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Abstract

This paper investigates an unknown input observer design for a large class of linear systems with unknown inputs and commensurate delays. A Luenberger-like observer is proposed by involving only the past and actual values of the system output. The required conditions for the proposed observer are considerably relaxed in the sense that they coincide with the necessary and sufficient conditions for the unknown input observer design of linear systems without delays.

Keywords: Delay systems, commensurate delays, observer, unknown inputs.

1. Introduction

Time delay systems are widely used to model many applications, ranging from chemical and biological process to sampled data effects Richard (2003). Many results have been published to treat this kind of systems for different aspects, such as stability Fridman (2014), observability Zheng et al. (2011) and identifiability Zheng et al. (2013).

The unknown input observer design for linear systems without delays has already been solved in Bhattacharyya (1978); Darouach et al. (1994); Yang and Wilde (1988); Hou and Muller (1992); Kudva et al. (1980); Wang et al. (1975); Hostetter and Meditch (1973). This problem becomes more complicated when the studied system involves delays, which might appear in the state, in the input and in the output. For this issue, different techniques have been proposed in the literature, such as infinite dimensional approach Salamon (1980), polynomial approach based on the ring theory Sename (1997); Emre and Khargonekar (1982), Lyapunov function based on LMI Darouach (2001); Seuret et al. (2007) and so on.

More precisely, Fattouh et al. (1999) proposed an unknown input observer with dynamic gain for linear systems with commensurate delays in state, input and output variables, while the output was not affected by the unknown inputs. Inspired by the technique of output injection Krener (1985), Hou et al. (2002) solved this problem by transforming the studied system into a higher dimensional observer canonical form with delayed output injection. In Darouach (2001, 2006), the unknown input observer was designed for the systems involving only one delay in the state, and no delay appears in the input and output. The other observers for some classes of time-delay systems can be found in Conte et al. (2003); Sename (2001); Fu et al. (2004) and the references therein.

Most of the existing works on unknown input observer are focused on time-delay systems whose outputs are not affected by unknown inputs. However, this situation might exist in many practical applications since most of the sensors involve computation and communication, thus introduce output delays. This motivates the work of this paper. Compared to the existing results in the literature, this paper deals with the unknown input observer design problem for a more general sort of linear time-delay systems where the commensurate delays are involved in the state, in the input as well as in the output. Moreover, the studied linear time-delay system admits more than one delay. As far as we known, there exist some methods to eliminate (or reduce the degree of) the delay, such as Lee et al. (1982), Germani et al. (2001) and Garate-Garcia et al. (2011). It has been proven in Garate-Garcia et al. (2011) that the elimination or the reduction of delay degree via a bicausal transformation with the same dimension is possible if some conditions on $A(\delta)$ and $B(\delta)$ are satisfied. Since this paper investigates the most general linear system with commensurate delays on the state, the input and the output, to impose those kinds of conditions will definitely restrict the contribution of this paper. Moreover, even for the general single delay system with **unknown input**, the problem to design an observer is still unsolved, thus the contribution of this paper does not depend on the degree of time delay involved in $A(\delta), B(\delta), C(\delta)$ and $D(\delta)$.

This paper adopts the polynomial method based on ring theory since it enables us to reuse some useful techniques developed for systems without delays. The following notations will be used in this paper. $\mathbb{R}$ is the field of real numbers. The set of nonnegative integers is denoted by $\mathbb{N}_0$. I_r means the $r \times r$ identity matrix. $\mathbb{R}[\delta]$ is the polynomial ring over the field $\mathbb{R}$. $\mathbb{R}^n[\delta]$

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is the $\mathbb{R}[\delta]$-module whose elements are the vectors of dimension n and whose entries are polynomials. By $\mathbb{R}^{q\times s}[\delta]$ we denote the set of matrices of dimension $q\times s$, whose entries are in $\mathbb{R}[\delta]$. For a matrix $M(\delta)$, $\text{rank}_{\mathbb{R}[\delta]}M(\delta)$ means the rank of the matrix $M(\delta)$ over $\mathbb{R}[\delta]$. $M(\delta)\sim N(\delta)$ means the similarity between two polynomial matrices $M(\delta)$ and $N(\delta)$ over $\mathbb{R}[\delta]$, i.e. there exist two unimodular[1] matrices $U_1(\delta)$ and $U_2(\delta)$ over $\mathbb{R}[\delta]$ such that $M(\delta)=U_1(\delta)N(\delta)U_2(\delta)$.

2. Problem statement

In this paper, we consider the following class of linear systems with commensurate delays:

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & \sum\limits_{i=0}^{k_a}A_i x(t-ih)+\sum\limits_{i=0}^{k_b}B_i u(t-ih)\\ y(t) & = & \sum\limits_{i=0}^{k_c}C_i x(t-ih)+\sum\limits_{i=0}^{k_e}D_i u(t-ih)\end{array}\right. \tag{1}$$

where the state vector $x(t)\in\mathbb{R}^{n_x}$, the system output vector $y(t)\in\mathbb{R}^{p}$, the unknown input vector $u(t)\in\mathbb{R}^{m}$, the initial condition $\varphi(t)$ is a piecewise continuous function $\varphi(t):[-kh,0]\to\mathbb{R}^{n}$ ($k=\max\{k_a,k_b,k_c,k_d\}$); thereby $x(t)=\varphi(t)$ on $[-kh,0]$. A_i, B_i, C_i and D_i are the matrices of appropriate dimension with entries in $\mathbb{R}$.

In order to simplify the analysis, let us introduce the delay operator $\delta: x(t)\to x(t-h)$ with $\delta^k x(t)=x(t-kh)$, $k\in\mathbb{N}_0$. Let $\mathbb{R}[\delta]$ be the polynomial ring of δ over the field $\mathbb{R}$, and it is obvious that $\mathbb{R}[\delta]$ is a commutative ring.

After having introduced the delay operator δ, system (1) may be then represented in the following compact form:

$$\left\{\begin{array}{rcl}\dot{x}(t) & = & A(\delta)x(t)+B(\delta)u(t)\\ y(t) & = & C(\delta)x(t)+D(\delta)u(t)\end{array}\right. \tag{2}$$

where $A(\delta)\in\mathbb{R}^{n_x\times n_x}[\delta]$, $B(\delta)\in\mathbb{R}^{n_x\times m}[\delta]$, $C(\delta)\in\mathbb{R}^{p\times n_x}[\delta]$, and $D(\delta)\in\mathbb{R}^{p\times m}[\delta]$ are matrices over the polynomial ring $\mathbb{R}[\delta]$, defined as $A(\delta):=\sum\limits_{i=0}^{k_a}A_i\delta^i$, $B(\delta):=\sum\limits_{i=0}^{k_b}B_i\delta^i$, $C(\delta):=\sum\limits_{i=0}^{k_c}C_i\delta^i$, and $D(\delta):=\sum\limits_{i=0}^{k_d}D_i\delta^i$.

Remark 1. *For the system without delay, i.e. $A(\delta)=A$, $B(\delta)=B$, $C(\delta)=C$ and $D(\delta)=D$ in (2), Hautus (1983) proposed the following unknown input Luenberger-like observer:*

$$\begin{array}{l}\dot{\xi}=P\xi+Qy\\ \hat{x}=\xi+Ky\end{array}$$

and it has been proven as well the above Luenberger-like observer exists only if the following rank condition:

$$\text{rank}\begin{bmatrix}CB & D\\ D & 0\end{bmatrix}=\text{rank}\begin{bmatrix}B\\ D\end{bmatrix}+\text{rank}D \tag{3}$$

is satisfied.

[1]refer to Definition 2 for the concept of unimodular matrix over $\mathbb{R}[\delta]$.

When considering the general linear system (2) with commensurate delays which can appear in the state, in the input and in the output, the problem to design a simple unknown input Luenberger-like observer is still open. The main idea of this paper is inspired by the method proposed in Hou et al. (2002) where only linear time-delay systems without input were studied. More precisely, we firstly try to decompose system (2) into a simpler form provided that some conditions are satisfied, and then transform it into a higher dimensional observer normal form with output (and the derivative of the output) injection and its delay. Finally we can design an unknown input observer for the obtained observer normal form.

3. Notations and definitions

When designing an unknown input observer for time-delay systems, it is desired to use only the actual and the past information (not the future information) of the measurements to estimate the states because of the causality. Therefore, by noting $x(t;\varphi,u)$ as the solution of (2) with the initial condition φ and the input u, we have the following observability definition stated in Bejarano and Zheng (2014).

Definition 1. *System (1) (or system (2)) is said to be backward unknown input observable on $[t_1,t_2]$ if for each $\tau\in[t_1,t_2]$ there exist $t_1'<t_2'\le\tau$ such that, for all input u and every initial condition φ, $y(t;\varphi,u)=0$ for all $t\in[t_1',t_2']$ implies $x(\tau;\varphi,u)=0$.*

Concerning the above definition of backward unknown input observability, Bejarano and Zheng (2014) analyzed it by following the ideas of Silverman (1969) and Molinari (1976). Define $\{\Delta_k(\delta)\}$ as the matrices generated by the following algorithm:

$$\begin{array}{c}\Delta_0\triangleq 0,\ G_0(\delta)\triangleq C(\delta),\ F_0(\delta)\triangleq D(\delta)\\ S_k(\delta)\triangleq\begin{bmatrix}\Delta_k(\delta)B(\delta)\\ F_k(\delta)\end{bmatrix},\quad k\ge 0\\ \begin{bmatrix}F_{k+1}(\delta) & G_{k+1}(\delta)\\ 0 & \Delta_{k+1}(\delta)\end{bmatrix}\triangleq P_k(\delta)\begin{bmatrix}\Delta_k(\delta)B(\delta) & \Delta_k(\delta)A(\delta)\\ F_k(\delta) & G_k(\delta)\end{bmatrix}\end{array} \tag{4}$$

where $P_k(\delta)$ is a unimodular matrix over $\mathbb{R}[\delta]$ that transforms $S_k(\delta)$ into its Hermite form. Moreover define $\{M_k(\delta)\}$ as follows:

$$\begin{array}{c}M_0(\delta)\triangleq N_0(\delta)\triangleq\Delta_0,\ N_{k+1}(\delta)\triangleq\begin{bmatrix}N_k(\delta)\\ \Delta_{k+1}(\delta)\end{bmatrix},\ \text{for } k\ge 0\\ \begin{bmatrix}M_{k+1}(\delta)\\ 0\end{bmatrix}\triangleq\begin{bmatrix}\mathscr{S}_{N_{k+1}(\delta)} & 0\\ 0 & 0\end{bmatrix}=\Lambda_{k+1}(\delta)N_{k+1}(\delta)\Sigma_{k+1}(\delta)\end{array} \tag{5}$$

where $\mathscr{S}_{N_{k+1}(\delta)}=\text{diag}\{\psi_1^{k+1}(\delta),\cdots,\psi_{r_{k+1}}^{k+1}(\delta)\}$ with $\Lambda_{k+1}(\delta)$ and $\Sigma_{k+1}(\delta)$ being unimodular matrices over $\mathbb{R}[\delta]$ that transform $N_{k+1}(\delta)$ into its Smith form, and $\left\{\psi_i^{k+1}(\delta)\right\}$ are called the invariant factors of $N_{k+1}(\delta)$.

Since we are going to analyze system (2) which is described by the polynomial matrices over $\mathbb{R}[\delta]$, therefore let us give some useful definitions of unimodular and change of coordinates over $\mathbb{R}[\delta]$.

Definition 2. *A given polynomial matrix* $A(\delta) \in \mathbb{R}^{n\times q}[\delta]$ *is said to be left (or right) unimodular over* $\mathbb{R}[\delta]$ *if there exists* $A_L^{-1}(\delta) \in \mathbb{R}^{q\times n}[\delta]$ *with* $n \geq q$ *(or* $A_R^{-1}(\delta) \in \mathbb{R}^{q\times n}[\delta]$ *with* $n \leq q$*), such that* $A_L^{-1}(\delta)A(\delta) = \mathrm{I}_q$ *(or* $A(\delta)A_R^{-1}(\delta) = \mathrm{I}_n$*). A square matrix* $A(\delta) \in \mathbb{R}^{n\times n}[\delta]$ *is said to be unimodular over* $\mathbb{R}[\delta]$ *if* $A_L^{-1}(\delta) = A_R^{-1}(\delta)$.

Definition 3. *Hou et al. (2002) For* $x(t)$ *defined in (2),* $z(t) = T(\delta)x(t)$ *with* $T(\delta) \in \mathbb{R}^{n_z\times n_x}[\delta]$ *and* $n_z \geq n_x$ *is said to be a causal generalized change of coordinates over* $\mathbb{R}[\delta]$ *if* $\mathrm{rank}_{\mathbb{R}[\delta]}T(\delta) = n_x$. *Moreover, it is said to be a bicausal generalized change of coordinates over* $\mathbb{R}[\delta]$ *if* $T(\delta)$ *is left unimodular over* $\mathbb{R}[\delta]$.

When applying the algorithm (4)-(5) to system (2), it has been proven in Bejarano and Zheng (2014) that there always exists a least integer k^*, which is independent of the choices of $\{P_k(\delta), \Lambda_k(\delta), \Sigma_k(\delta)\}$, such that $M_{k^*+1}(\delta) = M_{k^*}(\delta)$, based on which the following assumption will be made.

Assumption 1. *For the quadruple* $(A(\delta), B(\delta), C(\delta), D(\delta))$ *of system (2), there exists a least integer* $k^* \in \mathbb{N}_0$ *such that* $\mathrm{rank}_{\mathbb{R}[\delta]}M_{k^*}(\delta) = n_x$*, and* $M_{k^*}(\delta)$ *is unimodular over* $\mathbb{R}[\delta]$.

Then the following theorem was stated in Bejarano and Zheng (2014).

Theorem 1. *If Assumption 1 is satisfied, then there exists a* t_1 *such that system (2) is backward unknown input observable on* $[t_1, t_2]$ *for all* $t_2 > t_1$.

For simplicity, for any polynomial matrix $D(\delta) \in \mathbb{R}^{p\times m}[\delta]$ with $\mathrm{rank}_{\mathbb{R}[\delta]}D(\delta) = r_D \leq \min\{p, m\}$, let us denote $\mathrm{Inv}_S[D(\delta)] = \{\psi_i(\delta)\}_{1\leq i\leq r_D}$ as the set of its invariant factors of the Smith form defined in (5). Thereby, the following statement, adapted from the result on the left unimodular stated in Hou et al. (2002), is obvious.

Lemma 1. *A polynomial matrix* $D(\delta) \in \mathbb{R}^{p\times m}[\delta]$ *is left (or right) unimodular over* $\mathbb{R}[\delta]$ *if and only if* $\mathrm{rank}_{\mathbb{R}[\delta]}D(\delta) = m \leq p$ *(or* $\mathrm{rank}_{\mathbb{R}[\delta]}D(\delta) = p \leq m$ *) and* $\mathrm{Inv}_S[D(\delta)] \subset \mathbb{R}$.
It is said to be unimodular over $\mathbb{R}[\delta]$ *if and only if* $\mathrm{rank}_{\mathbb{R}[\delta]}D(\delta) = p = m$ *and* $\mathrm{Inv}_S[D(\delta)] \subset \mathbb{R}$.

4. Main result

Before proposing an unknown input observer for the general system (2), we will firstly decompose system (2) into a simpler form under some additional conditions.

4.1. Preliminary result

In order to transform the general system (2) into a simpler form, let us make the following assumption.

Assumption 2. *For the polynomial matrices* $B(\delta)$*,* $C(\delta)$ *and* $D(\delta)$ *in system (2), it is assumed that*

$$\mathrm{Inv}_S\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} = \mathrm{Inv}_S\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} \tag{6}$$

Remark 2. *When treating linear systems without delay, the conditions imposed in Assumption 1 is equivalent to:*

$$\mathrm{rank}\begin{bmatrix} sI-A & -B \\ C & D \end{bmatrix} = n + \mathrm{rank}\begin{bmatrix} B \\ D \end{bmatrix} \text{ for all } s \in \mathbb{C} \tag{7}$$

which is exactly the necessary and sufficient condition such that the system is strongly observable Trentelman et al. (2001).

The condition (6) imposed in Assumption 2 is equivalent to:

$$\mathrm{rank}\begin{bmatrix} CB & D \\ D & 0 \\ B & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} CB & D \\ D & 0 \end{bmatrix} = \mathrm{rank}\begin{bmatrix} B \\ D \end{bmatrix} + \mathrm{rank}D$$

and it is exactly the necessary condition to ensure the existence of such a Luenberger-like observer (see Remark 1*). As we are going to propose an unknown input observer for the general time-delay system (2) with the same structure, thus the condition (6) imposed in Assumption 2 is not restrictive.*

Lemma 2. *Suppose Assumption 2 is satisfied, then there exists a matrix* $W(\delta) \in \mathbb{R}^{(n_x+p)\times 2p}[\delta]$ *satisfying the following conditions:*

1) $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$;

2) *for any matrix* $J(\delta)$ *such that* $J(\delta)\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} = 0$*, then* $J(\delta)W(\delta) = 0$.

Proof.

For simplicity, let us denote $C_{BD} = \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix}$ and $C_D = \begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}$.

It is easy to see that if Assumption 2 is satisfied, then we have

$$\mathrm{rank}_{\mathbb{R}[\delta]}\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \mathrm{rank}_{\mathbb{R}[\delta]}\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} = r$$

where r denotes the number of invariant factors. Moreover, this implies as well that there exist the unimodular matrices $P_1(\delta)$, $P_2(\delta)$, $Q_1(\delta)$ and $Q_2(\delta)$ over $\mathbb{R}[\delta]$ such that $P_1(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}P_2(\delta) = \begin{bmatrix} \mathscr{S}_{C_D} & 0 \\ 0 & 0 \end{bmatrix}$ and $Q_1(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix}Q_2(\delta) = \begin{bmatrix} \mathscr{S}_{C_{BD}} & 0 \\ 0 & 0 \end{bmatrix}$ with $\mathscr{S}_{C_{BD}} = \mathscr{S}_{C_D} = \mathrm{diag}\{\psi_1, \cdots, \psi_r\}$. Then, we obtain

$$\begin{aligned}
&\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} \\
&\sim \begin{bmatrix} P_1(\delta) & 0 \\ 0 & \mathrm{I}_{n_x} \end{bmatrix}\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix}P_2(\delta) \\
&\sim \begin{bmatrix} P_1(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}P_2(\delta) \\ \begin{bmatrix} B(\delta) & 0 \end{bmatrix}P_2(\delta) \end{bmatrix}
\end{aligned}$$

since $\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} \sim \begin{bmatrix} \mathscr{S}_{C_D} & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} \sim \begin{bmatrix} \mathscr{S}_{C_{BD}} & 0 \\ 0 & 0 \end{bmatrix}$, therefore, $\begin{bmatrix} B(\delta) & 0 \end{bmatrix} P_2(\delta)$ should be spanned by $P_1(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} P_2(\delta)$ over $\mathbb{R}[\delta]$, otherwise the condition (6) is violated. Thus, there exists a matrix $\bar{W}(\delta)$ over $\mathbb{R}[\delta]$ such that $\begin{bmatrix} B(\delta) & 0 \end{bmatrix} = \bar{W}(\delta)P_1(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}$, this implies as well that there exists a matrix $W(\delta) \in \mathbb{R}^{(n_x+p)\times 2p}[\delta]$ such that $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$.

In order to prove the second condition of Lemma 2, let $U(\delta)$ be a unimodular matrix over $\mathbb{R}[\delta]$ which transforms the following matrix into its Hermite form:

$$U(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} V(\delta) \\ 0 \end{bmatrix}$$

where $V(\delta)$ is full row rank over $\mathbb{R}[\delta]$. Since there exists a matrix $W(\delta) \in \mathbb{R}^{(n_x+p)\times 2p}[\delta]$ such that $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$, then we can find a matrix $\bar{V}(\delta)$ such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Finally we can select $W(\delta) = [\bar{V}(\delta), 0]U(\delta)$ with which we still have $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Moreover, since $V(\delta)$ is full row rank over $\mathbb{R}[\delta]$, then for any matrix $J(\delta)$ such that $J(\delta)\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} = 0$ we always have $J(\delta)\bar{V}(\delta) = 0$, which implies $J(\delta)W(\delta) = 0$. ■

4.2. System decomposition

Suppose that $W(\delta) \in \mathbb{R}^{(n_x+p)\times 2p}[\delta]$ is a matrix over $\mathbb{R}[\delta]$ such that Lemma 2 is satisfied. Decompose $W(\delta) = [W_1(\delta), W_2(\delta)]$ with $W_1(\delta), W_2(\delta) \in \mathbb{R}^{(n_x+p)\times p}[\delta]$, then we have $W_1(\delta)D(\delta) = 0$ since $W(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Thus, we obtain

$$W_1(\delta)y = W_1(\delta)C(\delta)x$$

which yields the following equation

$$W_1(\delta)\dot{y} = W_1(\delta)C(\delta)A(\delta)x + W_1(\delta)C(\delta)B(\delta)u$$

Since $W(\delta)\begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} = \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix}$, then we have

$$\begin{aligned} & W_1(\delta)\dot{y} + W_2(\delta)y \\ & = W(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + W(\delta)\begin{bmatrix} C(\delta)B(\delta) \\ D(\delta) \end{bmatrix} u \\ & = W(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} x + \begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u \end{aligned}$$

Decompose again the matrix $W(\delta)$ as $W(\delta) = \begin{pmatrix} K(\delta) \\ \Gamma(\delta) \end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix}$, where $K_i(\delta) \in \mathbb{R}^{n_x\times p}[\delta]$ and $\Gamma_i(\delta) \in \mathbb{R}^{p\times p}[\delta]$ for $1 \le i \le 2$. Thereby, we can substitute $\begin{bmatrix} B(\delta) \\ D(\delta) \end{bmatrix} u$ into the original system (2) in order to replace the unknown input u. That is, if Assumption 2 is satisfied, then system (2) can be put into the following simpler form:

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x + K_1(\delta)\dot{y} + K_2(\delta)y \\ y &= \tilde{C}(\delta)x + \Gamma_1(\delta)\dot{y} + \Gamma_2(\delta)y \end{aligned} \tag{8}$$

where $\bar{A}(\delta) = A(\delta) - K(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} \in \mathbb{R}^{n_x\times n_x}[\delta]$ and $\tilde{C}(\delta) = C(\delta) - \Gamma(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix} \in \mathbb{R}^{p\times n_x}[\delta]$.

Suppose $\mathrm{rank}_{\mathbb{R}[\delta]}\tilde{C}(\delta) = r \le p$, then there exists a unimodular matrix $\Lambda(\delta)$ over $\mathbb{R}[\delta]$ such that

$$\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix} \tag{9}$$

with $\bar{C}(\delta) \in \mathbb{R}^{r\times n_x}[\delta]$ being full row rank over $\mathbb{R}[\delta]$. By noting $\bar{y} = \Lambda(\delta)y$, finally system (8) can be written into the following decomposed form:

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x + K_1(\delta)\Lambda^{-1}(\delta)\dot{\bar{y}} + K_2(\delta)\Lambda^{-1}(\delta)\bar{y} \\ \bar{y} &= \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix} x + \bar{\Gamma}_1(\delta)\dot{\bar{y}} + \bar{\Gamma}_2(\delta)\bar{y} \end{aligned} \tag{10}$$

where

$$\begin{aligned} \bar{\Gamma}_1(\delta) &= \Lambda(\delta)\Gamma_1(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{p\times p}[\delta] \\ \bar{\Gamma}_2(\delta) &= \Lambda(\delta)\Gamma_2(\delta)\Lambda^{-1}(\delta) \in \mathbb{R}^{p\times p}[\delta] \end{aligned} \tag{11}$$

4.3. Observer normal form

This subsection is devoted to designing a Luenberger-Like observer for the deduced simple form (10). Before this, define the following polynomial matrix over $\mathbb{R}[\delta]$:

$$\bar{\mathscr{O}}_l(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ \bar{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \bar{C}(\delta)\bar{A}^{l-1}(\delta) \end{bmatrix} \in \mathbb{R}^{rl\times n_x}[\delta] \tag{12}$$

where $l \in \mathbb{N}_0$, and let us recall a useful result stated in Hou and Muller (1992).

Theorem 2. *Hou and Muller (1992) There exists a bicausal generalized change of coordinates $z = T(\delta)x$ which transforms the following system:*

$$\begin{aligned} \dot{x} &= \bar{A}(\delta)x \\ \bar{y} &= \bar{C}(\delta)x \end{aligned} \tag{13}$$

with $\mathrm{rank}_{\mathbb{R}[\delta]}\bar{C}(\delta) = r$ into the following observer normal form:

$$\begin{cases} \dot{z} = A_0 z + F(\delta)\bar{y} \\ \bar{y} = C_0 z \end{cases}$$

where $F(\delta) = \left[F_1^T(\delta), \cdots, F_{l^*}^T(\delta)\right]^T$ *and*

$$A_0 = \begin{bmatrix} 0 & \mathrm{I}_r & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{I}_r \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{rl^* \times rl^*} \tag{14}$$
$$C_0 = [\mathrm{I}_r, 0, \cdots, 0] \in \mathbb{R}^{r \times rl^*}$$

if and only if there exists a least integer $l^* \in \mathbb{N}_0$ *such that* $\bar{\mathscr{O}}_{l^*}(\delta)$ *defined in (12) is left unimodular over* $\mathbb{R}[\delta]$.

Moreover, the bicausal generalized change of coordinates $z = T(\delta)x$ *with* $T(\delta) = \mathrm{col}\{T_1(\delta), \cdots, T_{l^*}(\delta)\}$ *is defined as follows:*

$$\begin{cases} T_1(\delta) = \bar{C}(\delta) \\ T_{i+1}(\delta) = T_i(\delta)\bar{A}(\delta) - F_i(\delta)\bar{C}(\delta), \text{ for } 1 \leq i \leq l^* - 1 \end{cases} \tag{15}$$

with $F_i(\delta)$ *being determined through the following equations:*

$$[F_{l^*}(\delta), \cdots, F_1(\delta)] = \bar{C}(\delta)\bar{A}^{l^*}(\delta)\left[\bar{\mathscr{O}}_{l^*}(\delta)\right]_L^{-1} \tag{16}$$

Remark 3. *Theorem 2 deduces a very simple observer normal form with constant matrices* A_0 *and* C_0 *plus a linear output delayed term* $F(\delta)\bar{y}$ *for a special form of system (10), which implies in fact that in the original system (2) the matrices* $B(\delta) = D(\delta) = 0$, *i.e. system (10) has no inputs. Theorem 2 can be also applied to design a Luenberger-like observer for system (2) with known input, however it cannot treat directly system (2) with unknown input, since the observability in this case depends as well on the matrices* $B(\delta)$ *and* $D(\delta)$.

It has been stated in Theorem 1 that system (2) is backward unknown input observable if Assumption 1 is satisfied. In the following it will be shown that Assumptions 1 and 2 imply that the observability matrix defined in (12) is left unimodular. For this, we need the following result.

Lemma 3. *If Assumption 1 is satisfied for the quadruple* $(A(\delta), B(\delta), C(\delta), D(\delta))$ *in system (2), then there exists a least integer* $l^* \in \mathbb{N}_0$ *such that*

$$\mathscr{O}_{l^*}(\delta) = \begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ \vdots \\ C(\delta)A^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \tag{17}$$

is left unimodular over $\mathbb{R}[\delta]$.

Proof.

If Assumption 1 is satisfied for the quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$ of system (2), then by applying the algorithm defined in (4)-(5) to system (2), we can find an integer $k^* \in \mathbb{N}_0$ such that $\mathrm{rank}_{\mathbb{R}[\delta]} M_{k^*}(\delta) = n$, and $M_{k^*}(\delta)$ is unimodular over $\mathbb{R}[\delta]$.

Now, consider the first iteration of the algorithm, we have

$$P_0(\delta)\begin{bmatrix} \Delta_0(\delta)B(\delta) & \Delta_0(\delta)A(\delta) \\ D(\delta) & C(\delta) \end{bmatrix} = \begin{bmatrix} F_1(\delta) & G_1(\delta) \\ 0 & \Delta_1(\delta) \end{bmatrix}$$

with $\Delta_0(\delta) = 0$. Thus, there exist two matrices $S_1(\delta)$ and $R_1(\delta)$ over $\mathbb{R}[\delta]$ such that $\Delta_1(\delta) = S_1(\delta)C(\delta)$ and $G_1(\delta) = R_1(\delta)C(\delta)$. In the second iteration, we obtain

$$\begin{aligned} &P_1(\delta)\begin{bmatrix} \Delta_1(\delta)B(\delta) & \Delta_1(\delta)A(\delta) \\ F_1(\delta) & G_1(\delta) \end{bmatrix} \\ &= P_1(\delta)\begin{bmatrix} S_1(\delta)C(\delta)B(\delta) & S_1(\delta)C(\delta)A(\delta) \\ F_1(\delta) & R_1(\delta)C(\delta) \end{bmatrix} \\ &= \begin{bmatrix} F_2(\delta) & G_2(\delta) \\ 0 & \Delta_2(\delta) \end{bmatrix} \end{aligned}$$

from which we can see that there exist the matrices $S_2(\delta)$ and $R_2(\delta)$ over $\mathbb{R}[\delta]$ such that $\Delta_2(\delta) = S_2(\delta)\begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \end{bmatrix}$ and $G_2(\delta) = R_2(\delta)\begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \end{bmatrix}$. The next iteration yields

$$P_2(\delta)\begin{bmatrix} \Delta_2(\delta)B(\delta) & \Delta_2(\delta)A(\delta) \\ F_2(\delta) & G_2(\delta) \end{bmatrix} = \begin{bmatrix} F_3(\delta) & G_3(\delta) \\ 0 & \Delta_3(\delta) \end{bmatrix}$$

with $\Delta_3(\delta) = S_3(\delta)\begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ C(\delta)A^2(\delta) \end{bmatrix}$ and $G_3(\delta) = R_3(\delta)\begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ C(\delta)A^2(\delta) \end{bmatrix}$, for some matrices $S_3(\delta)$ and $R_3(\delta)$ over $\mathbb{R}[\delta]$.

Thus, by induction we can see that in the k-th iteration there exists the matrix $S_k(\delta)$ such that $\Delta_k(\delta) = S_k(\delta)\begin{bmatrix} C(\delta) \\ \vdots \\ C(\delta)A^k(\delta) \end{bmatrix}$ with $k \in \mathbb{N}_0$. Hence, the matrix $N_l(\delta)$ defined in (5) can be expressed as $N_l(\delta) = S(\delta)\mathscr{O}_l(\delta)$ where $S(\delta)$ depends on $S_i(\delta)$ and $\mathscr{O}_l(\delta)$ is defined in (17).

Therefore, if Assumption 1 is satisfied for system (2), i.e. there exists a least integer $l^* \in \mathbb{N}_0$ such that $\mathrm{rank} M_{l^*}(\delta) = n$ and $M_{l^*}(\delta)$ is unimodular over $\mathbb{R}[\delta]$, where the unimodular matrix $M_{l^*}(\delta)$ over $\mathbb{R}[\delta]$ is the corresponding Smith form of $N_{l^*}(\delta)$, then $N_{l^*}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$. Thus, $N_{l^*}(\delta)$ admits a left inverse matrix $[N_{l^*}(\delta)]_L^{-1}$ such that

$$\mathrm{I}_{n_x} = [N_{l^*}(\delta)]_L^{-1} N_{l^*}(\delta) = [N_{l^*}(\delta)]_L^{-1} S(\delta)\mathscr{O}_{l^*}(\delta)$$

which implies $[N_{l^*}(\delta)]_L^{-1} S(\delta)$ is a left inverse of $\mathscr{O}_{l^*}(\delta)$. Therefore $\mathscr{O}_{l^*}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$. ■

Based on Lemma 3, we have the following result for the deduced system (8).

Lemma 4. *If Assumption 1 is satisfied for the quadruple* $(A(\delta), B(\delta), C(\delta), D(\delta))$ *defined in (2), then for the deduced system (8) there exists a least integer* $l^* \in \mathbb{N}_0$ *such that*

$$\tilde{\mathscr{O}}_{l^*}(\delta) = \begin{bmatrix} \tilde{C}(\delta) \\ \tilde{C}(\delta)\bar{A}(\delta) \\ \vdots \\ \tilde{C}(\delta)\bar{A}^{l^*-1}(\delta) \end{bmatrix} \in \mathbb{R}^{pl^* \times n_x}[\delta] \tag{18}$$

is left unimodular over $\mathbb{R}[\delta]$.

Proof.

According to Lemma 3, we need only to prove that, if Assumption 1 is satisfied for the quadruple $(A(\delta),B(\delta),C(\delta),D(\delta))$, then this assumption is also satisfied for the quadruple $\left(\bar{A}(\delta),B(\delta),\tilde{C}(\delta),D(\delta)\right)$, which implies that $\tilde{\mathscr{O}}_{l^*}(\delta)$ is left unimodular over $\mathbb{R}[\delta]$.

For this, by keeping the same matrix $P_0(\delta)$ as in the proof of Lemma 3 for the quadruple $(A(\delta),B(\delta),C(\delta),D(\delta))$, we obtain:

$$
\begin{aligned}
&P_0(\delta)\begin{bmatrix}\Delta_0(\delta)B(\delta) & \Delta_0(\delta)\bar{A}(\delta)\\ D(\delta) & \tilde{C}(\delta)\end{bmatrix}\\
&=P_0(\delta)\begin{bmatrix}\Delta_0(\delta)B(\delta) & \Delta_0(\delta)A(\delta)\\ D(\delta) & C(\delta)\end{bmatrix}\\
&\quad -P_0(\delta)\begin{bmatrix}0 & \Delta_0(\delta)K(\delta)\begin{bmatrix}C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}\\ 0 & \Gamma(\delta)\begin{bmatrix}C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}\end{bmatrix}\\
&=\begin{bmatrix}S_1(\delta)D(\delta) & G_1(\delta)-S_1(\delta)\Gamma(\delta)\begin{bmatrix}C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}\\ 0 & \Delta_1(\delta)\end{bmatrix}
\end{aligned}
$$

for some matrix $S_1(\delta)$ over $\mathbb{R}[\delta]$. In the last equality we use the fact that, by the result in Lemma 2, any annihilator of $\begin{bmatrix}B(\delta)\\ D(\delta)\end{bmatrix}$ is also an annihilator of $W(\delta)=\begin{bmatrix}K(\delta)\\ \Gamma(\delta)\end{bmatrix}$. Hence $\Delta_1(\delta)$ for $\left(\bar{A}(\delta),B(\delta),\tilde{C}(\delta),D(\delta)\right)$ is the same as for $(A(\delta),B(\delta),C(\delta),D(\delta))$. For the second step of the same algorithm, we obtain

$$
\begin{aligned}
&P_1(\delta)\begin{bmatrix}\Delta_1(\delta)B(\delta) & \Delta_1(\delta)\bar{A}(\delta)\\ S_1(\delta)D(\delta) & G_1(\delta)-S_1(\delta)\Gamma(\delta)\begin{bmatrix}C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}\end{bmatrix}\\
&=P_1(\delta)\begin{bmatrix}\Delta_1(\delta)B(\delta) & \Delta_1(\delta)A(\delta)\\ S_1(\delta)D(\delta) & G_1(\delta)\end{bmatrix}\\
&\quad -P_1(\delta)\begin{bmatrix}0 & \Delta_1(\delta)K(\delta)\begin{bmatrix}C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}\\ 0 & S_1(\delta)\Gamma(\delta)\begin{bmatrix}C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}\end{bmatrix}\\
&=\begin{bmatrix}S_2(\delta)\begin{bmatrix}B(\delta)\\ D(\delta)\end{bmatrix} & G_2(\delta)-S_2(\delta)\begin{bmatrix}K(\delta)\\ \Gamma(\delta)\end{bmatrix}\begin{bmatrix}C(\delta)A(\delta)\\ C(\delta)\end{bmatrix}\\ 0 & \Delta_2(\delta)\end{bmatrix}.
\end{aligned}
$$

Thus, by induction, it is easy to see that the matrix $N_k(\delta)$ for the quadruple $\left(\bar{A}(\delta),B(\delta),\tilde{C}(\delta),D(\delta)\right)$ is the same as the matrix $N_k(\delta)$ calculated for $(A(\delta),B(\delta),C(\delta),D(\delta))$. Therefore, according to Lemma 3, there exists a least integer $l^*\in\mathbb{N}_0$ such that $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (12) is left unimodular over $\mathbb{R}[\delta]$. ■

Lemma 5. *If there exists a least integer $l^*\in\mathbb{N}_0$ such that $\tilde{\mathscr{O}}_{l^*}(\delta)$ defined in (18) is left unimodular over $\mathbb{R}[\delta]$, then $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (12) is left unimodular over $\mathbb{R}[\delta]$.*

Proof.

Since there exists a unimodular matrix $\Lambda(\delta)$ over $\mathbb{R}[\delta]$ such that $\Lambda(\delta)\tilde{C}(\delta)=\begin{bmatrix}\bar{C}(\delta)\\ 0\end{bmatrix}$, then we can define the unimodular matrix $\Lambda_d(\delta)=\mathrm{diag}\{\Lambda(\delta),\cdots,\Lambda(\delta)\}\in\mathbb{R}^{pl^*\times pl^*}[\delta]$, and there exists an elementary matrix $E\in\mathbb{R}^{pl^*\times pl^*}$ such that

$$
E\Lambda_d(\delta)\tilde{\mathscr{O}}_{l^*}(\delta)=\begin{bmatrix}\bar{\mathscr{O}}_{l^*}(\delta)\\ 0\end{bmatrix}
$$

which shows that the left unimodularity of $\tilde{\mathscr{O}}_{l^*}(\delta)$ over $\mathbb{R}[\delta]$ defined in (18) implies the left unimodularity of $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (12) over $\mathbb{R}[\delta]$. ■

Theorem 3. *If Assumption 1 and Assumption 2 are both satisfied for system (2), then for the deduced system (10) there exists a least integer $l^*\in\mathbb{N}_0$ such that $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (12) is left unimodular over $\mathbb{R}[\delta]$.*

Proof. If Assumption 2 is satisfied for system (2), then it can be transformed into system (10). Thus, the proof can be easily achieved by combing the results stated in Lemma 3, Lemma 4 and Lemma 5. ■

After having proved the left unimodularity of $\bar{\mathscr{O}}_{l^*}(\delta)$ defined in (12) over $\mathbb{R}[\delta]$, the following corollary is obvious due to Theorem 2.

Corollary 1. *If Assumption 1 and Assumption 2 are both satisfied for system (2), then there exists a bicausal generalized change of coordinates $z=T(\delta)x$ defined in (15) such that system (10) can be transformed into the following observer normal form:*

$$
\begin{cases}\dot{z} &= A_0z+[F(\delta),0]\,\bar{y}+\bar{K}_1(\delta)\dot{\bar{y}}+\bar{K}_2(\delta)\bar{y}\\ \bar{y} &= \begin{bmatrix}C_0\\ 0\end{bmatrix}z+\bar{\Gamma}_1(\delta)\dot{\bar{y}}+\bar{\Gamma}_2(\delta)\bar{y}\end{cases}\tag{19}
$$

where $\bar{\Gamma}_1(\delta)$, $\bar{\Gamma}_2(\delta)$, A_0, C_0 and $F(\delta)$ are defined in (11), (14) and (16) respectively, with

$$
\begin{aligned}\bar{K}_1(\delta)&=T(\delta)K_1(\delta)\Lambda^{-1}(\delta)\in\mathbb{R}^{n_z\times p}[\delta]\\ \bar{K}_2(\delta)&=T(\delta)K_2(\delta)\Lambda^{-1}(\delta)\in\mathbb{R}^{n_z\times p}[\delta]\end{aligned}\tag{20}
$$

where $n_z=rl^*$.

Proof. The proof of this corollary just needs to apply the results stated in Theorem 2 and Theorem 3. ■

4.4. Unknown input observer design

For the obtained observer normal form (19), we are ready to present our main result.

Theorem 4. *If Assumption 1 and Assumption 2 are both satisfied for system (2), then the following dynamics:*

$$
\begin{cases}\dot{\xi} &= L_0\xi+J(\delta)\Lambda(\delta)y\\ \hat{z} &= \xi+H(\delta)\Lambda(\delta)y\\ \hat{x} &= T_L^{-1}(\delta)\hat{z}\end{cases}\tag{21}
$$

with $T_L^{-1}(\delta)$ being defined in (15), and

$$
\begin{aligned}L_0 &= A_0-G_0C_0\\ H(\delta) &= \bar{K}_1(\delta)-[G_0,0]\,\bar{\Gamma}_1(\delta)\\ J(\delta) &= [F(\delta),0]+\bar{K}_2(\delta)+L_0H(\delta)-[G_0,0]\,\bar{\Gamma}_2(\delta)+[G_0,0]\end{aligned}\tag{22}
$$

where G_0 is a constant matrix which makes $(A_0-G_0C_0)$ Hurwitz, is an exponential unknown input observer for system (2).

Proof. As we have proved that, if Assumption 1 and Assumption 2 are both satisfied for system (2), then it can be transformed into the observer normal form (19). Denote $e_z = z - \hat{z}$, $e_x = x - \hat{x}$ and note $\bar{G}_0 = [G_0, 0]$, $\bar{F}(\delta) = [F(\delta), 0]$, since $\bar{y} = \Lambda(\delta)y$, then we have

$$\begin{aligned} \dot{e}_z &= A_0 z - L_0 \hat{z} + [\bar{F}(\delta) + \bar{K}_2(\delta) + L_0 H(\delta) - J(\delta)]\bar{y} \\ &\quad + [\bar{K}_1(\delta) - H(\delta)]\dot{\bar{y}} \end{aligned}$$

Since the pair (A_0, C_0) is observable, then there exists a constant matrix G_0 such that $(A_0 - G_0C_0)$ is Hurwitz. With the chosen matrix G_0, we can determine the matrices L_0, $H(\delta)$ and $J(\delta)$ defined in (22), with which the above equation becomes:

$$\begin{aligned} \dot{e}_z &= A_0 z - L_0 \hat{z} + \left[\bar{G}_0 \bar{\Gamma}_2(\delta) - \bar{G}_0\right]\bar{y} + \bar{G}_0 \bar{\Gamma}_1(\delta)\dot{\bar{y}} \\ &= [A_0 - G_0 C_0]\, e_z \end{aligned}$$

Since $x = T_L^{-1}(\delta) z$, then $e_x = x - \hat{x} = T_L^{-1}(\delta) e_z$ is governed by:

$$\dot{e}_x = T_L^{-1}(\delta)\,(A_0 - G_0C_0)\,T(\delta) e_x$$

Due to the fact that $(A_0 - G_0C_0)$ is a constant Hurwitz matrix, so $T_L^{-1}(\delta)\,(A_0 - G_0C_0)\,T(\delta)$ is Hurwtiz as well. Consequently, we proved that system (21) is an exponential unknown input observer of system (2). ■

Remark 4. *When treating linear systems without delay, Assumption 1 and Assumption 2 are necessary and sufficient for the existence of a Luenberger-like observer. In this sense, when studying linear systems with delay, the conditions required by Assumption 1 and Assumption 2 are not restrictive to guarantee the existence of a Luenberger-like observer.*

Remark 5. *The proposed method is based on the output injection (delayed) technique. It can be seen that the observation error dynamics $\dot{e}_z = [A_0 - G_0C_0]e_z$ is independent of the delay, which implies that this method can be applied to any commensurate and constant delay.*

For the given quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$, if Theorem 4 is valid, the following summarizes the procedure to design the proposed unknown input observer for system (2):

Step 1: Compute the unimodular matrix $U(\delta)$ over $\mathbb{R}[\delta]$ which transforms the following matrix into its Hermite form:

$$U(\delta)\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} = \begin{bmatrix} V(\delta) \\ 0 \end{bmatrix}$$

with $V(\delta)$ being full row rank over $\mathbb{R}[\delta]$, and calculate $\bar{V}(\delta)$ such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$. Then we obtain the gain matrix $W(\delta) = [\bar{V}(\delta), 0]U(\delta)$;

Step 2: With the obtained matrix $W(\delta)$, decompose it as $W(\delta) = \begin{pmatrix} K(\delta) \\ \Gamma(\delta) \end{pmatrix} = \begin{bmatrix} K_1(\delta) & K_2(\delta) \\ \Gamma_1(\delta) & \Gamma_2(\delta) \end{bmatrix}$, then transform system (2) into (10) with $\bar{A}(\delta) = A(\delta) - K(\delta)\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix}$, $\tilde{C}(\delta) = C(\delta) - \Gamma\begin{bmatrix} C(\delta)A(\delta) \\ C(\delta) \end{bmatrix}$, and find the unimodular matrix $\Lambda(\delta)$ over $\mathbb{R}[\delta]$ such that $\Lambda(\delta)\tilde{C}(\delta) = \begin{bmatrix} \bar{C}(\delta) \\ 0 \end{bmatrix}$;

Step 3: After having obtained $\bar{A}(\delta)$ and $\bar{C}(\delta)$, deduce $T(\delta)$ defined in (15) and $F(\delta)$ defined in (16);

Step 4: Deduce A_0 and C_0 defined in (14), $\bar{\Gamma}_1(\delta)$ and $\bar{\Gamma}_2(\delta)$ defined in (11), $\bar{K}_1(\delta)$ and $\bar{K}_2(\delta)$ defined in (20);

Step 5: Design the observer of the form (21) by choosing the matrices L_0, $H(\delta)$ and $J(\delta)$ defined in (22).

Remark 6. *The above procedure involves the computations over polynomial matrices. There exist lots of packages already implemented in Matlab and Maple which enable us to easily realize those calculations by computer.*

5. Illustrative example

Consider the following example:

$$A(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \delta & 1 & 1 & -1 \end{bmatrix}, B(\delta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & \delta \end{bmatrix}$$

$$C(\delta) = \begin{bmatrix} \delta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, D(\delta) = \begin{pmatrix} 1 & \delta \\ 0 & 0 \\ 1 & \delta \end{pmatrix}$$

For the given quadruple $(A(\delta), B(\delta), C(\delta), D(\delta))$, by applying the algorithm (4)-(5), we find that there exist $k^* = 3$ such that $M_{k^*} = M_{k^*+1} = I_4$, thus Assumption 1 is satisfied. Moreover, by calculating the invariant factors we have

$$\begin{aligned} \mathrm{Inv_S}\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \\ B(\delta) & 0 \end{bmatrix} &= \mathrm{Inv_S}\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix} \\ &= \{1, 1, 1\} \end{aligned}$$

therefore Assumption 2 is satisfied as well. According to Theorem 4, there exists a Luenberger-like observer to exponentially estimate the state of the studied system.

Step 1:

In order to transform the matrix $\begin{bmatrix} C(\delta)B(\delta) & D(\delta) \\ D(\delta) & 0 \end{bmatrix}$ into its Hermite form, we can find

$$U(\delta) = \begin{bmatrix} \delta & 0 & -\delta & 1 & 0 & \delta \\ -1 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $V(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \delta \end{bmatrix}$, then we can find

$$\bar{V}(\delta) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & \delta & \delta & 0 & \delta \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

such that $\bar{V}(\delta)V(\delta) = \begin{bmatrix} B(\delta) & 0 \\ D(\delta) & 0 \end{bmatrix}$, which gives us

$$W(\delta) = \left[\begin{array}{c|c} K_1(\delta) & K_2(\delta) \\ \hline \Gamma_1(\delta) & \Gamma_2(\delta) \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \delta-1 & 0 & 1-\delta & 1 & 0 & \delta-1 \\ -\delta & 0 & \delta & 0 & 0 & -\delta \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

Step 2:

With the above deduced $W(\delta)$, we obtain:

$$\bar{A}(\delta) = \begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & \delta & 0 & 0 \\ \delta^2-\delta & \delta^2-1 & -\delta^2+2\delta & -2\delta+2 \\ -\delta^2 & 1-\delta-\delta^2 & 1-\delta+\delta^2 & 2\delta-1 \end{bmatrix}$$

and $\tilde{C}(\delta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$, thus we can choose the unimodular matrix over $\mathbb{R}[\delta]$ as $\Lambda(\delta) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, which gives $\bar{C}(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$.

Step 3:

With the deduced $\bar{A}(\delta)$ and $\bar{C}(\delta)$, we can check that there exists $l^* = 3$ such that $\bar{\mathcal{O}}_3(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & \delta & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -\delta & 1+\delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$ is left unimodular over $\mathbb{R}[\delta]$, with $\left[\bar{\mathcal{O}}_3(\delta)\right]_L^{-1} = \begin{bmatrix} \delta & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \delta & 0 & -1 & 0 \\ \delta^2-1 & 1 & -2\delta & 0 & 1 & 0 \end{bmatrix}$, which gives $[F_3(\delta), F_2(\delta), F_1(\delta)] = \bar{C}(\delta)\bar{A}^3(\delta)\left[\bar{\mathcal{O}}_3(\delta)\right]_L^{-1}$ with

$$F(\delta) = \left[\begin{array}{c} F_1(\delta) \\ \hline F_2(\delta) \\ \hline F_3(\delta) \end{array}\right] = \left[\begin{array}{cc} -2-\delta^2+5\delta & 0 \\ 0 & 0 \\ \hline 1+3\delta-5\delta^2+\delta^3 & 0 \\ 0 & 0 \\ \hline 3-4\delta-\delta^2+\delta^3 & 2\delta-2 \\ 0 & 1 \end{array}\right] \tag{23}$$

Then we obtain the bicausal generalized change of coordinates $z = T(\delta)x$ where

$$T(\delta) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -1 & 2-4\delta+\delta^2 & 0 & 0 \\ -\delta & 0 & 1 & 1 \\ -2+4\delta-\delta^2 & -\delta+\delta^2 & -1 & 0 \\ -\delta & 0 & 1 & 1 \end{bmatrix}$$

with

$$[T(\delta)]_L^{-1} = \begin{bmatrix} 2-4\delta+\delta^2 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ -19\delta^2+15\delta+8\delta^3-4-\delta^4 & 0 & -4\delta+2+\delta^2 & 0 & -1 & 0 \\ 15\delta^2-13\delta-7\delta^3+4+\delta^4 & 0 & 3\delta-2-\delta^2 & 1 & 1 & 0 \end{bmatrix}$$

Step 4:

With the deduced change of coordinates, the studied system can be transformed into the simple observer form (19) with $F(\delta)$ given in (23) and

$$A_0 = \left[\begin{array}{cc|cc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], C_0 = \left[\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{c|c} \bar{K}_1(\delta) & \bar{K}_2(\delta) \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & \delta-1 & 1-\delta & 0 & 1-\delta & -1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{c|c} \bar{\Gamma}_1(\delta) & \bar{\Gamma}_2(\delta) \end{array}\right] = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

For the simulation setting, we can choose

$$G_0 = \begin{bmatrix} 85 & 0 & 2000 & 0 & 12500 & 0 \\ 0 & 85 & 0 & 2000 & 0 & 12500 \end{bmatrix}^T$$

such that $(A_0 - G_0C_0)$ has negative eigenvalues $(-10, -10, -25, -25, -50, -50)$. And finally we obtain the following gain matrices:

$$L_0 = \begin{bmatrix} -85 & 0 & 1 & 0 & 0 & 0 \\ 0 & -85 & 0 & 1 & 0 & 0 \\ -2000 & 0 & 0 & 0 & 1 & 0 \\ 0 & -2000 & 0 & 0 & 0 & 1 \\ -12500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12500 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H(\delta) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & \delta-1 & 1-\delta \\ 0 & 1 & -1 \end{bmatrix}$$

$$J(\delta) = \begin{bmatrix} 83+5\delta-\delta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 2001-5\delta^2+3\delta+\delta^3 & \delta-1 & 1-\delta \\ 0 & 0 & 0 \\ 12503-4\delta-\delta^2+\delta^3 & \delta-1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 5:

With the deduced L_0, $H(\delta)$ and $J(\delta)$, one can easily design the unknown input observer described in (21). In the simulation of the studied system, we set the unknown input as $u_1 = -10\sin 100t$ and $u_2 = 20\sin 20t$ (see Fig. 1). The simulation step is 0.001s, and the basic delay $h = 0.01$s. By choosing the calculated gain matrices L_0, $H(\delta)$ and $J(\delta)$, the observation errors (in log scale) are given in Fig. 2, from which we can notice not only the convergence of the proposed observer, but also the delay effect in the observer which is equal to 0.04s and it is due to the term δ^4 in $[T(\delta)]_L^{-1}$. The singularity in the figure is due to the fact that the observation error passes zero and changes the sign. In order to show that the proposed method is independent of the time-delay involved in the studied system, another simulation was made with the same gains and a bigger delay $h = 0.1s$, whose results (again in log scale) were depicted in Fig. 3. Compared Fig. 2 with Fig. 3, we can conclude that, with two distinct delays, the resulting estimation errors converge to 0 with the same speed, depending on the eigenvalues of $T_L^{-1}(\delta)(A_0 - G_0C_0)T(\delta)$. Moreover, with a bigger delay, the estimations will have a bigger delay as well (in Fig. 3 the delay effect in the observer equals to $0.4s$, corresponding to the term δ^4 in $[T(\delta)]_L^{-1}$).

In order to show the robustness of the proposed observer, the third simulation (the same gains with $h = 0.01s$) was made by adding a mean-zero random disturbance in the output belonging to $[-2, 2]$. The estimation errors are depicted in Fig. 4 and it can be noticed that the estimation error is always bounded.

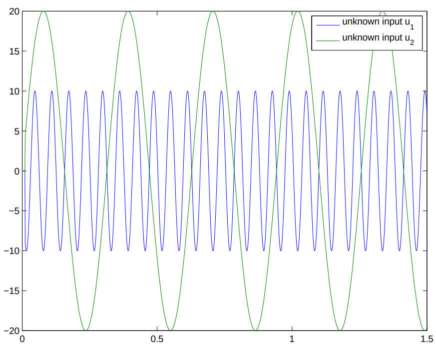

Figure 1: Unknown input u of the studied system.

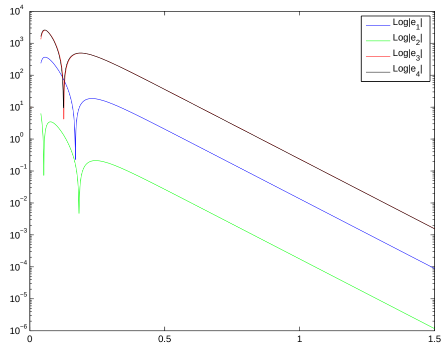

Figure 2: The observation error (in log scale) for $h = 0.01s$.

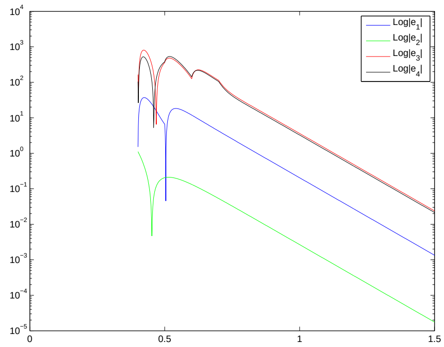

Figure 3: The observation error (in log scale) for $h = 0.1s$.

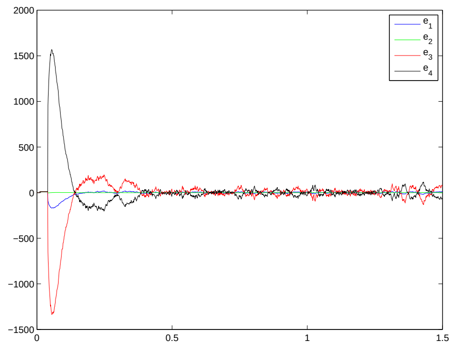

Figure 4: The observation error for $h = 0.01s$ with noisy measurement.

6. Conclusion

The class of linear time-delay systems investigated in this paper is quite larger than that those in the literature since we consider unknown inputs in both the state equation and in the system output. Moreover, commensurate delays are allowed to appear in the state, input, and in the output also. We have matched the backward unknown input observability condition recently obtained in Bejarano and Zheng (2014), with the observability condition required in Hou et al. (2002) for the observer design of linear time-delay systems without inputs. The required conditions for the observer design are considerably relaxed in the sense that they coincide with the necessary and sufficient conditions for the unknown input observer design of linear systems **without delays**.

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