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Abstract. Extending earlier work on supertropical adjoints and applying symmetrization, we provide a symmetrized supertropical version SL_n of the special linear group, which we partition into submonoids, based on "quasi-identity" matrices, and we display maximal sub-semigroups of SL_n. We also study the monoid generated by SL_n. Several illustrative examples are given of unexpected behavior. We describe the action of elementary matrices on SL_n, which enables one to connect different matrices in SL_n, but in a weaker sense than the classical situation.

Introduction

This paper rounds out [11, 12], its main objective being to lay out the foundations of the theory of SL_n in tropical linear algebra. Given any semiring R, one can define the matrix semiring, comprised of matrices $A = (a_{i,j})$ with entries in R, where the addition and multiplication of matrices are induced from R as in the familiar ring-theoretic matrix construction.

Tropical algebra is based on the max-plus algebra, for which negation does not exist and its underlying semiring structure is idempotent. Thus, the classical determinant is no longer available, and one of the challenges of tropical matrix theory has been to introduce a viable analog of the general and special linear groups. In [11, 12, 13, 15] the permanent was used as a variant of the determinant, given as

$$\det(A) = \sum_{\pi \in \mathrm{S}_n} \prod_i a_{i,\pi(i)}.$$

It leads to the adjoint matrix and $A^{\nabla} := \det(A)^{-1} \operatorname{adj}(A)$, and was used to build a theory parallel to the classical theory. On the other hand, Akian, Gaubert, and A Guterman [2] refined this further by distinguishing between the even and odd permutations in defining the determinant.

We rely on these notions to provide a symmetrized version of SL_n. The natural definition of SL_n is to take all matrices with symmetrized determinant $\mathbb{1}$, in which case $A^{\nabla} = \operatorname{adj}(A)$. Unfortunately, SL_n need not be closed under multiplication, but it does yield a monoid under "ghost surpasses," whose subset of nonsingular elements is precisely SL_n. In particular, when the product of two elements of SL_n is nonsingular, then it is in SL_n. Our first goal is to find the smallest natural monoid $\overline{\mathrm{SL}}_n$ which contains SL_n. Towards this end, we turn to Akian, Gaubert, and A Guterman [2], who refined the tropical determinant further by distinguishing between the even and odd permutations in its definition. For $n = 2$, $\overline{\mathrm{SL}}_n$ is the set of matrices symmetrically surpassing $\overline{\mathrm{SL}}_n$, in the sense of Definition 3.3, but for $n \geq 4$, there are matrices in $\overline{\mathrm{SL}}_n$ that are not products of matrices from SL_n, as seen in Corollary 3.8.

On the other hand, SL_n itself has a host of interesting submonoids, given in Example 3.11, especially (iii), built from "strictly normal" matrices (cf. Definition 2.8), which are nonsingular matrices all of whose entries are bounded by $\mathbb{1}$, and which is a maximal nonsingular submonoid, cf. Theorems 3.18 and 3.19. In the process, we obtain some restrictive behavior in Examples 2.26, 2.32, 2.50, 2.51, 2.52, 2.53, 2.59, 5.1, and 5.2, as well as in the proof of Proposition 3.10.

But fortunately there are also positive results. Perhaps the most interesting monoids arise via Lemma 4.4, which for any $A \in \mathrm{SL}_n$ defines a sub-semigroup of $\overline{\mathrm{SL}}_n$ with left unit element $\mathcal{I}_A^{\ell} := A \operatorname{adj}(A)$,

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which contains $\mathcal{I}_A^{\ell} A$. This reflects the important role of quasi-identities $\mathcal{I}_A^{\ell}$ in [12], and "almost" partitions SL_n naturally into submonoids.

Next we consider the natural conjugation $B \mapsto A^{\nabla} BA$ in §5. Although some basic properties expected for conjugation fail in this setting, they do hold when A is "strictly normal".

In the last section we bring in the role of elementary matrices, which is rather subtle. In Lemma 6.2, we see that a Gaussian transformation can turn a nonsingular matrix into a singular matrix; then we show in Theorem 6.4 that although not every matrix in SL_n is itself a product of elementary matrices, all matrices in SL_n are equivalent with respect to multiplication by elementary matrices.

1. Supertropical structures

1.1. Supertropical semirings and semifields.

We review some basic notions from [10].

Definition 1.1. *A **supertropical semiring** is a quadruple $R := (R, \mathcal{T}, \mathcal{G}, \nu)$ where R is a semiring, $\mathcal{T} \subset R$ is a multiplicative submonoid, and $\mathcal{G} \cup \{\mathbb{0}\} \subset R$ is an ordered semiring ideal, together with a map $\nu : R \to \mathcal{G} \cup \{\mathbb{0}\}$, satisfying $\nu^2 = \nu$ as well as the conditions:*

$$\begin{array}{lll} a + b = a & \text{whenever} & \nu(a) > \nu(b), \\ a + b = \nu(a) & \text{whenever} & \nu(a) = \nu(b). \end{array}$$

Note that R contains the "absorbing" element $\mathbb{0}$, satisfying $a + \mathbb{0} = a$ and $a\mathbb{0} = \mathbb{0}a = \mathbb{0}$ for all $a \in R$. The tropical theory works for $R \setminus \{\mathbb{0}\}$, but it is convenient to assume the existence of $\mathbb{0}$ when working with matrices.

We write a^{ν} for $\nu(a)$; $a \cong_{\nu} b$ stands for $a^{\nu} = b^{\nu}$. We define the ν-order on R by

$$\begin{array}{lll} a \geq_{\nu} b & \Leftrightarrow & a^{\nu} \geq b^{\nu}, \\ a >_{\nu} b & \Leftrightarrow & a^{\nu} > b^{\nu}, \end{array}$$

The **ghost surpassing relation** on R is given by defining

$$a \underset{\text{gs}}{\models} b \quad \text{if} \quad a = b + g \text{ for some } g \in \mathcal{G}_{\mathbb{0}}.$$

(When b is tangible, $b \underset{\text{gs}}{\models} a$ collapses to the standard equality $b = a$.)

The monoid $\mathcal{T}$ is called the monoid of **tangible elements**, while the elements of $\mathcal{G}$ are called **ghost elements**, and $\nu : R \to \mathcal{G} \cup \{\mathbb{0}\}$ is called the **ghost map**. Intuitively, the tangible elements correspond to the original max-plus algebra, although now $a + a = a^{\nu}$ instead of $a + a = a$. We denote the multiplicative unit of R by $\mathbb{1}$ and its zero element by $\mathbb{0}$. We write $\mathcal{T}_{\mathbb{0}}$ for $\mathcal{T} \cup \{\mathbb{0}\}$ and $\mathcal{G}_{\mathbb{0}}$ for $\mathcal{G} \cup \{\mathbb{0}\}$, where the zero element is considered both as tangible and ghost.

Definition 1.2. *A supertropical semiring R is a **supertropical domain** when the multiplicative monoid $(R \setminus \{\mathbb{0}\}, \cdot)$ is commutative and cancellative with respect to $\mathcal{T}$, $R \setminus \{\mathbb{0}\} = \mathcal{T} \cup \mathcal{G}$, and the restriction $\nu|_{\mathcal{T}} : \mathcal{T} \to \mathcal{G}$ is onto. If $\mathcal{T}$ (and thus also $\mathcal{G}$) is an Abelian group, we call R a **supertropical semifield**.*

*R is **dense** if whenever $a >_{\nu} b$ there is $c \in R$ for which $a >_{\nu} c >_{\nu} b$.*

For example, $\mathbb{Q} \cup \mathbb{Q}^{\nu}$ is a dense supertropical semifield. For each a in a supertropical domain R we choose an element $\widehat{a} \in \mathcal{T}$ such that $\widehat{a}^{\nu} = a^{\nu}$. (Thus $a \mapsto \widehat{a}$ defines a section from $\mathcal{G}$ to $\mathcal{T}$, which we call the **tangible lift**.)

Example 1.3. *Our main supertropical example is the **extended tropical semiring** (cf. [5]) that is,*

$$R = \mathbb{R} \cup \{-\infty\} \cup \mathbb{R}^{\nu},$$

*with $\mathcal{T} = \mathbb{R}$, $\mathcal{G} = \mathbb{R}^{\nu}$, where the restriction of the ghost map $\nu|_{\mathcal{T}} : \mathbb{R} \to \mathbb{R}^{\nu}$ is the identity map, whose addition and multiplication are induced by maximum and standard summation of the real numbers [5]. This supertropical semifield extends the familiar max-plus semifield [1], and serves in our examples in **logarithmic notation** (in particular $\mathbb{1} = 0$ and $\mathbb{0} = -\infty$).*

2. Matrices

2.1. Invertible matrices.

Definition 2.1. *We define three special types of matrices:*

(i) *A* ***permutation matrix*** *(corresponding to the permutation $\pi \in S_n$) is the matrix*
$$P_\pi = (p_{i,j}) \in \mathrm{Mat}_n(\mathcal{T}_{\mathbb{0}})$$
such that
$$p_{i,j} = \begin{cases} \mathbb{0}, & j \neq \pi(i), \\ \mathbb{1}, & j = \pi(i). \end{cases}$$

(ii) *A* ***diagonal matrix*** *is a matrix $D = (d_{i,j})$ in which $d_{i,j} = \mathbb{0}$ for all $i \neq j$.*

(iii) *A* ***generalized permutation matrix*** *$Q_\pi = (q_{i,j})$ is the product of a permutation matrix P_π and an invertible diagonal matrix D; i.e., $q_{i,j} \in \mathcal{T}$ if $j = \pi(i)$, and otherwise $q_{i,j} = \mathbb{0}$.*

We denote by $S^\times$ the subset of invertible matrices in a set S.

Remark 2.2.

(i) *Every permutation matrix is invertible, since $P_{\pi^{-1}} = (P_\pi)^{-1}$.*

(ii) *A diagonal matrix D is invertible if and only if $\det(D)$ is invertible, which is the case iff each of the diagonal entries is in $\mathcal{T}$.*

2.2. **Supertropical matrices.** In this paper we fix a supertropical semifield F, and work exclusively in the monoid $\mathrm{Mat}_n(F)$ of all $n \times n$ matrices over F. We consider it as a multiplicative monoid, whose matrix multiplication is induced from F. Its unit element is the **identity matrix** I with $\mathbb{1}$ on the main diagonal and whose off-diagonal entries are $\mathbb{0}$. We say that a matrix is **tangible** if its entries are all in $\mathcal{T}_{\mathbb{0}}$, and **ghost** if its entries are all in $\mathcal{G}_{\mathbb{0}}$. We write $\mathrm{Mat}(\mathcal{T}_{\mathbb{0}})$ for the set of all tangible matrices, and $\mathrm{Mat}_n(\mathcal{G}_{\mathbb{0}})$ for the monoid of all ghost matrices.

The **tropical determinant** of a matrix $A = (a_{i,j})$ is defined as the permanent:
$$\det(A) = \sum_{\pi \in \mathrm{S}_n} \prod_i a_{i,\pi(i)},$$
where S_n is the set of permutations of $\{1, \dots, n\}$.

Given matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ in $\mathrm{Mat}_n(F)$, we write $B \geq_\nu A$ if $b_{i,j} \geq_\nu a_{i,j}$ for all i, j, and $B \cong_\nu A$ if $B \geq_\nu A$ and $B \leq_\nu A$. The ghost surpassing relation extends naturally to matrices, defined as $A \underset{\mathrm{gs}}{\models} B$ if $A = B + G$ for some ghost matrix $G \in \mathrm{Mat}_n(\mathcal{G}_{\mathbb{0}})$. (When A is tangible, $\underset{\mathrm{gs}}{\models}$ collapses to the standard equality $A = B$.)

Lemma 2.3. *If $A_1 \underset{\mathrm{gs}}{\models} A_2$ and $B_1 \underset{\mathrm{gs}}{\models} B_2$, then $A_1 + B_1 \underset{\mathrm{gs}}{\models} A_2 + B_2$ and $A_1 B_1 \underset{\mathrm{gs}}{\models} A_2 B_2$. In particular, $AB \underset{\mathrm{gs}}{\models} A$ and $BA \underset{\mathrm{gs}}{\models} A$ if $B \underset{\mathrm{gs}}{\models} I$.*

Proof. Check the components in the multiplication. □

2.2.1. *Supertropical singularity.* Invertibility of matrices (in its classical sense) is limited in the (super)tropical setting.

Remark 2.4. *The only invertible matrices over a supertropical domain are the generalized permutation matrices, a venerable result going back to* [18]*; the supertropical version is in* [11, Proposition 3.9].

In view of Remark 2.2, limiting nonsingularity to invertible matrices is too restrictive for a viable matrix theory, and leads to following definition.

Definition 2.5. *We define a matrix $A \in \mathrm{Mat}_n(F)$ to be* ***(supertropically) nonsingular*** *if $\det(A) \in \mathcal{T}$; otherwise A is* ***(supertropically) singular*** *(in which case $\det(A) \in \mathcal{G}_{\mathbb{0}}$).*

Consequently, a matrix $A \in \mathrm{Mat}_n(F)$ is singular if $\det(A) \underset{\mathrm{gs}}{\models} \mathbb{0}$. This definition does not match the semigroup notion of regularity.

2.3. Dominant permutations.

Definition 2.6. *A permutation $\pi \in S_n$ is* ***dominant*** *for A if*

$$\det(A) \cong_\nu a_{1,\pi(1)} a_{2,\pi(2)} \cdots a_{n,\pi(n)}.$$

A dominant permutation π is ***strictly dominant*** *if $\prod_i a_{i,\pi(i)} >_\nu \prod_i a_{i,\sigma(i)}$ for any $\sigma \neq \pi$ in S_n. A strictly dominant permutation $\pi \in S_n$ is* ***uniformly dominant*** *if $a_{1,\pi(1)} = \cdots = a_{n,\pi(n)}$ and $a_{i,j} <_\nu a_{i,\pi(i)}$ for all $j \neq \pi(i)$.*

Clearly the matrix A is nonsingular if and only if it has a strictly dominant permutation, all of whose corresponding entries are tangible.

Example 2.7. *The permutation π is uniformly dominant for the permutation matrix P_π.*

We specify some useful classes of matrices, to be used in the present paper.

Definition 2.8. *A matrix is* ***definite*** *if its dominant permutation is the identity with $a_{i,i} = \mathbb{1}$ for all i. A definite matrix with $a_{i,j} \leq_\nu \mathbb{1}$ (resp. $a_{i,j} < \mathbb{1}$) for every $i \neq j$ is called a* ***normal*** *(resp.* ***strictly normal****) matrix.*

Accordingly a strictly normal matrix is always nonsingular, while a normal matrix (and thus also a definite matrix) can be singular. However, for any of these matrices we have $\det(A) \cong_\nu \mathbb{1}$.

Lemma 2.9. *If the permutations π_t are uniformly dominant for matrices A_t for $1 \leq t \leq \ell$, then $\pi := \pi_\ell \circ \cdots \circ \pi_1$ is uniformly dominant for $A = A_1 \cdots A_\ell$, and $\det(A) = \prod_{t=1}^{\ell} \det(A_t)$.*

Proof. If $\alpha_t = a_{i,\pi_t(i)}$ is the entry of A_t for its uniformly dominant permutation π_t (for $1 \leq i \leq n$), then $\det(A_t) = \alpha_t^n$. On the other hand, the entries contributing to $\det(A)$ are all of the form

$$a_{i,\pi_1(i)} a_{\pi_1(i),\pi_2\pi_1(i)} a_{\pi_2\pi_1(i),\pi_3\pi_2\pi_1(i)} \cdots = \alpha_1 \cdots \alpha_\ell,$$

since all other entries are clearly less. Hence $\pi := \pi_\ell \circ \cdots \circ \pi_1$ is uniformly dominant for A, and $\det(A) = \alpha_1^n \cdots \alpha_\ell^n$. □

Trying to weaken the hypothesis brings one into confrontation with Proposition 3.10 below.

Lemma 2.10. *$BA \geq_\nu A$ and $AB \geq_\nu A$, for any matrix A and any matrix $B \geq_\nu I$.*

Proof. Let $A = (a_{i,j})$, $B = (b_{i,j})$, and $BA = (c_{i,j})$. Then

$$c_{i,j} = \sum_k b_{i,k} a_{k,j} \geq_\nu b_{i,i} a_{i,j} \geq_\nu \mathbb{1} a_{i,j} = a_{i,j}.$$

□

Lemma 2.11. *For any matrix A and any normal matrix B, if $B \geq_\nu BA$, then $B \geq_\nu A$.*

Proof. Write $BA = (c_{i,j})$. Then, since $b_{i,i} = \mathbb{1}$ for every i, the hypothesis yields

$$b_{i,j} \geq_\nu c_{i,j} = \sum_k b_{i,k} a_{k,j} \geq_\nu b_{i,i} a_{i,j} = a_{i,j}.$$

□

A surprising phenomenon is that the product of two nonsingular matrices might be singular (cf. [5, Remark 2.4] or [11, Example 6.11]), but we do have:

Theorem 2.12 ([11, Theorem 3.5])**.** *Given matrices $A, B \in \mathrm{Mat}_n(F)$, then*

$$\det(AB) \underset{\text{gs}}{\models} \det(A)\det(B),$$

and $\det(AB) = \det(A)\det(B)$ whenever AB is nonsingular.

This result also is proved in [3, Proposition 2.1.7] by means of transfer principles (see [2, Theorems 3.3 and 3.4]), and likewise is sharpened in [2, Corollary 4.18], as to be discussed below.

Corollary 2.13. $\det(AB) = \det(A)\det(B) = \det(BA)$ *whenever B is a generalized permutation matrix (i.e., B is invertible).*

Proof. $\det(AB) \underset{\mathrm{gs}}{\models} \det(A)\det(B)$, and

$$\det(A) = \det(ABB^{-1}) \underset{\mathrm{gs}}{\models} \det(AB)\det(B^{-1}) = \det(AB)\det(B)^{-1}$$

since B is a generalized permutation matrix. Hence

$$\det(A)\det(B) \underset{\mathrm{gs}}{\models} \det(AB),$$

and thus $\det(AB) = \det(A)\det(B)$. The proof that $\det(BA) = \det(A)\det(B)$ is analogous. $\square$

2.4. **Symmetrization.** Following [2, Example 4.11], we define the **symmetrized semiring** $\widehat{R}$, defined to have the same module structure as $R \times R$, but with multiplication

$$(a_1, a_2)(a_1', a_2') = (a_1a_1' + a_2a_2', a_1a_2' + a_2a_1')$$

(motivated by viewing the second component to be the negative of the first component). Define

$$R^\circ = \{(a_1, a_2) \in R : a_1 \cong_\nu a_2\},$$

easily seen to be an ideal of $\widehat{R}$. The following relation is reminiscent of $\underset{\mathrm{gs}}{\models}$.

Definition 2.14. $(a_1, a_2) \succeq_\circ (b_1, b_2)$ *in $\widehat{R}$ if there are $c_i \in R$ with $c_1 \cong_\nu c_2$, such that $a_i = b_i + c_i$ for $i = 1, 2$.*

In other words, $(a_1, a_2) = (b_1, b_2) + (c_1, c_2)$ where $(c_1, c_2) \in R^\circ$. (This is a bit weaker than the definition given in [2], since we want to permit $(a^\nu, a) \succeq_\circ (a, a)$, but the arguments are the same.)

Lemma 2.15. *When R is a dense supertropical domain $(a_1, a_2) \succeq_\circ (b_1, b_2)$ for $a_1, a_2 \in \mathcal{T}$, iff one of the following possibilities arises:*

(i) $a_1 \cong_\nu a_2$ *with* $a_i \geq_\nu b_i$ *for* $i = 1, 2$;
(ii) $a_1 = b_1$ *and* $a_1 >_\nu a_2$;
(iii) $a_2 = b_2$ *and* $a_2 >_\nu a_1$;
(iv) $a_1 = b_1$ *and* $a_2 = b_2$.

Proof. If $c_i \geq_\nu b_i$ then $a_1 \cong_\nu c_1 \cong_\nu c_2 \cong_\nu a_2$, yielding (i). Thus we may assume that $c_i <_\nu b_i$ for some i, say $i = 1$. Now $a_1 = b_1$. If $c_2 \geq_\nu b_2$, then $a_2 = b_2 \leq_\nu c_1 <_\nu a_1$, yielding (ii). If $c_2 <_\nu b_2$, then $a_2 = b_2$, yielding (iv). We get (iii) symmetrically.

Conversely, if (i) or (iv) hold then clearly $(a_1, a_2) \succeq_\circ (b_1, b_2)$ (taking $c_i = a_i$ in (i) and $c_i = \mathbb{0}$ in (iv).) If say (ii) holds then take $a_1 >_\nu c >_\nu b_1$. $\square$

Lemma 2.16. *If $(a_1, a_2) \succeq_\circ (b_1, b_2)$, then $a_1 + a_2 \underset{\mathrm{gs}}{\models} b_1 + b_2$.*

Proof. Suppose $a_1 \geq_\nu a_2$. If $a_1 \cong_\nu a_2$, then $a_1 + a_2 \in \mathcal{G}$ and the assertion is clear. Thus, we may assume that $a_1 >_\nu a_2$, and again the assertion is clear unless $a_1 \in \mathcal{T}$. But then the same argument as in Lemma 2.15 shows that $a_1 = b_1$. Now $a_1 + a_2 = a_1 = b_1 + b_2$. $\square$

2.4.1. *The symmetrized determinant.* Accordingly, one defines

$$\det{}^+(A) = \sum_{\pi \in \mathrm{S}_n,\ \mathrm{sgn}(\pi) = +1} \prod_i a_{i,\pi(i)},$$

$$\det{}^-(A) = \sum_{\pi \in \mathrm{S}_n,\ \mathrm{sgn}(\pi) = -1} \prod_i a_{i,\pi(i)},$$

and the **symmetrized determinant**

$$\mathrm{bidet}(A) = (\det{}^+(A),\ \det{}^-(A)).$$

Note that $\mathrm{bidet}(A) = \det^+(A) + \det^-(A)$.

Using that they call the **weak transfer principle**, Akian, Gaubert, and Guterman proved:

Theorem 2.17 ([2, Corollary 4.18])**.** $\mathrm{bidet}(AB) \succeq_\circ \mathrm{bidet}(A)\,\mathrm{bidet}(B)$.

Interpreted in terms of Lemma 2.16, this result yields Theorem 2.12, and leads us to a more refined definition of nonsingular matrix.

2.4.2. *Symmetric singularity.*

Definition 2.18. *A matrix in $M_n(R)$ is* ***symmetrically singular*** *if its symmetrized determinant* $\mathrm{bidet}(A)$ *is in* R°.

Lemma 2.19. *Every symmetrically singular matrix is singular.*

Proof. $\mathrm{bidet}(A) \in R^\circ$ implies $\det(A) \in \mathcal{G}$. □

But a singular matrix A can be symmetrically nonsingular. In the 3×3 case, A can be tangible.

Example 2.20. *The singular matrix*

$$A = \begin{pmatrix} \mathbb{1} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{1} \\ \mathbb{1} & \mathbb{0} & \mathbb{1} \end{pmatrix}$$

has $\mathrm{bidet}(A) = (\mathbb{1}^\nu, \mathbb{0}) \notin R^\circ$.

2.5. **The adjoint matrix.** As in classical theory of matrices over a field, the adjoint matrix has a major role in supertropical matrix algebra in [11].

Definition 2.21. *The (i,j)-minor $A_{i,j}$ of a matrix $A = (a_{i,j})$ is obtained by deleting the i row and the j column of A. The* ***adjoint matrix*** $\mathrm{adj}(A)$ *of A is defined as $(a'_{i,j})$, where* $a'_{i,j} = \det(A_{j,i})$.

Proposition 2.22 ([11, Proposition 4.8])**.** $\mathrm{adj}(AB) \underset{\mathrm{gs}}{\models} \mathrm{adj}(B)\,\mathrm{adj}(A)$ *for any* $A, B \in \mathrm{Mat}_n(F)$.

One need not have equality, as indicated in [11, Example 4.7]. This leads us to examine under what situation equality does hold. By Proposition 2.22, $\mathrm{adj}(AB) = \mathrm{adj}(B)\,\mathrm{adj}(A)$ when each entry of $\mathrm{adj}(AB)$ is tangible, but this condition is rather restrictive.

Lemma 2.23.

(i) $\mathrm{adj}(P_\pi) = P_\pi^{-1} = P_{\pi^{-1}}$ *for any permutation matrix* P_π.

(ii) [16, Lemma 5.7] $\mathrm{adj}(AB) = \mathrm{adj}(B)\,\mathrm{adj}(A)$ *and* $\mathrm{adj}(BA) = \mathrm{adj}(A)\,\mathrm{adj}(B)$ *for any generalized permutation matrix B.*

Proof. The proof of (i) is just as in the classical case. □

Remark 2.24. *Assume $A = PA'$ and $B = B'Q$, where A' and B' are definite matrices and P and Q are generalized permutation matrices. Then, by Lemma 2.23,*

$$\mathrm{adj}(AB) = \mathrm{adj}(PA'B'Q) = \mathrm{adj}(Q)\,\mathrm{adj}(A'B')\,\mathrm{adj}(P),$$

so to check that $\mathrm{adj}(AB) = \mathrm{adj}(B)\,\mathrm{adj}(A)$*, it suffices to verify the case of definite matrices A, B.*

Proposition 2.25 ([11, Example 4.7])**.** $\mathrm{adj}(AB) = \mathrm{adj}(B)\,\mathrm{adj}(A)$ *for any 2×2 matrices.*

(Note, in view of Remark 2.24, we could assume in the proof that A and B are strictly normal.) The key in general is to consider the case when AB is nonsingular.

This result leads us to the question as to whether AB nonsingular implies BA is nonsingular. But this fails, even when $B = A^{\mathrm{t}}$:

Example 2.26. *(Inspired by an idea of Guy Blachar.)*

Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$, given in logarithmic notation. Then $A^{\mathrm{t}} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$ and $AA^{\mathrm{t}} = \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}$ which is nonsingular of determinant 10*, whereas $A^{\mathrm{t}}A = \begin{pmatrix} 4 & 6 \\ 6 & 8 \end{pmatrix}$ is singular (even symmetrically singular) of determinant 12^ν.*

We recall another result from [11]:

Theorem 2.27 ([11, Theorem 4.12.]). *$(A\,\mathrm{adj}(A))^2 = \det(A)A\,\mathrm{adj}(A)$ for every $A \in \mathrm{Mat}_n(F)$.*

In particular $A\,\mathrm{adj}(A)$ is idempotent only when $\det(A) \in \{\mathbb{0}, \mathbb{1}, \mathbb{1}^\nu\}$.
Modifying the proof of Theorem 2.27, we also obtain:

Theorem 2.28. *$(A\,\mathrm{adj}(A))^2 = \widehat{\det(A)}A\,\mathrm{adj}(A)$ for every $A \in \mathrm{Mat}_n(F)$.*

Proof. Adapting the proof of [11, Theorem 4.12], we only need to deal with the case when the diagonal (i,i)-entry of $(A\,\mathrm{adj}(A))^2$ is tangible while the (i,i)-entry of $A\,\mathrm{adj}(A)$ is ghost. But this is impossible as all off-diagonal entries of $A\,\mathrm{adj}(A)$ are ghosts, so squaring $A\,\mathrm{adj}(A)$ can not make the (i,i)-entry tangible. □

2.6. **Quasi-identity matrices and the ∇-operation.** Since so few matrices are invertible, we need to replace the identity matrix by a more general notion.

Definition 2.29. *A matrix E is **(multiplicatively) idempotent** if $E^2 = E$. A **quasi-identity matrix** is a nonsingular idempotent matrix.*

We define the set of all quasi-identity matrices

$$\mathrm{QI}_n(F) := \{\mathcal{I} \text{ is a quasi-identity matrix}\} \subset \mathrm{Mat}_n(F),$$

each simulating the role of the identity matrix.

Remark 2.30. *The fact that a quasi-identity matrix $\mathcal{I}$ is idempotent implies that its off-diagonal entries are in $\mathcal{G}_{\mathbb{0}}$, and its diagonal entries are all $\mathbb{1}$.*

Remark 2.31 ([11, Proposition 4.17]). *$\mathrm{adj}(\mathcal{I}) = \mathcal{I}$ for every quasi-identity matrix $\mathcal{I}$.*

On the other hand, $\mathrm{QI}_n(F)$ is not a monoid.

Example 2.32.
Let $\mathcal{I}_1 = \begin{pmatrix} \mathbb{1} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & b^\nu \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix}$, $\mathcal{I}_2 = \begin{pmatrix} \mathbb{1} & a^\nu & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix}$ with $a^\nu, b^\nu \neq \mathbb{0}$. These matrices are quasi-identities, but $\mathcal{I}_1\mathcal{I}_2 = \begin{pmatrix} \mathbb{1} & a^\nu & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & b^\nu \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix}$ is not idempotent, even though it is nonsingular.

Definition 2.33. *A matrix A is **quasi-invertible** (with respect to quasi-invertible matrices $\mathcal{I}$ and $\mathcal{I}'$) if there are matrices B and B' such that $BA = \mathcal{I}$ and $AB' = \mathcal{I}'$.*

(It may well turn out that $B \neq B'$.) Our next task is to find a "quasi-inverse." As in classical theory, the following matrices have an important role in supertropical matrix algebra.

Definition 2.34. *When $\det(A) \neq \mathbb{0}$, we define the matrix*

$$A^\nabla := \frac{\mathbb{1}}{\widehat{\det(A)}}\mathrm{adj}(A).$$

A^∇ is nonsingular if and only if A is nonsingular by [11, Theorem 4.9].

Lemma 2.35 ([12, Lemma 2.17]). *$\det(A)\,\mathrm{adj}(A) \geq_\nu \mathrm{adj}(A)A\,\mathrm{adj}(A)$ for any matrix $A \in \mathrm{Mat}_n(F)$, and thus $A^\nabla \geq_\nu A^\nabla A A^\nabla$ when $\det(A) \neq \mathbb{0}$.*

Remark 2.36 ([15, Remark 2.18]). *A^∇ is definite (resp. strictly normal) whenever the matrix A is definite (resp. strictly normal), since the identity remains the unique dominant permutation.*

Proposition 2.37 ([12, Proposition 4.17]). *$\mathcal{I}^\nabla = \mathcal{I}$ for every quasi-identity matrix $\mathcal{I}$.*

We write $A^{\nabla\nabla}$ for $\left(A^\nabla\right)^\nabla$.

Remark 2.38 ([12, Remark 4.2])**.** *Although $A^{\nabla\nabla} \neq A$ in general, by* [11, Example 4.16]*, one does get $A^{\nabla\nabla} \underset{\rm gs}{\models} A$, by* [15, Theorem 3.5]*.*

Lemma 2.39. *Any generalized permutation matrix Q satisfies $Q^{\nabla} = Q^{-1}$, and hence*

$$Q^{\nabla\nabla} = \left(Q^{-1}\right)^{\nabla} = \left(Q^{-1}\right)^{-1} = Q.$$

Proof. By Lemma 2.23(i). □

Definition 2.40. *For any $A \in \mathrm{Mat}_n(F)$ with $\det(A) \neq \mathbb{0}$, we define*

$$\mathcal{I}_A^{\ell} = AA^{\nabla}, \qquad \mathcal{I}_A^{r} = A^{\nabla}A.$$

Then $\mathcal{I}_A^{\ell}$ is idempotent by Theorem 2.27. Furthermore, [12, Remark 2.21] shows that $\det(\mathcal{I}_A^{\ell}) \cong_\nu \mathbb{1}$ and $\det(\mathcal{I}_A^{r}) \cong_\nu \mathbb{1}$.

The following fact is crucial.

Theorem 2.41 ([11, Theorem 4.3])**.** *$\mathcal{I}_A^{\ell}$ and $\mathcal{I}_A^{r}$ are quasi-identities, for any nonsingular matrix A.*

Lemma 2.42. *If A is singular, then $\mathcal{I}_A^{\ell}$ and $\mathcal{I}_A^{r}$ are both ghost matrices, i.e., $\mathcal{I}_A^{\ell}, \mathcal{I}_A^{r} \in \mathrm{Mat}_n(\mathcal{G}_{\mathbb{0}})$.*

Proof. Each diagonal entry of $\mathcal{I}_A^{\ell}$ (and $\mathcal{I}_A^{r}$) is $\mathbb{1}^{\nu}$, and all their off-diagonal entries are ghost. □

Although we need not have $\mathcal{I}_A^{\ell} = \mathcal{I}_A^{r}$ in general, it does hold in the following situation:

Lemma 2.43. *$\mathcal{I}_A^{\ell} = \mathcal{I}_A^{r}$ for any definite nonsingular matrix A.*

Proof. By Lemma 2.10, we have

$$\mathcal{I}_A^{\ell} = (AA^{\nabla})^2 = AA^{\nabla}AA^{\nabla} \geq_\nu AA^{\nabla}A \geq_\nu A^{\nabla}A = \mathcal{I}_A^{r},$$

and, by symmetry, $\mathcal{I}_A^{\ell} \geq_\nu \mathcal{I}_A^{r}$, implying that $\mathcal{I}_A^{\ell} \cong_\nu \mathcal{I}_A^{r}$. The off-diagonal entries of $\mathcal{I}_A^{\ell}$ and $\mathcal{I}_A^{r}$ are the same (since they are all ghosts), whereas the diagonal entries are $\mathbb{1}$; thus $\mathcal{I}_A^{\ell} = \mathcal{I}_A^{r}$. □

In general, we have:

Proposition 2.44 ([12, Corollary 4.7])**.** *$\mathcal{I}_A^{\ell} = \mathcal{I}_{A^{\nabla}}^{r}$.*

In this way, A^{∇} is a right quasi-inverse with respect to $\mathcal{I}_A^{\ell}$, and a left quasi-inverse with respect to $\mathcal{I}_A^{r}$. In order to cope with the general situation, we introduce a weaker condition.

Definition 2.45. *For any A with $\det(A) \neq \mathbb{0}$, we define*

$$\mathcal{I}_A = \mathcal{I}_A^{\ell}\mathcal{I}_A^{r}\mathcal{I}_A^{\ell}, \qquad \widetilde{\mathcal{I}_A} = \mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{r}.$$

*We say that A is **reversible** if $\mathcal{I}_A = \widetilde{\mathcal{I}_A}$.*

$\mathcal{I}_A$ need not be a quasi-identity, although it is under reversibility:

Lemma 2.46. *If A is reversible, then $\mathcal{I}_A$ is idempotent.*

Proof. $\mathcal{I}_A^2 = \mathcal{I}_A^{\ell}\mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{r}\mathcal{I}_A^{\ell} = (\mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{r})\mathcal{I}_A^{r}\mathcal{I}_A^{\ell} = (\mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{r})\mathcal{I}_A^{\ell} = \mathcal{I}_A^{\ell}\mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{\ell} = \mathcal{I}_A$. □

Proposition 2.47. *If A is reversible and $\mathcal{I}_A$ is nonsingular, then $\mathcal{I}_A$ is a quasi-identity with respect to which (on each side) A is quasi-invertible.*

Proof. The determinant of $\mathcal{I}_A$ is $\mathbb{1}^3 = \mathbb{1}$, so its leading summand must come from the diagonal, with every entry $\mathbb{1}$. Finally, $\mathcal{I}_A = A(A^{\nabla}\mathcal{I}_A^{r}\mathcal{I}_A^{\ell})$ and $\mathcal{I}_A = \widetilde{\mathcal{I}_A} = (\mathcal{I}_A^{r}\mathcal{I}_A^{\ell}A^{\nabla})A$. □

Lemma 2.48. *$\widetilde{\mathcal{I}_A} = \mathcal{I}_{A^{\nabla}}$.*

Proof. $\mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{r} = \mathcal{I}_{A^{\nabla}}^{\ell}\mathcal{I}_{A^{\nabla}}^{r}\mathcal{I}_{A^{\nabla}}^{\ell} = \mathcal{I}_{A^{\nabla}}$. □

In particular, any quasi-identity is reversible, but we aim for a stronger result.

Proposition 2.49. *If $\mathcal{I}_A^{\ell}\mathcal{I}_A^{r} = \mathcal{I}_A^{r}\mathcal{I}_A^{\ell}$, then A is reversible, and $\mathcal{I}_A = \mathcal{I}_A^{\ell}\mathcal{I}_A^{r}$.*

Proof. $\mathcal{I}_A^{\ell}\mathcal{I}_A^{r}\mathcal{I}_A^{\ell} = \mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{\ell} = \mathcal{I}_A^{r}\mathcal{I}_A^{\ell} = \mathcal{I}_A^{\ell}\mathcal{I}_A^{r} = \mathcal{I}_A^{\ell}\mathcal{I}_A^{r}\mathcal{I}_A^{r} = \mathcal{I}_A^{r}\mathcal{I}_A^{\ell}\mathcal{I}_A^{r}$. □

Thus we see that in this situation we could replace $\mathcal{I}_A^\ell$ and $\mathcal{I}_A^r$ by the common quasi-identity $\mathcal{I}_A$ in developing the theory. When $\mathcal{I}_A^\ell = \mathcal{I}_A^r$ then $\mathcal{I}_A = \mathcal{I}_A^\ell$ and everything simplifies. But here are some examples showing the complexity of the situation in general.

Example 2.50.
For the matrix $A = \begin{pmatrix} a & \mathbb{1} \\ \mathbb{1} & b \end{pmatrix}$, *we have* $\operatorname{adj}(A) = \begin{pmatrix} b & \mathbb{1} \\ \mathbb{1} & a \end{pmatrix}$ *and*

$$A\operatorname{adj}(A) = \begin{pmatrix} \mathbb{1}+ab & a^\nu \\ b^\nu & \mathbb{1}+ab \end{pmatrix}, \qquad \operatorname{adj}(A)A = \begin{pmatrix} \mathbb{1}+ab & b^\nu \\ a^\nu & \mathbb{1}+ab \end{pmatrix}.$$

In particular, if $ab <_\nu \mathbb{1}$, *we have* $\operatorname{adj}(A) = A^\nabla$ *and*

$$\mathcal{I}_A^\ell = \begin{pmatrix} \mathbb{1} & a^\nu \\ b^\nu & \mathbb{1} \end{pmatrix}, \qquad \mathcal{I}_A^r = \begin{pmatrix} \mathbb{1} & b^\nu \\ a^\nu & \mathbb{1} \end{pmatrix}$$

are distinct.

Note that

$$\mathcal{I}_A^\ell\mathcal{I}_A^r = \begin{pmatrix} (a^\nu)^2+\mathbb{1} & a^\nu+b^\nu \\ a^\nu+b^\nu & (b^\nu)^2+\mathbb{1} \end{pmatrix}, \qquad \mathcal{I}_A^r\mathcal{I}_A^\ell = \begin{pmatrix} (b^\nu)^2+\mathbb{1} & a^\nu+b^\nu \\ a^\nu+b^\nu & (a^\nu)^2+\mathbb{1} \end{pmatrix} = (\mathcal{I}_A^\ell\mathcal{I}_A^r)^\nabla.$$

When $a^\nu, b^\nu <_\nu \mathbb{1}$, *then*

$$\mathcal{I}_A^\ell\mathcal{I}_A^r = \mathcal{I}_A^r\mathcal{I}_A^\ell = \begin{pmatrix} \mathbb{1} & a^\nu+b^\nu \\ a^\nu+b^\nu & \mathbb{1} \end{pmatrix} = \mathcal{I}_A^\ell\mathcal{I}_A^r\mathcal{I}_A^\ell = \mathcal{I}_A^r\mathcal{I}_A^\ell\mathcal{I}_A^r.$$

Likewise,

$$\mathcal{I}_A^\ell\mathcal{I}_A^r\mathcal{I}_A^\ell = \begin{pmatrix} (a^\nu)^2+b^\nu+\mathbb{1} & (a^3)^\nu+a^\nu+b^\nu \\ (b^3)^\nu+a^\nu+b^\nu & a^\nu+(b^\nu)^2+\mathbb{1} \end{pmatrix}$$

whereas

$$\mathcal{I}_A^r\mathcal{I}_A^\ell\mathcal{I}_A^r = \begin{pmatrix} a^\nu+(b^\nu)^2+\mathbb{1} & (a^3)^\nu+a^\nu+b^\nu \\ (b^3)^\nu+a^\nu+b^\nu & (a^\nu)^2+b^\nu+\mathbb{1} \end{pmatrix} = (\mathcal{I}_A^\ell\mathcal{I}_A^r\mathcal{I}_A^r\mathcal{I}_A^\ell)^\nabla.$$

It follows that $\mathcal{I}_A \neq \widetilde{\mathcal{I}_A}$ *in general, but equality holds when either side is nonsingular, since then we must have* $a, b <_\nu \mathbb{1}$, *so each side is* $\begin{pmatrix} \mathbb{1} & a^\nu+b^\nu \\ a^\nu+b^\nu & \mathbb{1} \end{pmatrix}$.

Even worse, the following example is obtained by modifying an example from [5]:

Example 2.51. *(logarithmic notation)*
Take $A = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}$ *whose determinant is* $0 = \mathbb{1}$, *whereas* $A^2 = \begin{pmatrix} -1 & 0 \\ 1 & 2 \end{pmatrix}$ *is singular. We have the quasi-inverse* $A^\nabla = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$, *and the quasi-identity matrices*

$$\mathcal{I}_A^\ell = AA^\nabla = \begin{pmatrix} 0 & -2^\nu \\ 1^\nu & 0 \end{pmatrix} \quad \neq \quad \mathcal{I}_A^r = A^\nabla A = \begin{pmatrix} 0 & 0^\nu \\ -1^\nu & 0 \end{pmatrix}.$$

We see that

$$\mathcal{I}_A^\ell\mathcal{I}_A^r = \begin{pmatrix} 0 & 0^\nu \\ 1^\nu & 1^\nu \end{pmatrix} \neq \begin{pmatrix} 1^\nu & 0^\nu \\ 1^\nu & 0 \end{pmatrix} = \mathcal{I}_A^r\mathcal{I}_A^\ell.$$

Here is the general situation for 2×2 matrices, in algebraic notation.

Example 2.52. *Take* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *whose determinant* $ad + bc$ *is* $\mathbb{1}$. *Then*

$$A^\nabla = \begin{pmatrix} d & b \\ c & a \end{pmatrix},$$

and we get the quasi-identity matrices

$$\mathcal{I}_A^\ell = AA^\nabla = \begin{pmatrix} \mathbb{1} & (ab)^\nu \\ (cd)^\nu & \mathbb{1} \end{pmatrix} \quad \neq \quad \mathcal{I}_A^r = A^\nabla A = \begin{pmatrix} \mathbb{1} & (bd)^\nu \\ (ac)^\nu & \mathbb{1} \end{pmatrix}.$$

We see that

$$\mathcal{I}_A^{\ell}\mathcal{I}_A^{r} = \begin{pmatrix} \mathbb{1} + (a^2bc)^{\nu} & b^{\nu}(a+d) \\ c^{\nu}(a+d) & \mathbb{1} + (bcd^2)^{\nu} \end{pmatrix}$$

whereas

$$\mathcal{I}_A^{r}\mathcal{I}_A^{\ell} = \begin{pmatrix} \mathbb{1} + (bcd^2)^{\nu} & b^{\nu}(a+d) \\ c^{\nu}(a+d) & \mathbb{1} + (a^2bc)^{\nu} \end{pmatrix} = (\mathcal{I}_A^{\ell}\mathcal{I}_A^{r})^{\nabla},$$

but when either is nonsingular, then they are both equal to $\begin{pmatrix} \mathbb{1} & b^{\nu}(a+d) \\ c^{\nu}(a+d) & \mathbb{1} \end{pmatrix}$.

This raises hope that the theory works well when we only encounter tangible matrices, but a troublesome example exists for 3×3 matrices.

Example 2.53. *(logarithmic notation)*

Take $A = \begin{pmatrix} - & 5 & 0 \\ 0 & - & - \\ - & 0 & - \end{pmatrix}$ *whose determinant is* $0 = \mathbb{1}$. *Then* $A^{\nabla} = \begin{pmatrix} - & 0 & - \\ - & - & 0 \\ 0 & - & 5 \end{pmatrix}$, *so*

$$\mathcal{I}_A^{\ell} = AA^{\nabla} = \begin{pmatrix} 0 & - & 5^{\nu} \\ - & 0 & - \\ - & - & 0 \end{pmatrix}, \qquad \mathcal{I}_A^{r} = A^{\nabla}A = \begin{pmatrix} 0 & - & - \\ - & 0 & - \\ - & 5^{\nu} & 0 \end{pmatrix},$$

which are both definite (and would be strictly normal if we took -5 *instead of* 5*). But*

$$\mathcal{I}_A^{\ell}\mathcal{I}_A^{r} = \begin{pmatrix} 0 & 10^{\nu} & 5^{\nu} \\ - & 0 & - \\ - & 5^{\nu} & 0 \end{pmatrix} \qquad \neq \qquad \begin{pmatrix} 0 & - & 5^{\nu} \\ - & 0 & - \\ - & 5^{\nu} & 0 \end{pmatrix} = \mathcal{I}_A^{r}\mathcal{I}_A^{\ell}.$$

Furthermore, $\mathcal{I}_A^{\ell}\mathcal{I}_A^{r}$ *is idempotent and nonsingular, and thus a quasi-identity, so* $\mathcal{I}_A^{\ell}\mathcal{I}_A^{r} = (\mathcal{I}_A^{\ell}\mathcal{I}_A^{r})^{\nabla}$ *which does not equal* ${\mathcal{I}_A^{r}}^{\nabla}{\mathcal{I}_A^{\ell}}^{\nabla} = \mathcal{I}_A^{r}\mathcal{I}_A^{\ell}$.

Nevertheless, $\mathcal{I}_A^{\ell}$ and $\mathcal{I}_A^{r}$ always do satisfy a nice relation in the 2×2 case. We say that 2×2 matrices $\mathcal{I} = \begin{pmatrix} \mathbb{1} & u^{\nu} \\ v^{\nu} & \mathbb{1} \end{pmatrix}$ and $\mathcal{I}' = \begin{pmatrix} \mathbb{1} & u'^{\nu} \\ v'^{\nu} & \mathbb{1} \end{pmatrix}$ (in algebraic notation) are **paired** if $uv = u'v'$.

Lemma 2.54. *For any* 2×2 *matrix* A *of determinant* $\mathbb{1}$, *the quasi-identities* $\mathcal{I}_A^{\ell}$ *and* $\mathcal{I}_A^{r}$ *are paired. Conversely, if* F *is closed under square roots then, given paired quasi-identity matrices* $\mathcal{I}$ *and* $\mathcal{I}'$, *there is a* 2×2 *matrix* A *of determinant* $\mathbb{1}$, *such that* $\mathcal{I} = \mathcal{I}_A^{\ell}$ *and* $\mathcal{I}' = \mathcal{I}_A^{r}$.

Proof. After a permutation, we may write $A = \begin{pmatrix} a & b \\ c & a^{-1} \end{pmatrix}$ with $bc <_{\nu} \mathbb{1}$. Then

$$\mathcal{I}_A^{\ell} = \begin{pmatrix} \mathbb{1} & (ab)^{\nu} \\ (a^{-1}c)^{\nu} & \mathbb{1} \end{pmatrix} \quad \text{and} \quad \mathcal{I}_A^{r} = \begin{pmatrix} \mathbb{1} & (a^{-1}b)^{\nu} \\ (ac)^{\nu} & \mathbb{1} \end{pmatrix}$$

are paired since

$$(ab)(a^{-1}c) = bc = (a^{-1}b)(ac).$$

Conversely, given $uv = u'v'$ we take $b = \sqrt{uu'}$, $c = \sqrt{vv'}$, and $a = \sqrt{\frac{u}{u'}}$ to get $ab = u$, $a^{-1}c = \sqrt{\frac{u'vv'}{u}} = \sqrt{v^2} = v$, $ac = v'$, and $a^{-1}b = u'$. □

Lemma 2.55 ([12, Lemma 2.17])**.** $A^{\nabla} \le A^{\nabla}AA^{\nabla}$ *for any matrix* $A \in \mathrm{Mat}_n(F)$.

Definition 2.56. *A matrix* A *is* ∇***-regular*** *if* $A = AA^{\nabla}A$.

Example 2.57. $AA^{\nabla}A = \mathcal{I}_A^{\ell}A = A\mathcal{I}_A^{r}$ *is* ∇*- regular (but not necessarily reversible, nor nonsingular). Every quasi-identity matrix is* ∇*-regular as well as reversible.*

Since $AA^{\nabla}A$ shares many properties with A (for example, yielding the same quasi-identities $\mathcal{I}_A^{\ell}$ and $\mathcal{I}_A^{r}$ and other properties concerning solutions of equations in [12]), they are of particular interest to us.

Lemma 2.58. *If a nonsingular matrix* A *is* ∇*-regular, then* A^{∇} *is* ∇*-regular.*

Proof. $A^\nabla = (AA^\nabla A)^\nabla \underset{\mathrm{gs}}{\models} A^{\nabla\nabla} A^\nabla A^{\nabla\nabla} \underset{\mathrm{gs}}{\models} A^\nabla$, cf. [11] and Lemma 2.10, so equality holds at each step. $\square$

Example 2.59. *A strictly normal matrix which is not ∇-regular (given in logarithmic notation). Take*

$$A = \begin{pmatrix} 0 & -3 & -6 \\ -1 & 0 & -2 \\ -1 & -1 & 0 \end{pmatrix} \quad \Rightarrow \quad A^\nabla = \begin{pmatrix} 0 & -3 & -5 \\ -1 & 0 & -2 \\ -1 & -1 & 0 \end{pmatrix} \quad \Rightarrow \quad A^{\nabla\nabla} = \begin{pmatrix} 0 & -3 & -5^\nu \\ -1 & 0 & -2 \\ -1 & -1 & 0 \end{pmatrix}.$$

Then

$$AA^\nabla A = \begin{pmatrix} 0 & -3^\nu & -5^\nu \\ -1^\nu & 0 & -2^\nu \\ -1^\nu & -1^\nu & 0 \end{pmatrix}.$$

Thus $A \neq AA^\nabla A \neq A^{\nabla\nabla}$. In fact, $A \not\cong_\nu AA^\nabla A$. Note that $A^2 = \begin{pmatrix} 0 & -3^\nu & -5 \\ -1^\nu & 0 & -2^\nu \\ -1^\nu & -1^\nu & 0 \end{pmatrix}$ also is nonsingular.

3. Special linear supertropical matrices

As stated earlier, our main objective is to pinpoint the most viable version of SL_n. The obvious attempt is the set

$$\mathrm{SL}_n(F) := \{A \in \mathrm{Mat}_n(F) : \det(A) = \mathbb{1}\}$$

of matrices A with supertropical determinant $\mathbb{1}$, which we call **special linear matrices**.

3.1. The monoid generated by $\mathrm{SL}_n(F)$.

$\mathrm{SL}_n(F)$ is not a monoid, as seen from Example 2.51. Thus, we would like to determine the monoid generated by $\mathrm{SL}_n(F)$, as well as the submonoids of $\mathrm{SL}_n(F)$.

Remark 3.1. *If the matrix $P \in \mathrm{SL}_n^\times$ and $A \in \mathrm{SL}_n$, then $PA \in \mathrm{SL}_n$, by Corollary 2.13.*

Thus, any difficulty would involve noninvertible matrices of $\mathrm{SL}_n(F)$. The following observation ties this discussion to definite matrices.

Lemma 3.2.

(i) *Any nonsingular matrix A is the product PA_1 of a generalized permutation matrix P with a nonsingular definite matrix A_1.*

(ii) *Any matrix A of $\mathrm{SL}_n(F)$ is the product PA_1 of a a generalized permutation matrix $P \in \mathrm{SL}_n(F)$ with a definite matrix $A_1 \in \mathrm{SL}_n(F)$. Likewise we can write $A = A_2Q$ for a generalized permutation matrix Q in SL_n and A_2 a definite matrix.*

Proof. Multiplying by a permutation matrix puts the dominant permutation of A on the diagonal, which we can make definite by multiplying by a diagonal matrix. If $A \in \mathrm{SL}_n(F)$ then $A_1 \in \mathrm{SL}_n(F)$, in view of Corollary 2.13. $\square$

The point of this lemma is that the process of passing a matrix of SL_n to definite form takes place entirely in SL_n, so the results of [15] are applicable in this paper, as we shall see.

In the spirit of [11, Proposition 3.9], but using symmetrization, we turn to $\succeq_\circ$, and define:

Definition 3.3.

$$\overline{\mathrm{SL}}_n(F) := \big\{A \in \mathrm{Mat}_n(F) : \mathrm{bidet}(A) \succeq_\circ (\alpha, \beta) \quad \textit{where} \quad \alpha + \beta = \mathbb{1}\big\}.$$

We write SL_n and $\overline{\mathrm{SL}}_n$ for $\mathrm{SL}_n(F)$ and $\overline{\mathrm{SL}}_n(F)$, when F is clear from the context.

Here is a generic sort of example.

Example 3.4. *Consider two rank 1 matrices* $\begin{pmatrix} a & \mathbb{0} \\ b & \mathbb{0} \end{pmatrix}$ *and* $\begin{pmatrix} c & d \\ \mathbb{0} & \mathbb{0} \end{pmatrix}$. *Their product is* $\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$ *whose symmetrized determinant is* $(abcd, abcd) \in F^{\circ}$.

Although the first two matrices are singular, they "explain" the following modification: The product of the matrices $\begin{pmatrix} a & \mathbb{0} \\ b & a^{-1} \end{pmatrix}$ *and* $\begin{pmatrix} c & d \\ \mathbb{0} & c^{-1} \end{pmatrix}$, *both from* $\mathrm{SL}_n(F)$, *is* $\begin{pmatrix} ac & ad \\ bc & bd + a^{-1}c^{-1}, \end{pmatrix}$ *which is* $\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$ *when* $abcd > \mathbb{1}$. *Put another way, given any* $u, v, u', v' \in F$ *satisfying* $uv = u'v'$, *we can find two matrices whose product is* $\begin{pmatrix} u & u' \\ v & v' \end{pmatrix}$, *namely take* $a = \mathbb{1}$, $c = u$, $d = u'$, *and* $b = \frac{v}{u'}$. *Thus, every* 2×2 *matrix in* $\overline{\mathrm{SL}}_2(F)$ *is a product of two matrices in* $\mathrm{SL}_2(F)$.

This yields:

Proposition 3.5. $\overline{\mathrm{SL}}_2(F)$ *is the submonoid of matrices generated by* $\mathrm{SL}_2(F)$.

Proof. The key computation is $\begin{pmatrix} \mathbb{1} & b \\ ab^{-1} & a \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ ab^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbb{1} & b \\ \mathbb{0} & \mathbb{1} \end{pmatrix}$, a special case of the previous example. □

On the other hand, for larger n, we have room for obstructions.

Example 3.6. *For* $n \geq 3$, *suppose* A *has the form*

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \mathbb{0} & \mathbb{0} & \dots & \mathbb{0} \\ \mathbb{0} & a_{2,2} & a_{2,3} & \mathbb{0} & \dots & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & a_{3,3} & a_{3,4} & \dots & \mathbb{0} \\ \vdots & & & \ddots & \ddots & \vdots \\ \mathbb{0} & \dots & \mathbb{0} & \mathbb{0} & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & \mathbb{0} & \dots & \mathbb{0} & \mathbb{0} & a_{n,n} \end{pmatrix}.$$

Then A *cannot be factored into* A_1A_2 *unless one of the* A_i *is invertible.*

More generally, Niv[16, Proposition 3.2] proved:

Proposition 3.7. *Suppose* $\pi, \sigma \in S_n$ *such that there exists an integer* $0 < t < \frac{n}{2}$ *for which, for all* i, $\pi(i) \equiv \sigma(i) + t \pmod n$. *Then any* $n \times n$ *matrix* $A = \sum_{i=1}^{n} (a_{i,\pi(i)} e_{i,\pi(i)} + a_{i,\sigma(i)} e_{i,\sigma(i)})$ *(with invertible coefficients* $a_{i,\pi(i)}, a_{i,\sigma(i)}$*) is not factorizable.*

For $n = 3$ this example is not so bad, since A has no odd permutations contributing to the determinant. But for n even, A has one odd permutation and one even permutation which contribute.

Corollary 3.8. *For even* $n \geq 4$, $\overline{\mathrm{SL}}_n(F)$ *is not a product of elements of* $\mathrm{SL}_n(F)$.

Proof. The permutation $(1\ 2\ \dots\ n)$ is odd, and so we get an element of $\overline{\mathrm{SL}}_n(F)$ which is not factorizable, and in particular is not a product of elements of $\mathrm{SL}_n(F)$. □

Example 3.9. *For* $n = 3$, *we have the factorization*

$$\begin{pmatrix} \mathbb{1} & b & c \\ ab^{-1} & a & d \\ e & f & g \end{pmatrix} = \begin{pmatrix} \mathbb{1} & \mathbb{0} & c \\ ab^{-1} & a & d \\ e & f & g \end{pmatrix} \begin{pmatrix} \mathbb{1} & b & \mathbb{0} \\ \mathbb{0} & \mathbb{1} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & \mathbb{1} \end{pmatrix},$$

whenever $f >_{\nu} be$.

We do not know of a general factorization when $f <_{\nu} be$.

3.2. **Submonoids of** $\mathrm{SL}_n(F)$**.** Non-normal matrices and permutation matrices do not mix well, as we see in the next result.

Proposition 3.10. *Suppose* $A \in \mathrm{SL}_n$ *is not strictly normal. Then some product of* A *(twice) with permutation matrices and one strictly normal matrix is symmetrically singular.*

Proof. First of all, by applying permutation matrices we may assume that the dominant permutation of A is on the diagonal, with $a_{1,1} \leq a_{2,2} \leq \cdots \leq a_{n,n}$. By hypothesis either there is some i such that $a_{i,i} < a_{i+1,i+1}$, or all the $a_{i,i} = \mathbb{1}$ and there is some $a_{i,j} \geq_\nu \mathbb{1}$. To simplify notation we assume that $i = 1$, $j = 2$, and also may assume that $n = 2$ by leaving everything fixed for indices ≥ 3. Thus, we take $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a \leq d$ and $\mathbb{1} = ad > bc$.

We may assume that $b \leq c$. We claim that we may assume that $c \geq \mathbb{1}$. Indeed, this is clear if $a = d = \mathbb{1}$, since A not strictly normal then implies $c \geq \mathbb{1}$. So assume that $a < d$. Then $d > \mathbb{1}$. Taking $u < 1$ such that $du > \mathbb{1}$, and multiplying by the strictly normal matrix $\begin{pmatrix} \mathbb{1} & - \\ u & \mathbb{1} \end{pmatrix}$, yields $\begin{pmatrix} a + bu & b \\ c + du & d \end{pmatrix}$. Thus we may replace c by $c+du$ and assume that $c > \mathbb{1}$, as desired. (Note that this matrix is singular, with determinant $(bdu)^\nu$, unless $bu < a$.)

Applying permutations to A yields $\begin{pmatrix} d & c \\ b & a \end{pmatrix}$ and the product $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & c \\ b & a \end{pmatrix}$ is $\begin{pmatrix} ad + b^2 & a(b+c) \\ d(b+c) & ad + c^2 \end{pmatrix}$. But $c^2 > \mathbb{1} = ad$, so our matrix product is $\begin{pmatrix} ad & ac \\ dc & c^2 \end{pmatrix}$, whose determinant is $c^2(ad)^\nu = (c^2)^\nu$. □

3.3. **Nonsingular submonoids.** Although SL_n is not a monoid, it does have interesting submonoids consisting only of nonsingular matrices.

Example 3.11.

(i) *The set of* $\mathrm{Mat}_n(F)^\times$ *of generalized permutation matrices in* SL_n *is a subgroup (with unit element I).*

(ii) *The upper triangular matrices of* SL_n *are a submonoid (with unit element I).*

(iii) *If A is strictly normal, then the monoid generated by A is nonsingular. (Indeed, A^k is nonsingular for any $k < n$, which means that there is only one way of getting a maximal diagonal element in any power of A, which is by taking a power of $a_{i,i} = \mathbb{1}$, and the non-diagonal elements will be smaller.)*

We continue with (iii), and appeal to a more restricted version of SL_n.

Definition 3.12. *A matrix monoid is* ***nonsingular*** *if it consists of nonsingular matrices. A subset $S \subset \mathrm{Mat}_n(F)$ is* ∇***-closed*** *if $A^\nabla \in S$ for all $A \in S$.*

Most of the sets we consider are ∇-closed. (Geometric and combinatorial characterizations of nonsingular tropical matrix monoids are provided in [4].)

Definition 3.13. SN_n *denotes the set of all strictly normal $n \times n$ matrices in* $\mathrm{Mat}_n(F)$.

Lemma 3.14. SN_n *is a nonsingular ∇-closed monoid, also closed under transpose.*

Proof. A straightforward verification. □

Lemma 3.15. *$A^\nabla \geq_\nu A$ for any strictly definite matrix A.*

Proof. Let $A^\nabla = (a'_{i,j})$, where $a'_{i,j} = \det(A_{j,i})$. Clearly the diagonal $\geq_\nu I$, so we need to check that $a'_{i,j} \geq_\nu a_{i,j}$ for each $i \neq j$. But one of the terms contributing to $a'_{i,j}$ is $a_{i,j}$ together with the product of entries $\mathbb{1}$ on the diagonal of the matrix obtained by eliminating the i and j rows and columns, i.e., $a_{i,j}$ itself. □

3.3.1. *The $\mathbb{1}$-special linear monoid.* We enlarge the monoid SN_n via the left and right action of permutation matrices.

Definition 3.16. *Given a set S, we define its* ***permutation closure*** *to be*

$$\{PJQ : J \in S \text{ and } P, Q \text{ are permutation matrices}\}.$$

The $\mathbb{1}$***-special linear monoid*** $\mathrm{SL}_n^{\mathbb{1}}$ *is the permutation closure of the monoid* SN_n *of strictly normal matrices.*

Remark 3.17. *$A \in \mathrm{SL}_n^{\mathbb{1}}$ iff $A = (a_{i,j})$ has a uniformly dominant permutation π with $a_{i,\pi(i)} = \mathbb{1}$ for all i.*

Theorem 3.18. *$\mathrm{SL}_n^{\mathbb{1}}$ is a ∇-closed submonoid of SL_n.*

Proof. Write $B = A_1A_2$ where $A_1 = P_{\pi_1}J_1Q_{\pi_2}$ and $A_2 = P_{\pi_3}J_2Q_{\pi_4}$ are in $\mathrm{SL}_n^{\mathbb{1}}$ with J_1, J_2 strictly normal. Writing $B = \big(P_{\pi_1}J_1Q_{\pi_2}\big)\big(P_{\pi_3}J_2Q_{\pi_4}\big)$, as a product of matrices with respective uniformly dominant permutations $\pi_1, \mathrm{id}, \pi_2, \pi_3, \mathrm{id}, \pi_4$, we see from Lemma 2.9 that the permutation

$$\tau = \pi_1 \,\mathrm{id}\, \pi_2\pi_3 \,\mathrm{id}\, \pi_4$$

is uniformly dominant for B, in which $b_{i,\tau(i)} = \mathbb{1}$ for all i. Hence, $B \in \mathrm{SL}_n^{\mathbb{1}}$, and we have proved that $\mathrm{SL}_n^{\mathbb{1}}$ is a monoid.

By Lemma 3.14 and Corollary 2.13 it follows that

$$A^{\nabla} = (P_{\pi_1}JQ_{\pi_2})^{\nabla} = Q_{\pi_2}^{\nabla}J^{\nabla}P_{\pi_1}^{\nabla} = Q_{\pi_2}^{-1}J^{\nabla}P_{\pi_1}^{-1},$$

for every $A \in \mathrm{SL}_n^{\mathbb{1}}$, and thus $\mathrm{SL}_n^{\mathbb{1}}$ is ∇-closed. □

Theorem 3.19. *The monoid $\mathrm{SL}_n^{\mathbb{1}}$ is a maximal nonsingular submonoid of SL_n.*

Proof. $\mathrm{SL}_n^{\mathbb{1}}$ is a nonsingular submonoid of SL_n by Theorem 3.18, and is maximal nonsingular by Proposition 3.10. □

But $\mathrm{SL}_n^{\mathbb{1}}$ is not the only maximal nonsingular submonoid of SL_n, since it does not contain the other monoids of Example 3.11.

4. Semigroup partitions in $\overline{\mathrm{SL}}_n$

Our objective here is to carve $\overline{\mathrm{SL}}_n$ into monoids, each of which has a multiplicative unit $\mathcal{I}$, where $\mathcal{I}$ is a quasi-identity. Although we cannot quite do this, the process works for ∇-regular matrices.

4.1. Sub-semigroups and subgroups arising from quasi-inverses.

Definition 4.1. *For any A with $\det(A) \neq \mathbb{0}$:*

(i) $\overline{\mathrm{SL}}^{\ell}_{A;n} = \{B \in \overline{\mathrm{SL}}_n : \mathcal{I}_A^{\ell}B = B\}$;

(ii) $\overline{\mathrm{SL}}^{r}_{A;n} = \{B \in \overline{\mathrm{SL}}_n : B\mathcal{I}_A^{r} = B\}$.

In particular, for a quasi-identity $\mathcal{I}$,

$\overline{\mathrm{SL}}^{\ell}_{\mathcal{I};n} = \{B \in \overline{\mathrm{SL}}_n : \mathcal{I}B = B\}$;

$\overline{\mathrm{SL}}^{r}_{\mathcal{I};n} = \{B \in \overline{\mathrm{SL}}_n : B\mathcal{I} = B\}$.

Lemma 4.2. *If A is ∇-regular, then $A \in \overline{\mathrm{SL}}^{\ell}_{A;n} \cap \overline{\mathrm{SL}}^{r}_{A;n}$.*

Proof. $A = AA^{\nabla}A = \mathcal{I}_A^{\ell}A = A\mathcal{I}_A^{r}$. □

Remark 4.3. *If $\mathcal{I}_A^{\ell}B_1 = B_1$ then for any B_2 we have $\mathcal{I}_A^{\ell}(B_1B_2) = B_1B_2$. Thus $\overline{\mathrm{SL}}^{\ell}_{A;n}$ is closed under multiplication on the right by any matrix.*

Lemma 4.4. *$\overline{\mathrm{SL}}_{A;n}$ is a sub-semigroup of $\overline{\mathrm{SL}}_n$ with left unit element $\mathcal{I}_A^{\ell}$ and right unit element $\mathcal{I}_A^{r}$.*

Proof. By Remark 4.3. The other assertion holds since $\mathcal{I}_A^{\ell}$ and $\mathcal{I}_A^{r}$ are idempotent. □

This provides the intriguing situation in which we have a natural semigroup with left and right identities which could be unequal. We also have a uniqueness result:

Lemma 4.5. *If $\mathcal{I}_A^{\ell}B = \mathcal{I}_A^{\ell}$ for $B \in \overline{\mathrm{SL}}^{\ell}_{A;n}$, then $B = \mathcal{I}_A^{\ell}$. Consequently, the multiplicative unit element $\mathcal{I}_A^{\ell}$ of $\overline{\mathrm{SL}}^{\ell}_{A;n}$ is unique.*

Proof. $B = \mathcal{I}_A^{\ell}B = \mathcal{I}_A^{\ell}$. □

Putting everything together, we have:

Theorem 4.6. *Every reversible element A of SL_n defines a submonoid $\overline{\mathrm{SL}}_{A;n}$ with unique unit element $\mathcal{I}_A$, and which contains $\mathcal{I}_A^{\ell}A$. The union of these submonoids contains every reversible ∇-regular element of SL_n, and in particular, every quasi-identity matrix.*

Proof. Lemma 4.5 provides $\overline{\mathrm{SL}}_{A;n}$. When A is reversible, then $\mathcal{I}_A \in \overline{\mathrm{SL}}_{A;n}$ is the unique unit element. Furthermore, $\mathcal{I}_A A = AA^\nabla A$ is in $\overline{\mathrm{SL}}_{A;n}$, and equals A when A is ∇-regular. □

5. The conjugate action

For any nonsingular matrix A and any matrix B, we define

$$^{A}B = A^\nabla BA.$$

This is the closest we have to conjugation by supertropical matrices. (Note that $^{A}I = A^\nabla IA = \mathcal{I}_A^r$.)

We continue with an example of a nonsingular matrix having a singular conjugate.

Example 5.1. *Take*

$$A = \begin{pmatrix} \alpha & \mathbb{1} \\ \mathbb{1} & \beta \end{pmatrix}, \qquad B = \begin{pmatrix} x & z \\ w & y \end{pmatrix},$$

where $x >_\nu y$, $xy >_\nu zw \geq_\nu \mathbb{0}$, and $\alpha, \beta <_\nu \mathbb{1}$ such that $\alpha\beta >_\nu \frac{y}{x}$. Thus $A^\nabla = \begin{pmatrix} \beta & \mathbb{1} \\ \mathbb{1} & \alpha \end{pmatrix}$, and

$$\begin{aligned} A^\nabla BA &= \begin{pmatrix} \beta & \mathbb{1} \\ \mathbb{1} & \alpha \end{pmatrix}\begin{pmatrix} x & z \\ w & y \end{pmatrix}\begin{pmatrix} \alpha & \mathbb{1} \\ \mathbb{1} & \beta \end{pmatrix} \\ &= \begin{pmatrix} \beta & \mathbb{1} \\ \mathbb{1} & \alpha \end{pmatrix}\begin{pmatrix} x\alpha + z & x + z\beta \\ w\alpha + y & w + y\beta \end{pmatrix} \\ &= \begin{pmatrix} x\alpha\beta + z\beta + w\alpha + y & x\beta + z\beta^2 + y\beta + w \\ x\alpha + w\alpha^2 + y\alpha + z & z\beta + y\alpha\beta + w\alpha + x \end{pmatrix}. \end{aligned}$$

Then

$$A^\nabla BA = \begin{pmatrix} x\alpha\beta + z\beta + w\alpha & x\beta + z\beta^2 + w \\ x\alpha + w\alpha\beta + z & z\beta + w\alpha + x \end{pmatrix}$$

for which

$$\begin{aligned} \det(A^\nabla BA) &= (x\alpha\beta + z\beta + w\alpha)(z\beta + w\alpha + x) + (x\beta + z\beta^2 + w)(x\alpha + w\alpha\beta + z) \\ &= (wx\alpha + w^2\alpha^2 + xz\beta + x^2\alpha\beta + wz\alpha\beta + wx\alpha^2\beta + z^2\beta^2 + xz\alpha\beta^2) \\ &\quad + (wz + wx\alpha + w^2\alpha^2 + xz\beta + x^2\alpha\beta + wx\alpha^2\beta + z^2\beta^2 + xz\alpha\beta^2 + wz\alpha^2\beta^2) \\ &= (wx\alpha + w^2\alpha^2 + xz\beta + x^2\alpha\beta + wx\alpha^2\beta + z^2\beta^2 + xz\alpha\beta^2) \\ &\quad + (wx\alpha + w^2\alpha^2 + xz\beta + x^2\alpha\beta + wx\alpha^2\beta + z^2\beta^2 + xz\alpha\beta^2), \\ &= (wx\alpha + w^2\alpha^2 + xz\beta + x^2\alpha\beta + wx\alpha^2\beta + z^2\beta^2 + xz\alpha\beta^2)^\nu, \end{aligned}$$

since

$$x^2\alpha\beta >_\nu xy \geq_\nu wz >_\nu wz\alpha\beta >_\nu wz\alpha^2\beta^2.$$

Thus $A^\nabla BA$ is singular.

Obviously, this holds for any nonsingular matrix B with $y = x^{-1}$, namely when $\det(B) = \mathbb{1}$.

Given a nonempty set $S \subset \mathrm{Mat}_n(F)$ of matrices and a matrix A, we write

$$^{A}S = \{A^\nabla BA \ : \ B \in S\}.$$

If S is a monoid and A is invertible, then ^{A}S also is a monoid. But when $A \in \overline{\mathrm{SL}}_n$ is not invertible, we get into difficulties, even in the 2×2 case.

Example 5.2. *(logarithmic notation)*

$B = \begin{pmatrix} 0 & -\infty \\ 1 & 0 \end{pmatrix}$ *is definite, and* $A = \begin{pmatrix} 0 & 5^\nu \\ -\infty & 0 \end{pmatrix}$ *is a quasi-identity matrix, but* $BAB = \begin{pmatrix} 6^\nu & 5^\nu \\ 7^\nu & 6^\nu \end{pmatrix}$ *is singular.*

Note that if $\mathcal{S}_n$ is a nonsingular matrix submonoid of $\mathrm{Mat}_n(F)$, then ${}^P\mathcal{S}_n$ also is a nonsingular submonoid of $\mathrm{Mat}_n(F)$, for any permutation matrix P. On the other hand, these often do not mix well, as seen in Lemma 6.2 below.

The following example also shows that nonsingularity in multiplication in $\mathrm{SL}_n^{\mathbb{1}}$ need not be preserved even when we conjugate by diagonal matrices.

Example 5.3. *(logarithmic notation)*
$B = \begin{pmatrix} 0 & -\infty \\ 1 & 0 \end{pmatrix}$, *and* $D = \begin{pmatrix} 1 & -\infty \\ -\infty & -1 \end{pmatrix}$, *and* $BDB^{\mathrm{t}} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ *is singular. In view of Corollary 2.13,* $B(DB^{\mathrm{t}}D^{-1})$ *is singular.*

Here is one consolation.

Lemma 5.4. *If* $\mathcal{I}_A \in \mathrm{SN}_n$, *then* $\{A^\nabla JA \ : \ J \in \mathrm{SN}_n\}$ *is a monoid.*

Proof. By Theorem 3.18. □

Furthermore, the situation improves significantly when we restrict our attention to the submonoid $\overline{\mathrm{SL}}^{\,\ell}_{A;n}$ and the space on which it acts.

We define
$$V_A = \{v \in F^{(n)} : \mathcal{I}_A^\ell v = v\}.$$

Lemma 5.5. $\overline{\mathrm{SL}}^{\,\ell}_{A;n}\, F^{(n)} = V_A = \overline{\mathrm{SL}}^{\,\ell}_{A;n}\, V_A$.

Proof. If $B \in \overline{\mathrm{SL}}^{\,\ell}_{A;n}$ and $v \in F^{(n)}$, then $\mathcal{I}_A^\ell(Bv) = (\mathcal{I}_A^\ell B)v = Bv$. On the other hand, if $v \in V_A$ then $v = \mathcal{I}_A^\ell v \in \overline{\mathrm{SL}}^{\,\ell}_{A;n}\, V_A$. Thus, we have
$$\overline{\mathrm{SL}}^{\,\ell}_{A;n}\, F^{(n)} \subseteq V_A \subseteq \overline{\mathrm{SL}}^{\,\ell}_{A;n}\, V_A \subseteq \overline{\mathrm{SL}}^{\,\ell}_{A;n}\, F^{(n)},$$
so equality holds. □

Proposition 5.6. *For any nonsingular* A, *left multiplication by* A^∇ *yields a module map from* V_A *to* V_{A^∇}, *which commutes with conjugation by* A.

Proof. If $B \in \overline{\mathrm{SL}}^{\,\ell}_{A;n}$, then letting $v' = A^\nabla v$ we have
$$(A^\nabla BA)v' = (A^\nabla BA)A^\nabla v = A^\nabla Bv.$$
□

6. Tropical elementary matrices

Unlike the situation over a field, the tropical concepts of singularity, invertibility, and factorability into elementary matrices do not coincide, cf. [17]. Over a field, the fact that a nonsingular matrix can be written as the product of elementary matrices means that one can pass between any two nonsingular matrices using elementary operations. In the tropical case, even though factorability fails, we show in Theorem 6.4 below that one still can pass between nonsingular matrices, in a certain sense.

In analogy with the classical definition, we define three types of tropical elementary matrices of SL_n:

Transposition matrices, which switch two rows (resp. columns).

Diagonal multipliers, which multiply a row (resp. a column) by some element of $\mathcal{T}$.

Gaussian matrices, which add one row (resp. column), multiplied by a scalar, to another row (resp. column).

Definition 6.1. *A nonsingular matrix is defined to be* ***(tropically) factorizable*** *if it can be written as a product of tropical elementary matrices.*

As noted earlier, the product of nonsingular matrices could be singular. In the next lemma we pinpoint the elementary operation that causes a nonsingular matrix which is non-invertible to become singular.

Lemma 6.2. *For every non-invertible matrix A in SL_n, there exists an elementary Gaussian matrix E such that EA is singular.*

Proof. First we recall that if A is a factorizable matrix, then we can find a factorization in which the Gaussian matrices are at the right of its factorization (see [16]). Therefore, in view of Lemma 3.2s it suffices to prove the lemma for a definite matrix A. Hence, $\det(A) = \mathbb{1}$ is attained solely by the diagonal.

Since A is non-invertible, there exists at least one nonzero off-diagonal entry $a_{i,j} \neq \mathbb{0}$. We fix E to be the Gaussian matrix with $a_{i,j}^{-1}$ in the (j,i)-position. Then

$$\begin{aligned}\det(EA) &= \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{i,\sigma(i)} \cdots a_{n,\sigma(n)} + \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots (a_{i,j}^{-1}) a_{i,\sigma(j)} \cdots a_{n,\sigma(n)} \\ &= \det(A) + \sum_{\sigma \in S_n} a_{1,\sigma(1)} \cdots (a_{i,j}^{-1}) a_{i,\sigma(j)} \cdots a_{n,\sigma(n)}.\end{aligned}$$

Since $a_{i,j}^{-1}$ is the scalar of E, the summand in the right side given by $\sigma = (i,j)$ is $\mathbb{1}$, which together with $\det(A)$ yields $\mathbb{1}^\nu$. Moreover, by Theorem 2.12, any larger dominant term on the right sum must be ghost. Since $\det(A) = \mathbb{1}$, the assertion follows. $\square$

We recall the well-known connection between tropical matrices and digraphs. Any $n \times n$ matrix A is associated with a weighted digraph G_A over n vertices having edge (i,j) of weight $a_{i,j}$ whenever $a_{i,j} \neq \mathbb{0}$ cf. [11, §3.2]. From this viewpoint the (i,j)-entry of the matrix $\mathrm{adj}(A)$ equals the maximal weight of all paths from i to j in the graph $G_{\mathrm{adj}(A)}$. We utilize this identification and work with nonsingular definite matrices, in which case $A^\nabla \cong_\nu A^{\nabla\nabla}$ by [15, Corollary 6.2]. A path is called **simple** if each vertex appears only once.

Proposition 6.3. *For any matrix $A \in \mathrm{SL}_n$ there exists a product E of elementary Gaussian matrices such that $A^{\nabla\nabla} = EA$.*

Proof. In view of Lemma 3.2 we may assume that A is definite; indeed, writing $A = PA_1$, for A_1 definite, we would have

$$(PA_1)^{\nabla\nabla} = (A_1^\nabla (P^{-1})^\nabla = PA_1^{\nabla\nabla} = PEA_1.$$

Now let $A = (a_{i,j})$ and let $A^{\nabla\nabla} = (a''_{i,j})$. Then $A^{\nabla\nabla} \underset{\mathrm{gs}}{\models} A$ by Remark 2.38. Since A is nonsingular definite, we have $\det(A) = \mathbb{1}$ where its dominant permutation is the identity. It follows that any cycle has weight ≤ 1, and can be removed. (It is superfluous when the weight is less than 1.) Thus, we may assume that each (i,j)-entry of $\mathrm{adj}(A)$ is ν-equivalent to the sum of weights of a simple path from i to j. If this is the only path, then the entry is tangible; otherwise, the entry is ghost. Thus,

$$a'_{i,j} \cong_\nu \sum_{\substack{\pi \in S_n : \\ \pi(j) = i \\ \pi^k(i) \neq i, \ \forall 1 \leq k \leq m}} a_{i,\pi(i)} a_{\pi(i),\pi^2(i)} \cdots a_{\pi^m(i),j}$$

(that is, the sum of simple paths from i to j). As a result, we have

$$A^{\nabla\nabla} = \Bigg(\prod_{\substack{j \neq i : \\ a''_{i,j} \neq a_{i,j}}} E_{i,j} \Bigg) A,$$

where the elementary operations $E_{i,j}$ (the Gaussian elementary matrix adding the j'th row multiplied by $a''_{i,j}$ to the i'th row) are applied only on positions where $a''_{i,j} \neq a_{i,j}$ in the following order: First $E_{k,\ell}$ for all $k < \ell$, followed by $E_{\ell,k}$ for every $k < \ell$ and all $\ell = 2, ..., n$, in increasing order.

Denoting

$$\Bigg(\prod_{\substack{j \neq i : \\ a''_{i,j} \neq a_{i,j}}} E_{i,j} \Bigg) A = (c_{i,j}),$$

we have that

$$c_{i,j} = a_{i,j} + \sum_{\substack{k \neq i,j\,: \\ a_{i,k} \neq a''_{i,k}}} a''_{i,k} a_{k,j} = \begin{cases} a''_{i,j}, & k = j, \\ a_{i,j} + \displaystyle\sum_{\substack{k \neq i,j\,: \\ a_{i,k} \neq a''_{i,k}}} a''_{i,k} a_{k,j}, & k \neq j. \end{cases}$$

For $k \neq j$, we have that

$$a_{i,j} + \sum_{\substack{k \neq i,j\,: \\ a_{i,k} \neq a''_{i,k}}} a''_{i,k} a_{k,j},$$

equals or is dominated by the weight of a simple path from i to j whose weight equals or is dominated by $a''_{i,j}$. Therefore $E_{i,j}$ replaces $a_{i,j}$ by $a''_{i,j}$ in A whenever they are different. □

Let A and B be nonsingular matrices. Over a field, in classical linear algebra, A and B can be written as products of elementary matrices. Thus, one can pass from A to B by applying elementary operations. In the tropical case, whereas we do not have factorizability into elementary matrices, cf. [16, Example 4.5], we do have the second implication, described in the following theorem.

Theorem 6.4. *For any two nonsingular matrices A, B, there exist matrices E_1, E_2, E_3, E_4 which are products of elementary matrices of SL_n, such that $E_1 A E_2 = E_3 B E_4$.*

Proof. Using Lemma 3.2, we write $A^{\nabla\nabla} = \overline{A}^{\nabla\nabla} P$ and $B^{\nabla\nabla} = \overline{B}^{\nabla\nabla} Q$ where $\overline{A}, \overline{B}$ are definite, and P, Q are invertible matrices. Recalling [16, Lemma 6.5], and noting that $P^{\nabla\nabla} = P$ and $Q^{\nabla\nabla} = Q$, we see by Lemma 2.10 that $\overline{A}^{\nabla\nabla}$ and $\overline{B}^{\nabla\nabla}$ are factorizable and respectively ghost-surpass $\overline{A}$ and $\overline{B}$, and

$$A^{\nabla\nabla} = \overline{A}^{\nabla\nabla} P \underset{\mathrm{gs}}{\models} A = \overline{A} P \quad \text{and} \quad B^{\nabla\nabla} = \overline{B}^{\nabla\nabla} Q \underset{\mathrm{gs}}{\models} B = \overline{B} Q,$$

.

Clearly $I A^{\nabla\nabla} B^{\nabla\nabla} = A^{\nabla\nabla} B^{\nabla\nabla} I$, which provides the assertion for $A^{\nabla\nabla}$ and $B^{\nabla\nabla}$. As in Proposition 6.3 we denote by E, E' the elementary products such that

$$\overline{A}^{\nabla\nabla} = E\overline{A} \quad \text{and} \quad \overline{B}^{\nabla\nabla} = E'\overline{B},$$

and have

$$EAB^{\nabla\nabla} = E\overline{A}PB^{\nabla\nabla} = \overline{A}^{\nabla\nabla} P B^{\nabla\nabla} = A^{\nabla\nabla} B^{\nabla\nabla} = A^{\nabla\nabla} \overline{B}^{\nabla\nabla} Q = A^{\nabla\nabla} E' \overline{B} Q = A^{\nabla\nabla} E' B.$$

But E, $B^{\nabla\nabla}$, $A^{\nabla\nabla}$, and E' are products of elementary matrices. □

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