrm{T}}]\cdot\mathrm{SL}_\kappa(\dot{\mathbf{u}}) &= 0\;,\\
[\gamma_{\mathrm{R}}]\cdot\mathrm{DL}_\kappa(\dot{\mathbf{u}}) &= 0\quad, & [\gamma_{\mathrm{R}}]\cdot\mathrm{SL}_\kappa(\dot{\mathbf{u}}) &= [\dot{\mathbf{u}}]\;.
\end{aligned}$$

Proof: As mentioned above, Lemma 7.4 directly implies $[\gamma_{\mathrm{T}}]\cdot\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = 0$. In addition Proposition 7.6 shows that $\mathbf{curl}\,\mathrm{DL}_\kappa(\dot{\mathbf{u}}) = \kappa^2\mathrm{SL}_\kappa(\dot{\mathbf{u}})$ in $\mathbb{R}^3\setminus\overline{\Gamma}$, so we conclude in particular that $[\gamma_{\mathrm{R}}]\cdot\mathrm{DL}_\kappa(\dot{\mathbf{u}}) = \kappa^2[\gamma_{\mathrm{T}}]\cdot\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = 0$. Now there only remains to prove that $[\gamma_{\mathrm{R}}]\cdot\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = [\dot{\mathbf{u}}]$ since this will automatically imply $[\gamma_{\mathrm{T}}]\cdot\mathrm{DL}_\kappa(\dot{\mathbf{u}}) = [\dot{\mathbf{u}}]$ according to Proposition 7.6.

Set $\boldsymbol{\psi} := \mathrm{SL}_\kappa(\dot{\mathbf{u}})$ and pick an arbitrary smooth test function $\boldsymbol{\varphi} \in (\mathcal{D}(\mathbb{R}^3))^3$. Since $\mathbf{curl}^2\,\boldsymbol{\psi} - \kappa^2\boldsymbol{\psi} = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$, see Corollary 7.7 above, we can apply (7.2), which yields

$$\begin{aligned}\langle \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{\psi} - \kappa^2\boldsymbol{\psi}, \boldsymbol{\varphi}\rangle &= \langle\!\langle \gamma_{\mathrm{R}}(\boldsymbol{\varphi}), \gamma_{\mathrm{T}}(\boldsymbol{\psi})\rangle\!\rangle_\times + \langle\!\langle \gamma_{\mathrm{T}}(\boldsymbol{\varphi}), \gamma_{\mathrm{R}}(\boldsymbol{\psi})\rangle\!\rangle_\times \\ &= \langle\!\langle \gamma_{\mathrm{T}}(\boldsymbol{\varphi}), \gamma_{\mathrm{R}}(\boldsymbol{\psi})\rangle\!\rangle_\times \ .\end{aligned}$$

In the calculus above, $\langle\cdot,\cdot\rangle$ is the duality pairing between $\mathcal{D}(\mathbb{R}^3)^3$ and $(\mathcal{D}'(\mathbb{R}^3))^3$. We have also used the fact that $[\gamma_{\mathrm{T}}(\boldsymbol{\psi})] = 0$, as $\gamma_{\mathrm{R}}(\boldsymbol{\varphi}) \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. In addition, applying directly the result of Corollary 7.7 yields $\langle \mathbf{curl}^2\,\boldsymbol{\psi} - \kappa^2\boldsymbol{\psi}, \boldsymbol{\varphi}\rangle = -\,\langle\!\langle \dot{\mathbf{u}}, \gamma_{\mathrm{T}}(\boldsymbol{\varphi})\rangle\!\rangle_\times$. Hence, we conclude that

$$\langle\!\langle (\dot{\mathbf{u}} - \gamma_{\mathrm{R}}(\boldsymbol{\psi})), \gamma_{\mathrm{T}}(\boldsymbol{\varphi})\rangle\!\rangle_\times = 0 \quad \forall \boldsymbol{\varphi} \in (\mathcal{D}(\mathbb{R}^3))^3 \ .$$

It remains to show that any vector single trace $\dot{\mathbf{v}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ is the limit (in the sense of $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$) of a sequence of traces of the form $\gamma_{\mathrm{T}}(\boldsymbol{\varphi}), \boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3)^3$. In light of the continuity of the trace operator γ_{T}, it is sufficient to prove density of $\mathcal{D}(\mathbb{R}^3)^3$ in $\mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$, which is a classical result (see for example [19, Chap.3]). □

The jump relations above admit the same form as in [9, Prop. 8.5]. The next result shows that vector single traces do not radiate when taken as arguments of the potential operators.

Lemma 7.9 (Kernels of potentials, cf. [9, Lemma 8.6]**).**

$$\mathrm{Ker}(\mathrm{DL}_\kappa) = \mathrm{Ker}(\mathrm{SL}_\kappa) = \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \ .$$

Proof: We will prove the result for SL_κ, since the same arguments will confirm the result for DL_κ. Take any $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ such that $\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = 0$. Applying the jump relations of Proposition 7.8, we obtain $[\gamma_{\mathrm{T}}(\dot{\mathbf{u}})] = \mathrm{SL}_\kappa(\dot{\mathbf{u}}) = 0$, which implies $\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ according to Proposition 4.5.

Now take an arbitrary $\dot{\mathbf{u}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, and consider any $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Gamma}$. Denote $\mathcal{G}_{\kappa,\boldsymbol{x}}(\boldsymbol{y}) := \mathcal{G}_\kappa(\boldsymbol{x} - \boldsymbol{y})$. For any constant vector $\mathbf{v} \in \mathbb{C}^3$ the vector field $\mathbf{v}\mathcal{G}_{\kappa,\boldsymbol{x}}$ is smooth in the neighbourhood of Γ, so that $\gamma_{\mathrm{T}}(\mathbf{v}\mathcal{G}_{\kappa,\boldsymbol{x}}) \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. As a consequence of the polarity stated in Proposition 4.5, we have $\mathbf{v} \cdot (\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}}))(\boldsymbol{x}) = \langle\!\langle \dot{\mathbf{u}}, \gamma_{\mathrm{T}}(\mathbf{v}\mathcal{G}_{\kappa,\boldsymbol{x}})\rangle\!\rangle_\times = 0$ for any $\mathbf{v} \in \mathbb{C}^3$, and $\boldsymbol{x} \in \mathbb{R}^3 \setminus \overline{\Gamma}$. This shows that $\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$. In a similar manner we prove that $\nabla(\mathcal{G}_\kappa * \gamma_{\mathrm{D}}'\mathrm{curl}_\Gamma(\dot{\mathbf{u}})) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$. Since $\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$ in the sense of distributions, and $\mathrm{SL}_\kappa(\dot{\mathbf{u}}) \in (\mathrm{L}^2(\mathbb{R}^3))^3$, we finally conclude that $\mathrm{SL}_\kappa(\dot{\mathbf{u}}) = 0$ in $\mathbb{R}^3$. □

According to Lemma 4.7 the jump space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ defined by (4.13) can be considered as a quotient space. Therefore the above lemma shows that the potential operators naturally induce injective continuous maps defined on the space of jumps:

$$\mathrm{SL}_\kappa, \mathrm{DL}_\kappa : \widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, [\Gamma]) \to \mathbf{H}_{\mathrm{loc}}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma}) \ .$$

For the sake of simplicity, we use the same notation for these quotient maps. This should not lead to any confusion.

7.3. Electric Field Integral Equation (EFIE)

In this paragraph, we come back to the boundary value problem (6.1)-(6.2) and show how to reformulate it as a boundary integral equation by means of the layer potentials (7.6). For this, assuming that $\mathbf{u}$ is solution to the scattering problem, we use the integral representation formula from Proposition 7.3 that gives us

$$\gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa(\gamma_{\mathrm{R}}(\mathbf{u})) = \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa(\,[\gamma_{\mathrm{R}}(\mathbf{u})]\,) \;=\; \mathbf{f} := \mathbf{g} - \gamma_{\mathrm{T}} \cdot \mathrm{DL}_\kappa(\mathbf{g})\;, \tag{7.9}$$

where the first equality is a consequence of Lemma 7.9. According to the jump relations given in Proposition 7.8 we have $\mathbf{f} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. Let us consider $\mathbf{p} = [\gamma_{\mathrm{R}}(u)] \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ as unknown in the equation above. Thanks to the representation formula of Proposition 7.3, determining $\mathbf{u}$ boils down to determining $\mathbf{p}$ for which we will derive a boundary integral equation. Since (7.9) is posed in $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, we obtain an equivalent variational form by testing with arbitrary functions in $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$:

$$\left\{\begin{array}{l} \text{Find } \mathbf{p} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \text{ such that} \\ \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa(\mathbf{p}), \mathbf{q} \rangle\!\rangle_\times \;=\; \langle\!\langle \mathbf{f}, \mathbf{q} \rangle\!\rangle_\times \qquad \forall \mathbf{q} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])\;. \end{array}\right. \tag{7.10}$$

This is our generalisation of the so-called *electric field integral equation* (EFIE) [7, Section 7.2]. Analogous to [7, Formula (36)], the bilinear form in (7.10) can be split into two parts. Indeed plugging the definition of the single layer potential provided by Proposition 7.3, as well as the surface Green's formula (5.5), we obtain the important identity

$$\begin{aligned} &\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa(\mathbf{p}), \mathbf{q} \rangle\!\rangle_\times \;= \\ &\qquad \kappa^{-2}\, \langle\!\langle \gamma_{\mathrm{D}} \cdot \mathcal{G}_\kappa * \gamma_{\mathrm{D}}'(\mathrm{curl}_\Gamma\, \mathbf{p}), \mathrm{curl}_\Gamma\, \mathbf{q} \rangle\!\rangle - \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\mathbf{p}), \mathbf{q} \rangle\!\rangle_\times\;. \end{aligned} \tag{7.11}$$

Based on this expression we give an explicit integral representation of the EFIE bilinear form. Let $\Gamma_j, j = 0, \dots n$, be the subsets of a decomposition of $\overline{\Gamma} = \cup_{j=0}^n \overline{\Gamma}_j$ where $\overline{\Gamma}_j = \overline{\Gamma} \cap \partial\Omega_j$ as in Definition 2.3. Take two arbitrary functions $\mathbf{p}, \mathbf{q} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ that are traces of smooth vector fields $\mathbf{u}, \mathbf{v}$. As in Section 4.4, we consider $\mathbf{u}_j = \mathbf{u}|_{\Omega_j}$, $\mathbf{v}_j = \mathbf{v}|_{\Omega_j}$ and $\mathbf{p}_j = (\mathbf{n}_j \times \mathbf{u}_j|_{\Gamma_j}) \times \mathbf{n}_j$ and $\mathbf{q}_j = (\mathbf{n}_j \times \mathbf{v}_j|_{\Gamma_j}) \times \mathbf{n}_j$. With these notations we have

$$\begin{aligned} &\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa(\mathbf{p}), \mathbf{q} \rangle\!\rangle_\times \;= \\ &\sum_{j=0}^n \sum_{k=0}^n \int_{\Gamma_j} \int_{\Gamma_k} \mathcal{G}_\kappa(\boldsymbol{x} - \boldsymbol{y}) \Big(\kappa^{-2}\, \mathbf{curl}_{\Gamma_j}\, \mathbf{p}_j(\boldsymbol{x})\; \mathbf{curl}_{\Gamma_k}\, \mathbf{q}_k(\boldsymbol{y}) \\ &\qquad\qquad - (\mathbf{n}_j(\boldsymbol{x}) \times \mathbf{p}_j(\boldsymbol{x})) \cdot (\mathbf{n}_k(\boldsymbol{y}) \times \mathbf{q}_k(\boldsymbol{y})) \Big) d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y}) \end{aligned} \tag{7.12}$$

Now, let us focus on the case of a standard orientable Lipschitz screen as in Sections 4.4.2 and 4.6.2. The above expression yields the customary definition of the EFIE operator for screens in the case where Γ is a standard Lipschitz screen. Indeed, according to Lemma 4.9, Proposition 7.8, Lemma

7.9, and (7.9), the identity (7.12) remains unchanged when adding any element of $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ to $\mathbf{p}, \mathbf{q}$. Moreover, in the case of standard Lipschitz screens, the space $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ can be identified with $\{(\dot{\mathbf{v}}, \dot{\mathbf{v}}) \mid \dot{\mathbf{v}} \in \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)\}$, see discussion in Section 4.6.

As a consequence, for each $\mathbf{p}, \mathbf{q}$ in (7.12), we can choose representatives of the form $\mathbf{p} = (\mathbf{p}_0, \mathbf{p}_1)$, $\mathbf{q} = (\mathbf{q}_0, \mathbf{q}_1)$ with $\mathbf{p}_1 = -\mathbf{p}_0$ and $\mathbf{q}_1 = -\mathbf{q}_0$, adding elements of $\mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, if necessary. This implies $\mathbf{n}_0 \times \mathbf{p}_0 = \mathbf{n}_1 \times \mathbf{p}_1$ and $\mathbf{n}_0 \times \mathbf{q}_0 = \mathbf{n}_1 \times \mathbf{q}_1$. So in the case of a standard Lipschitz screen, Expression (7.12) contains four terms that are all equal, which yields

$$\begin{aligned}\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa(\mathbf{p}), \mathbf{q} \rangle\!\rangle_\times \; = \; 4 \int_\Gamma \int_\Gamma \mathcal{G}_\kappa(\boldsymbol{x} - \boldsymbol{y}) \Big(& \kappa^{-2}\, \mathbf{curl}_\Gamma\, \mathbf{p}_0(\boldsymbol{x})\; \mathbf{curl}_\Gamma\, \mathbf{q}_0(\boldsymbol{y}) \\ & -(\mathbf{n}_0(\boldsymbol{x}) \times \mathbf{p}_0(\boldsymbol{x})) \cdot (\mathbf{n}_0(\boldsymbol{y}) \times \mathbf{q}_0(\boldsymbol{y})) \Big) d\sigma(\boldsymbol{x}) d\sigma(\boldsymbol{y})\,.\end{aligned}$$

8. An Equivalent Norm on Jump Spaces

Since jump traces provide the natural variational space of the EFIE integral equation, as this appears in Formulation (7.10), we will dedicate the present section to a more detailed study of this space. To begin with, the next result shows that the operators SL_κ with purely imaginary κ induces a scalar product on $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$.

Lemma 8.1 (Coercivity of single layer potential, cf. [9, Proposition 8.7]**).** *With* $\imath := \sqrt{-1}$*, the sesquilinear form* $(\mathbf{p}, \mathbf{q}) \mapsto \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\mathbf{p}), \overline{\mathbf{q}} \rangle\!\rangle_\times$ *provides an inner product on* $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$*. In particular, there exists a constant* $C > 0$ *such that*

$$\langle\!\langle \overline{\mathbf{p}}, \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\mathbf{p}) \rangle\!\rangle_\times \;\geq\; C\, \|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])} \qquad \forall \mathbf{p} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])\,.$$

Proof: Continuity of the above sesquilinear form is a direct consequence of the continuity of the map $\mathrm{SL}_\imath : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \to \mathbf{H}(\mathbf{curl}, \mathbb{R}^3)$. Take any $\mathbf{p}, \mathbf{q} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. Since $\mathbf{curl}^2\, \mathrm{SL}_\imath(\overline{\mathbf{q}}) + \mathrm{SL}_\imath(\overline{\mathbf{q}}) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$ and $\overline{\mathbf{q}} = [\gamma_{\mathrm{R}}] \cdot \mathrm{SL}_\imath(\overline{\mathbf{q}})$, Green's Formula (4.6) yields

$$\begin{aligned}\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\mathbf{p}), \overline{\mathbf{q}} \rangle\!\rangle_\times &= \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\mathbf{p}), \gamma_{\mathrm{R}} \cdot \mathrm{SL}_\imath(\overline{\mathbf{q}}) \rangle\!\rangle_\times \\ &= \int_{\mathbb{R}^3 \setminus \overline{\Gamma}} \mathbf{curl}\, \mathrm{SL}_\imath(\mathbf{p}) \cdot \mathbf{curl}\, \mathrm{SL}_\imath(\overline{\mathbf{q}}) + \mathrm{SL}_\imath(\mathbf{p}) \cdot \mathrm{SL}_\imath(\overline{\mathbf{q}})\, d\boldsymbol{x}\end{aligned} \tag{8.1}$$

Symmetry clearly follows from this expression, so we only need to check coercivity. The expression above also implies that

$$\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\mathbf{p}), \overline{\mathbf{p}} \rangle\!\rangle_\times = \|\mathrm{SL}_\imath(\mathbf{p})\|^2_{\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})} \tag{8.2}$$

Temporarily set $\Psi := \mathrm{SL}_\imath(\mathbf{p})$ and observe that $\|\,\mathbf{curl}(\Psi)\|_{\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})} = \|\Psi\|_{\mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \overline{\Gamma})}$, since $\mathrm{SL}_\imath(\mathbf{p})$ solves the homogeneous Maxwell equations and enjoys an exponential decay towards ∞. Continuity of the trace operator γ_{T}

and of the jump operator show that, for some constant $C > 0$,

$$\begin{aligned} \| [\gamma_{\mathrm{R}}(\mathbf{v})] \|_{\widetilde{\mathrm{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])} \leq C \| \mathbf{curl}(\mathbf{v}) \|_{\mathbf{H}(\mathbf{curl}^2,\Gamma)} \\ \forall \mathbf{v} \in \mathbf{H}(\mathbf{curl}^2, \mathbb{R}^3 \setminus \overline{\Gamma}) \; . \end{aligned} \tag{8.3}$$

Since the jump formulas of Proposition 7.8 yield that $\mathbf{p} = [\gamma_{\mathrm{R}}(\Psi)]$, (8.3) together with (8.2) concludes the proof. □

As indicated by the notation $\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma)$, usually (jump) spaces of tangential traces of vector fields in $\mathbf{H}(\mathbf{curl}, \Omega)$ on $\partial\Omega$ or a standard Lipschitz screen $\Gamma \subset \partial\Omega$ are introduced as graph space

$$\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \Gamma) := \{ \mathbf{v} \in (\mathbf{H}_\times^{1/2}(\Gamma))' \mid \mathrm{curl}_\Gamma \mathbf{v} \in \widetilde{\mathrm{H}}^{-\frac{1}{2}}(\Gamma) \} \; , \tag{8.4}$$

equipped with the corresponding graph norm (notations will be explained shortly). In this section we demonstrate that (8.4) carries over to multi-screens. As a by-product we will derive a continuity result for $\gamma_{\mathrm{T}} \cdot \mathcal{G}_\kappa * \gamma_{\mathrm{T}}'$. As a tool, consider the following tangential trace space

$$\begin{aligned} \mathbf{H}_\times^{\frac{1}{2}}(\Gamma) \; &:= \; \pi_{\mathrm{T}}\big((\mathrm{H}^1(\mathbb{R}^3))^3 \big) \quad \text{endowed with} \\ \|\dot{\mathbf{u}}\|_{\mathbf{H}_\times^{1/2}(\Gamma)} &:= \inf \big\{ \; \|\mathbf{u}\|_{\mathrm{H}^1(\mathbb{R}^3)^3} \mid \pi_{\mathrm{T}}(\mathbf{u}) = \dot{\mathbf{u}}, \; \mathbf{u} \in (\mathrm{H}^1(\mathbb{R}^3))^3 \; \big\} \; . \end{aligned} \tag{8.5}$$

This is clearly a Banach space, so we may consider its dual $\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma) := (\mathbf{H}_\times^{1/2}(\Gamma))'$ and equip it with the dual norm. In what follows, the duality pairing between $\mathbf{H}_\times^{1/2}(\Gamma)$ and $\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$ shall be denoted $\langle \cdot, \cdot \rangle$. Observe that $\|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3)} \leq C \|\mathbf{v}\|_{\mathrm{H}^1(\mathbb{R}^3)^3}$ $\forall \mathbf{v} \in \mathrm{H}^1(\mathbb{R}^3)$ for some constant $C > 0$. As a consequence for any $\dot{\mathbf{u}} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, and any $\dot{\mathbf{v}} \in \mathbf{H}_\times^{1/2}(\Gamma)$, we have

$$\begin{aligned} | \langle\!\langle \dot{\mathbf{v}}, \dot{\mathbf{u}} \rangle\!\rangle_\times | \; &\leq \; C \; \|\dot{\mathbf{u}}\|_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])} \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl},\mathbb{R}^3)} \\ &\leq \; C \; \|\dot{\mathbf{u}}\|_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])} \|\mathbf{v}\|_{\mathrm{H}^1(\mathbb{R}^3)^3} \end{aligned}$$

and the inequality above holds for any $\mathbf{v} \in (\mathrm{H}^1(\mathbb{R}^3))^3$ that satisfies $\pi_{\mathrm{T}}(\mathbf{v}) = \dot{\mathbf{v}}$. This clearly establishes that $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \subset \widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$ with continuous embedding.

It turns out that the second part of the EFIE bilinear form (7.11) is continuous on $\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma) \times \widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$.

Lemma 8.2 (Continuity of vector single layer potential, cf. [7, Corollary 3]**).** *The mapping* $(\mathbf{p}, \mathbf{q}) \mapsto \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\mathbf{p}), \mathbf{q} \rangle\!\rangle_\times$ *that is a priori well defined for* $\mathbf{p}, \mathbf{q} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ *can be extended to a continuous bilinear form on* $\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$.

Proof: As discussed in Remark 7.2, for any $\dot{\mathbf{u}} \in \mathbb{H}^{-1/2}(\mathrm{curl}_\Gamma, \Gamma)$ we have $\gamma_{\mathrm{T}}'(\dot{\mathbf{u}}) \in (\mathrm{H}_{\mathrm{loc}}^1(\mathbb{R}^3)')^3$. More precisely, what precedes shows that, for any bounded open ball $\mathrm{B}_r \subset \mathbb{R}^3$ centred at 0 with radius $r > 0$ large enough to guarantee $\overline{\Gamma} \subset \mathrm{B}_r$, there exists a constant $C > 0$ such that

$$\begin{aligned} \left|\langle \gamma_{\mathrm{T}}(\mathbf{v}), \dot{\mathbf{u}}\rangle\right| &\leq C\, \|\dot{\mathbf{u}}\|_{\mathbf{H}_\times^{-1/2}(\Gamma)} \|\gamma_{\mathrm{T}}(\mathbf{v})\|_{\mathbf{H}_\times^{1/2}(\Gamma)} \\ &\leq C\, \|\dot{\mathbf{u}}\|_{\mathbf{H}_\times^{-1/2}(\Gamma)} \|\mathbf{v}\|_{\mathrm{H}^1(\mathrm{B}_r)^3} \qquad \forall \mathbf{v} \in (\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3))^3\ . \end{aligned}$$

As a consequence $\mathbf{u} \mapsto \gamma_{\mathrm{T}}'(\mathbf{u})$ induces a continuous linear mapping from $\mathbf{H}_\times^{-1/2}(\Gamma)$ into $(\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)')^3$. In addition, the convolution operator $\mathcal{G}_\kappa *$ maps continuously $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)'$ into $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)$, which finally proves that $\mathbf{u} \mapsto \gamma_{\mathrm{T}}(\mathcal{G}_\kappa * \gamma_{\mathrm{T}}'(\mathbf{u}))$ maps continuously $\mathbf{H}_\times^{-1/2}(\Gamma)$ into $\mathbf{H}_\times^{1/2}(\Gamma)$. □

Since the operator $\gamma_{\mathrm{T}} \cdot \mathcal{G}_\kappa * \gamma_{\mathrm{T}}'$ is close to the classical Dirichlet trace of the single layer operator, it inherits its positivity properties.

Lemma 8.3 (Coercivity of vector single layer boundary integral operator, cf. [7, Lemma 8]**).** *The sesquilinear form* $(\mathbf{p}, \mathbf{q}) \mapsto \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{T}}'(\mathbf{p}), \overline{\mathbf{q}}\rangle\!\rangle_\times$ *defines an equivalent norm on* $\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$*: there exists a constant* $C > 0$ *such that*

$$|\,\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{T}}'(\mathbf{p}), \overline{\mathbf{p}}\rangle\!\rangle_\times| \;\geq\; C\, \|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)} \quad \forall \mathbf{p} \in \widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)\ .$$

Proof: The only thing to be checked here is coercivity. We follow the proof of [5, Theorem 4]. Take any $\mathbf{p} \in \widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$. Let $\mathbf{e}_j$, $j = 1, 2, 3$, stand for the canonical basis vectors of $\mathbb{R}^3$. The map $u \mapsto \langle \mathbf{p}, \pi_{\mathrm{T}}(u\, \mathbf{e}_j)\rangle$ is continuous on $\mathrm{H}^1(\mathbb{R}^3)$ and vanishes whenever $u \in \mathrm{H}^1_{0,\Gamma}(\mathbb{R}^3)$. So it induces an element $p_j \in \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$. This establishes a continuous embedding of $\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$ into $\widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \times \widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \times \widetilde{\mathrm{H}}^{-1/2}([\Gamma])$. Continuity implies that there exists a constant $C > 0$ such that

$$\|\mathbf{p}\|^2_{\mathbf{H}_\times^{-1/2}(\Gamma)} \leq C \sum_{j=1}^{3} \|p_j\|^2_{\widetilde{\mathrm{H}}^{-1/2}([\Gamma])}\ . \tag{8.6}$$

We can abbreviate this decomposition by writing $\pi_{\mathrm{T}}(\mathbf{p}) = p_1\mathbf{e}_1 + p_2\mathbf{e}_2 + p_3\mathbf{e}_2$. Plugging this decomposition into the sesquilinear form mentioned in the statement of the lemma we obtain the decomposition

$$\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{T}}'(\mathbf{p}), \overline{\mathbf{p}}\rangle\!\rangle_\times = \sum_{j=1}^{3} \langle\!\langle \gamma_{\mathrm{D}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{D}}'(p_j), \overline{p}_j\rangle\!\rangle\ . \tag{8.7}$$

In [9, Proposition 8.7] we showed that there exists a constant $C > 0$ such that $\langle\!\langle \gamma_{\mathrm{D}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{D}}'(q), \overline{q}\rangle\!\rangle \geq C\|q\|^2_{\widetilde{\mathrm{H}}^{-1/2}([\Gamma])}$. This together with (8.7) and (8.6) leads to the conclusion of the proof. □

Proposition 8.4 (Jump space as graph space). *There exist constants* $C_\pm > 0$ *such that, for any* $\mathbf{p} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$*, we have* $C_-\|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])} \leq$

$\|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}^{-1/2}_{\times}(\Gamma)}+\|\mathrm{curl}_\Gamma(\mathbf{p})\|^2_{\widetilde{\mathrm{H}}^{-1/2}([\Gamma])}\leq C_+\|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])}$. *Thus, we can identify, algebraically and topologically,*

$$\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma]) = \{\mathbf{p}\in\widetilde{\mathbf{H}}^{-1/2}_{\times}(\Gamma)\mid \mathrm{curl}_\Gamma(\mathbf{p})\in\widetilde{\mathrm{H}}^{-1/2}([\Gamma])\}\ .$$

Proof: The continuous embedding $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])\subset\widetilde{\mathbf{H}}^{-1/2}_{\times}(\Gamma)$ is an immediate consequence of the continuous embedding $\mathbf{H}^{1/2}_{\times}(\Gamma)\subset\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,[\Gamma])$, which follows from $(\mathrm{H}^1(\mathbb{R}^3))^3\subset\mathbf{H}(\mathbf{curl},\mathbb{R}^3)$. Moreover, from Lemma 5.3 we conclude $\mathrm{curl}_\Gamma\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])\subset\widetilde{\mathrm{H}}^{-1/2}([\Gamma])$. To confirm the reverse embedding $\{\mathbf{p}\in\widetilde{\mathbf{H}}^{-1/2}_{\times}(\Gamma)\mid\mathrm{curl}_\Gamma\mathbf{p}\in\widetilde{\mathrm{H}}^{-1/2}([\Gamma])\}\subset\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])$ we have to show that

$$\left\{\begin{array}{ccc}\mathbf{H}(\mathbf{curl},\mathbb{R}^3) & \to & \mathbb{C}\ ,\\ \mathbf{q} & \mapsto & \langle\!\langle\mathbf{p},\pi_{\mathrm{T}}(\mathbf{q})\rangle\!\rangle_{\times}\ ,\end{array}\right. \tag{8.8}$$

is continuous, if $\mathbf{p}\in\widetilde{\mathbf{H}}^{-1/2}_{\times}(\Gamma)$ and $\mathrm{curl}_\Gamma\mathbf{p}\in\widetilde{\mathrm{H}}^{-1/2}([\Gamma])$. This can be inferred from the *regular decomposition* of $\mathbf{H}(\mathbf{curl},\mathbb{R}^3)$, see [1, Lemma 3.5] or [15, Lemma 2.4], which guarantees a stable decomposition

$$\mathbf{H}(\mathbf{curl},\mathbb{R}^3) = (\mathrm{H}^1(\mathbb{R}^3))^3+\nabla\mathrm{H}^1(\mathbb{R}^3)\ .$$

Plugging the resulting splitting $\mathbf{q}=\mathbf{q}^\perp+\nabla w$, $\mathbf{q}^\perp\in(\mathrm{H}^1(\mathbb{R}^3))^3$, $w\in\mathrm{H}^1(\mathbb{R}^3)$, into (8.8), and using Green's Formula (5.5), we get

$$\langle\!\langle\mathbf{p},\pi_{\mathrm{T}}(\mathbf{q})\rangle\!\rangle = \langle\!\langle\mathbf{p},\pi_{\mathrm{T}}(\mathbf{q}^\perp)\rangle\!\rangle+\langle\!\langle\mathrm{curl}_\Gamma(\mathbf{p}),\pi_{\mathrm{D}}(w)\rangle\!\rangle$$

which is obviously continuous. □

9. Generalised Gårding Inequality for EFIE bilinear form

From (7.11) it is obvious that the EFIE bilinear form cannot be coercive on $\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$, because curl_Γ features a kernel of infinite dimension. This compounds the difficulties of the analysis of the variational EFIE (7.10) compared to the corresponding 1st-kind single layer BIE for scalar problems. If Γ is the boundary of a domain [7, Lemma 10], [16, Section 3] or a simple screen [4, Theorem 3.4], this key challenge has been successfully tackled by showing a generalised Gårding inequality, sometimes called "T-coercivity". Its proof in [4] relies on an $\mathrm{L}^2(\Gamma)$-orthogonal Hodge decomposition of $\widetilde{\mathbf{H}}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma,\Gamma)$, which is not available in a multi-screen setting. Yet, the proof in [7] needs only a regular vector potential *in the volume*, [7, Lemma 1]. This perfectly fits our policy of understanding function spaces on Γ from the volume $\mathbb{R}^3\setminus\overline{\Gamma}$. Thus, in this section we exploit vector potential liftings in $\mathbb{R}^3\setminus\overline{\Gamma}$ to analyse the EFIE bilinear form on multi-screens. We shall make use of the following spaces, *cf.* (6.3),

$$\mathbf{E} := \mathbf{H}(\mathbf{curl}, \mathrm{B} \setminus \overline{\Gamma}) \cap \mathbf{H}(\mathrm{div}, \mathrm{B} \setminus \overline{\Gamma}) \,, \tag{9.1a}$$

$$\mathbf{E}_{\mathrm{N}} := \{\mathbf{v} \in \mathbf{E} \mid \pi_{\mathrm{N}}(\mathbf{v}) = 0 \text{ on } \Gamma \;\text{ and }\; \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\mathrm{B}\,\} \tag{9.1b}$$

$$\mathbf{E}_{\mathrm{T}} := \{\mathbf{v} \in \mathbf{E} \mid \pi_{\mathrm{T}}(\mathbf{v}) = 0 \text{ on } \Gamma \;\text{ and }\; \mathbf{v}\times\mathbf{n} = 0 \text{ on } \partial\mathrm{B}\,\} \tag{9.1c}$$

$$\mathcal{E}_{\mathrm{T}} := \{\mathbf{v} \in \mathbf{E}_{\mathrm{T}} \mid \mathbf{curl}(\mathbf{v}) = 0 \text{ and } \mathrm{div}(\mathbf{v}) = 0\} \,, \tag{9.1d}$$

all equipped with the graph norm $\|\cdot\|_{\mathbf{E}}$ of $\mathbf{E}$. Here, $\mathrm{B} \subset \mathbb{R}^3$ is a fixed ball sufficiently large to satisfy $\overline{\Gamma} \subset \mathrm{B}$.

Throughout this section we consider only multi-screens, for which Assumption 6.2 *holds, which puts the Maxwell compactness property of Theorem* 6.5 *at our disposal.*

9.1. Vector potential lifting operator

An important consequence of the Maxwell compactness property asserted in Theorem 6.5 (together with Assumption 6.2) is the finite dimensionality of the co-homology space $\mathcal{E}_{\mathrm{T}}$. A more precise description is given in the following lemma that extends Lemma 3 of [20] for the present geometrical setting.

Lemma 9.1 (Harmonic Dirichlet vector fields). *The space $\mathcal{E}_{\mathrm{T}}$ is finite dimensional and*

$$\mathcal{E}_{\mathrm{T}} = \{\nabla\varphi \mid \varphi \in \mathrm{H}^1(\mathrm{B}\setminus\overline{\Gamma}),\ \Delta\varphi = 0,\ \nabla_\Gamma\pi_{\mathrm{D}}(\varphi) = 0,\ \nabla_{\partial\mathrm{B}}\varphi = 0\} \,. \tag{9.2}$$

Proof: Consider the form $(\mathbf{v},\mathbf{w}) \mapsto \int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{curl}(\mathbf{v})\cdot\mathbf{curl}(\mathbf{w}) + \mathrm{div}(\mathbf{w})\mathrm{div}(\mathbf{w})d\boldsymbol{x}$ for $\mathbf{v},\mathbf{w} \in \mathbf{E}_{\mathrm{T}}$ that has $\mathcal{E}_{\mathrm{T}}$ as kernel. Since $\mathbf{E}_{\mathrm{T}}$ is compactly embedded into $\mathrm{L}^2(\mathrm{B})$, finite dimensionality of $\mathcal{E}_{\mathrm{T}}$ is a direct consequence of Fredholm's alternative applied to this bilinear form.

Next, it is clear that we have the inclusion "$\supset$" in (9.2). On the other hand, consider an element $\mathbf{w} \in \mathcal{E}_{\mathrm{T}}$. Since $\pi_{\mathrm{T}}(\mathbf{w}) = 0$, we have $[\pi_{\mathrm{T}}(\mathbf{w})] = 0$ hence $\mathbf{w} \in \mathbf{H}(\mathbf{curl}, \mathrm{B})$ with $\mathbf{curl}(\mathbf{w}) = 0$ in B. The trivial topology of B then ensure the existence of a scalar potential: $\mathbf{w} = \nabla\varphi$ for some $\varphi \in \mathrm{H}^1(\mathrm{B})$, see [14, Thm.2.9]. The fact $\Delta\varphi = 0$ follows from $\mathrm{div}\mathbf{w} = 0$. Finally, we have $\mathbf{n}\times\mathbf{w}|_{\partial\mathrm{B}} = 0$ so $\nabla_{\partial\mathrm{B}}\varphi = 0$, and $0 = \pi_{\mathrm{T}}(\mathbf{w}) = \nabla_\Gamma\pi_{\mathrm{D}}(\varphi)$ according to Lemma 5.1. □

Let us give an example of a geometrical configuration that is covered by the present analysis but is outside the scope of [20, Lemma 3]. Figure 2 depicts a multi-screen scatterer in this situation.

Lemma 3.5 of [1] and variants of it are instrumental in domain based approaches tackling the EFIE bilinear form on boundaries of a domain Ω. This lemma asserts the existence of vector potentials in $(\mathrm{H}^1(\Omega))^3$ for divergence free functions with vanishing flux through any closed surface. Alas, [1, Lemma 3.5] assumes that Ω is Lipschitz. Unfortunately we cannot use this result here because $\mathbb{R}^3\setminus\overline{\Gamma}$ is not Lipschitz and, in particular, it lies on both sides of its own boundary. Providing a substitute for [1, Lemma 3.5] turned out to be a major challenge.

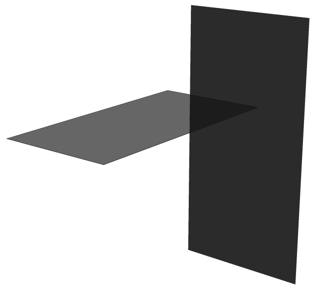

FIGURE 2. Connected multi-screen

Lemma 9.2 (Vector potential lifting operator). *There exists a continuous operator* $\mathrm{S} : (\mathrm{L}^2(\mathrm{B}))^3 \mapsto \mathbf{E}_{\mathrm{N}}$ *such that* $\mathrm{div}\,\mathrm{S}(\mathbf{u}) = 0$ *in* $\mathrm{B} \setminus \overline{\Gamma}$ *and*

$$\int_{\mathrm{B}\setminus\overline{\Gamma}} \big(\mathbf{u} - \mathbf{curl}\ \mathrm{S}(\mathbf{u})\big)\,\mathbf{curl}(\mathbf{v})\,d\boldsymbol{x} = 0 \qquad \forall \mathbf{v} \in \mathbf{E}_{\mathrm{N}}\ .$$

Proof: For the sake of conciseness, let us temporarily set

$$a(\mathbf{v},\mathbf{w}) := \int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{curl}(\mathbf{v})\cdot\mathbf{curl}(\mathbf{w}) + \mathrm{div}(\mathbf{v})\mathrm{div}(\mathbf{w})\,d\boldsymbol{x} \quad \forall \mathbf{v},\mathbf{w} \in \mathbf{E}_{\mathrm{N}}\ .$$

According to Theorem 6.5 the space $\mathbf{E}_{\mathrm{N}}$ is compactly embedded into $(\mathrm{L}^2(\mathrm{B}))^3$. As a consequence, the bilinear form $a(\cdot,\cdot)$ induces a Fredholm operator with index 0. For any $\mathbf{u} \in (\mathrm{L}^2(\mathrm{B}))^3$, consider the following variational problem

$$\begin{aligned}&\text{Find } \ \mathbf{v} \in \mathbf{E}_{\mathrm{N}} \ \text{ such that}\\ &a(\mathbf{v},\mathbf{w}) = \int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{u}\cdot\mathbf{curl}(\mathbf{w})d\boldsymbol{x} \quad \forall \mathbf{w} \in \mathbf{E}_{\mathrm{N}}.\end{aligned}$$

If $\mathbf{w} \in \mathbf{E}_{\mathrm{N}}$ such that $a(\mathbf{w},\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{E}_{\mathrm{N}}$ then $\mathbf{curl}(\mathbf{w}) = 0$ and $\mathrm{div}(\mathbf{w}) = 0$ obviously, so that $\int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{u}\cdot\mathbf{curl}(\mathbf{w})d\boldsymbol{x} = 0$. As a consequence, although the bilinear form of Problem (9.1) may have a non-trivial kernel, the compatibility conditions of the Fredholm alternative are satisfied (see case (ii) of [18, Theorem 2.27], for example). Hence Problem (9.1) admits a solution (that is a priori not unique), and we may define $\mathrm{S}(\mathbf{u})$ as the unique solution satisfying $\|\mathrm{S}(\mathbf{u})\|_{\mathbf{E}} = \min\{\|\mathbf{v}\|_{\mathbf{E}} \mid \mathbf{v} \text{ solves } (9.1)\ \}$, so that $\mathrm{S} : \mathrm{L}^2(\mathrm{B})^3 \to \mathbf{E}_{\mathrm{N}}$ is continuous.

Now set $f := \mathrm{div}(\mathrm{S}(\mathbf{u})) \in \mathrm{L}^2(\mathrm{B})$. Since, by assumption, $\pi_{\mathrm{N}}(\mathrm{S}(\mathbf{u})) = 0$ and $\mathrm{S}(\mathbf{u})\cdot\mathbf{n} = 0$ on ∂B, Green's formula yields $\int_{\mathrm{B}} f\,d\boldsymbol{x} = 0$. Let $\psi \in \mathrm{H}^1(\mathrm{B}\setminus\overline{\Gamma})$ satisfy $\Delta\psi = f$ in $\mathrm{B}\setminus\overline{\Gamma}$, and $\pi_{\mathrm{N}}(\nabla\psi) = 0$, $\nabla\psi\cdot\mathbf{n} = 0$ on $\partial\mathrm{B}$. By construction $\nabla\psi \in \mathbf{E}_{\mathrm{N}}$. Taking $\mathbf{w} = \nabla\psi$ in the variational problem (9.1) satisfied by $\mathrm{S}(\mathbf{u})$, we obtain $0 = \int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathrm{div}(\mathrm{S}(\mathbf{u}))\Delta\psi\,d\boldsymbol{x} = \int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathrm{div}(\mathrm{S}(\mathbf{u}))f\,d\boldsymbol{x} = \|\mathrm{div}\,\mathrm{S}(\mathbf{u})\|^2_{\mathrm{L}^2(\mathrm{B})}$. From this, we conclude that $\mathrm{div}\mathrm{S}(\mathbf{u}) = 0$ in $\mathrm{B}\setminus\overline{\Gamma}$. Plugging this into the variational problem (9.1) satisfied by $\mathrm{S}(\mathbf{u})$ leads to the variational identity stated in the lemma. $\square$

We continue writing S for a mapping $\mathrm{S} : (\mathrm{L}^2(\mathrm{B}))^3 \mapsto \mathbf{E}_{\mathrm{N}}$ provided by the previous lemma. The following corollary gives sufficient conditions for a vector field over $\mathrm{B} \setminus \overline{\Gamma}$ to be the **curl** of another vector field. This result is weaker than Lemma 3.5 of [1], because we fail to obtain extra Sobolev regularity of the vector potential. On the other hand, we can handle more general geometries

Corollary 9.3 (Existence of vector potentials). *For all* $\mathbf{u} \in \mathbf{H}(\mathrm{div}, \mathrm{B} \setminus \overline{\Gamma})$ *such that* $\mathrm{div}(\mathbf{u}) = 0$ *in* $\mathrm{B} \setminus \overline{\Gamma}$ *and* $\int_{\mathrm{B}} \mathbf{u} \cdot \mathbf{v}\, d\boldsymbol{x} = 0\ \forall \mathbf{v} \in \mathcal{E}_{\mathrm{T}}$ *we have* $\mathbf{curl}\ \mathrm{S}(\mathbf{u}) = \mathbf{u}$.

Proof: Let $\mathbf{u} \in (\mathrm{L}^2(\mathrm{B}))^3$ satisfy the assumptions of the corollary. Set $\mathbf{w} := \mathbf{u} - \mathbf{curl}\,\mathrm{S}(\mathbf{u})$, so that $\int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{w} \cdot \mathbf{curl}(\mathbf{v}) d\boldsymbol{x} = 0$ for any $\mathbf{v} \in \mathbf{E}_{\mathrm{N}}$ and $\mathrm{div}(\mathbf{w}) = 0$ in $\mathrm{B} \setminus \overline{\Gamma}$. In particular we have $\int_{\mathrm{B}\setminus\overline{\Gamma}} \overline{\mathbf{w}} \cdot \mathbf{curl}\,\mathrm{S}(\mathbf{u}) d\boldsymbol{x} = 0$.

Take an arbitrary $\mathbf{p} \in \mathbf{H}(\mathbf{curl}, \mathrm{B} \setminus \overline{\Gamma})$ and let ψ refer to the unique element of $\mathrm{H}^1(\mathrm{B} \setminus \overline{\Gamma})$ satisfying $\int_{\mathrm{B}} \psi\, d\boldsymbol{x} = 0$ and $\int_{\mathrm{B}\setminus\overline{\Gamma}} \nabla\psi \cdot \nabla\varphi\, d\boldsymbol{x} = \int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{p} \cdot \nabla\varphi\, d\boldsymbol{x}$ for all $\varphi \in \mathrm{H}^1(\mathrm{B} \setminus \overline{\Gamma})$. Then we have $\mathbf{p} - \nabla\psi \in \mathbf{E}_{\mathrm{N}}$, so $\int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{w} \cdot \mathbf{curl}(\mathbf{p}) d\boldsymbol{x} = \int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{w} \cdot \mathbf{curl}(\mathbf{p} - \nabla\psi) d\boldsymbol{x} = 0$. Since $\mathbf{p}$ was chosen arbitrarily in $\mathbf{H}(\mathbf{curl}, \mathrm{B}\setminus\overline{\Gamma})$, we deduce that $\mathbf{curl}(\mathbf{w}) = 0$ in $\mathrm{B}\setminus\overline{\Gamma}$, $\gamma_{\mathrm{T}}(\mathbf{w}) = 0$ and $\mathbf{w} \times \mathbf{n} = 0$ on $\partial\mathrm{B}$. As a consequence $\mathbf{w} \in \mathcal{E}_{\mathrm{T}}$, and we have $\int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{u} \cdot \overline{\mathbf{w}}\, d\boldsymbol{x} = 0$. This yields $\int_{\mathrm{B}\setminus\overline{\Gamma}} |\mathbf{w}|^2 d\boldsymbol{x} = \int_{\mathrm{B}\setminus\overline{\Gamma}} \overline{\mathbf{w}} \cdot \mathbf{u} d\boldsymbol{x} - \int_{\mathrm{B}\setminus\overline{\Gamma}} \overline{\mathbf{w}} \cdot \mathbf{curl}\ \mathrm{S}(\mathbf{u}) d\boldsymbol{x} = 0$. So, finally, $\mathbf{w} = 0$. □

9.2. Hodge-type decomposition of jump space

For the boundary of a Lipschitz domain Ω Hodge-type decompositions refer to splittings of $\mathbf{H}^{-\frac{1}{2}}(\mathrm{curl}_\Gamma, \partial\Omega)$ into the kernel of curl_Γ and a complement space that is *compactly embedded* in $\widetilde{\mathbf{H}}_\times^{-1/2}(\partial\Omega)$. As hinted above, we pursue a domain based approach to construct a Hodge-type decomposition of the jump space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, taking the cue from [7, Lemma 2 and (21)]. A key tool will be the lifting operator S introduced in Lemma 9.2.

Let $\mathrm{H}_0^1(\mathrm{B}) = \{v \in \mathrm{H}^1(\mathrm{B}) \mid v|_{\partial\mathrm{B}} = 0\}$ and recall that $v \mapsto \|\nabla v\|_{\mathrm{L}^2(\mathrm{B})}$ is a norm on $\mathrm{H}_0^1(\mathrm{B})$. We define the continuous operator $\mathrm{T} : \widetilde{\mathrm{H}}^{-1/2}([\Gamma]) \to \mathbf{H}_0(\mathbf{curl}, \mathrm{B}) \cap \mathbf{H}(\mathrm{div}, \mathrm{B} \setminus \overline{\Gamma})$ by

$$\begin{aligned} &\mathrm{T}(q) := \nabla\psi_q \quad \text{where} \quad \psi_q \in \mathrm{H}_0^1(\mathrm{B}) \ \text{ and} \\ &\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \nabla\psi_q \nabla v\, d\boldsymbol{x} = \langle\!\langle q, \pi_{\mathrm{D}}(v) \rangle\!\rangle \quad \forall v \in \mathrm{H}_0^1(\mathrm{B})\,. \end{aligned} \tag{9.3}$$

Lemma 9.4 (Projection onto complement of kernel of curl_Γ**, cf.** [7, Lemma 2]**).** *Let* S *be a mapping satisfying the properties stated in Lemma* 9.2*, and let* T *be defined by (*9.3*). Define the continuous mapping*

$$\begin{aligned} &\mathfrak{R} : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \to \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]), \\ &\mathfrak{R} := [\gamma_{\mathrm{T}}] \cdot \mathrm{S} \cdot \mathrm{T} \cdot \mathrm{curl}_\Gamma\,. \end{aligned}$$

This map is a projection *satisfying* $\mathrm{curl}_\Gamma(\mathbf{v}) = \mathrm{curl}_\Gamma(\mathfrak{R}(\mathbf{v}))$ *and* $\overline{\mathfrak{R}(\mathbf{v})} = \mathfrak{R}(\overline{\mathbf{v}})$*, for all for all* $\mathbf{v} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$.

Proof: Note that $\mathrm{div}(\mathrm{T}(q)) = 0$ in $\mathbb{R}^3 \setminus \overline{\Gamma}$ and $[\pi_{\mathrm{N}}] \cdot \mathrm{T}(q) = q$ for all $q \in \widetilde{\mathrm{H}}^{-1/2}(\Gamma)$. Also observe that

$$\int_{\mathbb{R}^3\setminus\overline{\Gamma}} \mathrm{T}(\mathrm{curl}_\Gamma \mathbf{v}) \cdot \mathbf{w}_j \, d\boldsymbol{x} = \langle\!\langle \mathrm{curl}_\Gamma \mathbf{v}, w_j \rangle\!\rangle = \langle\!\langle \mathbf{v}, \nabla_\Gamma(\pi_{\mathrm{D}} w_j) \rangle\!\rangle_\times = 0 \;,$$

where $\mathbf{w}_j = \nabla w_j$, $w_j \in \mathrm{H}^1(\mathrm{B})$, are the basis functions of $\mathcal{E}_{\mathrm{T}}$ defined in Lemma 9.1. To understand the last equality remember that $\nabla_\Gamma(\pi_{\mathrm{D}} w_j) = 0$ on Γ. Summing up, we have found that the vector fields $\mathrm{T}(\mathrm{curl}_\Gamma \mathbf{v})$ satisfy the assumptions of Corollary 9.3. Thus, we can use Corollary 9.3 together with Lemma 5.3 and its commuting diagram involving jumps and surface differential operators:

$$\begin{aligned}\mathrm{curl}_\Gamma \mathfrak{R}(\mathbf{v}) &= \mathrm{curl}_\Gamma \cdot [\gamma_{\mathrm{T}}] \cdot \mathrm{S} \cdot \mathrm{T} \cdot \mathrm{curl}_\Gamma \mathbf{v} = [\mathrm{curl}_\Gamma \gamma_{\mathrm{T}} \cdot \mathrm{S} \cdot \mathrm{T} \cdot \mathrm{curl}_\Gamma \mathbf{v}] \\ &= [\gamma_{\mathrm{N}} \cdot \mathbf{curl} \cdot \mathrm{S} \cdot \mathrm{T} \cdot \mathrm{curl}_\Gamma \mathbf{v}] = [\gamma_{\mathrm{N}} \cdot \mathrm{T} \cdot \mathrm{curl}_\Gamma \mathbf{v}] = \mathrm{curl}_\Gamma \mathbf{v} \;.\end{aligned}$$

The projection property is immediate from this. Finally, we clearly have $\overline{\mathfrak{R}(\mathbf{v})} = \mathfrak{R}(\overline{\mathbf{v}})$ simply because this property is satisfied by S and T. □

The operator $\mathfrak{R}$ induces a decomposition of the space $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. In particular the operator $\mathbf{v} \mapsto \mathfrak{R}\mathbf{v} - (\mathrm{Id} - \mathfrak{R})\mathbf{v}$ is an involution of $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. In addition, the range of $\mathfrak{R}$ satisfies a crucial compactness property.

Lemma 9.5 (Compactness of $\mathfrak{R}$). *The operator $\mathfrak{R}$ from* (9.4) *is compact as a mapping* $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \to \widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)$.

Proof: Consider once again the ball B introduced in the beginning of this section. During construction of $\mathfrak{R}$ we built a continuous operator $\mathfrak{T} : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \to \mathbf{E}_{\mathrm{N}}$ such that $\mathfrak{R} = [\gamma_{\mathrm{T}}] \cdot \mathfrak{T}$. For any $\mathbf{p} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ and any $\mathbf{v} \in (\mathrm{H}_0^1(\mathrm{B}))^3$, we have

$$\begin{aligned}|\, \langle\!\langle \gamma_{\mathrm{T}}(\mathbf{v}), \gamma_{\mathrm{T}} \cdot \mathfrak{T}(\mathbf{p}) \rangle\!\rangle_\times | &= \left| \int_{\mathrm{B}\setminus\overline{\Gamma}} \mathbf{curl}(\mathbf{v}) \cdot \mathfrak{T}(\mathbf{p}) - \mathbf{v} \cdot \mathbf{curl}(\mathfrak{T}(\mathbf{p}))\, d\boldsymbol{x} \right| \\ &\leq \|\mathbf{v}\|_{\mathrm{H}^1(\mathrm{B})^3} \|\mathfrak{T}(\mathbf{p})\|_{\mathrm{L}^2(\mathrm{B})^3} + \|\mathbf{v}\|_{\mathrm{H}^1(\mathrm{B})^3} \|\,\mathbf{curl}\,\mathfrak{T}(\mathbf{p})\|_{\mathrm{H}^{-1}(\mathrm{B})^3} \;.\end{aligned}$$

Dividing the inequality above by $\|\mathbf{v}\|_{\mathrm{H}^1(\mathrm{B})^3}$ and taking the supremum over all $\mathbf{v} \in (\mathrm{H}_0^1(\mathrm{B}))^3$, we obtain that $\|\mathfrak{R}(\mathbf{p})\|_{\widetilde{\mathbf{H}}_\times^{-1/2}(\Gamma)} \leq \|\mathfrak{T}(\mathbf{p})\|_{\mathrm{L}^2(\mathrm{B})^3} + \|\,\mathbf{curl}\,\mathfrak{T}(\mathbf{p})\|_{\mathrm{H}^{-1}(\mathrm{B})^3}$. Now recall that $\mathfrak{T}$ maps continuously into $\mathbf{E}_{\mathrm{N}}$. Besides $\mathbf{E}_{\mathrm{N}}$ is compactly embedded into $(\mathrm{L}^2(\mathrm{B}))^3$ and $(\mathrm{L}^2(\mathrm{B}))^3$ is compactly embedded into $(\mathrm{H}^{-1}(\mathrm{B}))^3$. This concludes the proof. □

Thus, from the continuity result of Lemma 8.2 we immediately infer the following compactness.

Corollary 9.6 (Compactness of vectorial single layer potential on the range of $\mathfrak{R}$). *The bilinear form* $(\mathbf{p}, \mathbf{q}) \mapsto \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_{\imath} * \gamma_{\mathrm{T}}'(\mathfrak{R}(\mathbf{p})), \mathfrak{R}(\mathbf{q}) \rangle\!\rangle_\times$ *is compact on* $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \times \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$.

We abbreviate $\Theta := 2\mathfrak{R} - \mathrm{Id}$, which clearly defines an "sign-flipping" isomorphism of $\widetilde{\mathbf{H}}(\mathrm{curl}_\Gamma, [\Gamma])$, *cf.* the operator X_Γ from [7, Eq. (38)]. We now proceed with the proof of a generalised Gårding inequality satisfied by the EFIE operator on multi-screens, which is the main finding of the present section.

Theorem 9.7 (Generalized Gårding inequality, cf. [7, Lemma 10]**).** *There exists a compact operator* $\mathrm{K} : \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ *and a constant* $C > 0$ *such that, for all* $\mathbf{p} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ *we have*

$$\big| \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa(\mathbf{p}), \Theta(\overline{\mathbf{p}})\rangle\!\rangle_\times + \langle\!\langle \mathrm{K}(\mathbf{p}), \overline{\mathbf{p}}\rangle\!\rangle_\times \big| \geq C \, \|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}(\mathrm{curl}_\Gamma, [\Gamma])}. \tag{9.4}$$

Proof: Denote $\imath := \sqrt{-1}$. According to Remark 3.1.3 in [24], the convolution operator $(\mathcal{G}_\imath - \mathcal{G}_\kappa)*$ is a pseudo-differential operator of order -4 mapping $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^3)'$ to $\mathrm{H}^3_{\mathrm{loc}}(\mathbb{R}^3)$. As a consequence, the bilinear form induced by the operator $\gamma_{\mathrm{T}} \cdot \mathrm{SL}_\kappa$ on $\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \times \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ differs only by a compact perturbation from the following symmetric bilinear form

$$\alpha(\mathbf{p}, \mathbf{q}) := \kappa^{-2} \, \langle\!\langle \gamma_{\mathrm{D}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{D}}'(\mathrm{curl}_\Gamma \mathbf{p}), \mathrm{curl}_\Gamma \mathbf{q}\rangle\!\rangle - \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{T}}'(\mathbf{p}), \mathbf{q}\rangle\!\rangle_\times \; ,$$

for $\mathbf{p}, \mathbf{q} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$. Thus it suffices to show that $\alpha(\mathbf{p}, \Theta(\overline{\mathbf{p}}))$ satisfies a Garding inequality of the form (9.4). First, we conclude from (7.11)

$$\begin{aligned} \alpha(\mathbf{p}, \mathbf{q}) = \quad & -\kappa^{-2} \, \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\mathbf{p}), \mathbf{q}\rangle\!\rangle_\times \\ & -(1 + \kappa^{-2}) \, \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{T}}'(\mathbf{p}), \mathbf{q}\rangle\!\rangle_\times \; . \end{aligned} \tag{9.5}$$

Next, set $\mathfrak{Z} := \mathrm{Id} - \mathfrak{R}$. According to Lemma 9.4, for all $\mathbf{p} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$ we have $\mathrm{curl}_\Gamma \mathfrak{Z}(\mathbf{p}) = \mathrm{curl}_\Gamma \mathbf{p} - \mathrm{curl}_\Gamma \mathfrak{R}(\mathbf{p}) = 0$, which involves

$$\alpha(\, \mathfrak{Z}(\mathbf{p}), \mathfrak{Z}(\overline{\mathbf{p}})\,) = -\, \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{T}}'(\mathbf{p}), \mathbf{q}\rangle\!\rangle_\times \; .$$

Thus, for $\mathbf{p} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])$, thanks to the symmetry of α and (9.5)

$$\begin{aligned} \mathrm{Re}\{\alpha(\, \mathbf{p}, \Theta(\overline{\mathbf{p}})\,)\} &= \alpha(\, \mathfrak{R}(\mathbf{p}), \mathfrak{R}(\overline{\mathbf{p}})\,) - \alpha(\, \mathfrak{Z}(\mathbf{p}), \mathfrak{Z}(\overline{\mathbf{p}})\,) \\ &= -\kappa^{-2} \, \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\, \mathfrak{R}(\mathbf{p})), \mathfrak{R}(\overline{\mathbf{p}})\rangle\!\rangle_\times \\ &\quad - \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\, \mathfrak{Z}(\mathbf{p})), \mathfrak{Z}(\overline{\mathbf{p}})\rangle\!\rangle_\times \\ &\quad -(1 + \kappa^{-2}) \, \langle\!\langle \gamma_{\mathrm{T}} \cdot \mathcal{G}_\imath * \gamma_{\mathrm{T}}'(\mathfrak{R}(\mathbf{p})), \mathfrak{R}(\overline{\mathbf{p}})\rangle\!\rangle_\times \end{aligned} \tag{9.6}$$

Let write $\beta(\mathbf{p}, \overline{\mathbf{q}})$ for the continuous sesquilinear form associated with the last term in (9.6). According to Lemma 9.6 above, the bilinear form β is compact. So it suffices to prove Garding inequality for $\mathrm{Re}\{\alpha(\, \mathbf{p}, \Theta(\overline{\mathbf{p}})\,) - \beta(\mathbf{p}, \overline{\mathbf{p}})\}$. To deal with the first two terms in (9.6) recall Lemma 8.1 that yields

$$\langle\!\langle \gamma_{\mathrm{T}} \cdot \mathrm{SL}_\imath(\mathbf{q}), \overline{\mathbf{q}}\rangle\!\rangle_\times \geq C \, \|\mathbf{q}\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])} \quad \forall \mathbf{q} \in \widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma]) \; .$$

Since $\mathrm{Id} = \mathfrak{R} + \mathfrak{Z}$ and $\mathfrak{R}$ is a continuous projector, we deduce that

$$\|\mathfrak{R}(\mathbf{p})\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])} + \|\mathfrak{Z}(\mathbf{p})\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])} \geq \tfrac{1}{2} \|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma, [\Gamma])} \; .$$

From this we conclude

$$
\begin{aligned}
-\operatorname{Re}\{\, &\alpha(\,\mathbf{p},\Theta(\overline{\mathbf{p}})\,) + \beta(\mathbf{p},\overline{\mathbf{p}})\,\} \\
&= \kappa^{-2}\,\langle\!\langle \gamma_{\mathrm{T}}\cdot \mathrm{SL}_{\imath}(\,\mathfrak{R}(\mathbf{p})),\mathfrak{R}(\overline{\mathbf{p}})\rangle\!\rangle_{\times} + \langle\!\langle \gamma_{\mathrm{T}}\cdot \mathrm{SL}_{\imath}(\,\mathfrak{Z}(\mathbf{p})\,),\mathfrak{Z}(\overline{\mathbf{p}})\rangle\!\rangle_{\times} \\
&\geq C\kappa^{-2}\|\mathfrak{R}(\mathbf{p})\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])} + C\|\mathfrak{Z}(\mathbf{p})\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])} \\
&\geq \tfrac{1}{2}C\min(1,\kappa^{-2})\|\mathbf{p}\|^2_{\widetilde{\mathbf{H}}^{-1/2}(\mathrm{curl}_\Gamma,[\Gamma])}\,,
\end{aligned}
$$

which completes the proof. □

Acknowledgement

The authors would like to thank Prof. Rainer Picard for valuable hints on the Maxwell compactness property.

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