Asymptotic expansion for the magnetic potential in the eddy current problem: the ferromagnetic case

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ASYMPTOTIC EXPANSION FOR THE MAGNETIC POTENTIAL IN THE EDDY CURRENT PROBLEM : THE FERROMAGNETIC CASE.

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ABSTRACT. We derive rigorously a multiscale expansion at any order for the solution of an eddy current problem which is addressed in a bidimensional setting where the conducting medium is magnetic. We make explicit the first terms of the expansion up to the fifth order of approximation with respect to the *skin depth* parameter. As an application of this expansion we derive impedance conditions up to the fifth order for the magnetic potential. This approach leads to solve only equations in the dielectric medium.

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1. Introduction

Many industrial applications in electrothermics require a precise knowledge of the Joule power density. However the so-called eddy current phenomenon is an obstacle to reach this accuracy since it reflects the flow of current near the surface of a metallic conductor. We intend to work in the context of these applications for which we need to compute the magnetic potential for a fairly wide range of frequencies. We consider that the medium of interest consists of a conducting region surrounded by a dielectric region such that the skin depth δ is a small parameter (typically with respect to the wavelength). This raises the difficulty of applying a finite element method on a mesh that combines fine cells inside the boundary layer zone of the conducting part and much larger cells in the dielectric part. To overcome this difficulty it is possible to adopt an asymptotic method to approximate the potential with the first terms of an expansion in power series of the complex parameter $\underline{\delta} = \delta \exp(-i\pi/4)$. As a result we obtain a parameterization of the magnetic potential with respect to the parameter $\underline{\delta}$. All the terms of this parameterization which are defined in the dielectric region are real valued functions and independent of $\underline{\delta}$. Thus it decreases the computational cost since these functions can be computed only one time for the different frequencies. Finally the asymptotics which are defined in the conducting region depend exponentially and polynomially of $\underline{\delta}$.

In this paper we consider a magneto-harmonic problem in 2D where the conducting medium is *magnetic* and we shall apply rigorously this asymptotic method which is based on a multiscale expansion with a scaling technique to describe the boundary layer.

As a consequence of this expansion it is also possible to derive Impedance Boundary Conditions (also called approximate, effective, or equivalent conditions). This concept is classical in the modeling of wave propagation phenomena. Impedance Conditions (ICs) are usually introduced to reduce the computational domain, see for instance [17, 13, 12, 18, 10, 1, 9] for scattering problems from highly absorbing obstacles. The main idea consists to replace the "exact" model inside a part of the domain by an approximate boundary condition. This idea is pertinent when the effective condition can be readily handled for numerical computations, for instance when this condition is local [7, 18, 2, 9]. For a given parameter δ it is possible to adopt this method to "approximate" the conducting medium by an equivalent boundary condition. This new boundary condition is then coupled with the Laplace equation and a finite element method can be applied to solve the resulting boundary value problem. The main drawback of this method regarding our applications is that this condition depends on δ in general (except for the first order condition, which is nothing but the Dirichlet boundary condition) which increases drastically the computational cost for a wide range of frequencies. Therefore we focus mainly in this paper on the validation of the parameterization technics with respect to $\underline{\delta}$ for the potential.

Since several decades the mathematical analysis of the eddy current is a hot research topic world wide [14, 15, 5, 6]. There are several similarities in this work and in the works in Ref. [15, 5, 6, 8]. In the work [15] the authors validate rigorously a multiscale expansion at any order for the solution of an eddy current problem which is addressed in a bidimensional setting where the conducting medium is non-magnetic, the domain is unbounded and the solution grows logarithmically at infinity. In [5, 6] the authors prove estimates when δ is sufficiently small (and a convergence estimate as $\delta \to 0$) and for moderate frequencies (i.e. when the ratio permittivity/conductivity tends to zero) for the electric field solution of the 3D problem time-harmonic eddy current problem. In the work in Ref. [8] the authors address the issue of impedance conditions for the skin effect problem. They derive rigorously ICs up to the fourth order of approximation for a scalar transmission problem (Helmholtz equation) in 3D. Such ICs are justified provided the parameter δ is sufficiently small.

In this paper we focus mainly on the derivation of a multiscale expansion in power series of $\underline{\delta}$ for the magnetic potential and we prove error estimates at any order. The model problem has a specific feature since the magnetic permeability is assumed to be piecewise constant on subdomains. There is a difference between this paper and the works in Ref. [5, 6, 8] since we can prove estimates for all $\delta > 0$. It is worth to notice that contrary to the *skin effect* problem our estimates are available without any cut-off frequencies. We make explicit the first asymptotics up to the order δ^4. The asymptotics defined in the dielectric region have a specific feature since all these functions are real when the data is real. As an application we derive also formally ICs up to the fifth order of approximation for the magnetic potential.

The outline of the paper proceeds as follows. In Section 2 we introduce the studied problem and give the main results: we prove uniform estimates for solutions of the problem for all $\delta > 0$ and we apply this result to the convergence study of an asymptotic expansion as δ tends to zero. In Section 3 we present the first terms of the asymptotic expansion. Then we prove existence and regularity results for the asymptotics and we present a synthesis of the multiscale expansion, Section 4. In Section 5 we prove a convergence result for this expansion. We present elements of derivations for the multiscale expansion, Sec. 6. In Appendix A we give a hierarchy of impedance boundary conditions up to the fifth order of approximation. Appendix B presents a formal derivation of impedance conditions.

2. Statement of the problem and main results

Let Ω be a smooth bounded simply connected domain of the plane $\mathbb{R}^2$ with boundary $\partial\Omega$, and $\Omega_- \subset\subset \Omega$ be a Lipschitz connected subdomain of Ω, with boundary Σ. We denote by Ω_+ the complementary of $\overline{\Omega_-}$ in Ω, see Fig. 2.

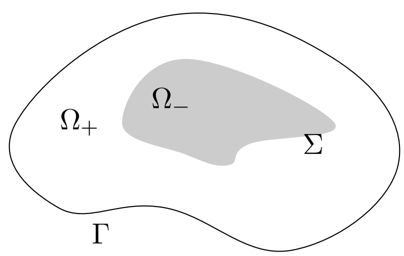

Figure 1 – The domain Ω and its subdomains Ω_-, Ω_+

The magnetic vector potential $\mathcal{A}$ reduced to a single scalar component satisfies the following problem

$$-\operatorname{div}\left(\frac{1}{\mu}\nabla\mathcal{A}\right) + \mathrm{i}\omega\sigma\mathbb{1}_{\Omega_-}\mathcal{A} = J \quad \text{in} \quad \Omega \tag{2.1}$$

Here the magnetic permeability μ is piecewise constant on $\Omega_\pm$ with positive real values : $\mu = \mu_\pm > 0$ in $\Omega_\pm$. In this paper we shall use the following notation

$$\mu_r = \mu_-/\mu_+ \,. \tag{2.2}$$

For the sake of simplicity, we assume that the right hand side J (which represents a current source term) has a support in Ω_+ and it is a smooth function.

Notation 2.1. *We shall denote by h^+ (resp. h^-) the restriction of any function h to Ω_+ (resp. Ω_-).*

We introduce the skin depth parameter defined as $\delta = \dfrac{1}{\sqrt{\omega\mu_-\sigma}} > 0$ inside the problem (2.1). Then the magnetic vector potential $\mathcal{A} = (\mathcal{A}_\delta^+, \mathcal{A}_\delta^-)$ satisfies the following boundary value problem

$$\left\{\begin{array}{llll} -\Delta\mathcal{A}_\delta^+ & = \mu_+ J & \text{in} & \Omega_+ \,, \\ -\Delta\mathcal{A}_\delta^- + \dfrac{\mathrm{i}}{\delta^2}\mathcal{A}_\delta^- & = 0 & \text{in} & \Omega_- \,, \\ \mathcal{A}_\delta^+ & = \mathcal{A}_\delta^- & \text{on} & \Sigma \,, \\ \dfrac{1}{\mu_+}\,\partial_{\mathbf{n}}\mathcal{A}_\delta^+ & = \dfrac{1}{\mu_-}\,\partial_{\mathbf{n}}\mathcal{A}_\delta^- & \text{on} & \Sigma \,, \\ \mathcal{A}_\delta^+ & = 0 & \text{on} & \Gamma \,, \end{array}\right. \tag{2.3}$$

where the differential operator Δ is the Laplace operator in cartesian coordinates $\mathbf{x} = (x, y) \in \mathbb{R}^2$. As a convention, the unit normal vector $\mathbf{n}$ on the interface Σ is inwardly oriented to Ω_-.

2.1. **Uniform estimates.** We present in this part uniform a priori estimates with respect to the parameter δ for the solution of the problem (2.3), when right-hand sides are supported in Ω and on Σ. Precisely, we consider the following boundary value problem

$$\left\{\begin{array}{llll} -\Delta\varphi_\delta^+ & = \mu_+ f^+ & \text{in} & \Omega_+ \,, \\ -\Delta\varphi_\delta^- + \mathrm{i}\delta^{-2}\varphi_\delta^- & = \mu_- f^- & \text{in} & \Omega_- \,, \\ \varphi_\delta^+ & = \varphi_\delta^- & \text{on} & \Sigma \,, \\ \dfrac{1}{\mu_+}\,\partial_{\mathbf{n}}\varphi_\delta^+ & = \dfrac{1}{\mu_-}\,\partial_{\mathbf{n}}\varphi_\delta^- + g & \text{on} & \Sigma \,, \end{array}\right.$$

associated with Dirichlet external boundary conditions on Γ. In this subsection, we assume that the domains Ω_- and Ω_+ are Lipschitz. The functional space suitable for its variational formulation is the Sobolev space $V = \mathrm{H}_0^1(\Omega)$. For any given function $\mu = (\mu_+, \mu_-)$ determined by the two real constants $\mu_\pm$ on $\Omega_\pm$, the variational problem writes: Find $\varphi \in V$ such that

(2.4)
$$\forall\psi \in V, \quad \int_{\Omega_+} \frac{1}{\mu_+}\nabla\varphi^+ \cdot \nabla\overline{\psi}^+ \,\mathrm{d}\mathbf{x} + \int_{\Omega_-} \frac{1}{\mu_-}\nabla\varphi^- \cdot \nabla\overline{\psi}^- \,\mathrm{d}\mathbf{x} + i\delta^{-2}\int_{\Omega_-} \frac{1}{\mu_-}\varphi^-\overline{\psi}^- \,\mathrm{d}\mathbf{x} = \int_\Omega f\,\overline{\psi}\,\mathrm{d}\mathbf{x} + \langle g, \psi\rangle_{\mathrm{H}^{-\frac12}(\Sigma),\,\mathrm{H}^{\frac12}(\Sigma)} \,,$$

where the right-hand sides f and g satisfy the regularity assumption

$$f \in \mathrm{L}^2(\Omega) \quad \text{and} \quad g \in \mathrm{H}^{-\frac12}(\Sigma) \,. \tag{2.5}$$

Our result is the following H^1 a priori estimate, uniform with respect to the small parameter δ.

Theorem 2.2. *For all $\delta > 0$, the problem (2.4) with data f and g satisfying (2.5) has a unique solution $\varphi \in V$ and satisfies the uniform estimate*

$$\|\varphi\|_{1,\Omega} + \delta^{-1}\|\varphi^-\|_{0,\Omega_-} \leqslant C(\|f\|_{0,\Omega} + \|g\|_{-\frac{1}{2},\Sigma}) \tag{2.6}$$

with a constant $C > 0$, independent of δ, f, and g.

Remark 2.1. It is worth noting that estimates (2.6) hold for all $\delta > 0$. This result is a difference with the results [5, Th. 2.3]-[6, Th. 1.2] for the 3D problem time-harmonic eddy current problem where the authors prove estimates for the electric field (and a convergence estimate as $\delta \to 0$) when δ is sufficiently small and for moderate frequencies (i.e. when the ratio permittivity/conductivity tends to zero). This is also a main difference between this eddy current problem and the skin effect problem [8] in the scalar case. Precisely in [8, Th. 2.1] the authors prove uniform estimates for the solution of the skin effect problem in the scalar case but such estimates hold only when the skin depth parameter is sufficiently small.

Proof of Theorem 2.2. We introduce the sesquilinear form a_δ associated with the variational formulation (2.4)

$$a_\delta(\varphi,\psi) := \int_{\Omega_+} \frac{1}{\mu_+} \nabla\varphi^+ \cdot \nabla\overline{\psi}^+ \,\mathrm{d}\mathbf{x} + \int_{\Omega_-} \frac{1}{\mu_-} \nabla\varphi^- \cdot \nabla\overline{\psi}^- \,\mathrm{d}\mathbf{x} + i\delta^{-2} \int_{\Omega_-} \varphi^- \overline{\psi}^- \,\mathrm{d}\mathbf{x} \,.$$

As a consequence of the Poincaré inequality in V

$$\exists C > 0\,, \quad \forall \varphi \in V \quad \int_\Omega |\nabla\varphi|^2 \,\mathrm{d}\mathbf{x} \geqslant C\|\varphi\|^2_{1,\Omega}\,,$$

the sesquilinear form a_δ is strongly coercive on V, i.e.

$$\exists C > 0\,, \quad \forall \delta > 0\,, \quad \forall \varphi \in V \quad \Re a_\delta(\varphi,\varphi) \geqslant \max\{\mu_-,\mu_+\}^{-1} C\|\varphi\|^2_{1,\Omega}\,,$$

and the constant $C > 0$ is independent of δ. Hence, according to the Lax-Milgram lemma, for all $\delta > 0$ the problem (2.4) has a unique solution $\varphi = \varphi_\delta \in V$ and there exists a constant $C_1 > 0$ ($C_1 = \max\{\mu_-,\mu_+\}C^{-1}$) independent of δ such that

$$\forall \delta > 0\,, \quad \|\varphi_\delta\|_{1,\Omega} \leqslant C_1(\|f\|_{0,\Omega} + \|g\|_{-\frac{1}{2},\Sigma})\,. \tag{2.7}$$

Since

$$\Im a_\delta(\varphi,\varphi) = \Im\left(\int_\Omega f\,\overline{\varphi}\,\mathrm{d}\mathbf{x} + \langle g,\varphi\rangle_{\mathrm{H}^{-\frac{1}{2}}(\Sigma),\,\mathrm{H}^{\frac{1}{2}}(\Sigma)}\right)$$

then using (2.7) we infer successively the following inequalities

$$\begin{aligned}\delta^{-2}\int_{\Omega_-} |\varphi^-|^2 \,\mathrm{d}\mathbf{x} &\leqslant \|f\|_{0,\Omega}\|\varphi\|_{0,\Omega} + \|g\|_{-\frac{1}{2},\Sigma}\|\varphi\|_{\frac{1}{2},\Sigma}\\ &\leqslant C_2(\|f\|_{0,\Omega} + \|g\|_{-\frac{1}{2},\Sigma})\|\varphi\|_{1,\Omega}\\ &\leqslant C_3(\|f\|_{0,\Omega} + \|g\|_{-\frac{1}{2},\Sigma})^2\end{aligned}$$

(the constants $C_i > 0$ are independent of δ) and the uniform estimate (2.6) is proved. ☐

2.2. **Application: Convergence of a multiscale expansion.** In this section we assume that Σ is a smooth curve. Then it is possible to derive a multiscale expansion for the solution $(\mathcal{A}_\delta^+, \mathcal{A}_\delta^-)$ of the model problem (2.3): it possesses an asymptotic expansion in power series of the small complex parameter $\underline{\delta} = \dfrac{\delta}{\lambda}$ where $\delta > 0$ and λ is the complex number defined as $\lambda = \frac{\sqrt{2}}{2}(1 + \mathrm{i})$.

$$\mathcal{A}_\delta^+(\mathbf{x}) = \mathcal{A}_0^+(\mathbf{x}) + \underline{\delta}\mathcal{A}_1^+(\mathbf{x}) + \underline{\delta}^2\mathcal{A}_2^+(\mathbf{x}) + \underline{\delta}^3\mathcal{A}_3^+(\mathbf{x}) + \cdots \quad \text{in} \quad \Omega_+\,, \tag{2.8}$$

$$\mathcal{A}_\delta^-(\mathbf{x}) = \mathcal{A}_0^-(\mathbf{x};\delta) + \underline{\delta}\mathcal{A}_1^-(\mathbf{x};\delta) + \underline{\delta}^2\mathcal{A}_2^-(\mathbf{x};\delta) + \underline{\delta}^3\mathcal{A}_3^-(\mathbf{x};\delta) + \cdots \quad \text{in} \quad \Omega_-\,, \tag{2.9}$$

$$\text{with} \quad \mathcal{A}_j^-(\mathbf{x};\delta) = \chi(\nu)\mathfrak{A}_j(\xi, \frac{\nu}{\delta})\,. \tag{2.10}$$

Here $\mathbf{x} \in \mathbb{R}^2$ are the cartesian coordinates and (ξ, ν) is a "*curvilinear coordinate system*" to the curve Σ on the manifold Ω_- : ξ is a curvilinear abscissa on Σ and $\nu \in (0, \nu_0)$ (where ν_0 is sufficiently small, see Section 6) is the distance to the curve Σ. In Eq. (2.10), χ is a smooth cut-off function, its support is included in a tubular neighborhood of Σ, and equal to 1 in a smaller neighborhood. The first-order terms $\mathfrak{A}_j$ and $\mathcal{A}_j^+$ are explicited in Sec. 3. Each term $\mathfrak{A}_j$ is a "*profile*" defined on $\Sigma \times I$ where $I = (0, +\infty)$. This expansion in power series of the complex parameter $\underline{\delta}$ has a specific feature since the terms $\mathcal{A}_j^+$ are real valued functions when the data J is a real valued function, see also [11]. Moreover, for any $j \in \mathbb{N}$

$$\mathcal{A}_j^+ \in \mathrm{H}^1(\Omega_+) \quad \text{and} \quad \mathfrak{A}_j \in \mathrm{H}^1(\Sigma \times \mathbb{R}^+)\,,$$

see Prop. 4.2 for precise results. The validation of the asymptotic expansion (2.8)-(2.9) consist to prove estimates for the remainder $r_{m;\delta}$ defined at any order $m \in \mathbb{N}^*$ as

$$r_{m;\delta} = \mathcal{A}_\delta - \sum_{n=0}^{m} \underline{\delta}^n\, \mathcal{A}_n \quad \text{in} \quad \Omega\,.$$

This is done by an evaluation of the right hand side when the operator $-\operatorname{div}(\frac{1}{\mu}\nabla) + \mathrm{i}\omega\sigma\mathbb{1}_{\Omega_-}$ is applied to the remainder $r_{m;\delta}$. By construction of the expansion (2.8)-(2.9) we obtain

$$\begin{cases} -\Delta r_{m;\delta}^+ & = & 0 & \text{in} \quad \Omega_+\,, \\ -\Delta r_{m;\delta}^- + \mathrm{i}\delta^{-2} r_{m;\delta}^- & = & \mu_- f_{m;\delta}^- & \text{in} \quad \Omega_-\,, \\ r_{m;\delta}^+ & = & r_{m;\delta}^- & \text{on} \quad \Sigma\,, \\ \dfrac{1}{\mu_+}\ \partial_{\mathbf{n}} r_{m;\delta}^+ & = & \dfrac{1}{\mu_-}\ \partial_{\mathbf{n}} r_{m;\delta}^- - \underline{\delta}^m \frac{1}{\mu_+}\partial_{\mathbf{n}}\mathcal{A}_m^+ & \text{on} \quad \Sigma\,, \\ r_{m;\delta}^+ & = & 0 & \text{on} \quad \Gamma\,. \end{cases} \tag{2.11}$$

Here the right hand side $f_{m;\delta}^-$ is (roughly) a residue of the order δ^{m-1}. The main result of this section is the following.

Theorem 2.3. *We assume that J is a smooth data and the interface Σ is a smooth curve. Then there exists a constant $C > 0$ independent of δ such that the remainder $r_{m;\delta}$ satisfies the optimal uniform estimates*

$$\|r_{m;\delta}^+\|_{1,\Omega_+} + \sqrt{\delta}\|\nabla r_{m;\delta}^-\|_{0,\Omega_-} + \delta^{-1}\|r_{m;\delta}^-\|_{0,\Omega_-} \leqslant C\delta^{m+1}\,. \tag{2.12}$$

This Theorem is proved in Section 5.

Remark 2.2. The error estimate (2.12) is available for all $\delta > 0$.

3. First terms of the multiscale expansion

Straightforward calculations lead to the first-order terms $\mathcal{A}_j^- = \chi\mathfrak{A}_j$ and $\mathcal{A}_j^+$. Hereafter we make explicit the asymptotics $(\mathfrak{A}_j, \mathcal{A}_j^+)$ when $j = 0, 1, 2, 3, 4$. Elements of formal derivations are given in Section 6. The first term is nothing but

$$\mathcal{A}_0^- = 0\,.$$

Then $\mathcal{A}_0^+$ solves the problem

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_0^+ &=& \mu_+ J & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_0^+ &=& 0 & \text{on} \quad \Sigma\,, \\ \mathcal{A}_0^+ &=& 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{3.1}$$

The next term which is determined in the asymptotics is the profile $\mathfrak{A}_1$

$$\mathfrak{A}_1(\xi, \Upsilon) = -\mu_r \partial_{\mathbf{n}}\mathcal{A}_0^+\big(\mathbf{X}(\xi)\big)\,\mathrm{e}^{-\lambda\Upsilon}\,. \tag{3.2}$$

Remind that μ_r is defined by (2.2). Then $\mathcal{A}_1^+$ solves

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_1^+ &=& 0 & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_1^+ &=& -\mu_r\partial_{\mathbf{n}}\mathcal{A}_0^+ & \text{on} \quad \Sigma\,, \\ \mathcal{A}_1^+ &=& 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{3.3}$$

The next term which is determined is the profile $\mathfrak{A}_2$

$$\mathfrak{A}_2(\xi, \Upsilon) = (a(\xi) + \Upsilon b(\xi))\mathrm{e}^{-\lambda\Upsilon}\,, \tag{3.4}$$

where

$$a(\xi) = -\mu_r\left(\partial_{\mathbf{n}}\mathcal{A}_1^+ + \frac{k(\xi)}{2}\partial_{\mathbf{n}}\mathcal{A}_0^+\right)\big(\mathbf{X}(\xi)\big) \quad \text{and} \quad b(\xi) = -\lambda\mu_r\frac{k(\xi)}{2}\partial_{\mathbf{n}}\mathcal{A}_0^+\big(\mathbf{X}(\xi)\big)\,.$$

Here $k(\xi)$ denotes a scalar curvature at the point $\mathbf{X}(\xi) \in \Sigma$.

Notation 3.1. *The scalar curvature $k(\xi)$ is well-defined by*

$$\frac{d\mathbf{n}}{d\xi} = -k(\xi)\frac{d\mathbf{X}}{d\xi}.$$

Here, $\mathbf{n}(\xi)$ denotes the unit inner normal at $\mathbf{X}(\xi) \in \Sigma$ which is a normal parameterization of Σ ($\xi \in \mathbb{R}/L\mathbb{Z} =: \mathbb{T}_L$ and L is the length of the curve Σ); $\mathbf{X}$ is a smooth function from $\mathbb{R}$ into $\mathbb{R}^2$.

Then $\mathcal{A}_2^+$ solves the problem

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_2^+ &=& 0 & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_2^+ &=& -\mu_r\left(\partial_{\mathbf{n}}\mathcal{A}_1^+ + \dfrac{k(\xi)}{2}\partial_{\mathbf{n}}\mathcal{A}_0^+\right) & \text{on} \quad \Sigma\,, \\ \mathcal{A}_2^+ &=& 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{3.5}$$

The next term which is determined is the profile $\mathfrak{A}_3$

$$\mathfrak{A}_3(\xi, \Upsilon) = (a_3(\xi) + \Upsilon b_3(\xi) + \Upsilon^2 c_3(\xi))\mathrm{e}^{-\lambda\Upsilon}\,, \tag{3.6}$$

where the functions a_3, b_3, c_3 are given by

$$\begin{cases} a_3 = -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_2^+ + \dfrac{k(\xi)}{2} \partial_{\mathbf{n}} \mathcal{A}_1^+ + \dfrac{1}{2} \left\{ \partial_\xi^2 + \dfrac{3}{4} k(\xi)^2 \mathbb{I} \right\} \partial_{\mathbf{n}} \mathcal{A}_0^+ \right) , \\ b_3 = -\dfrac{\lambda \mu_r}{2} \left(k(\xi) \partial_{\mathbf{n}} \mathcal{A}_1^+ + \left\{ \partial_\xi^2 + \frac{3}{4} k(\xi)^2 \mathbb{I} \right\} \partial_{\mathbf{n}} \mathcal{A}_0^+ \right) , \\ c_3 = -\dfrac{3\lambda^2 \mu_r}{8} k(\xi)^2 \partial_{\mathbf{n}} \mathcal{A}_0^+ . \end{cases}$$

Then $\mathcal{A}_3^+$ solves the problem

(3.7)
$$\begin{cases} -\Delta \mathcal{A}_3^+ = 0 & \text{in} \quad \Omega_+ , \\ \mathcal{A}_3^+ = -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_2^+ + \dfrac{k(\xi)}{2} \partial_{\mathbf{n}} \mathcal{A}_1^+ + \dfrac{1}{2} \left\{ \partial_\xi^2 + \dfrac{3}{4} k(\xi)^2 \mathbb{I} \right\} \partial_{\mathbf{n}} \mathcal{A}_0^+ \right) & \text{on} \quad \Sigma , \\ \mathcal{A}_3^+ = 0 & \text{on} \quad \Gamma . \end{cases}$$

The next term which is determined is the profile $\mathfrak{A}_4$

$$\mathfrak{A}_4(\xi, \Upsilon) = (a_4(\xi) + \Upsilon b_4(\xi) + \Upsilon^2 c_4(\xi) + \Upsilon^3 d_4(\xi)) \mathrm{e}^{-\lambda \Upsilon} , \tag{3.8}$$

where the functions a_4, b_4, c_4, d_4 are given by

$$a_4 = -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_3^+ + \tfrac{k}{2} \partial_{\mathbf{n}} \mathcal{A}_2^+ + \tfrac{1}{2} \left\{ \partial_\xi^2 + \tfrac{3}{4} k^2 \right\} \partial_{\mathbf{n}} \mathcal{A}_1^+ + \tfrac{1}{4} \left\{ \tfrac{11}{2} k \partial_\xi^2 + 7k' \partial_\xi + 3k'' + \tfrac{3}{2} k^3 \right\} \partial_{\mathbf{n}} \mathcal{A}_0^+ \right)$$

$$b_4 = -\frac{\lambda \mu_r}{2} \left(k \partial_{\mathbf{n}} \mathcal{A}_2^+ + \left\{ \partial_\xi^2 + \tfrac{3}{4} k^2 \right\} \partial_{\mathbf{n}} \mathcal{A}_1^+ + \tfrac{1}{2} \left\{ \tfrac{11}{2} k \partial_\xi^2 + 7k' \partial_\xi + 3k'' + \tfrac{3}{2} k^3 \right\} \partial_{\mathbf{n}} \mathcal{A}_0^+ \right) ,$$

$$c_4 = -\frac{\lambda^2 \mu_r}{4} \left(\frac{3k(\xi)^2}{2} \partial_{\mathbf{n}} \mathcal{A}_1^+ + \left\{ \frac{9k(\xi)}{2} \partial_\xi^2 + 5k'(\xi) \partial_\xi + 2k''(\xi) + \frac{3}{2} k(\xi)^3 \right\} \partial_{\mathbf{n}} \mathcal{A}_0^+ \right) (\mathbf{X}(\xi)) ,$$

$$d_4(\xi) = -\frac{5\lambda^3 \mu_r}{16} k(\xi)^3 \partial_{\mathbf{n}} \mathcal{A}_0^+ (\mathbf{X}(\xi)) .$$

Then, $\mathcal{A}_4^+$ solves the problem

$$\begin{cases} -\Delta \mathcal{A}_4^+ = 0 & \text{in} \quad \Omega_+ , \\ \mathcal{A}_4^+ = a_4(\xi) & \text{on} \quad \Sigma , \\ \mathcal{A}_4^+ = 0 & \text{on} \quad \Gamma , \end{cases} \tag{3.9}$$

where the function a_4 is given by

$$a_4 = -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_3^+ + \frac{k}{2} \partial_{\mathbf{n}} \mathcal{A}_2^+ + \frac{1}{2} \left\{ \partial_\xi^2 + \frac{3}{4} k^2 \right\} \partial_{\mathbf{n}} \mathcal{A}_1^+ + \frac{1}{4} \left\{ \frac{11}{2} k \partial_\xi^2 + 7k' \partial_\xi + 3k'' + \frac{3}{2} k^3 \right\} \partial_{\mathbf{n}} \mathcal{A}_0^+ \right)$$

4. Existence and regularity of the asymptotics

4.1. **Regularity results for the first terms.** It is possible to prove that problem (3.1) (resp. (3.3), (3.5), (3.7), (3.9)) has a unique weak solution $\mathcal{A}_0^+$ in the Sobolev space $\mathrm{H}_0^1(\Omega_+)$ (resp. $\mathcal{A}_l^+$ in the space $V = \{u \in \mathrm{H}^1(\Omega_+) \mid u = 0 \quad \text{on} \quad \Gamma\}$, $l = 1, 2, 3, 4$) when both the data J and the domain Ω_+ are sufficiently smooth. We first describe briefly the derivate consuming process of the expansion (2.8) for L^2 data.

The derivate consuming process of the expansion. We assume that the data $J \in \mathrm{L}^2(\Omega)$. Then, the problem (3.1) is solvable in $\mathrm{H}^1_0(\Omega_+)$. Since this problem is elliptic, the solution $\mathcal{A}^+_0$ has an optimal regularity which depends on the regularity of the data. Hence, $\mathcal{A}^+_0 \in \mathrm{H}^2(\Omega_+) \cap \mathrm{H}^1_0(\Omega_+)$. Moreover, this solution satisfies an a priori estimate:

$$\|\mathcal{A}^+_0\|_{2,\Omega_+} \leqslant C\|J\|_{0,\Omega_+} \tag{4.1}$$

with a constant $C > 0$ independent of J. Then, the trace $\partial_{\mathbf{n}}\mathcal{A}^+_0\big|_\Sigma$ has to be inserted as a Dirichlet data into problem (3.3) defining $\mathcal{A}^+_1$. Since $\partial_{\mathbf{n}}\mathcal{A}^+_0\big|_\Sigma$ belongs to $\mathrm{H}^{1/2}(\Sigma)$, the problem (3.3) is solvable in $\mathrm{H}^1(\Omega_+)$. Moreover, the solution satisfies the estimate:

$$\|\mathcal{A}^+_1\|_{1,\Omega_+} \leqslant C\|\partial_{\mathbf{n}}\mathcal{A}^+_0\|_{1/2,\Sigma}$$

and, from (4.1), we infer the estimate:

$$\|\mathcal{A}^+_1\|_{1,\Omega_+} \leqslant C_1\|J\|_{0,\Omega_+} \,. \tag{4.2}$$

Hence we loose one rank of regularity to derive $\mathcal{A}^+_1$ from the first asymptotic $\mathcal{A}^+_0$: this is due to the fact that only one derivative of $\mathcal{A}^+_0$ is consumed to derive $\mathcal{A}^+_1$. The problem (3.5) is not solvable in $\mathrm{H}^1(\Omega_+)$ since the right hand side on Σ does not belong to the space $\mathrm{H}^{1/2}(\Sigma)$ ($\partial_{\mathbf{n}}\mathcal{A}^+_1\big|_\Sigma$ belongs a priori only to $\mathrm{H}^{-1/2}(\Sigma)$).

Using elliptic regularity results it is also possible to prove that the problem (3.5) (resp. (3.7)) has a unique weak solution $\mathcal{A}^+_2$ (resp. $\mathcal{A}^+_3$) in the Sobolev space $W = \{u \in \mathrm{H}^1(\Omega_+) \mid u = 0 \quad \text{on} \quad \Gamma\}$) when the data J belongs to $\mathrm{H}^1(\Omega)$ (resp. $\mathrm{H}^2(\Omega)$) and when the domain Ω_+ is sufficiently smooth. The next proposition ensures regularity results in Sobolev spaces H^s for each term $\mathcal{A}^+_l$ and $\mathfrak{A}_l$, $l = 0, 1, 2, 3$.

Proposition 4.1. *Let k be an integer such that $k \geqslant 2$. We assume that the domain Ω_+ is of class $\mathcal{C}^{k+2}$. If J belongs to $\mathrm{H}^k(\Omega_+)$ then $\mathcal{A}^+_l$ belongs to $\mathrm{H}^{k+2-l}(\Omega_+)$ for $l = 0, 1, 2, 3$ and $\mathfrak{A}_l$ belongs to $\mathrm{H}^{k+\frac{3}{2}-l}(\Sigma \times I)$ for $l = 1, 2, 3$.*

Proof. We prove successively that $\mathcal{A}^+_0 \in \mathrm{H}^{k+2}(\Omega_+)$, $\mathfrak{A}_1 \in \mathrm{H}^{k+\frac{1}{2}}(\Sigma \times I)$, $\mathcal{A}^+_1 \in \mathrm{H}^{k+1}(\Omega_+)$, $\mathfrak{A}_2 \in \mathrm{H}^{k-\frac{1}{2}}(\Sigma \times I)$, $\mathcal{A}^+_2 \in \mathrm{H}^{k}(\Omega_+)$, $\mathfrak{A}_3 \in \mathrm{H}^{k-\frac{3}{2}}(\Sigma \times I)$ and $\mathcal{A}^+_3 \in \mathrm{H}^{k-1}(\Omega_+)$. The regularity of $\mathcal{A}^+_0$, $\mathcal{A}^+_1$, $\mathcal{A}^+_2$ and $\mathcal{A}^+_3$ is a consequence of a general elliptic regularity result in Sobolev spaces for the Laplace operator. ☐

It is also possible to prove regularity results in analytic classes. We assume that Ω_+ is analytic. If J is an analytic function on Ω_+ then each term $\mathcal{A}^+_l$ is an analytic function on Ω_+, see for instance [4, Th 4.1.1].

4.2. **Synthesis of the multiscale expansion.** Following the process of construction described above it is possible to derive the asymptotics $\mathcal{A}^\pm_j$ at any order when the data J is smooth (of class $\mathcal{C}^\infty$). In this section we specify this result in Prop. 4.2.

The following definition for the resolvent operator $\mathcal{R}$ makes sense and define a bounded operator

$$\begin{array}{rccc} \mathcal{R} \; : & \mathrm{H}^{\frac{3}{2}}(\Sigma) & \to & \{\varphi \in \mathrm{H}^2(\Omega_+) \,|\, \varphi = 0 \quad \text{on} \quad \Gamma\} \\ & h & \mapsto & \varphi \end{array}$$

where φ is the unique solution (in distributional sense) of the boundary value problem

$$\left\{\begin{array}{lcll}\Delta\varphi & = & 0 & \text{in} \quad \Omega_+ \\ \varphi & = & h & \text{on} \quad \Sigma \\ \varphi & = & 0 & \text{on} \quad \Gamma\end{array}\right.$$

For all $t \geqslant 2$, the resolvent operator $\mathcal{R}$ is still a bounded operator in the following spaces

$$\mathrm{H}^{t-\frac{1}{2}}(\Sigma) \rightarrow \{\varphi \in \mathrm{H}^t(\Omega_+) \,|\, \varphi = 0 \quad \text{on } \Gamma\}$$

Proposition 4.2. *Let $s \geqslant 2$ and assume that $J \in \mathrm{H}^{s-2}(\Omega)$ has a support in Ω_+. Assume that the interface Σ is a curve of class $\mathcal{C}^\infty$. For all $j = 0, \cdots, \lfloor s \rfloor - 1$, it is possible to derive successively the following terms which are independent of δ*

$$\mathfrak{A}_j \in \mathrm{H}^{s-j-\frac{1}{2}}\big(\Sigma, \mathcal{C}^\infty(I)\big) \quad \textit{and} \quad \mathcal{A}_j^+ \in \mathrm{H}^{s-j}(\Omega_+)\,.$$

For $j = 0$, $\mathfrak{A}_0 = 0$ and $\mathcal{A}_0^+ \in \mathrm{H}^s(\Omega_+) \cap \mathrm{H}^1_0(\Omega_+)$ is the solution of problem (3.1). For all $j = 1, \cdots, \lfloor s \rfloor - 1$, there exists a tangential operator $\mathfrak{T}_j$ of order j which is defined as follow

$$\begin{array}{cccc}\mathfrak{T}_j & : & \mathrm{H}^s(\Omega_+) & \rightarrow & \mathrm{H}^{s-j-\frac{1}{2}}(\Sigma) \\ & & \mathcal{A}_0^+ & \mapsto & \mathcal{A}_j^+|_\Sigma\end{array}$$

such that

$$\mathcal{A}_j^+ = \mathcal{R} \circ \mathfrak{T}_j(\mathcal{A}_0^+)\,. \tag{4.3}$$

In particular for $j = 1, 2, 3$, the operator $\mathfrak{T}_j$ is defined as

$$\begin{aligned}&\mathfrak{T}_1(\mathcal{A}) = -\mu_r \partial_{\mathbf{n}} \mathcal{A}|_\Sigma \\ &\mathfrak{T}_2 = \left(\mathfrak{T}_1 \circ \mathcal{R} + \frac{k(\xi)}{2}\mathbb{I}\right) \circ \mathfrak{T}_1 \\ &\mathfrak{T}_3 = \mathfrak{T}_1 \circ \mathcal{R} \circ \left(\mathfrak{T}_2 + \frac{k(\xi)}{2}\mathfrak{T}_1\right) + \frac{1}{2}\left(\partial_\xi^2 + \frac{3}{4}k(\xi)^2\mathbb{I}\right) \circ \mathfrak{T}_1\end{aligned} \tag{4.4}$$

Furthermore $\mathfrak{A}_1$ (resp. $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$) is defined by (3.2) (resp. (3.4), (3.6), (3.8)) and for all $n \geqslant 5$, the unique solution of the ODE (6.9) such that $\mathfrak{A}_n(\cdot, \Upsilon) \longrightarrow 0$ as $\Upsilon \to \infty$ writes

$$\mathfrak{A}_n(\xi, \Upsilon) = (a_n(\xi) + \Upsilon b_n(\xi) + \Upsilon^2 c_n(\xi) + \cdots + \Upsilon^{n-1} d_n(\xi))\mathrm{e}^{-\lambda\Upsilon}\,. \tag{4.5}$$

The solution $\mathcal{A}_\delta$ of problem (2.3) possesses the following asymptotic expansion

$$\mathcal{A}_\delta^+(\mathbf{x}) = \mathcal{A}_0^+(\mathbf{x}) + \underline{\delta}\mathcal{A}_1^+(\mathbf{x}) + \underline{\delta}^2\mathcal{A}_2^+(\mathbf{x}) + \underline{\delta}^3\mathcal{A}_3^+(\mathbf{x}) + \cdots \quad \textit{in} \quad \Omega_+\,,$$

$$\mathcal{A}_\delta^-(\mathbf{x}) = \underline{\delta}\chi(\nu)\mathfrak{A}_1(\xi, \frac{\nu}{\delta}) + \underline{\delta}^2\chi(\nu)\mathfrak{A}_2(\xi, \frac{\nu}{\delta}) + \underline{\delta}^3\chi(\nu)\mathfrak{A}_3(\xi, \frac{\nu}{\delta}) + \cdots \quad \textit{in} \quad \Omega_-\,,$$

(2.8)-(2.9), *see Theorem 2.3 for precise estimates.*

5. Convergence of the asymptotic expansion

The aim of this section is to present a proof of Theorem 2.3 and error estimates (2.12). We first focus on the validation of the asymptotic expansion (2.8)-(2.9) up to the order δ^3. It consist in proving estimates for the remainder $r_{3;\delta}$ defined as

$$r_{3;\delta} = \mathcal{A}_\delta - \sum_{n=0}^{3} \underline{\delta}^n \mathcal{A}_n \quad \text{in} \quad \Omega\,.$$

Theorem 5.1. *We assume that the data J is smooth (at least in $\mathrm{H}^{\frac{9}{2}}(\Omega_+)$) and the domain Ω_+ is smooth (at least of class $\mathcal{C}^7$). There exists a constant $C > 0$ independent of δ such that for all $\delta > 0$ the remainder $r_{3;\delta}$ satisfies the optimal uniform estimates*

$$\|r^+_{3;\delta}\|_{1,\Omega_+} + \delta^{\frac{1}{2}}\|\nabla r^-_{3;\delta}\|_{0,\Omega_-} + \delta^{-1}\|r^-_{3;\delta}\|_{0,\Omega_-} \leqslant C\delta^4 \,.$$

Proof. We first evaluate the right hand side when the operator $-\operatorname{div}(\frac{1}{\mu}\nabla) + \mathrm{i}\omega\sigma\mathbb{1}_{\Omega_-}$ is applied to the remainder $r_{3;\delta}$. By construction of the expansion (2.8)-(2.9) we obtain

$$\begin{cases} -\Delta r^+_{3;\delta} = 0 & \text{in} \quad \Omega_+ \,, \\ -\Delta r^-_{3;\delta} + \mathrm{i}\delta^{-2} r^-_{3;\delta} = \mu_- f^-_{3;\delta} & \text{in} \quad \Omega_- \,, \\ r^+_{3;\delta} = r^-_{3;\delta} & \text{on} \quad \Sigma \,, \\ \dfrac{1}{\mu_+}\,\partial_{\mathbf{n}} r^+_{3;\delta} = \dfrac{1}{\mu_-}\,\partial_{\mathbf{n}} r^-_{3;\delta} - \underline{\delta}^3 \frac{1}{\mu_+}\partial_{\mathbf{n}}\mathcal{A}^+_3 & \text{on} \quad \Sigma \,, \\ r^+_{3;\delta} = 0 & \text{on} \quad \Gamma \,. \end{cases}$$

Here the right hand side $f^-_{3;\delta}$ is a residue of the order δ^2, we refer the reader to [3, Rem. 5.1] and [16, Prop. 7.4] for a similar context. According to Proposition 4.1 we can assume that $f^-_{3;\delta}$ belongs to $\mathrm{L}^2(\Omega_-)$ since the data J is sufficiently smooth. Hence there exists a constant $C > 0$ independent of δ such that

$$\|f^-_{3;\delta}\|_{0,\Omega_-} \leqslant C\delta^2 \,. \tag{5.1}$$

Since the domain Ω_+ is (at least) of class $\mathcal{C}^4$ and since J belongs to $\mathrm{H}^2(\Omega_+)$, then according to Proposition 4.1 we deduce that the term $\mathcal{A}^+_3$ belongs to $\mathrm{H}^1(\Omega_+)$ and $\partial_{\mathbf{n}}\mathcal{A}^+_3$ belongs to $\mathrm{H}^{-\frac{1}{2}}(\Sigma)$. Hence we can apply Theorem 2.2 to $r_{3;\delta}$ and we obtain

$$\|r_{3;\delta}\|_{1,\Omega} + \delta^{-1}\|r^-_{3;\delta}\|_{0,\Omega_-} \leqslant C(\|f^-_{3;\delta}\|_{0,\Omega} + \delta^3\|\partial_{\mathbf{n}}\mathcal{A}^+_3\|_{-\frac{1}{2},\Sigma}) \tag{5.2}$$

with a constant $C > 0$ independent of δ. Combining this estimate with (5.1) we deduce

$$\|r_{3;\delta}\|_{1,\Omega} + \delta^{-1}\|r^-_{3;\delta}\|_{0,\Omega_-} \leqslant C\delta^2 \,.$$

For a smoother right hand side $J \in \mathrm{H}^{\frac{9}{2}}(\Omega_+)$ and a smooth domain Ω_+ (at least of class $\mathcal{C}^7$) we can push two steps forward the asymptotic expansion (up to the term $\mathcal{A}_5$ and Proposition 4.1 is still true when $l = 4, 5$). We obtain, similarly to the previous estimate, the following one for the remainder $r_{5;\delta}$

$$\|r_{5;\delta}\|_{1,\Omega} + \delta^{-1}\|r^-_{5;\delta}\|_{0,\Omega_-} \leqslant C\delta^4 \,.$$

We then obtain optimal uniform estimates for $r_{3;\delta} = r_{5;\delta} + \underline{\delta}^4\,\mathcal{A}_4 + \underline{\delta}^5\,\mathcal{A}_5$

$$\|r^+_{3;\delta}\|_{1,\Omega_+} + \delta^{\frac{1}{2}}\|\nabla r^-_{3;\delta}\|_{0,\Omega_-} + \delta^{-1}\|r^-_{3;\delta}\|_{0,\Omega_-} \,, \leqslant C\delta^4$$

since as a consequence of the estimate

$$\delta^{-\frac{1}{2}}\|\mathcal{A}^-_j\|_{0,\Omega_-} + \delta^{\frac{1}{2}}\|\nabla\mathcal{A}^-_j\|_{0,\Omega_-} \leqslant C\|\mathfrak{A}_j\|_{1,\Sigma\times\mathbb{R}^+} \,,$$

we can deduce the following estimates for $\mathcal{A}_j$ (which is available for any $j \leqslant 5$)

$$\|\mathcal{A}^+_j\|_{1,\Omega_+} + \delta^{-\frac{1}{2}}\|\mathcal{A}^-_j\|_{0,\Omega_-} + \delta^{\frac{1}{2}}\|\nabla\mathcal{A}^-_j\|_{0,\Omega_-} \leqslant C_j \,.$$

□

Proof of Theorem 2.3. The right hand side $f^-_{m;\delta}$ in (2.11) is a residue of the order δ^{m-1}, we refer the reader to [3, Rem. 5.1] and [16, Prop. 7.4] for a similar context. Then it is possible to adapt the proof of Theorem 5.1 which is given in this section since the data J is smooth and the interface Σ is smooth.

6. Elements of derivations for the multiscale expansion

We assume that Σ is a smooth ($\mathcal{C}^\infty$) curve of length L. Let $\mathbf{X}(\xi)$, $\xi \in \mathbb{R}/L\mathbb{Z} =: \mathbb{T}_L$, be a normal parameterization of Σ, where $\mathbf{X}$ is a smooth function from $\mathbb{R}$ into $\mathbb{R}^2$. By abuse of notation, we write $\xi \in \Sigma$ the point $\mathbf{X}(\xi) \in \Sigma$.

For $\upsilon_0 > 0$ small enough, the tubular neighborhood $\mathcal{V}(\Sigma)$ of Σ of diameter υ_0 in the inner domain Ω_- is parameterized by

$$\mathcal{V}(\Sigma) = \{\mathbf{x}(\xi,\upsilon) = \mathbf{X}(\xi) + \upsilon\mathbf{n}(\xi), \quad (\xi,\upsilon) \in \Sigma \times (0,\upsilon_0)\} \ .$$

Here, $\mathbf{n}(\xi)$ denotes the unit inner normal at $\mathbf{X}(\xi)$. This system of coordinates is orthogonal, and the first fundamental form writes

$$Q^2(\xi,\upsilon)\,(d\xi)^2 + (d\upsilon)^2, \quad \text{with} \quad Q(\xi,\upsilon) = (1 - \upsilon k(\xi)),$$

where $k(\xi)$ denotes the curvature at the point $\mathbf{X}(\xi)$ defined by

$$\frac{d\mathbf{n}}{d\xi} = -k(\xi)\frac{d\mathbf{X}}{d\xi}.$$

For $\upsilon_0 < 1/\|k\|_\infty$ the change of coordinates

$$\Psi : \mathbf{x} = (x,y) \longmapsto (\xi,\upsilon) \tag{6.1}$$

is a $\mathcal{C}^\infty$-diffeomorphism from $\mathcal{V}(\Sigma)$ into the cylinder $\mathbb{T}_L \times (0,\upsilon_0)$:

$$\mathcal{V}(\Sigma) = \Psi^{-1}\big(\mathbb{T}_L \times (0,\upsilon_0)\big) \ .$$

Hereafter, we set the ansatz for the solution of problem (2.3):

$$\begin{aligned} \mathcal{A}^+_\delta &\sim \sum_{j\geqslant 0} \underline{\delta}^j \ \mathcal{A}^+_j(\mathbf{x}) \\ \mathcal{A}^-_\delta &\sim \sum_{j\geqslant 0} \underline{\delta}^j \ \mathcal{A}^-_j(\mathbf{x};\delta) \quad \text{with} \quad \mathcal{A}^-_j(\mathbf{x};\delta) = \chi(\upsilon)\mathfrak{A}_j(\xi,\frac{\upsilon}{\delta}) \end{aligned}$$

where $\underline{\delta} = \dfrac{\delta}{\lambda}$ and

$$\lambda = \frac{\sqrt{2}}{2}(1+\mathrm{i}) \ . \tag{6.2}$$

The function $(\xi,\upsilon) \mapsto \chi(\upsilon)$ is a smooth cut-off function whose support is included in $\overline{\mathcal{V}(\Sigma)}$, and equal to 1 in a smaller tubular neighborhood of the interface.

6.1. **Expansion of the operators in power series of δ.** In order to exhibit the formal expansion of $\mathcal{A}_\delta^-$ we first use the change of variables Ψ in order to write the equations in the cylinder $\mathbb{T}_L \times (0, \upsilon_0)$. We then perform the rescaling

$$\Upsilon = \frac{\upsilon}{\delta} \tag{6.3}$$

in the equations set in Ω_- and Σ in order to make appear the small parameter δ in the equations. The Laplace operator writes in coordinates (ξ, ν) as

$$\Delta = Q^{-1}\left[\delta^{-2}\partial_\Upsilon(Q\partial_\Upsilon) + \partial_\xi(Q^{-1}\partial_\xi)\right] , \tag{6.4}$$

where $Q = (1 - \delta\Upsilon k(\xi))$. This operator Δ (6.4) expands in power series of δ with coefficients intrinsic operators with respect to Σ

$$\Delta = \frac{1}{\delta^2}\partial_\Upsilon^2 + \sum_{n\geqslant -1} \delta^n D_n . \tag{6.5}$$

Note that this expansion (6.5) is a convergent power series expansion when $\Upsilon \in I_\delta = (0, \frac{\nu_0}{\delta})$. Here for all $n \geq -1$, $D_n = D_n(\xi, \Upsilon; \partial_\xi, \partial_\Upsilon)$ is an operator with at most one derivative in the Υ-variable, which has smooth coefficients in ξ, and polynomial coefficients of degree $n-1$ in Υ. Note that in particular $D_{-1} = -k(\xi)\partial_\Upsilon$. In the following we shall use the operator $\mathrm{A}_n := D_{n-2}$

$$\delta^2\Delta = \partial_\Upsilon^2 + \sum_{n\geqslant 1} \delta^n \mathrm{A}_n.$$

Observe that

$$\mathrm{A}_1 = -k(\xi)\partial_\Upsilon , \tag{6.6a}$$

$$\mathrm{A}_2 = \partial_\xi^2 - k(\xi)^2\Upsilon\partial_\Upsilon , \tag{6.6b}$$

$$\mathrm{A}_3 = -k(\xi)^3\Upsilon^2\partial_\Upsilon + \Upsilon k'(\xi)\partial_\xi + 2\Upsilon k(\xi)\partial_\xi^2 . \tag{6.6c}$$

Similarly we write

$$\partial_{\mathbf{n}}(\xi; \partial_\Upsilon) = \frac{1}{\delta}\partial_\Upsilon$$

on the interface Σ. Define v_δ by

$$\mathrm{v}_\delta(\xi, \Upsilon) = \mathcal{A}_\delta^-(\mathbf{x}) .$$

After the scaling $\upsilon \mapsto \Upsilon = \dfrac{\upsilon}{\delta}$ in $\mathcal{V}(\Sigma)$, the problem (2.3) writes (with $\mu_r = \mu_-/\mu_+$)

$$\left\{\begin{array}{lcll} (-\partial_\Upsilon^2 + \mathrm{i})\mathrm{v}_\delta - \displaystyle\sum_{n\geqslant 1}\delta^n \mathrm{A}_n \mathrm{v}_\delta & = & 0 & \text{in} \quad \mathbb{T}_L \times (0, +\infty), \\ \partial_\Upsilon \mathrm{v}_\delta & = & \delta\mu_r\partial_{\mathbf{n}}\mathcal{A}_\delta^+ & \text{on} \quad \mathbb{T}_L \times \{0\} , \end{array}\right. \tag{6.7}$$

and

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_\delta^+ & = & \mu_+ J & \text{in} \quad \Omega_+ , \\ \mathcal{A}_\delta^+ & = & \mathrm{v}_\delta & \text{on} \quad \Sigma , \\ \mathcal{A}_\delta^+ & = & 0 & \text{on} \quad \Gamma . \end{array}\right. \tag{6.8}$$

6.2. **Equations for the coefficients of the magnetic potential.** We insert the Ansatz

$$\mathcal{A}_\delta^+ \sim \sum_{n\geqslant 0} \underline{\delta}^n\, \mathcal{A}_n^+(\mathbf{x}) \quad \text{and} \quad \mathrm{v}_\delta \sim \sum_{n\geqslant 0} \underline{\delta}^n\, \mathfrak{A}_n(\xi, \frac{\upsilon}{\delta})$$

with $\mathfrak{A}_n(\cdot,\Upsilon) \longrightarrow 0$ as $\Upsilon \to \infty$, in equations (6.8) and (6.7), and we perform the identification of terms with the same power in δ. The terms $\mathfrak{A}_n$ and $\mathcal{A}_n^+$ satisfy the following family of problems coupled by their conditions on the interface Σ (corresponding to $\Upsilon = 0$):

$$\left\{\begin{array}{lcll} -\partial_\Upsilon^2\mathfrak{A}_n + \lambda^2\mathfrak{A}_n &=& \displaystyle\sum_{p=1}^{n} \lambda^p \mathrm{A}_p \mathfrak{A}_{n-p} & \text{in} \quad \Sigma\times(0,+\infty)\,, \\ \partial_\Upsilon \mathfrak{A}_n &=& \lambda\mu_r\partial_{\mathbf{n}}\mathcal{A}_{n-1}^+ & \text{on} \quad \Sigma\,, \end{array}\right. \tag{6.9}$$

and

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_n^+ &=& \mu_+ J\delta_n^0 & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_n^+ &=& \mathfrak{A}_n & \text{on} \quad \Sigma\,, \\ \mathcal{A}_n^+ &=& 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{6.10}$$

In (6.9), we use the convention $\mathcal{A}_{-1}^+ = 0$ and in (6.10) δ_n^0 denotes the Kronecker symbol. Hereafter, we make explicit the first asymptotics $\mathfrak{A}_n$ and $\mathcal{A}_n^+$ for $n = 0, 1, 2, 3, 4$ by induction.

6.3. **First terms of the asymptotics for the magnetic potential.**

$$\left\{\begin{array}{lcll} -\partial_\Upsilon^2\mathfrak{A}_0(\cdot,\Upsilon) + \lambda^2\mathfrak{A}_0(\cdot,\Upsilon) &=& 0 & \text{for} \quad \Upsilon\in(0,+\infty)\,, \\ \partial_\Upsilon\mathfrak{A}_0(\cdot,0) &=& 0\,. \end{array}\right. \tag{6.11}$$

The unique solution of (6.11) such that $\mathfrak{A}_0 \to 0$ when $\Upsilon \to \infty$ is

$$\mathfrak{A}_0 = 0\,.$$

Hence, according to (6.9) for $n = 0$, $\mathcal{A}_0^+$ solves the problem

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_0^+ &=& \mu_+ J & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_0^+ &=& 0 & \text{on} \quad \Sigma\,, \\ \mathcal{A}_0^+ &=& 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{6.12}$$

The next term which is determined in the asymptotics is the profile $\mathfrak{A}_1$. According to (6.9) for $n = 1$ and since $\mathfrak{A}_0 = 0$, $\mathfrak{A}_1$ solves the ODE

$$\left\{\begin{array}{lcll} -\partial_\Upsilon^2\mathfrak{A}_1(\cdot,\Upsilon) + \lambda^2\mathfrak{A}_1(\cdot,\Upsilon) &=& 0 & \text{for} \quad \Upsilon\in(0,+\infty)\,, \\ \partial_\Upsilon\mathfrak{A}_1(\cdot,0) &=& \lambda\mu_r\partial_{\mathbf{n}}\mathcal{A}_0^+(\cdot,0)\,. \end{array}\right. \tag{6.13}$$

The unique solution of (6.13) such that $\mathfrak{A}_1 \to 0$ when $\Upsilon \to \infty$ is

$$\mathfrak{A}_1(\xi,\Upsilon) = -\mu_r\partial_{\mathbf{n}}\mathcal{A}_0^+\big(\mathbf{X}(\xi)\big)\,\mathrm{e}^{-\lambda\Upsilon}\,. \tag{6.14}$$

Hence, according to (6.10) for $n = 1$, $\mathcal{A}_1^+$ satisfies the problem

$$\left\{\begin{array}{lcll} -\Delta\mathcal{A}_1^+ &=& 0 & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_1^+ &=& -\mu_r\partial_{\mathbf{n}}\mathcal{A}_0^+ & \text{on} \quad \Sigma\,, \\ \mathcal{A}_1^+ &=& 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{6.15}$$

The next term which is determined is the profile $\mathfrak{A}_2$. According to (6.9) for $n = 2$, $\mathfrak{A}_2$ solves the ODE

$$\left\{\begin{array}{lcll} -\partial_\Upsilon^2 \mathfrak{A}_2(\cdot,\Upsilon) + \lambda^2 \mathfrak{A}_2(\cdot,\Upsilon) & = & -\lambda k(\xi)\partial_\Upsilon \mathfrak{A}_1(\cdot,\Upsilon) & \text{for} \quad \Upsilon \in (0,+\infty)\,, \\ \partial_\Upsilon \mathfrak{A}_2(\cdot,0) & = & \lambda \mu_r \partial_{\mathbf{n}} \mathcal{A}_1^+(\cdot,0)\,. & \end{array}\right. \tag{6.16}$$

According to (6.14) the previous right-hand side writes

$$-\lambda k(\xi)\partial_\Upsilon \mathfrak{A}_1(\cdot,\Upsilon) = -\mu_r \lambda^2 k(\xi)\partial_{\mathbf{n}} \mathcal{A}_0^+ \left(\mathbf{X}(\xi)\right) \mathrm{e}^{-\lambda\Upsilon}\,,$$

hence the unique solution of (6.16) such that $\mathfrak{A}_2 \to 0$ when $\Upsilon \to \infty$ is (see (3.4))

$$\mathfrak{A}_2(\xi,\Upsilon) = \left(a(\xi) + \Upsilon b(\xi)\right) \mathrm{e}^{-\lambda\Upsilon}\,, \tag{6.17}$$

where

$$a(\xi) = -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_1^+ + \frac{k(\xi)}{2}\partial_{\mathbf{n}} \mathcal{A}_0^+\right)\left(\mathbf{X}(\xi)\right) \quad \text{and} \quad b(\xi) = -\lambda\mu_r \frac{k(\xi)}{2}\partial_{\mathbf{n}} \mathcal{A}_0^+ \left(\mathbf{X}(\xi)\right)\,.$$

Hence, according to (6.10) for $n = 2$, $\mathcal{A}_2^+$ solves the problem (see (3.5))

$$\left\{\begin{array}{lcll} -\Delta \mathcal{A}_2^+ & = & 0 & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_2^+ & = & -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_1^+ + \dfrac{k(\xi)}{2}\partial_{\mathbf{n}} \mathcal{A}_0^+\right) & \text{on} \quad \Sigma\,, \\ \mathcal{A}_2^+ & = & 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{6.18}$$

The next term which is determined is the profile $\mathfrak{A}_3$. According to (6.9) for $n = 3$, $\mathfrak{A}_3$ solves the ODE for $\Upsilon \in (0,+\infty)$

$$\left\{\begin{array}{lcl} -\partial_\Upsilon^2 \mathfrak{A}_3(\cdot,\Upsilon) + \lambda^2 \mathfrak{A}_3(\cdot,\Upsilon) & = & -\lambda k(\xi)\partial_\Upsilon \mathfrak{A}_2(\cdot,\Upsilon) + \lambda^2 \left(\partial_\xi^2 - k(\xi)^2 \Upsilon \partial_\Upsilon\right) \mathfrak{A}_1(\cdot,\Upsilon) \\ \partial_\Upsilon \mathfrak{A}_3(\cdot,0) & = & \lambda \mu_r \partial_{\mathbf{n}} \mathcal{A}_2^+(\cdot,0)\,. \end{array}\right. \tag{6.19}$$

According to (6.14)-(6.17), the unique solution of (6.19) such that $\mathfrak{A}_3 \to 0$ when $\Upsilon \to \infty$ is (see (3.6))

$$\mathfrak{A}_3(\xi,\Upsilon) = (a_3(\xi) + \Upsilon b_3(\xi) + \Upsilon^2 c_3(\xi))\mathrm{e}^{-\lambda\Upsilon}\,, \tag{6.20}$$

where the functions a_3, b_3, c_3 are given by

$$\left\{\begin{array}{l} a_3 = -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_2^+ + \dfrac{k(\xi)}{2}\partial_{\mathbf{n}} \mathcal{A}_1^+ + \dfrac{1}{2}\left\{\partial_\xi^2 + \dfrac{3}{4}k(\xi)^2 \mathbb{I}\right\}\partial_{\mathbf{n}} \mathcal{A}_0^+\right)\,, \\ b_3 = -\dfrac{\lambda\mu_r}{2}\left(k(\xi)\partial_{\mathbf{n}} \mathcal{A}_1^+ + \left\{\partial_\xi^2 + \frac{3}{4}k(\xi)^2 \mathbb{I}\right\}\partial_{\mathbf{n}} \mathcal{A}_0^+\right)\,, \\ c_3 = -\dfrac{3\lambda^2\mu_r}{8}k(\xi)^2 \partial_{\mathbf{n}} \mathcal{A}_0^+\,. \end{array}\right.$$

Then, according to (6.10) for $n = 3$, $\mathcal{A}_3^+$ satisfies the problem (see (3.7))

$$\left\{\begin{array}{lcll} -\Delta \mathcal{A}_3^+ & = & 0 & \text{in} \quad \Omega_+\,, \\ \mathcal{A}_3^+ & = & -\mu_r \left(\partial_{\mathbf{n}} \mathcal{A}_2^+ + \dfrac{k(\xi)}{2}\partial_{\mathbf{n}} \mathcal{A}_1^+ + \dfrac{1}{2}\left\{\partial_\xi^2 + \dfrac{3}{4}k(\xi)^2 \mathbb{I}\right\}\partial_{\mathbf{n}} \mathcal{A}_0^+\right) & \text{on} \quad \Sigma\,, \\ \mathcal{A}_3^+ & = & 0 & \text{on} \quad \Gamma\,. \end{array}\right. \tag{6.21}$$

The next term which are determined in a similar way are the profile $\mathfrak{A}_4$ (3.8) and the term $\mathcal{A}_4^+$ (3.9).

Appendix A. Impedance boundary conditions

In the framework above it is possible to derive high-order impedance boundary conditions set on the interface Σ. In this section we present a hierarchy of impedance boundary conditions up to the fifth order of approximation.

Order 1*: Perfect conductor b.c.*

$$\mathcal{A}_0 = 0 \quad \text{on} \quad \Sigma \tag{A.1}$$

Order 2*: Leontovich b.c.*

$$\mathcal{A}_1^\delta + \mu_r \underline{\delta} \partial_{\mathbf{n}} \mathcal{A}_1^\delta = 0 \quad \text{on} \quad \Sigma \tag{A.2}$$

Order 3*.*

$$\mathcal{A}_2^\delta + \mu_r \underline{\delta} \left(1 + \underline{\delta} \frac{k(\xi)}{2} \right) \partial_{\mathbf{n}} \mathcal{A}_2^\delta = 0 \quad \text{on} \quad \Sigma \tag{A.3}$$

Order 4*.*

$$\mathcal{A}_3^\delta + \mu_r \underline{\delta} \left(\left(1 + \underline{\delta} \frac{k(\xi)}{2} \right) \mathbb{I} + \frac{\underline{\delta}^2}{2} \left\{ \partial_\xi^2 + \frac{3}{4} k(\xi)^2 \mathbb{I} \right\} \right) \partial_{\mathbf{n}} \mathcal{A}_3^\delta = 0 \quad \text{on} \quad \Sigma \tag{A.4}$$

Order 5*.*

$$\mathcal{A}_4^\delta + \mu_r \underline{\delta} \left(\left(1 + \underline{\delta} \frac{k}{2} \right) + \frac{\underline{\delta}^2}{2} \left\{ \partial_\xi^2 + \frac{3}{4} k^2 \right\} + \frac{\underline{\delta}^3}{4} \left\{ \frac{11}{2} k \partial_\xi^2 + 7k' \partial_\xi + 3k'' + \frac{3}{2} k^3 \right\} \right) \partial_{\mathbf{n}} \mathcal{A}_4^\delta = 0 \quad \text{on} \quad \Sigma \tag{A.5}$$

It is possible to justify rigorously these boundary conditions, we refer the reader to [8] for a similar context. The perfect conductor condition and the Leontovich condition are justified for all $\delta > 0$ whereas the higher order conditions are justified provided δ is sufficiently small. The condition of order 3 is justified for all $\delta \in \left(0, \dfrac{\sqrt{2}}{\|k\|_{\infty,\Sigma}} \right]$.

Appendix B. Formal derivation of impedance conditions

In this section we first present briefly a formal derivation of impedance conditions. The construction of impedance conditions is detailed in Section B.1.

Construction of impedance conditions of order $k+1$. This step consist to identify for all $k \in \mathbb{N}$ a simpler problem satisfied by the truncated expansion

$$\mathcal{A}^+_{k,\delta} := \mathcal{A}^+_0 + \underline{\delta}\mathcal{A}^+_1 + \cdots + \underline{\delta}^k \mathcal{A}^+_k$$

up to a residual term in $\mathcal{O}(\delta^{k+1})$. The simpler problem writes

$$\left\{ \begin{array}{lcll} -\Delta \mathcal{A}^\delta_k & = & \mu_+ J & \text{in} \quad \Omega_+ \,, \\ \mathcal{A}^\delta_k + \mathsf{B}_{k,\delta}(\partial_{\mathbf{n}} \mathcal{A}^\delta_k) & = & 0 & \text{on} \quad \Sigma \,, \\ \mathcal{A}^\delta_k & = & 0 & \text{on} \quad \Gamma \,, \end{array} \right.$$

and we say that the impedance condition set on Σ is of order $k+1$. Here J is the data of the model problem (2.3) and $\mathsf{B}_{k,\delta}$ is a differential operator acting on functions defined on Γ and which depends on δ. For all $k \in \{0,1,2,3,4\}$, the impedance operator $\mathsf{B}_{k,\delta}$ writes as (see Section A)

$$\mathsf{B}_{0,\delta} = 0 \,,$$

$$\mathsf{B}_{1,\delta} = \mu_r \underline{\delta}\, \mathbb{I} \,,$$

$$\mathsf{B}_{2,\delta} = \mu_r \underline{\delta} \left(1 + \underline{\delta} \frac{k(\xi)}{2} \right) \mathbb{I} \,,$$

$$\mathsf{B}_{3,\delta} = \mu_r \underline{\delta} \left(\left(1 + \underline{\delta} \frac{k(\xi)}{2} \right) \mathbb{I} + \frac{\underline{\delta}^2}{2} \left\{ \partial^2_\xi + \frac{3}{4} k(\xi)^2 \mathbb{I} \right\} \right) ,$$

$$\mathsf{B}_{4,\delta} = \mu_r \underline{\delta} \left(\left(1 + \underline{\delta} \frac{k}{2} \right) + \frac{\underline{\delta}^2}{2} \left\{ \partial^2_\xi + \frac{3}{4} k^2 \right\} + \frac{\underline{\delta}^3}{4} \left\{ \frac{11}{2} k \partial^2_\xi + 7k' \partial_\xi + 3k'' + \frac{3}{2} k^3 \right\} \right) .$$

Remark B.1. For all $k \in \{0,1,2\}$, $\mathsf{B}_{k,\delta}$ are scalar operators whereas $\mathsf{B}_{3,\delta}$ and $\mathsf{B}_{4,\delta}$ are second order differential operators.

B.1. **Construction of impedances conditions.** In this section, we detail the formal derivation of impedance boundary conditions (Sec. A).

Order 1. Since $\mathcal{A}^+_0$ solves the problem (3.1), the condition of order 1 is the perfect conductor boundary condition, see (A.1)

$$\mathcal{A}_0 = 0 \quad \text{on} \quad \Sigma$$

Order 2. According to (3.1)-(3.3), the truncated expansion $\mathcal{A}^+_{1,\delta} := \mathcal{A}^+_0 + \underline{\delta}\mathcal{A}^+_1$ solves the Laplace equation $-\Delta \mathcal{A}^+_{1,\delta} = \mu_+ J$ in Ω_+ together with the boundary condition set on Σ

$$\mathcal{A}^+_{1,\delta} + \mu_r \underline{\delta} \partial_{\mathbf{n}} \mathcal{A}^+_{1,\delta} = \mu_r \underline{\delta}^2 \partial_{\mathbf{n}} \mathcal{A}^+_1 \quad \text{on} \quad \Sigma$$

Neglecting the term of order 2 in the previous right-hand side, we infer the Leontovich boundary condition, see (A.2)

$$\mathcal{A}^\delta_1 + \mu_r \underline{\delta} \partial_{\mathbf{n}} \mathcal{A}^\delta_1 = 0 \quad \text{on} \quad \Sigma$$

Order 3. According to (3.1)-(3.3)-(3.5), the truncated expansion $\mathcal{A}^+_{2,\delta} := \mathcal{A}^+_0 + \underline{\delta}\mathcal{A}^+_1 + \underline{\delta}^2\mathcal{A}^+_2$ solves the Laplace equation $-\Delta\mathcal{A}^+_{2,\delta} = \mu_+ J$ in Ω_+ together with the boundary condition set on Σ

$$\mathcal{A}^+_{2,\delta} + \mu_r\underline{\delta}\left(1 + \underline{\delta}\frac{k(\xi)}{2}\right)\partial_{\mathsf{n}}\mathcal{A}^+_{2,\delta} = \mu_r\underline{\delta}^3\left(\frac{k(\xi)}{2}\partial_{\mathsf{n}}\mathcal{A}^+_1 + \left(1 + \underline{\delta}\frac{k(\xi)}{2}\right)\partial_{\mathsf{n}}\mathcal{A}^+_2\right) \quad \text{on} \quad \Sigma$$

We neglect the terms of order 3 in the previous right-hand side and we infer the impedance boundary condition of order 3, see (A.3)

$$\mathcal{A}^\delta_2 + \mu_r\underline{\delta}\left(1 + \underline{\delta}\frac{k(\xi)}{2}\right)\partial_{\mathsf{n}}\mathcal{A}^\delta_2 = 0 \quad \text{on} \quad \Sigma$$

Order 4. According to (3.1)-(3.3)-(3.5)-(3.7), the truncated expansion $\mathcal{A}^+_{3,\delta} := \mathcal{A}^+_0 + \underline{\delta}\mathcal{A}^+_1 + \underline{\delta}^2\mathcal{A}^+_2 + \underline{\delta}^3\mathcal{A}^+_3$ solves the Laplace equation $-\Delta\mathcal{A}^+_{3,\delta} = \mu_+ J$ in Ω_+ together with the boundary condition set on Σ

$$\text{(B.1)} \quad \mathcal{A}^+_{3,\delta} + \mu_r\underline{\delta}\left(\left(1 + \underline{\delta}\frac{k(\xi)}{2}\right)\mathbb{I} + \frac{\underline{\delta}^2}{2}\left\{\partial^2_\xi + \frac{3}{4}k(\xi)^2\mathbb{I}\right\}\right)\partial_{\mathsf{n}}\mathcal{A}^+_{3,\delta} =$$
$$\mu_r\frac{\underline{\delta}^4}{2}\left(\partial^2_\xi + \frac{3}{4}k(\xi)^2\mathbb{I}\right)\partial_{\mathsf{n}}\mathcal{A}^+_1 + \mu_r\underline{\delta}^3\left(\underline{\delta}\frac{k(\xi)}{2}\mathbb{I} + \frac{\underline{\delta}^2}{2}\left\{\partial^2_\xi + \frac{3}{4}k(\xi)^2\mathbb{I}\right\}\right)\partial_{\mathsf{n}}\mathcal{A}^+_2$$
$$+ \mu_r\underline{\delta}^4\left(\left(1 + \underline{\delta}\frac{k(\xi)}{2}\right)\mathbb{I} + \frac{\underline{\delta}^2}{2}\left\{\partial^2_\xi + \frac{3}{4}k(\xi)^2\mathbb{I}\right\}\right)\partial_{\mathsf{n}}\mathcal{A}^+_3 \quad \text{on} \quad \Sigma$$

We neglect the terms of order 4 in the previous right-hand side and we infer the impedance boundary condition of order 4, see (A.4)

$$\mathcal{A}^\delta_3 + \mu_r\underline{\delta}\left(\left(1 + \underline{\delta}\frac{k(\xi)}{2}\right)\mathbb{I} + \frac{\underline{\delta}^2}{2}\left\{\partial^2_\xi + \frac{3}{4}k(\xi)^2\mathbb{I}\right\}\right)\partial_{\mathsf{n}}\mathcal{A}^\delta_3 = 0 \quad \text{on} \quad \Sigma$$

Order 5. According to (3.1)-(3.3)-(3.5)-(3.7)-(3.9), the truncated expansion $\mathcal{A}^+_{4,\delta} := \mathcal{A}^+_0 + \underline{\delta}\mathcal{A}^+_1 + \underline{\delta}^2\mathcal{A}^+_2 + \underline{\delta}^3\mathcal{A}^+_3 + \underline{\delta}^4\mathcal{A}^+_4$ solves the Laplace equation $-\Delta\mathcal{A}^+_{4,\delta} = \mu_+ J$ in Ω_+ together with the boundary condition set on Σ

$$\text{(B.2)} \quad \underline{\delta}\left(\left(1 + \underline{\delta}\frac{k}{2}\right) + \frac{\underline{\delta}^2}{2}\left\{\partial^2_\xi + \frac{3}{4}k^2\right\} + \frac{\underline{\delta}^3}{4}\left\{\frac{11}{2}k\partial^2_\xi + 7k'\partial_\xi + 3k'' + \frac{3}{2}k^3\right\}\right)\partial_{\mathsf{n}}\mathcal{A}^+_{4,\delta}$$
$$+ \mathcal{A}^+_{4,\delta} = \mathcal{O}(\delta^5)$$

We neglect the terms of order 5 in the previous right-hand side and we infer the impedance boundary condition of order 5 (A.5).

References

[1] X. Antoine, H. Barucq, and L. Vernhet. High-frequency asymptotic analysis of a dissipative transmission problem resulting in generalized impedance boundary conditions. *Asymptot. Anal.*, 26(3-4):257–283, 2001.

[2] A. Bendali and K. Lemrabet. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM J. Appl. Math.*, 56(6):1664–1693, 1996.

[3] G. Caloz, M. Dauge, and V. Péron. Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism. *Journal of Mathematical Analysis and Applications*, 370(2):555–572, 2010.

[4] M. Costabel, M. Dauge, and S. Nicaise. Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth domains. 211 pages v2: Improvement of layout.

[5] M. Costabel, M. Dauge, and S. Nicaise. Singularities of eddy currents problems. Preprint of the Newton Institute, 2003.

[6] M. Costabel, M. Dauge, and S. Nicaise. Corner singularities of Maxwell interface and eddy current problems. In *Operator theoretical methods and applications to mathematical physics*, volume 147 of *Oper. Theory Adv. Appl.*, pages 241–256. Birkhäuser, Basel, 2004.

[7] B. Engquist and J.C. Nédélec. Effective boundary condition for acoustic and electromagnetic scattering in thin layers. *Technical Report of CMAP*, 278, 1993.

[8] H. Haddar, P. Joly, and H.-M. Nguyen. Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case. *Math. Models Methods Appl. Sci.*, 15(8):1273–1300, 2005.

[9] H. Haddar, P. Joly, and H.-M. Nguyen. Generalized impedance boundary conditions for scattering problems from strongly absorbing obstacles: the case of Maxwell's equations. *Math. Models Methods Appl. Sci.*, 18(10):1787–1827, 2008.

[10] N. Ida and S. Yuferev. Impedance boundary conditions for transient scattering problems. *Magnetics, IEEE Transactions on*, 33(2):1444–1447, 1997.

[11] L. Krähenbühl, V. Péron, R. Perrussel, and C. Poignard. On the asymptotic expansion of the magnetic potential in eddy current problem: a practical use of asymptotics for numerical purposes. Research Report RR-8749, INRIA Bordeaux, June 2015.

[12] O. D. Lafitte and G. Lebeau. Équations de Maxwell et opérateur d'impédance sur le bord d'un obstacle convexe absorbant. *C. R. Acad. Sci. Paris Sér. I Math.*, 316(11):1177–1182, 1993.

[13] M. A. Leontovich. Approximate boundary conditions for the electromagnetic field on the surface of a good conductor. In *Investigations on radiowave propagation*, volume 2, pages 5–12. Printing House of the USSR Academy of Sciences, Moscow, 1948.

[14] R. C. MacCamy and E. Stephan. Solution procedures for three-dimensional eddy current problems. *J. Math. Anal. Appl.*, 101(2):348–379, 1984.

[15] R. C. MacCamy and E. Stephan. A skin effect approximation for eddy current problems. *Arch. Rational Mech. Anal.*, 90(1):87–98, 1985.

[16] V. Péron. *Modélisation mathématique de phénomènes électromagnétiques dans des matériaux à fort contraste*. PhD thesis, Université Rennes 1, 2009.

[17] S. M. Rytov. Calcul du skin effect par la méthode des perturbations. *Journal of Physics*, 11(3):233–242, 1940.

[18] T.B.A. Senior, J.L. Volakis, and Institution of Electrical Engineers. *Approximate Boundary Conditions in Electromagnetics*. IEE Electromagnetic Waves Series. Inst of Engineering & Technology, 1995.

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