Equivalent Robin Boundary Conditions for Acoustic and Elastic Media

Julien Diaz, Victor Péron

EQUIVALENT ROBIN BOUNDARY CONDITIONS FOR ACOUSTIC AND ELASTIC MEDIA*

JULIEN DIAZ† AND VICTOR PÉRON‡

Abstract. We present equivalent conditions and asymptotic models for a diffraction problem of acoustic and elastic waves. The mathematical problem is set with a Robin boundary condition. Elastic and acoustic waves propagate in a solid medium surrounded by a thin layer of fluid medium. Due to the thinness of the layer with respect to the wavelength, this problem is well suited for the notion of equivalent conditions and the effect of the fluid medium on the solid is as a first approximation local. This approach leads to solve only elastic equations. We derive and validate equivalent conditions up to the third order for the elastic displacement. The construction of equivalent conditions is based on a multiscale expansion in power series of the thickness of the layer for the solution of the transmission problem.

Key words. Robin Boundary Conditions, Elasto-Acoustic Coupling, Asymptotic Expansions, Equivalent Conditions

AMS subject classifications. 35C20, 35J25, 41A60, 74F10

1. Introduction.

The concept of Equivalent Boundary Conditions (also called approximate, effective, or impedance conditions) is classical in the modeling of wave propagation phenomena. Equivalent Conditions (ECs) are usually introduced to reduce the computational domain. The main idea consists in replacing a reference model inside a part of the domain (for instance a thin layer of dielectric or a highly absorbing material) by an approximate boundary condition. This idea is pertinent when the effective condition can be readily handled for numerical computations, for instance when this condition is local [16, 35, 7, 8]. In the 1990's Engquist–Nédélec [16], Abboud–Ammari [1], Bendali–Lemrabet [7], Ammari–Nédélec [3], and Lafitte [25] derived equivalent conditions for acoustic and electromagnetic scattering problems to approximate an obstacle coated by a thin layer of dielectric (absorbing) material inside the domain of interest.

The main application of this work is the mathematical modeling of earthquake on the Earth's surface. The simulation of large-scale geophysics phenomena represents a main challenge for our society. Seismic activities worldwide have shown how crucial it is to enhance our understanding of the impact of earthquakes. In this context, the coupling of elastic and acoustic waves equations is essential if we want to reproduce real physical phenomena such as an earthquake. We can thus take into account the propagation of seismic waves in the subsurface together with the propagation of acoustic waves in a part of the ocean. Elasto-acoustic coupling problems are rather classical in the mathematical modeling of wave propagation phenomena, in particular in harmonic domain. The well-posedness of the direct problem has been studied in [22, 27, 6, 14] and in the monography [12, §5.4.e] for the theoretical analysis of the problem. Various other works have been devoted to the numerical solution of the problem, using for instance BEM/FEM type method, where Boundary Element Method (BEM) are used to discretize the fluid and Finite Element Method (FEM) to discretize the solid [11, 19, 18, 30, 28, 34]; plane waves based methods, as in [15, 21]

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or Discontinuous Galerkin Methods [5]. The transient problem is mostly studied for geophysical applications such as numerical simulation of earthquakes and many works have been devoted to the numerical discretization of fluid-structure problems using Finite Difference methods [33], Spectral Element Methods [24] or Discontinuous Galerkin Methods allowing for non-conforming meshes [23]. Finally, we refer to [29, 2, 17] for the study of inverse problems of shape reconstruction of solid body immersed in a fluid.

We intend to work in the context of geophysical applications for which we consider that the medium consists of land areas surrounded by fluid zones whose thickness is very small, typically with respect to the wavelength. This raises the difficulty of applying a finite element method on a mesh that combines fine cells in the fluid zone and much larger cells in the solid zone. To overcome this difficulty and to solve this problem, we adopt an asymptotic method (based on a multiscale expansion) which consists in "approximating" the fluid portion by an equivalent boundary condition. This boundary condition is then coupled with the elastic wave equation and a finite element method can be applied to solve the resulting boundary value problem.

In this paper, one considers a problem of elasto-acoustic coupling set with an external Robin boundary condition. One presents elements of derivation together with mathematical justifications for equivalent boundary conditions, which appear as a first, second or third order approximations with respect to the small parameter ε (the thickness of the fluid layer) and which are satisfied by the elastic displacement $\mathbf{u}$. This work is concerned essentially with theoretical objectives. The numerical pertinence of these ECs up to the second order have already been shown for the two-dimensional problem [13]. In the context of geophysical applications one derives also ECs up to the second order when the thickness of the layer is not constant with respect to the tangential variable.

There are several similarities in this work and in the work in Ref. [32] where ECs (up to the fourth order) are derived for a problem of elasto-acoustic coupling set with an external Dirichlet boundary condition. As in [32], the ECs derived in this work are of "$\mathbf{u}\cdot\mathbf{n}$–to–$\mathbf{T}(\mathbf{u})$" nature for elasto-acoustics since a local impedance operator links the normal traces of $\mathbf{u}$ and the stress vector $\mathbf{T}(\mathbf{u})$. However, for a given order of approximation in ε, the new impedance operator is a partial differential operator of higher order than the impedance operator derived in [32]. There are additional differences between the results of this paper and the work in Ref. [32] and which appear in the different proofs to validate the new ECs. One difficulty to validate the ECs lies in the proof of uniform energy estimates for the solution of the transmission problem. One overcomes this difficulty by removing a discret set of resonant frequencies (which are known as Jone's frequencies) and by using a compactness argument. We revisit and adapt the proof of uniform estimates in Ref. [32]. One can not apply straightforwardly this proof to the transmission problem of interest since the external Robin boundary condition plays a crucial role. Then one proves well-posedness and convergence results for ECs up to the third order using a compactness argument. .

The outline of the paper proceeds as follows. In Section 2, we introduce the mathematical model and the framework for the elasto-acoustic problem and one presents briefly a formal derivation of ECs. Then one states uniform estimates for the solution of the transmission problem. In Section 3, one presents ECs and asymptotic models associated with the solution of the exact problem. In Section 4, one proves uniform estimates for the solution of the elasto-acoustic problem. In Section 5, one derives and validates a two-scale asymptotic expansion at any order for the solution of the problem, and we construct formally ECs. In Section 6, one proves stability results for ECs and the convergence of ECs towards the exact model. In Appendix B, one derives ECs in the bidimensional case when the layer has a non-constant thickness.

2. The Mathematical Model. In this section, one introduces the model problem (§2.2) and the framework for the elasto-acoustic problem. Then we remind the definition of equivalent conditions and we state uniform estimates for the solution of the exact problem. We start this section with a formal derivation of the approximate boundary conditions.

2.1. Formal derivation of equivalent conditions. In this section, one presents briefly a formal derivation of equivalent conditions. We summarize this process in two steps. All the details and formal calculi are presented in Section 5.

First step : a multiscale expansion. The first step consists in deriving a multiscale expansion for the solution $(\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon)$ of the model problem (2.3) (Sec. 2.2): it possesses an asymptotic expansion in power series of the small parameter ε

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \varepsilon\mathbf{u}_1(\mathbf{x}) + \varepsilon^2\mathbf{u}_2(\mathbf{x}) + \cdots \quad \text{in} \quad \Omega_{\mathsf{s}}\,,$$
$$\mathsf{p}_\varepsilon(\mathbf{x}) = \mathsf{p}_0(\mathbf{x};\varepsilon) + \varepsilon\mathsf{p}_1(\mathbf{x};\varepsilon) + \varepsilon^2\mathsf{p}_2(\mathbf{x};\varepsilon) + \cdots \quad \text{in} \quad \Omega_{\mathsf{f}}^\varepsilon\,,$$
$$\text{with} \quad \mathsf{p}_j(\mathbf{x};\varepsilon) = \mathfrak{p}_j(y_\alpha, \frac{y_3}{\varepsilon})\,.$$

Here, $\mathbf{x} \in \mathbb{R}^3$ are the cartesian coordinates and (y_α, y_3) is a "*normal coordinate system*" [10, 31] to the surface $\Gamma = \partial\Omega_{\mathsf{s}}$ on the manifold $\Omega_{\mathsf{f}}^\varepsilon$: y_α ($\alpha \in \{1,2\}$) is a tangential coordinate on Γ and $y_3 \in (0,\varepsilon)$ is the distance to the surface Γ. The term $\mathfrak{p}_j$ is a "*profile*" defined on $\Gamma \times (0,1)$. Formal calculi are presented in Section 5.1 and the first terms $(\mathfrak{p}_j, \mathbf{u}_j)$ for $j = 0,1,2$ are explicited in Section 5.2.

Second step : derivation of equivalent conditions. The second step consists in identifying for $k \in \{0,1,2\}$ a simpler problem satisfied by the truncated expansion

$$\mathbf{u}_{k,\varepsilon} := \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2 + \cdots + \varepsilon^k\mathbf{u}_k$$

up to a residual term in $\mathcal{O}(\varepsilon^{k+1})$. The simpler problem writes

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k) + \omega^2\rho\mathbf{u}_\varepsilon^k = \mathbf{f} & \text{in} \quad \Omega_{\mathsf{s}} \\ \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k)\mathbf{n} + \mathsf{F}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k \cdot \mathbf{n})\,\mathbf{n} = 0 & \text{on} \quad \Gamma\,. \end{cases}$$

Here, $\mathbf{f}$ is the data of the model problem (2.3) and $\mathsf{F}_{k,\varepsilon}$ is a surfacic differential operator acting on functions defined on Γ and which depends on ε when $k \neq 0$:

$$\mathsf{F}_{0,\varepsilon}(= \mathsf{F}_0) = -i\omega c\rho_{\mathsf{f}}\,\mathbb{I}\,,$$
$$\mathsf{F}_{1,\varepsilon} = -i\omega c\rho_{\mathsf{f}}\,(\mathbb{I} + \varepsilon P_1(D))\;,$$
$$\mathsf{F}_{2,\varepsilon} = -i\omega c\rho_{\mathsf{f}}\left(\mathbb{I} + \varepsilon P_1(D) + \varepsilon^2 P_2(D)\right)\;,$$

where the operators $P_1(D)$ and $P_2(D)$ are defined respectively as

$$P_1(D) = -2\mathcal{H}\,\mathbb{I} + i\kappa^{-1}\Delta_\Gamma \tag{2.1}$$

$$\begin{aligned} P_2(D) = -2i\kappa\mathcal{H}\,\mathbb{I} + \left((4\mathcal{H}^2 - \mathcal{K})\,\mathbb{I} - \Delta_\Gamma\right) \\ + (i\kappa)^{-1}\left[2\mathcal{H}\Delta_\Gamma - \mathrm{div}_\Gamma(\mathcal{H}\mathbb{I} - \mathcal{R})\nabla_\Gamma) + \Delta_\Gamma(2\mathcal{H})\right] + (i\kappa)^{-2}\Delta_\Gamma^2\;. \end{aligned} \tag{2.2}$$

Here, $\mathcal{H}$ and $\mathcal{K}$ denote respectively the *mean curvature* and the *Gaussian curvature* of the surface Γ, Δ_Γ is the Laplace-Beltrami operator along Γ and $\mathcal{R}$ is an intrinsic symmetric linear operator defined on the tangent plane $\mathbf{T}_{\mathbf{x}_\Gamma}(\Gamma)$ to Γ at the point $\mathbf{x}_\Gamma \in \Gamma$ which characterizes the curvature of Γ at the point $\mathbf{x}_\Gamma$. Equivalent conditions are stated in Section 3.1. The construction of these conditions is detailed in Section 5.3.

2.2. The model problem. Our interest lies in an elasto-acoustic wave propagation problem in time-harmonic regime set in a domain with a thin layer. We consider the following transmission problem

$$\begin{cases} \Delta \mathsf{p}_\varepsilon + \kappa^2 \mathsf{p}_\varepsilon = 0 & \text{in} \quad \Omega_{\mathsf{f}}^\varepsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in} \quad \Omega_{\mathsf{s}} \\ \partial_{\mathbf{n}} \mathsf{p}_\varepsilon = \rho_{\mathsf{f}} \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} & \text{on} \quad \Gamma \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -\mathsf{p}_\varepsilon \mathbf{n} & \text{on} \quad \Gamma \\ \partial_{\mathbf{n}} \mathsf{p}_\varepsilon - i\kappa \mathsf{p}_\varepsilon = 0 & \text{on} \quad \Gamma^\varepsilon \ , \end{cases} \tag{2.3}$$

set in a smooth bounded simply connected domain Ω^ε in $\mathbb{R}^3$ made of a solid, elastic object occupying a smooth connected subdomain Ω_{s} entirely immersed in a fluid region occupying the subdomain $\Omega_{\mathsf{f}}^\varepsilon$. The domain $\Omega_{\mathsf{f}}^\varepsilon$ is a thin layer of uniform thickness ε (i.e. the euclidean distance between surfaces Γ and Γ^ε is ε), see figure 1. We denote by Γ^ε the boundary of the domain Ω^ε, and by Γ the interface between the subdomains $\Omega_{\mathsf{f}}^\varepsilon$ and Ω_{s}. We denote by $\mathbf{n}$ the unit normal to Γ oriented from Ω_{s} to $\Omega_{\mathsf{f}}^\varepsilon$.

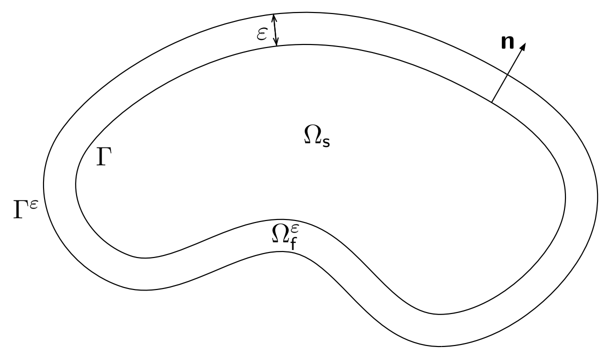

FIG. 1. *A cross-section of the domain Ω^ε and its subdomains Ω_{s} and $\Omega_{\mathsf{f}}^\varepsilon$*

In the elasto-acoustic system (2.3), we denote the unknowns by $\mathbf{u}_\varepsilon$ for the elastic displacement and by p_ε for the acoustic pressure. The time-harmonic wave field with angular frequency ω is characterized by using the Helmholtz equation for the pressure p_ε, and by using an anisotropic discontinuous linear elasticity system for the displacement $\mathbf{u}_\varepsilon$. These equations contain several physical constants: $\kappa = \omega/c$ is the acoustic wave number, c is the speed of the sound in the fluid, ρ is the density of the solid, and ρ_{f} is the density of the fluid. All these constants are independent of ε.

In the linear elastic equation, $\nabla\cdot$ denotes the divergence operator for tensors, $\underline{\underline{\sigma}}(\mathbf{u})$ is the stress tensor given by Hooke's law

$$\underline{\underline{\sigma}}(\mathbf{u}) = \underline{\underline{\underline{C}}}\ \underline{\underline{\epsilon}}(\mathbf{u}) \ .$$

Here $\underline{\epsilon}(\mathbf{u}) = \frac{1}{2}(\underline{\underline{\nabla}}\mathbf{u} + \underline{\underline{\nabla}}\mathbf{u}^T)$ is the strain tensor where $\underline{\underline{\nabla}}$ denotes the gradient operator for tensors, and $\underline{\underline{C}} = \underline{\underline{C}}(\mathbf{x})$ is the elasticity tensor. The components of $\underline{\underline{C}}$ are the elasticity moduli C_{ijkl} : $\underline{\underline{C}} = (C_{ijkl}(\mathbf{x}))$. The *traction operator* $\mathbf{T}$ is a surfacic differential operator defined on Γ as

$$\mathbf{T}(\mathbf{u}) = \underline{\underline{\sigma}}(\mathbf{u})\mathbf{n} \ .$$

The right-hand side $\mathbf{f}$ is a data with support in Ω_{s}. The first transmission condition set on Γ is a kinematic interface condition whereas the second one is a dynamic interface condition. The kinematic condition requires that the normal velocity of the fluid match the normal velocity of the solid on the interface Γ. The dynamic condition results from the equilibrium of forces on the interface Γ. The transmission conditions are natural. Furthermore the pressure satisfies a Robin boundary condition set on Γ^ε.

Remark 2.1. *The boundary Γ^ε represents a physical absorbing boundary and the Robin boundary condition set on Γ^ε can be seen as a low order approximation of the outgoing radiation condition at infinity for the exterior scattering problem set in $\mathbb{R}^3 \setminus \overline{\Omega_{\mathsf{s}}}$.*

In the above framework, we address the issue of Equivalent Conditions (ECs) for the elastic displacement $\mathbf{u}_\varepsilon$ as $\varepsilon \to 0$, see Section 2.4. This issue is linked with the question of Uniform Estimates for the couple $(\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon)$ solution of the problem (2.3) as $\varepsilon \to 0$ (Section 2.5) since it is a main ingredient in the mathematical justification of ECs. To answer these questions, we make hereafter several assumptions on the data and on the regularity of the surface Γ.

2.3. Framework. We will work under usual assumptions (symmetry and positiveness) on the elasticity tensor.

Assumption 2.2. (i) *The elasticity moduli $C_{ijkl}(\mathbf{x})$ are real valued smooth functions in $\overline{\Omega_{\mathsf{s}}}$.*
(ii) *The tensor $\underline{\underline{C}}$ is symmetric :*

$$C_{ijkl} = C_{jikl} = C_{klij} \quad \textit{almost everywhere in}\, \Omega_{\mathsf{s}} \ .$$

(iii) *The tensor $\underline{\underline{C}}$ is positive :*

$$\exists \alpha > 0, \quad \forall \xi = (\xi_{ij}) \textit{ symmetric tensor}, \sum_{i,j,k,l} C_{ijkl}\xi_{ij}\overline{\xi_{kl}} \geqslant \alpha \sum_{i,j} |\xi_{ij}|^2 \ .$$

Some resonant frequencies may appear in the solid domain. However, we prove uniform estimates for the elasto-acoustic field $(\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon)$ as well as ECs for $\mathbf{u}_\varepsilon$ when $\varepsilon \to 0$ under the following spectral assumption, compare with [32, Assumption 2.3].

Assumption 2.3. *The angular frequency ω is non-zero and is not an eigenfrequency of the problem*

$$\left\{ \begin{array}{lll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \textit{in} & \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}) = 0 \quad \textit{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 & \textit{on} & \Gamma \ . \end{array} \right. \tag{2.4}$$

Remark 2.4. *For axisymmetric bodies Ω_{s} (balls, ellipsoids, ...) and for a discrete set of frequencies ω, there exist non trivial solutions $\mathbf{u}$ to* (2.4)*, [22, 27]. Such a solution $\mathbf{u}$, resp. frequency ω, is called a Jones mode, resp. a Jones frequency. However, Jones eigenmodes do not exist for generic domains, [20].*

Our whole analysis is valid under the following assumption on the surfaces Γ and Γ^ε.

ASSUMPTION 2.5. *The fluid-solid interface* Γ *and the surface* Γ^ε *are smooth.*

For the sake of simplicity in the asymptotic modeling, we will work under the following assumption on the data $\mathbf{f}$.

ASSUMPTION 2.6. *The right-hand side* $\mathbf{f}$ *in* (2.3) *is a smooth* ε*-independent data.*

In the framework above, we prove in this paper that it is possible to replace the fluid region $\Omega_\mathsf{f}^\varepsilon$ by appropriate boundary conditions called equivalent conditions and set on Γ.

2.4. Validation of equivalent conditions. In this paper we derive surfacic differential operators F_ε

$$\mathsf{F}_\varepsilon : \mathcal{C}^\infty(\Gamma) \to \mathcal{C}^\infty(\Gamma) \ ,$$

together with $\tilde{\mathbf{u}}_\varepsilon$ which is a solution of the boundary value problem

$$\left\{ \begin{array}{lll} \nabla \cdot \underline{\underline{\sigma}}(\tilde{\mathbf{u}}_\varepsilon) + \omega^2 \rho \tilde{\mathbf{u}}_\varepsilon = \mathbf{f} & \text{in} & \Omega_\mathsf{s} \\ \mathbf{T}(\tilde{\mathbf{u}}_\varepsilon) + \mathsf{F}_\varepsilon(\tilde{\mathbf{u}}_\varepsilon \cdot \mathbf{n})\mathbf{n} = 0 & \text{on} & \Gamma \ . \end{array} \right. \tag{2.5}$$

Then, in the framework of Section 2.3, we prove uniform estimates for the error between the exact solution $\mathbf{u}_\varepsilon$ in (2.3) and $\tilde{\mathbf{u}}_\varepsilon$ provided ε is small enough:

$$\|\mathbf{u}_\varepsilon - \tilde{\mathbf{u}}_\varepsilon\|_{1,\Omega_\mathsf{s}} \leqslant C\varepsilon^{k+1} \ , \tag{2.6}$$

when $k \in \{0, 1, 2\}$, see Th. 3.4 for the main result and precise estimates. Here, we denote by $\|\cdot\|_{1,\Omega_\mathsf{s}}$ the norm in the Sobolev space $\mathbf{H}^1(\Omega_\mathsf{s}) = \mathrm{H}^1(\Omega_\mathsf{s})^3$. We say that the equivalent condition is of order $k+1$ when such an a priori estimate (2.6) holds. Then we define $\mathbf{u}_\varepsilon^k = \tilde{\mathbf{u}}_\varepsilon$ and we denote by $\mathsf{B}_{k,\varepsilon}$ the operator B_ε corresponding to the order $k+1$, Sec. 3. The validation of ECs relies on uniform estimates for solutions $(\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon)$ of (2.3) as $\varepsilon \to 0$. This issue is developped in Section 2.5.

2.5. Uniform estimates. We introduce a suitable variational framework for the solution of the problem (2.3) with more general right-hand sides. This framework is useful to prove error estimates (2.6).

Weak solutions. For given data $(\mathbf{f}, \mathsf{f}, \mathsf{g}, \mathsf{h})$ we consider the boundary value problem

$$\left\{ \begin{array}{lll} \Delta \mathsf{p}_\varepsilon + \kappa^2 \mathsf{p}_\varepsilon = \mathsf{f} & \text{in} & \Omega_\mathsf{f}^\varepsilon \\ \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in} & \Omega_\mathsf{s} \\ \partial_\mathbf{n} \mathsf{p}_\varepsilon = \rho_\mathsf{f} \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} + \mathsf{g} & \text{on} & \Gamma \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -\mathsf{p}_\varepsilon \mathbf{n} & \text{on} & \Gamma \\ \partial_\mathbf{n} \mathsf{p}_\varepsilon - i\kappa \mathsf{p}_\varepsilon = \mathsf{h} & \text{on} & \Gamma^\varepsilon \ . \end{array} \right. \tag{2.7}$$

Hereafter, we explicit a weak formulation of the problem (2.7). The variational problem writes : Find $(\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon) \in V_\varepsilon = \mathbf{H}^1(\Omega_\mathsf{s}) \times \mathrm{H}^1(\Omega_\mathsf{f}^\varepsilon)$ such that

$$\forall (\mathbf{v}, \mathsf{q}) \in V_\varepsilon, \quad a_\varepsilon\left((\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon), (\mathbf{v}, \mathsf{q})\right) = \langle F, (\mathbf{v}, \mathsf{q}) \rangle_{V_\varepsilon', V_\varepsilon} \ , \tag{2.8}$$

where the sesquilinear form a_ε is defined as

$$a_\varepsilon\left((\mathbf{u}, \mathsf{p}), (\mathbf{v}, \mathsf{q})\right) = \int_{\Omega_\mathsf{f}^\varepsilon} \left(\nabla \mathsf{p} \cdot \nabla \bar{\mathsf{q}} - \kappa^2 \mathsf{p} \bar{\mathsf{q}}\right) \mathrm{d}\mathbf{x} + \int_{\Omega_\mathsf{s}} \left(\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}\right) \mathrm{d}\mathbf{x}$$
$$+ \int_\Gamma \left(\omega^2 \rho_\mathsf{f} \mathbf{u} \cdot \mathbf{n}\, \bar{\mathsf{q}} + \mathsf{p} \bar{\mathbf{v}} \cdot \mathbf{n}\right) \mathrm{d}\sigma - i\kappa \int_{\Gamma^\varepsilon} \mathsf{p}\, \bar{\mathsf{q}}\, \mathrm{d}\sigma \ ,$$

and the right-hand side F is defined as

$$\langle F,(\mathbf{v},\mathsf{q})\rangle_{V'_\varepsilon,V_\varepsilon} = -\int_{\Omega_\mathsf{f}^\varepsilon} \mathsf{f}\bar{\mathsf{q}}\,\mathrm{d}\mathbf{x} - \int_{\Omega_\mathsf{s}} \mathbf{f}\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x} - \int_\Gamma \mathsf{g}\bar{\mathsf{q}}\,\mathrm{d}\sigma - \int_{\Gamma^\varepsilon} \mathsf{h}\,\bar{\mathsf{q}}\,\mathrm{d}\sigma\ .$$

We assume that the data $(\mathbf{f},\mathsf{f},\mathsf{g},\mathsf{h})$ are smooth enough such that the right-hand side F belongs to the space V'_ε. The space V_ε is endowed with the piecewise H^1 norm in Ω_s and $\Omega_\mathsf{f}^\varepsilon$.

Statement of uniform estimates. In the framework of Section 2.3 we prove ε-uniform a priori estimates for the solution of problem (2.8). The following theorem is the main result in this section.

THEOREM 2.7. *Under Assumptions 2.2-2.3-2.5, there exists constants $\varepsilon_0, C>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, the problem* (2.8) *with data $F\in V'_\varepsilon$ has a unique solution $(\mathbf{u}_\varepsilon,\mathsf{p}_\varepsilon)\in V_\varepsilon$ which satisfies*

$$\|\mathsf{p}_\varepsilon\|_{1,\Omega_\mathsf{f}^\varepsilon} + \|\mathbf{u}_\varepsilon\|_{1,\Omega_\mathsf{s}} \leqslant C\|F\|_{V'_\varepsilon}\ . \tag{2.9}$$

This result is proved in Section 4. The proof is based on a formulation of the problem set in a fixed domain. This formulation is obtained through a scaling along the thickness of the layer. As an application of uniform estimates (2.9), we prove the convergence result (2.6) in Section 6.

3. Equivalent Conditions. In the framework above, we derive for all $k\in\{0,1,2\}$ a boundary condition set on Γ which is associated with the problem (2.3) and satisfied by $\mathbf{u}_\varepsilon^k$, i.e. $\mathbf{u}_\varepsilon^k$ solves the problem

$$\left\{\begin{array}{ll} \nabla\cdot\underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k)+\omega^2\rho\mathbf{u}_\varepsilon^k=\mathbf{f} & \text{in}\quad \Omega_\mathsf{s} \\ \mathbf{T}(\mathbf{u}_\varepsilon^k)+\mathsf{F}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k\cdot\mathbf{n})\,\mathbf{n}=0 & \text{on}\quad \Gamma\ . \end{array}\right. \tag{3.1}$$

Here $\mathsf{F}_{k,\varepsilon}$ is a surfacic differential operator acting on functions defined on Γ and which depends on ε. In this section, we present Equivalent Conditions (ECs) up to the second order and asymptotic models for the solution of the exact problem, Sec. 3.1. Then, we present well-posedness and convergence results, Sec. 3.2. Elements of derivation and mathematical validations for ECs are presented in Section 5 and Section 6.

3.1. Statement of Equivalent conditions. We obtain a hierarchy of boundary-value problems. Each one gives a model with a different order of accuracy in ε and reflects the effect of the thin layer on the elastic displacement. We derive in Section 5.3 the following boundary conditions in problem (3.1) :

Order 1.

$$\mathbf{T}(\mathbf{u}_0) - i\omega c\rho_\mathsf{f}\mathbf{u}_0\cdot\mathbf{n}\,\mathbf{n}=0 \quad \text{on}\quad \Gamma \tag{3.2}$$

Order 2.

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c\rho_\mathsf{f}\left(\mathbb{I}+\varepsilon P_1(D)\right)(\mathbf{u}_\varepsilon^1\cdot\mathbf{n})\,\mathbf{n}=0 \quad \text{on}\quad \Gamma \tag{3.3}$$

Here the operator $P_1(D)$ is given by (2.1) .

Order 3.

$$\mathbf{T}(\mathbf{u}_\varepsilon^2) - i\omega c\rho_{\mathsf{f}}\left(\mathbb{I} + \varepsilon P_1(D) + \varepsilon^2 P_2(D)\right)(\mathbf{u}_\varepsilon^2\cdot\mathbf{n})\,\mathbf{n} = 0 \quad \text{on} \quad \Gamma \tag{3.4}$$

Here the operator $P_2(D)$ is defined in (2.2).

Successive corrections appear in these conditions when increasing the order of approximation. The Order 1 condition involves only partial derivatives of first order in the operator $\mathbf{T}$, whereas the Order 2 condition is a kind of "Ventcel's condition" [4, 26] since it involves partial derivatives of second order and the Order 3 condition involves partial derivatives up to the fourth order.

REMARK 3.1. *The "background" solution $\mathbf{u}_0$ $(= \mathbf{u}_\varepsilon^0)$ is independent of ε. It corresponds to a model where the effect of the fluid part appears through the density ρ_{f}. The influence of the geometry of the surface Γ appears from the order 2 model through the mean curvature of Γ.*

REMARK 3.2 (Layer with non-constant thickness). *In the context of geophysical applications, it is relevant to consider that the thickness of the layer is no longer constant with respect to the tangential variable. In this framework it is still possible to derive equivalent conditions and one derives ECs* (B.1)-(B.2) *which appear as first and second order of approximations. One presents elements of derivation for these conditions in the bidimensional case in appendix, see Section B. The order* 1 *condition is still* (3.2) *but the order* 2 *and* 3 *conditions are different from* (3.3) *and* (3.4). *The derivation of the asymptotics are more tedious since the change of variables (or scaling) lead to additional terms which come from the determinant of the metric of the layer.*

3.2. Stability and convergence of Equivalent conditions. Our goal in the next sections is to validate ECs set on Γ (Sec. 3.1) proving estimates for $\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k$ for all $k \in \{0,1,2\}$, where $\mathbf{u}_\varepsilon^k$ is the solution of the approximate model (3.1), and $\mathbf{u}_\varepsilon$ satisfies the problem (2.3). The functional setting for $\mathbf{u}_\varepsilon^k$ is described by the Hilbert space $\mathbf{V}^k$:

NOTATION 3.3. *$\mathbf{V}^k$ denotes the space $\mathbf{H}^1(\Omega_{\mathsf{s}})$ when $k=0$, and $\mathbf{V}^k$ denotes the space $\{\mathbf{u}\in\mathbf{H}^1(\Omega_{\mathsf{s}}) \mid \quad \mathbf{u}\cdot\mathbf{n}|_\Gamma \in \mathrm{H}^k(\Gamma)\}$ when $k = 1, 2$.*

THEOREM 3.4. *Under Assumptions 2.2-2.3-2.5-2.6, for all $k \in \{0,1,2\}$ there exists constants $\varepsilon_k, C_k > 0$ such that for all $\varepsilon \in (0,\varepsilon_k)$, the problem* (3.1) *with data $\mathbf{f}\in\mathbf{L}^2(\Omega_{\mathsf{s}})$ has a unique solution $\mathbf{u}_\varepsilon^k \in \mathbf{V}^k$ which satisfies uniform estimates*

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_{\mathsf{s}}} \leqslant C_k \varepsilon^{k+1} \ . \tag{3.5}$$

The well-posedness result for the problem (3.1) is stated in Thm. 5.3 and is proved in Section 6.1. It appears nontrivial to work straightforwardly with the difference $\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k$. A usual method consists in using the truncated series $\mathbf{u}_{k,\varepsilon}$ introduced in Section 5.3 as intermediate quantities [32]. Then, the error analysis is splitted into two steps detailed in the next sections :

1. We prove uniform estimates for the difference $\mathbf{u}_\varepsilon - \mathbf{u}_{k,\varepsilon}$ in Thm. 5.2 (Sec. 5.4)
2. We prove uniform estimates for the difference $\mathbf{u}_{k,\varepsilon} - \mathbf{u}_\varepsilon^k$ (Sec. 6.2).

REMARK 3.5. *The first step of the proof is independent of ECs and is valid for any integer k. The second step for $k = 0$ is useless since $\mathbf{u}_{0,\varepsilon} = \mathbf{u}_\varepsilon^0$.*

4. Uniform Estimates. In this section, we prove uniform estimates for the exact solution of the elasto-acoustic problem. Since the functional setting of the variational problem (2.8) depends on the small parameter ε, it is not well suited to prove uniform estimates for solutions $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V_\varepsilon$. To overcome this difficulty, we adapt an idea developed in [7, 32] writing equivalently the problem (2.8) in a common functional framework as ε varying, Sec. 4.1. We state uniform estimates in this new framework, Th. 4.1. One uses a compactness argument to prove estimates, Sec. 4.2.

4.1. The scaled problem. We write the variational problem (2.8) in a fixed domain through the scaling $S = \varepsilon^{-1}\nu$ where $\nu \in (0, \varepsilon)$ is the distance to the surface Γ. The fixed domain writes $\Omega_\mathsf{s} \times \Omega_\mathfrak{f}$ where $\Omega_\mathfrak{f} := \Gamma \times (0, 1)$ and the ad-hoc functional space writes

$$V = \mathbf{H}^1(\Omega_\mathsf{s}) \times \mathrm{H}^1(\Omega_\mathfrak{f}) \ .$$

Then the variational problem writes : Find $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ such that for all $(\mathbf{v}, \mathfrak{q}) \in V$,

$$\tag{4.1} \begin{aligned} \varepsilon a_\mathfrak{f}(\varepsilon; \mathfrak{p}_\varepsilon, \mathfrak{q}) + a_\mathsf{s}(\mathbf{u}_\varepsilon, \mathbf{v}) + \int_\Gamma \left(\omega^2 \rho_\mathfrak{f} \mathbf{u} \cdot \mathbf{n}\, \bar{\mathfrak{q}} + \mathfrak{p} \bar{\mathbf{v}} \cdot \mathbf{n}\right) \mathrm{d}\Gamma \\ - i\kappa \int_\Gamma \mathfrak{p}_\varepsilon(\,.\,, 1)\, \bar{\mathfrak{q}}(\,.\,, 1) \det(\mathbb{I} + \varepsilon\mathcal{R})\ \mathrm{d}\Gamma = \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathfrak{q}) \rangle_{V', V} \ , \end{aligned}$$

where

$$a_\mathfrak{f}(\varepsilon; \mathfrak{p}, \mathfrak{q}) = \int_0^1 \int_\Gamma \Big\{ (\mathbb{I} + \varepsilon S \mathcal{R})^{-2}\, \nabla_\Gamma \mathfrak{p} \nabla_\Gamma \bar{\mathfrak{q}} + \varepsilon^{-2} \partial_S \mathfrak{p} \partial_S \bar{\mathfrak{q}} - \kappa^2 \mathfrak{p} \bar{\mathfrak{q}} \Big\} \det(\mathbb{I} + \varepsilon S \mathcal{R})\, \mathrm{d}\Gamma \mathrm{d}S \ ,$$

$$a_\mathsf{s}(\mathbf{u}, \mathbf{v}) = \int_{\Omega_\mathsf{s}} \left(\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}} \right) \mathrm{d}\mathbf{x} \ ,$$

and

$$\begin{aligned} \langle \mathfrak{F}_\varepsilon, (\mathbf{v}, \mathfrak{q}) \rangle_{V', V} = -\varepsilon \int_{\Omega_\mathfrak{f}} \mathfrak{f} \bar{\mathfrak{q}} \det(\mathbb{I} + \varepsilon S \mathcal{R})\, \mathrm{d}\Gamma \mathrm{d}S - \int_{\Omega_\mathsf{s}} \mathbf{f} \cdot \bar{\mathbf{v}}\, \mathrm{d}\mathbf{x} \\ - \int_\Gamma \mathfrak{g} \bar{\mathfrak{q}}\, \mathrm{d}\Gamma - \int_\Gamma \mathfrak{h}(\,.\,, 1)\, \bar{\mathfrak{q}}(\,.\,, 1) \det(\mathbb{I} + \varepsilon \mathcal{R})\ \mathrm{d}\Gamma \ . \end{aligned}$$

Here, $\mathcal{R}$ is an intrinsic symmetric linear operator defined on the tangent plane $\mathbf{T}_{\mathbf{x}_\Gamma}(\Gamma)$ to Γ at the point $\mathbf{x}_\Gamma \in \Gamma$ which characterizes the curvature of Γ at the point $\mathbf{x}_\Gamma$. We refer to [7, 32]-[31, §4.1] for the introduction of geometrical tools and more details. The parameter ε weighting the form $a_\mathfrak{f}(\varepsilon; \mathfrak{p}, \mathfrak{q})$ in formulation (4.1) may lead to a solution $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ such that the surface gradient $\nabla_\Gamma \mathfrak{p}_\varepsilon$ can be unbounded as $\varepsilon \to 0$. This is a similarity with the works in Ref. [7, 32]. Furthermore, the sign of the left-hand side of the problem (4.1) for $(\mathbf{v}, \mathfrak{q}) = (\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon)$ cannot be controlled. Hence due to the lack of strong coerciveness of the variational formulation (4.1) one cannot get straightforwardly estimates. Our main result for the problem (4.1) is the following a priori estimate, uniform as $\varepsilon \to 0$.

THEOREM 4.1. *Under Assumptions 2.2-2.3-2.5, there exists constants $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the problem* (4.1) *with data $\mathfrak{F}_\varepsilon \in V'$ has a unique solution $(\mathbf{u}_\varepsilon, \mathfrak{p}_\varepsilon) \in V$ which satisfies the uniform estimates*

$$\tag{4.2} \sqrt{\varepsilon} \|\nabla_\Gamma \mathfrak{p}_\varepsilon\|_{0,\Omega_\mathfrak{f}} + \sqrt{\varepsilon}^{-1} \|\partial_S \mathfrak{p}_\varepsilon\|_{0,\Omega_\mathfrak{f}} + \|\mathfrak{p}_\varepsilon\|_{0,\Omega_\mathfrak{f}} + \|\mathfrak{p}_\varepsilon\|_{0,\Gamma} + \|\mathbf{u}_\varepsilon\|_{1,\Omega_\mathsf{s}} \leqslant C \|\mathfrak{F}_\varepsilon\|_{V'} \ .$$

This theorem is the key for the proof of Thm. 2.7 : as a consequence of the following estimates

$$\forall \mathsf{p} \in \mathrm{L}^2(\Omega_{\mathfrak{f}}^{\varepsilon}) \ , \quad \|\mathsf{p}\|_{0,\Omega_{\mathfrak{f}}^{\varepsilon}} \simeq \sqrt{\varepsilon}\|\mathfrak{p}\|_{0,\Omega_{\mathfrak{f}}} \ , \tag{4.3a}$$

$$\forall \mathsf{p} \in \mathrm{H}^1(\Omega_{\mathfrak{f}}^{\varepsilon}) \ , \quad \|\nabla\mathsf{p}\|_{0,\Omega_{\mathfrak{f}}^{\varepsilon}} \simeq \sqrt{\varepsilon}\|\nabla_{\Gamma}\mathfrak{p}\|_{0,\Omega_{\mathfrak{f}}} + \sqrt{\varepsilon}^{-1}\|\partial_S\mathfrak{p}\|_{0,\Omega_{\mathfrak{f}}} \ , \tag{4.3b}$$

available for ε small enough (see [32]) we obtain estimates (2.9). In (4.3a)-(4.3b), for any function p defined in $\Omega_{\mathfrak{f}}^{\varepsilon}$, the function $\mathfrak{p}$ is defined in the domain $\Omega_{\mathfrak{f}}$ as

$$\mathfrak{p}(\mathbf{x}_{\Gamma}, S) = \mathsf{p}(\mathbf{x}) \ , \quad (\mathbf{x}_{\Gamma}, S = \frac{\nu}{\varepsilon}) \in \Gamma \times (0,1) \ .$$

In (4.3a)-(4.3b), the symbol $\simeq$ means that quantities $\|\mathsf{p}\|_{0,\Omega_{\mathfrak{f}}^{\varepsilon}}$ and $\sqrt{\varepsilon}\|\mathfrak{p}\|_{0,\Omega_{\mathfrak{f}}}$ are equal up to a multiplicative constant which is independent of ε.

The proof of Thm. 4.1 is based on the following statement.

LEMMA 4.2. *Under Assumptions 2.2-2.3-2.5, there exists constants $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, any solution $(\mathbf{u}_{\varepsilon}, \mathfrak{p}_{\varepsilon}) \in V$ of problem (4.1) with a data $\mathfrak{F}_{\varepsilon} \in V'$ satisfies the uniform estimates*

$$\|(\mathbf{u}_{\varepsilon}, \mathfrak{p}_{\varepsilon})\|_{0,\Omega_{\mathsf{s}}\times\Omega_{\mathfrak{f}}} + \|\mathbf{u}_{\varepsilon}\cdot\mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_{\varepsilon}\|_{0,\Gamma} \leqslant C\|\mathfrak{F}_{\varepsilon}\|_{V'} \ . \tag{4.4}$$

The proof of this Lemma, which is given in Section 4.2, involves both a compactness argument and the spectral Assumption 2.3. As a consequence of estimates (4.4), we infer estimates (4.2). Since the problem (4.1) is of Fredholm type, Thm. 4.1 is then obtained as a consequence of the Fredholm alternative. We refer for instance to the work in Ref [32] for a similar context.

4.2. Proof of Lemma 4.2 : Uniform estimate of $(\mathbf{u}_{\varepsilon}, \mathfrak{p}_{\varepsilon})$. We prove this lemma by contradiction : We assume that there exists a sequence $(\mathbf{u}_m, \mathfrak{p}_m) \in V$, $m \in \mathbb{N}$, of solutions of problem (4.1) associated with a parameter ε_m and a right-hand side $\mathfrak{F}_m \in V'$:

$$\begin{aligned}\forall(\mathbf{v},\mathfrak{q}) \in V, \quad \varepsilon_m a_{\mathfrak{f}}(\varepsilon_m;\mathfrak{p}_m,\mathfrak{q}) + a_{\mathsf{s}}(\mathbf{u}_m,\mathbf{v}) + \int_{\Gamma}\left(\omega^2\rho_{\mathfrak{f}}\mathbf{u}_m\cdot\mathbf{n}\,\bar{\mathfrak{q}} + \mathfrak{p}_m\bar{\mathbf{v}}\cdot\mathbf{n}\right)\,\mathrm{d}\Gamma \\ - i\kappa\int_{\Gamma}\mathfrak{p}_m(\,.\,,1)\,\bar{\mathfrak{q}}(\,.\,,1)\det(\mathbb{I}+\varepsilon_m\mathcal{R})\,\mathrm{d}\Gamma = \langle\mathfrak{F}_m,(\mathbf{v},\mathfrak{q})\rangle_{V',V} \ ,\end{aligned} \tag{4.5}$$

satisfying the following conditions

$$\varepsilon_m \to 0 \quad \text{as } m \to \infty, \tag{4.6a}$$

$$\|(\mathbf{u}_m, \mathfrak{p}_m)\|_{0,\Omega_{\mathsf{s}}\times\Omega_{\mathfrak{f}}} + \|\mathbf{u}_m\cdot\mathbf{n}\|_{0,\Gamma} + \|\mathfrak{p}_m\|_{0,\Gamma} = 1 \quad \text{for all } m \in \mathbb{N}, \tag{4.6b}$$

$$\|\mathfrak{F}_m\|_{V'} \to 0 \quad \text{as } m \to \infty. \tag{4.6c}$$

Choosing tests functions $(\mathbf{v},\mathfrak{q}) = (\mathbf{u}_m,\mathfrak{p}_m)$ in (4.5), one obtains

$$\begin{aligned}\varepsilon_m a_{\mathfrak{f}}(\varepsilon_m;\mathfrak{p}_m,\mathfrak{p}_m) + a_{\mathsf{s}}(\mathbf{u}_m,\mathbf{u}_m) + \int_{\Gamma}\left(\omega^2\rho_{\mathfrak{f}}\mathbf{u}_m\cdot\mathbf{n}\,\overline{\mathfrak{p}_m} + \mathfrak{p}_m\overline{\mathbf{u}_m}\cdot\mathbf{n}\right)\,\mathrm{d}\Gamma \\ - i\kappa\int_{\Gamma}|\mathfrak{p}_m(\,.\,,1)|^2\det(\mathbb{I}+\varepsilon_m\mathcal{R})\,\mathrm{d}\Gamma = \langle\mathfrak{F}_m,(\mathbf{u}_m,\mathfrak{p}_m)\rangle_{V',V} \ .\end{aligned} \tag{4.7}$$

First, taking the real part of (4.7), one obtains with the help of conditions (4.6a)-(4.6b)-(4.6c) the following uniform bounds :
(i) The sequence $\{\mathbf{u}_m\}$ is bounded in the Sobolev space $\mathbf{H}^1(\Omega_\mathsf{s})$.
(ii) The sequences $\{\sqrt{\varepsilon_m}\nabla_\Gamma \mathfrak{p}_m\}$ and $\{\sqrt{\varepsilon_m}^{-1}\partial_S \mathfrak{p}_m\}$ are bounded in the space $\mathrm{L}^2(\Omega_\mathfrak{f})$.

As a consequence, the sequence $\{\mathfrak{p}_m\}$ is bounded in the space $\mathrm{H} = \mathrm{H}^1(0,1;\mathrm{L}^2(\Gamma))$ and the sequence $\{\partial_S \mathfrak{p}_m\}$ converges to 0 in $\mathrm{L}^2(\Omega_\mathfrak{f})$.

REMARK 4.3. *Remind that* H *is the space of distributions* $\mathfrak{p} \in \mathcal{D}'(0,1;\mathrm{L}^2(\Gamma))$ *such that* $\mathfrak{p}$ *and* $\mathfrak{p}'$ *belong to* $\mathrm{L}^2(0,1;\mathrm{L}^2(\Gamma))$. *Subsequently, we identify the space* $\mathrm{L}^2(0,1;\mathrm{L}^2(\Gamma))$ *and* $\mathrm{L}^2(\Omega_\mathfrak{f})$.

Then taking the imaginary part of (4.7), one obtains with the help of conditions (4.6b)-(4.6c) the following uniform bound :
(iii) The sequence $\{\mathfrak{p}_m(\cdot,1)\}$ is bounded in the space $\mathrm{L}^2(\Gamma)$.

Limit of the sequence $\{\mathbf{u}_m, \mathfrak{p}_m\}$. As a consequence of (4.6b) and *(i)*-*(ii)*, the sequence $\{\mathbf{u}_m, \mathfrak{p}_m\}$ is bounded in the space W defined as $W = \mathbf{H}^1(\Omega_\mathsf{s}) \times \mathrm{H}^1(0,1;\mathrm{L}^2(\Gamma))$:

$$\|(\mathbf{u}_m, \mathfrak{p}_m)\|_W \leqslant C. \tag{4.8}$$

Since the domain Ω_s is bounded, the embedding of $\mathbf{H}^1(\Omega_\mathsf{s})$ into $\mathbf{L}^2(\Omega_\mathsf{s})$ is compact. As a consequence of (4.8), using the Rellich Lemma one can extract a subsequence of $\{\mathbf{u}_m, \mathfrak{p}_m\}$ (still denoted by $\{\mathbf{u}_m, \mathfrak{p}_m\}$) which is strongly converging in $\mathbf{L}^2(\Omega_\mathsf{s}) \times \mathrm{L}^2(\Omega_\mathfrak{f})$, and one can assume that the sequence $\{\underline{\underline{\nabla}}\mathbf{u}_m\}$ is weakly converging in $\mathsf{L}^2(\Omega_\mathsf{s})$.

Summarizing these convergence results, one deduces that there exists $(\mathbf{u}, \mathfrak{p}) \in \mathbf{L}^2(\Omega_\mathsf{s}) \times \mathrm{L}^2(\Omega_\mathfrak{f})$ such that

$$\mathbf{u}_m \to \mathbf{u} \quad \text{in} \quad \mathbf{L}^2(\Omega_\mathsf{s})\,, \tag{4.9}$$

$$\mathfrak{p}_m \to \mathfrak{p} \quad \text{in} \quad \mathrm{L}^2(\Omega_\mathfrak{f})\,, \tag{4.10}$$

and

$$\underline{\underline{\epsilon}}(\mathbf{u}_m) \rightharpoonup \underline{\underline{\epsilon}}(\mathbf{u}) \quad \text{in} \quad \mathsf{L}^2(\Omega_\mathsf{s})\,, \tag{4.11}$$

$$\partial_S \mathfrak{p}_m \to \partial_S \mathfrak{p} = 0 \quad \text{in} \quad \mathrm{L}^2(\Omega_\mathfrak{f})\ , \tag{4.12}$$

since the sequence $\{\partial_S \mathfrak{p}_m\}$ converges to 0 in $\mathrm{L}^2(\Omega_\mathfrak{f})$.

Another consequence of (4.8) is that the sequence $\{\mathbf{u}_m \cdot \mathbf{n}\}$ is bounded in $\mathrm{H}^{\frac{1}{2}}(\Gamma)$. Therefore (up to the extraction of a subsequence) we can assume that the sequence $\{\mathbf{u}_m \cdot \mathbf{n}\}$ is strongly converging in $\mathrm{L}^2(\Gamma)$.

$$\mathbf{u}_m \cdot \mathbf{n} \to \mathbf{u} \cdot \mathbf{n} \quad \text{in} \quad \mathrm{L}^2(\Gamma)\ . \tag{4.13}$$

Furthermore, as a consequence of the convergence results (4.10)-(4.12), the sequence $\{\gamma_0 \mathfrak{p}_m\}$ converges to $\gamma_0 \mathfrak{p}$ in $\mathrm{L}^2(\Gamma)$

$$\gamma_0 \mathfrak{p}_m \to \gamma_0 \mathfrak{p} \quad \text{in} \quad \mathrm{L}^2(\Gamma)\ . \tag{4.14}$$

To prove (4.14), one applies the following trace inequality to $\mathfrak{q} = \mathfrak{p}_m - \mathfrak{p}$ (which is available in the Hilbert space H) together with (4.10)-(4.12):

PROPOSITION 4.4 (Trace inequality in H). *There exists* $C > 0$ *such that*

$$\forall \mathfrak{q} \in \mathrm{H}\ , \quad \|\gamma_0 \mathfrak{q}\|_{0,\Gamma} \leqslant C(\|\mathfrak{q}\|_{0,\Omega_\mathfrak{f}} + \|\partial_S \mathfrak{q}\|_{0,\Omega_\mathfrak{f}})\ . \tag{4.15}$$

Proof. Let $\mathfrak{p} \in \mathrm{H}$ and $Y \in (0,1)$. There holds

$$\mathfrak{p}(\mathbf{x}_\Gamma, 0) = \mathfrak{p}(\mathbf{x}_\Gamma, Y) - \int_0^Y \partial_S \mathfrak{p}(\mathbf{x}_\Gamma, S) \mathrm{d}S \ .$$

Note that since $\mathfrak{p}(\mathbf{x}_\Gamma, .)$ is a continuous function, the trace $\mathfrak{p}(\mathbf{x}_\Gamma, 0)$ on Γ is well defined. Integrating over Γ and using a Jensen inequality, one infers

$$\int_\Gamma |\mathfrak{p}(\mathbf{x}_\Gamma, 0)|^2 \,\mathrm{d}\Gamma \leqslant C \left(\int_\Gamma |\mathfrak{p}(\mathbf{x}_\Gamma, Y)|^2 \,\mathrm{d}\Gamma + \|\partial_S \mathfrak{p}\|^2_{0,\Omega_\mathsf{f}} \right) \ .$$

One concludes the proof by integrating over $Y \in (0,1)$. □

Finally, as a consequence of *(iii)* (the sequence $\{\mathfrak{p}_m(\cdot, 1)\}$ is bounded in the space $\mathrm{L}^2(\Gamma)$),x and up to the extraction of a subsequence, one can assume that the sequence $\{\mathfrak{p}_m(\cdot, 1)\}$ is weakly converging in $\mathrm{L}^2(\Gamma)$

$$\mathfrak{p}_m(\cdot, 1) \rightharpoonup \mathfrak{p}(\cdot, 1) \quad \text{in} \quad \mathrm{L}^2(\Gamma) \ . \tag{4.16}$$

As a consequence of the convergence results (4.9)-(4.10)-(4.13)-(4.14) together with (4.6b), we infer

$$\|(\mathbf{u}, \mathfrak{p})\|_{0,\Omega_\mathsf{s} \times \Omega_\mathsf{f}} + \|\mathbf{u} \cdot \mathbf{n}\|_{0,\Gamma} + \|\gamma_0 \mathfrak{p}\|_{0,\Gamma} = 1 \ . \tag{4.17}$$

Conclusion. Using Assumption 2.3, we are going to prove hereafter that $\mathbf{u} = 0$ and $\mathfrak{p} = 0$, which will contradict (4.17), and finally prove estimate (4.4).

One first characterizes $\mathbf{u}$ as a solution of a variational problem and then one defines $\mathfrak{p}$ from the Dirichlet trace $\mathbf{u} \cdot \mathbf{n}$ on Γ

PROPOSITION 4.5. *The vector field* $\mathbf{u} \in \mathbf{H}^1(\Omega_\mathsf{s})$ *satisfies : for all* $\mathbf{v} \in \mathbf{H}^1(\Omega_\mathsf{s})$*,*

$$\int_{\Omega_\mathsf{s}} \left(\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}} \right) \,\mathrm{d}\mathbf{x} - i\omega c \rho_\mathsf{f} \int_\Gamma \mathbf{u} \cdot \mathbf{n} \ \bar{\mathbf{v}} \cdot \mathbf{n} \,\mathrm{d}\sigma = 0 \ . \tag{4.18}$$

Furthermore $\mathfrak{p}$ *is a function constant in* S *in* Ω_f

$$\mathfrak{p}(\cdot, S) = -i\omega c \rho_\mathsf{f} \mathbf{u} \cdot \mathbf{n} \ . \tag{4.19}$$

The proof of this proposition is postponed to the end of this section. Then taking $\mathbf{v} = \mathbf{u}$ as test function in (4.18) and then taking the imaginary part one infers

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma \ .$$

Furthermore, according to (4.19) one obtains

$$\mathfrak{p} = 0 \quad \text{in} \quad \Omega_\mathsf{f} \ .$$

Finally, integrating by parts the first term in the sesquilinear form (4.18) as

$$-\int_{\Omega_\mathsf{s}} \underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) \,\mathrm{d}\mathbf{x} = \int_{\Omega_\mathsf{s}} \left(\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) \right) \cdot \bar{\mathbf{v}} \,\mathrm{d}\mathbf{x} - \langle \mathbf{T}(\mathbf{u}), \mathbf{v} \rangle_\Gamma \ , \tag{4.20}$$

one finds that $\mathbf{u}$ satisfies the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in} \quad \Omega_\mathsf{s} \\ \mathbf{T}(\mathbf{u}) = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on} \quad \Gamma \ . \end{cases}$$

Using Assumption 2.3 one deduces that $\mathbf{u} = 0$. Finally, one gets a contradiction with (4.17) and ends the proof of Lemma 4.2.

We are going to prove hereafter Proposition 4.5.

Proof of Proposition 4.5. One uses $(\mathbf{v}, \mathfrak{q} = 0) \in V$ as test functions in (4.5): there holds

$$\int_{\Omega_{\mathsf{s}}} \left(\underline{\underline{\sigma}}(\mathbf{u}_m) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \cdot \bar{\mathbf{v}}\right) \, \mathrm{d}\mathbf{x} + \int_\Gamma \mathfrak{p}_m \bar{\mathbf{v}} \cdot \mathbf{n} \, \mathrm{d}\Gamma = \langle \mathfrak{F}_m, (\mathbf{v}, 0) \rangle_{V',V} \ .$$

According to the convergence results (4.9)-(4.11)-(4.14) and (4.6c), taking limits as $m \to +\infty$, one deduces from the previous equalities $\mathbf{u} \in \mathbf{H}^1(\Omega_{\mathsf{s}})$ satisfies : for all $\mathbf{v} \in \mathbf{H}^1(\Omega_{\mathsf{s}})$,

$$\int_{\Omega_{\mathsf{s}}} \left(\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}}\right) \, \mathrm{d}\mathbf{x} + \int_\Gamma \mathfrak{p} \ \bar{\mathbf{v}} \cdot \mathbf{n} \, \mathrm{d}\Gamma = 0 \ . \tag{4.21}$$

Furthermore $\mathfrak{p}$ is a function constant in S in Ω_{f} since $\partial_S \mathfrak{p} = 0$ (4.12).

Then, one uses $(\mathbf{v} = 0, \mathfrak{q}) \in V$, such that $\partial_S \mathfrak{q} = 0$, as tests functions in (4.5); one obtains

$$\begin{aligned} \varepsilon_m a_{\mathsf{f}}(\varepsilon_m; \mathfrak{p}_m, \mathfrak{q}) + \omega^2 \rho_{\mathsf{f}} \int_\Gamma \mathbf{u}_m \cdot \mathbf{n} \, \bar{\mathfrak{q}} \, \mathrm{d}\Gamma \\ - i\kappa \int_\Gamma \mathfrak{p}_m(\,.\,,1) \bar{\mathfrak{q}} \det\left(\mathbb{I} + \varepsilon_m \mathcal{R}\right) \, \mathrm{d}\Gamma = \langle \mathfrak{F}_m, (0, \mathfrak{q}) \rangle_{V',V} \ , \end{aligned} \tag{4.22}$$

According to the convergence results (4.10)-(4.12), (4.6a), and since $\partial_S \mathfrak{q} = 0$ there holds

$$\varepsilon_m a_{\mathsf{f}}(\varepsilon_m; \mathfrak{p}_m, \mathfrak{q}) \to 0 \quad \text{as} \quad m \to \infty \ .$$

Hence, according to the convergence results (4.13)-(4.16), (4.6c), taking limits as $m \to +\infty$, one deduces from the previous equalities (4.22)

$$\omega^2 \rho_{\mathsf{f}} \int_\Gamma \mathbf{u} \cdot \mathbf{n} \, \bar{\mathfrak{q}} \, \mathrm{d}\Gamma - i\kappa \int_\Gamma \mathfrak{p}(\,.\,,1) \bar{\mathfrak{q}} \, \mathrm{d}\Gamma = 0 \tag{4.23}$$

for smooth functions $\mathfrak{q}$ on Γ. Since $\mathfrak{p} = \mathfrak{p}(\,.\,,1)$ is a function constant in S, one infers

$$\omega^2 \rho_{\mathsf{f}} \mathbf{u} \cdot \mathbf{n} - i\kappa \mathfrak{p} = 0 \quad \text{on} \quad \Gamma$$

which proves (4.19). Furthermore one deduces (4.18) from (4.21), which ends the proof of Proposition 4.5.

5. Derivation of Equivalent Conditions. In this section, we exhibit an asymptotic expansion for $\mathbf{u}_\varepsilon$ and p_ε, §5.1. We explicit the first terms in asymptotics, §5.2. Then we construct formally equivalent conditions, §5.3. In §5.4 we validate the asymptotic expansion with estimates for the remainders. The main result of this section is the Theorem 5.3 in §5.5 which proves the stability of equivalent conditions.

5.1. Multiscale expansion. We can exhibit series expansions in powers of ε for $\mathbf{u}_\varepsilon$ and p_ε :

$$\mathbf{u}_\varepsilon(\mathbf{x}) \approx \sum_{j \geqslant 0} \varepsilon^j \mathbf{u}_j(\mathbf{x}) \, , \tag{5.1}$$

$$\mathsf{p}_\varepsilon(\mathbf{x}) \approx \sum_{j \geqslant 0} \varepsilon^j \mathsf{p}_j(\mathbf{x}; \varepsilon) \quad \text{with} \quad \mathsf{p}_j(\mathbf{x}; \varepsilon) = \mathfrak{p}_j(y_\alpha, \frac{y_3}{\varepsilon}) \, , \tag{5.2}$$

see Sec. 5.4 for precise estimates. Here (y_α, y_3) is a "*normal coordinate system*" [10, 31] to the surface Γ on the manifold $\Omega_{\mathsf{f}}^\varepsilon$: y_α $(\alpha \in \{1,2\})$ is a tangential coordinate on Γ and $y_3 \in (0,\varepsilon)$ is the distance to the surface Γ. The term $\mathfrak{p}_j$ is a "*profile*" defined on $\Gamma \times (0,1)$. The formal calculi concerning the problem are presented in Section 5.1.1 and the first terms $(\mathfrak{p}_j, \mathbf{u}_j)$ for $j = 0,1,2$ are explicited in Section 5.2.

Expansion of the Helmholtz operator. It is possible to write the three dimensional Helmholtz operator in the layer $\Omega_{\mathsf{f}}^\varepsilon$ through the local coordinates (y_α, y_3) [31, Prop. B.1]. Then we make the scaling $Y_3 = \varepsilon^{-1} y_3 \in (0,1)$ into the normal coordinate and we expand formally the Helmholtz operator in power series of ε with coefficient intrinsic operators :

$$\Delta + \kappa^2 \mathrm{Id} = \varepsilon^{-2} \left(\sum_{n=0}^{N-1} \varepsilon^n \mathrm{L}^n + \varepsilon^N \mathrm{R}_\varepsilon^N \right) \quad \text{for all} \quad N \in \mathbb{N}^* \ .$$

The remainder R_ε^N has smooth coefficients in y_α and Y_3 which are bounded in ε, and the first operators L^n $(n = 0,1,2)$ are explicited in [32]-[31, Prop. B.3] :

$$\mathrm{L}^0 = \partial_3^2 \ , \quad \mathrm{L}^1 = 2\mathcal{H}(y_\alpha)\partial_3 \ , \quad \mathrm{L}^2 = \Delta_\Gamma + \kappa^2 \mathbb{I} - 2\left(2\mathcal{H}^2 - \mathcal{K}\right)(y_\alpha) Y_3 \partial_3 \ .$$

Here, ∂_3 is the partial derivative with respect to Y_3. We remind that Δ_Γ is the Laplace-Beltrami operator along Γ, $\mathcal{H}$ and $\mathcal{K}$ are respectively the *mean* and the *Gaussian curvature* of the surface Γ. It is also possible to explicit the operator L^3. Tedious calculi lead to

$$\mathrm{L}^3 = 2Y_3[\mathrm{div}_\Gamma \left(\mathcal{H}(y_\alpha)\mathbb{I} - \mathcal{R}\right)\nabla_\Gamma - \mathcal{H}(y_\alpha)\Delta_\Gamma] + 4Y_3^2 \mathcal{H}(y_\alpha)\left(3\mathcal{K} + 2\mathcal{H}^2\right)(y_\alpha)\partial_3$$

The operator $\mathcal{R}$ is a tangent linear operator which characterizes the curvature of Γ (see section 4.1).

5.1.1. Elementary problems. After the change of variables $y_3 \mapsto Y_3 = \varepsilon^{-1} y_3$ in the thin layer $\Omega_{\mathsf{f}}^\varepsilon$, the problem (4.9) becomes :

$$(5.3) \qquad \left\{ \begin{array}{lll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon) + \omega^2 \rho \mathbf{u}_\varepsilon = \mathbf{f} & \text{in} & \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_\varepsilon) = -\mathfrak{p}_\varepsilon \mathbf{n} & \text{on} & \Gamma \\ \varepsilon^{-2}[\partial_3^2 \mathfrak{p}_\varepsilon + \displaystyle\sum_{n \geqslant 1} \varepsilon^n \mathrm{L}^n \mathfrak{p}_\varepsilon] = 0 & \text{in} & \Gamma \times (0,1) \\ \varepsilon^{-1} \partial_3 \mathfrak{p}_\varepsilon = \rho_{\mathsf{f}} \omega^2 \mathbf{u}_\varepsilon \cdot \mathbf{n} & \text{on} & \Gamma \times \{0\} \\ \varepsilon^{-1} \partial_3 \mathfrak{p}_\varepsilon - i\kappa \mathfrak{p}_\varepsilon = 0 & \text{on} & \Gamma \times \{1\} \ . \end{array} \right.$$

Inserting the Ansatz (5.1)-(5.2) in equations (5.3), we get the following two families of problems, coupled by their boundary conditions on Γ (i.e. when $Y_3 = 0$):

$$(5.4) \qquad \left\{ \begin{array}{lll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_n) + \omega^2 \rho \mathbf{u}_n = \mathbf{f}\delta_0^n & \text{in} & \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_n) = -\mathfrak{p}_n \mathbf{n} & \text{on} & \Gamma \end{array} \right.$$

$$(5.5) \qquad \left\{ \begin{array}{lll} \partial_3^2 \mathfrak{p}_n = - \displaystyle\sum_{l+p=n, l \geqslant 1} \mathrm{L}^l \mathfrak{p}_p & \text{for} & Y_3 \in (0,1) \\ \partial_3 \mathfrak{p}_n = \rho_{\mathsf{f}} \omega^2 \mathbf{u}_{n-1} \cdot \mathbf{n} & \text{for} & Y_3 = 0 \\ \partial_3 \mathfrak{p}_n = i\kappa \mathfrak{p}_{n-1} & \text{for} & Y_3 = 1 \ . \end{array} \right.$$

In (5.4), δ_0^n denotes the Kronecker symbol and in (5.5) one uses the convention $\mathbf{u}_{-1} = 0$.

5.2. First terms. One finds successively from (5.4)-(5.5) when $n = 0, 1, 2$: $\mathbf{u}_0$ solves the problem

$$\left\{ \begin{array}{lll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} & \Omega_\mathsf{s} \\ \mathbf{T}(\mathbf{u}_0) - i\omega c \rho_\mathsf{f} \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = 0 & \text{on} & \Gamma \ , \end{array} \right. \tag{5.6}$$

and then one obtains

$$\mathfrak{p}_0 (= \mathfrak{p}_0(y_\alpha)) = -i\omega c \rho_\mathsf{f} \mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma} \ . \tag{5.7}$$

At the step $n = 1$, one finds that $\mathbf{u}_1$ solves the boundary value problem

$$\left\{ \begin{array}{lll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} & \Omega_\mathsf{s} \\ \mathbf{T}(\mathbf{u}_1) - i\omega c \rho_\mathsf{f} \mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} = i\omega c \rho_\mathsf{f} P_1(D)(\mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma})\mathbf{n} & \text{on} & \Gamma \end{array} \right. \tag{5.8}$$

where $P_1(D) = -\left(2\mathcal{H} + (i\kappa)^{-1}\Delta_\Gamma\right)$ and

$$\mathfrak{p}_1(y_\alpha, Y_3) = \rho_\mathsf{f} \omega^2 \mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma} Y_3 + b_1(y_\alpha) \ , \tag{5.9}$$

where

$$b_1(y_\alpha) = -i\omega c \rho_\mathsf{f} P_1(D)(\mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma}) - i\omega c \rho_\mathsf{f} \mathbf{u}_1 \cdot \mathbf{n}_{|\Gamma} \ .$$

At step $n = 2$ one finds that $\mathbf{u}_2$ solves the boundary value problem

$$\left\{ \begin{array}{lll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in} & \Omega_\mathsf{s} \\ \mathbf{T}(\mathbf{u}_2) - i\omega c \rho_\mathsf{f} \mathbf{u}_2 \cdot \mathbf{n}\,\mathbf{n} = i\omega c \rho_\mathsf{f} \left(P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n})\right) \mathbf{n} & \text{on} & \Gamma \end{array} \right. \tag{5.10}$$

where $P_2(D)$ is defined by (2.2) (Section 3.1). Then one obtains $\mathfrak{p}_2$ which writes

$$\mathfrak{p}_2(y_\alpha, Y_3) = a_2(y_\alpha) Y_3^2 + b_2(y_\alpha) Y_3 + c_2(y_\alpha) \ , \tag{5.11}$$

where

$$\begin{aligned} a_2 &= \omega \rho_f \left(-\omega \mathcal{H} + \tfrac{1}{2} ic[\Delta_\Gamma + \kappa^2 \mathbb{I}]\right) (\mathbf{u}_0 \cdot \mathbf{n}) \\ b_2 &= \omega^2 \rho_f \mathbf{u}_1 \cdot \mathbf{n} \\ c_2 &= -i\omega c \rho_\mathsf{f} \left(\mathbf{u}_2 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n})\right) \ . \end{aligned} \tag{5.12}$$

We refer the reader to Appendix A for more details. The whole construction of the asymptotics comes from an induction argument : if the sequences $(\mathbf{u}_n)$ and $(\mathfrak{p}_n)$ are known until rank $n = N - 1$, then the problem (5.4) and the Sturm-Liouville problem (5.5) uniquely define $\mathbf{u}_N$ and $\mathfrak{p}_N$. The next proposition ensures existence and uniqueness results together with regularity results for the first terms $\mathbf{u}_k$ in $\mathbf{H}^1(\Omega_\mathsf{s})$ $k = 0, 1, 2$.

Proposition 5.1. *Under Assumptions 2.2-2.3-2.5, if $\mathbf{f} \in \mathbf{L}^2(\Omega_\mathsf{s})$ then the boundary value problem* (5.6) *(resp.* (5.8)*) has a unique solution $\mathbf{u}_0$ (resp. $\mathbf{u}_1$) in $\mathbf{H}^1(\Omega_\mathsf{s})$; furthermore if Ω_s is of class $\mathcal{C}^4$ and $\mathbf{f} \in \mathbf{H}^2(\Omega_\mathsf{s})$ then the boundary value problem* (5.10) *has a unique solution $\mathbf{u}_2$ in $\mathbf{H}^1(\Omega_\mathsf{s})$.*

Let l be a non-negative integer. Under Assumptions 2.2-2.3, if Ω_s is of class $\mathcal{C}^{l+2}$ and $\mathbf{f} \in \mathbf{H}^l(\Omega_\mathsf{s})$, then $\mathbf{u}_0$, $\mathbf{u}_1$ belong to $\mathbf{H}^{l+2}(\Omega_\mathsf{s})$; furthermore if $l \geqslant 1$, Ω_s is of class $\mathcal{C}^{l+4}$ and $\mathbf{f} \in \mathbf{H}^{l+2}(\Omega_\mathsf{s})$ then $\mathbf{u}_2$ belongs to $\mathbf{H}^{l+2}(\Omega_\mathsf{s})$.

Here the regularity of $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{u}_2$ is obtained by a general shift result available in Sobolev spaces, see for instance [12, Th. 3.4.5].

5.3. Construction of equivalent conditions. In this section, we derive formally ECs (Sec. 3.1).

Order 1. Since the equations in (5.6) are independent of ε, the condition of order 1 writes

$$\mathbf{T}(\mathbf{u}_0) - i\omega c\rho_{\mathsf{f}}\mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = 0 \quad \text{on} \quad \Gamma\,. \quad \text{on} \quad \Gamma\,.$$

Order 2. According to (5.6) and (5.8), the truncated expansion $\mathbf{u}_{1,\varepsilon} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1$ solves the elastic equation in Ω_{s} together with the boundary condition

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - i\omega c\rho_{\mathsf{f}}\mathbf{u}_{1,\varepsilon} \cdot \mathbf{n}\,\mathbf{n} = \varepsilon i\omega c\rho_{\mathsf{f}} P_1(D)(\mathbf{u}_0 \cdot \mathbf{n})_{|\Gamma}\mathbf{n} \quad \text{on} \quad \Gamma\,.$$

Writting $\mathbf{u}_0 = \mathbf{u}_{1,\varepsilon} - \varepsilon\mathbf{u}_1$, there holds

$$\mathbf{T}(\mathbf{u}_{1,\varepsilon}) - i\omega c\rho_{\mathsf{f}}\mathbf{u}_{1,\varepsilon} \cdot \mathbf{n}\,\mathbf{n} - \varepsilon i\omega c\rho_{\mathsf{f}} P_1(D)(\mathbf{u}_{1,\varepsilon} \cdot \mathbf{n})\ \mathbf{n} = -\varepsilon^2 i\omega c\rho_{\mathsf{f}} P_1(D)(\mathbf{u}_1 \cdot \mathbf{n})\ \mathbf{n}\,.$$

Neglecting the term of order ε^2 in the previous right-hand side, we infer the condition (3.3).

Order 3. According to (5.6)-(5.8) and (5.10), the truncated expansion $\mathbf{u}_{2,\varepsilon} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \varepsilon^2\mathbf{u}_2$ solves the elastic equation in Ω_{s} together with the boundary condition

$$\begin{aligned}\mathbf{T}(\mathbf{u}_{2,\varepsilon}) - i\omega c\rho_{\mathsf{f}}\left(\mathbb{I} + \varepsilon P_1(D) + \varepsilon^2 P_2(D)\right)(\mathbf{u}_{2,\varepsilon} \cdot \mathbf{n})\,\mathbf{n} &= \\ - \varepsilon^3 i\omega c\rho_{\mathsf{f}}\left(P_1(D)(\mathbf{u}_2 \cdot \mathbf{n}) + P_2(D)\left((\varepsilon\mathbf{u}_2 + \mathbf{u}_1) \cdot \mathbf{n}\right)\right)\mathbf{n} &\quad \text{on} \quad \Gamma\,.\end{aligned} \tag{5.13}$$

Then, neglecting the term of order ε^3 in the previous right-hand side, one deduces the condition (3.4).

5.4. Estimates for the remainders. The validation of the asymptotic expansion (5.1)-(5.2) consists in proving estimates for remainders $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$ defined in Ω_{s} and $\Omega_{\mathsf{f}}^\varepsilon$ as

$$\mathbf{r}_\varepsilon^N = \mathbf{u}_\varepsilon - \sum_{n=0}^N \varepsilon^n\mathbf{u}_n \quad \text{in} \quad \Omega_{\mathsf{s}}\,, \quad \text{and} \quad r_\varepsilon^N(\mathbf{x}) = \mathsf{p}_\varepsilon(\mathbf{x}) - \sum_{n=0}^N \varepsilon^n\mathfrak{p}_n(y_\alpha, \frac{y_3}{\varepsilon}) \quad \text{for all}\,\mathbf{x} \in \Omega_{\mathsf{f}}^\varepsilon\,. \tag{5.14}$$

The convergence result is the following statement.

THEOREM 5.2. *Under Assumptions 2.2-2.3-2.5-2.6 and for ε small enough, the solution $(\mathbf{u}_\varepsilon, \mathsf{p}_\varepsilon)$ of problem (2.3) has a two-scale expansion which can be written in the form (5.1)-(5.2), with $\mathbf{u}_j \in \mathbf{H}^1(\Omega_{\mathsf{s}})$ and $\mathfrak{p}_j \in \mathrm{H}^1(\Gamma \times (0,1))$. For each $N \in \mathbb{N}$, the remainders $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$ satisfy*

$$\|\mathbf{r}_\varepsilon^N\|_{1,\Omega_{\mathsf{s}}} + \sqrt{\varepsilon}\|r_\varepsilon^N\|_{1,\Omega_{\mathsf{f}}^\varepsilon} \leqslant C_N\varepsilon^{N+1} \tag{5.15}$$

with a constant C_N independent of ε.

The error estimate (5.15) is obtained through an evaluation of the right-hand sides when applying Theorem 2.7 to the couple $(\mathbf{u}, \mathsf{p}) = (\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$.

Proof. The proof is rather standard, see for instance the proof of [9, Th. 2.1] where the authors consider an interface problem for the Laplacian operator set in a domain with a thin layer. The error estimate (5.15) is obtained through an evaluation of the right-hand sides when the elasto-acoustic operator is applied to $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$. By construction, the remainder $(\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$ is solution of problem

$$\left\{\begin{array}{lll}\Delta r_\varepsilon^N + \kappa^2 r_\varepsilon^N = f_{N,\varepsilon} & \text{in} & \Omega_{\mathsf{f}}^\varepsilon \\ \nabla\cdot\underline{\underline{\sigma}}(\mathbf{r}_\varepsilon^N) + \omega^2\rho\mathbf{r}_\varepsilon^N = 0 & \text{in} & \Omega_{\mathsf{s}} \\ \partial_{\mathbf{n}} r_\varepsilon^N = \rho_{\mathsf{f}}\omega^2\mathbf{r}_\varepsilon^N\cdot\mathbf{n} + g_{N,\varepsilon} & \text{on} & \Gamma \\ \mathbf{T}(\mathbf{r}_\varepsilon^N) = -r_\varepsilon^N\mathbf{n} & \text{on} & \Gamma \\ \partial_{\mathbf{n}} r_\varepsilon^N - i\kappa r_\varepsilon^N = h_{N,\varepsilon} & \text{on} & \Gamma^\varepsilon\ .\end{array}\right. \tag{5.16}$$

Here, the right-hand sides are explicit :

$$f_{N,\varepsilon} = \varepsilon^{N-1}[F_N - \sum_{l=0}^{N}\varepsilon^l R_\varepsilon^{N+1}\mathfrak{p}_l] \quad \text{in} \quad \Omega_{\mathsf{f}}^\varepsilon\ ,$$

$$g_{N,\varepsilon} = \rho_{\mathsf{f}}\omega^2\varepsilon^N\mathbf{u}^N\cdot\mathbf{n} \quad \text{on} \quad \Gamma\ , \quad \text{and} \quad h_{N,\varepsilon} = i\kappa\varepsilon^N\mathfrak{p}_N \quad \text{on} \quad \Gamma^\varepsilon\ .$$

We have the following estimates for the residues $f_{N,\varepsilon}$ and $g_{N,\varepsilon}$

$$\|f_{N,\varepsilon}\|_{0,\Omega_{\mathsf{f}}^\varepsilon} = \mathcal{O}(\varepsilon^{N-\frac{1}{2}})\ , \quad \|g_{N,\varepsilon}\|_{0,\Gamma} = \mathcal{O}(\varepsilon^N) \quad \text{and} \quad \|h_{N,\varepsilon}\|_{0,\Gamma^\varepsilon} = \mathcal{O}(\varepsilon^N)\ .$$

We can apply Theorem 2.7 to the couple $(\mathbf{u},\mathsf{p}) = (\mathbf{r}_\varepsilon^N, r_\varepsilon^N)$, and we obtain

$$\|\mathbf{r}_\varepsilon^N\|_{1,\Omega_{\mathsf{s}}} + \|r_\varepsilon^N\|_{1,\Omega_{\mathsf{f}}^\varepsilon} \leqslant C_N\varepsilon^{N-\frac{1}{2}}\ .$$

Writting $\mathbf{r}_\varepsilon^N = \mathbf{r}_\varepsilon^{N+2} + \varepsilon^{N+2}\mathbf{u}_{N+2} + \varepsilon^{N+1}\mathbf{u}_{N+1}$ and $r_\varepsilon^N = r_\varepsilon^{N+2} + \varepsilon^{N+2}\mathfrak{p}_{N+2} + \varepsilon^{N+1}\mathfrak{p}_{N+1}$, we apply the previous estimate to the couple $(\mathbf{r}_\varepsilon^{N+2}, r_\varepsilon^{N+2})$ and we use estimates

$$\|\mathbf{u}_l\|_{1,\Omega_{\mathsf{s}}} = \mathcal{O}(1) \quad \text{and} \quad \|\mathfrak{p}_l\|_{1,\Omega_{\mathsf{f}}^\varepsilon} = \mathcal{O}(\varepsilon^{-\frac{1}{2}})$$

to infer the optimal estimates (5.15). □

5.5. Validation of equivalent conditions. We consider the problem (3.1) with an equivalent condition and at a fixed non-zero frequency ω satisfying Assumption 2.3. The main result of this section is the following statement, that is for all $k \in \{0,1,2\}$ the problem (3.1) is well-posed in the space $\mathbf{V}^k$ (Not. 3.3), and its solution satisfies uniform $\mathbf{H}^1$ estimates.

THEOREM 5.3. *Under Assumptions 2.2-2.3-2.5, for all $k \in \{0,1,2\}$ there are constants $\varepsilon_k, C_k > 0$ such that for all $\varepsilon \in (0,\varepsilon_k)$, the problem (3.1) with a data $\mathbf{f} \in \mathbf{L}^2(\Omega_{\mathsf{s}})$ has a unique solution $\mathbf{u}_\varepsilon^k \in \mathbf{V}^k$ which satisfies the uniform estimates:*

$$\|\mathbf{u}_0\|_{1,\Omega_{\mathsf{s}}} \leqslant C_0\|\mathbf{f}\|_{0,\Omega_{\mathsf{s}}} \quad (\mathbf{u}_0 := \mathbf{u}_\varepsilon^0)\ , \tag{5.17a}$$

$$\|\mathbf{u}_\varepsilon^1\|_{1,\Omega_{\mathsf{s}}} + \|\nabla_\Gamma(\mathbf{u}_\varepsilon^1\cdot\mathbf{n})\|_{0,\Gamma} \leqslant C_1\|\mathbf{f}\|_{0,\Omega_{\mathsf{s}}}\ , \tag{5.17b}$$

$$\|\mathbf{u}_\varepsilon^2\|_{1,\Omega_{\mathsf{s}}} + \|\nabla_\Gamma(\mathbf{u}_\varepsilon^2\cdot\mathbf{n})\|_{0,\Gamma} + \|\Delta_\Gamma(\mathbf{u}_\varepsilon^2\cdot\mathbf{n})\|_{0,\Gamma} \leqslant C_2\|\mathbf{f}\|_{0,\Omega_{\mathsf{s}}}\ . \tag{5.17c}$$

The key for the proof of Thm. 5.3 is the following Lemma.

LEMMA 5.4. *Under Assumptions 2.2-2.3-2.5, for all $k \in \{0,1,2\}$ there exists constants $\varepsilon_k, C_k > 0$ such that for all $\varepsilon \in (0, \varepsilon_k)$, any solution $\mathbf{u}_\varepsilon^k \in \mathbf{V}^k$ of problem (3.1) with a data $\mathbf{f} \in \mathbf{L}^2(\Omega_\mathsf{s})$ satisfies the uniform estimate:*

$$\|\mathbf{u}_0\|_{0,\Omega_\mathsf{s}} \leqslant C_0 \|\mathbf{f}\|_{0,\Omega_\mathsf{s}} \,, \tag{5.18a}$$

$$\|\mathbf{u}_\varepsilon^1\|_{0,\Omega_\mathsf{s}} + \|\nabla_\Gamma(\mathbf{u}_\varepsilon^1 \cdot \mathbf{n})\|_{0,\Gamma} \leqslant C_1 \|\mathbf{f}\|_{0,\Omega_\mathsf{s}} \,, \tag{5.18b}$$

$$\|\mathbf{u}_\varepsilon^2\|_{0,\Omega_\mathsf{s}} + \|\nabla_\Gamma(\mathbf{u}_\varepsilon^2 \cdot \mathbf{n})\|_{0,\Gamma} + \|\Delta_\Gamma(\mathbf{u}_\varepsilon^2 \cdot \mathbf{n})\|_{0,\Gamma} \leqslant C_2 \|\mathbf{f}\|_{0,\Omega_\mathsf{s}} \,. \tag{5.18c}$$

REMARK 5.5. *For $k = 0$, the Theorem 5.3 and the Lemma 5.4 hold for all $\varepsilon > 0$. For $k = 1, 2$, using a compactness argument one proves uniform estimates provided ε is small enough.*

The Lemma 5.4 is proved in Section 6.1. As a consequence of this Lemma, each solution of the problem (3.1) satisfies uniform $\mathbf{H}^1$-estimates (5.17a), (5.17b), (5.17c) respectively when $k = 0, 1, 2$. Then, the proof of Thm. 5.3 is obtained as a consequence of the Fredholm alternative since the problem (3.1) is of Fredholm type. One passes from Lemma 5.4 to Theorem 5.3 as one passes from Lemma 4.2 to Theorem 4.1. We refer also to the work in Ref [32] for a similar context.

6. Analysis of Equivalent Conditions. In this section, one first proves the Lemma 5.4, i.e. uniform $\mathbf{L}^2$-estimate (5.18a) for the solution of problem (3.1). In Section 6.2, we prove that the solution $\mathbf{u}_\varepsilon^k$ of problem (3.1) satisfies uniform $\mathbf{H}^1$ error estimates (3.5) and we infer the Theorem 3.4. We focus on the proof of Lemma 5.4 for $k = 2$ since the proofs when $k = 0, 1$ are simpler. Hence we consider the problem (here $\mathbf{u} = \mathbf{u}_\varepsilon^2$)

$$\left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{f} & \text{in} \quad \Omega_\mathsf{s} \\ \mathbf{T}(\mathbf{u}) + \mathsf{F}_\varepsilon(\mathbf{u} \cdot \mathbf{n})\mathbf{n} = 0 & \text{on} \quad \Gamma \,, \end{array} \right. \tag{6.1}$$

where the operator F_ε is defined as

$$\mathsf{F}_\varepsilon = \mathcal{J}_\varepsilon \mathbb{I} + \beta_\varepsilon \Delta_\Gamma + \gamma_\varepsilon \operatorname{div}_\Gamma(\mathcal{H}\mathbb{I} - \mathcal{R})\nabla_\Gamma + \delta_\varepsilon \Delta_\Gamma^2 \,.$$

Here $\mathcal{J}_\varepsilon$ and β_ε are smooth scalar functions defined on Γ as

$$\mathcal{J}_\varepsilon = -i\omega c \rho_\mathsf{f} \left\{ 1 - 2\varepsilon\mathcal{H} + \varepsilon^2 \left(4\mathcal{H}^2 - \mathcal{K} - 2i \left[\kappa^{-1} \Delta_\Gamma \mathcal{H} + \kappa \mathcal{H} \right] \right) \right\} \tag{6.2a}$$

$$\beta_\varepsilon = \varepsilon c^2 \rho_\mathsf{f} \left(1 + i\kappa\varepsilon \left[1 + 2i\kappa^{-1}\mathcal{H} \right] \right) \,, \tag{6.2b}$$

and $\gamma_\varepsilon = \varepsilon^2 c^2 \rho_\mathsf{f} > 0$ and $\delta_\varepsilon = -i\varepsilon^2 \kappa^{-1} c^2 \rho_\mathsf{f}$.

To prepare for the proof, we introduce the variational formulation for $\mathbf{u}$. If $\mathbf{u} \in \mathbf{V}^2$ is a solution of (6.1), then it satisfies for all $\mathbf{v} \in \mathbf{V}^2$:

$$\begin{aligned} &\int_{\Omega_\mathsf{s}} \left(\underline{\underline{\sigma}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u} \cdot \bar{\mathbf{v}} \right) \, \mathrm{d}\mathbf{x} + \int_\Gamma \mathcal{J}_\varepsilon \mathbf{u} \cdot \mathbf{n} \; \bar{\mathbf{v}} \cdot \mathbf{n} \, \mathrm{d}\sigma \\ &- \int_\Gamma \nabla_\Gamma(\mathbf{u} \cdot \mathbf{n}) \cdot \nabla_\Gamma(\beta_\varepsilon \bar{\mathbf{v}} \cdot \mathbf{n}) \, \mathrm{d}\sigma - \gamma_\varepsilon \int_\Gamma (\mathcal{H}\mathbb{I} - \mathcal{R}) \nabla_\Gamma(\mathbf{u} \cdot \mathbf{n}) \cdot \nabla_\Gamma(\bar{\mathbf{v}} \cdot \mathbf{n}) \, \mathrm{d}\sigma \\ &\qquad + \delta_\varepsilon \int_\Gamma \Delta_\Gamma(\mathbf{u} \cdot \mathbf{n}) \Delta_\Gamma(\bar{\mathbf{v}} \cdot \mathbf{n}) \, \mathrm{d}\sigma = - \int_{\Omega_\mathsf{s}} \mathbf{f} \cdot \bar{\mathbf{v}} \, \mathrm{d}\mathbf{x} \,. \end{aligned} \tag{6.3}$$

6.1. Proof of Lemma 5.4 : Uniform $\mathbf{L}^2$-estimate of the elastic displacement. Reductio ad absurdum: We assume that there is a sequence $(\mathbf{u}_m) \in \mathbf{V}^2$, $m \in \mathbb{N}$, of solutions of the problem (6.1) associated with a parameter ε_m and a right-hand side $\mathbf{f}_m \in \mathbf{L}^2(\Omega_\mathsf{s})$:

$$\nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_m) + \omega^2 \rho \mathbf{u}_m = \mathbf{f}_m \quad \text{in} \quad \Omega_\mathsf{s}\,, \tag{6.4a}$$

$$\mathbf{T}(\mathbf{u}_m) + \big(\mathcal{J}_m \mathbb{I} + \beta_m \Delta_\Gamma + \gamma_m \operatorname{div}_\Gamma(\mathcal{H}\mathbb{I} - \mathcal{R})\nabla_\Gamma + \delta_m \Delta_\Gamma^2\big)\,(\mathbf{u}_m \cdot \mathbf{n})\mathbf{n} = 0 \quad \text{on} \quad \Gamma \tag{6.4b}$$

(with $\mathcal{J}_m := \mathcal{J}_{\varepsilon_m}$, $\beta_m = \beta_{\varepsilon_m}$, $\gamma_m = \gamma_{\varepsilon_m}$ and $\delta_m = \delta_{\varepsilon_m}$) satisfying the following conditions

$$\varepsilon_m \to 0 \qquad \text{as} \;\; m \to \infty \tag{6.5a}$$

$$\|\mathbf{u}_m\|_{0,\Omega_\mathsf{s}} + \|\nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\|_{0,\Gamma} + \|\Delta_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\|_{0,\Gamma} = 1 \qquad \text{for all } m \in \mathbb{N} \tag{6.5b}$$

$$\|\mathbf{f}_m\|_{0,\Omega_\mathsf{s}} \to 0 \qquad \text{as} \;\; m \to \infty\,. \tag{6.5c}$$

6.1.1. Estimates of the sequence $\{\mathbf{u}_m\}$. We first prove that the sequence $\{\mathbf{u}_m\}$ is bounded in $\mathbf{V}^2$. We particularize the elastic variational formulation (6.3) for the sequence $\{\mathbf{u}_m\}$: For all $\mathbf{v} \in \mathbf{V}^2$:

$$\begin{aligned}
&\int_{\Omega_\mathsf{s}} \big(\underline{\underline{\sigma}}(\mathbf{u}_m) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2 \rho \mathbf{u}_m \cdot \bar{\mathbf{v}}\big)\; \mathrm{d}\mathbf{x} + \int_\Gamma \mathcal{J}_m \mathbf{u}_m \cdot \mathbf{n}\bar{\mathbf{v}} \cdot \mathbf{n}\, \mathrm{d}\sigma \\
&\quad - \int_\Gamma \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \cdot \nabla_\Gamma(\beta_m \bar{\mathbf{v}} \cdot \mathbf{n})\, \mathrm{d}\sigma - \gamma_m \int_\Gamma (\mathcal{H}\mathbb{I} - \mathcal{R})\nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \cdot \nabla_\Gamma(\bar{\mathbf{v}} \cdot \mathbf{n})\, \mathrm{d}\sigma \\
&\qquad + \delta_m \int_\Gamma \Delta_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \Delta_\Gamma(\bar{\mathbf{v}} \cdot \mathbf{n})\, \mathrm{d}\sigma = -\int_{\Omega_\mathsf{s}} \mathbf{f}_m \cdot \bar{\mathbf{v}}\; \mathrm{d}\mathbf{x}\,.
\end{aligned} \tag{6.6}$$

We first take the imaginary part of the previous equality (6.6) when $\mathbf{v} = \mathbf{u}_m$. According to Assumption 2.2, there holds

$$\begin{aligned}
&-\int_\Gamma \operatorname{Im}(\mathcal{J}_m) |\mathbf{u}_m \cdot \mathbf{n}|^2\, \mathrm{d}\sigma + \operatorname{Im}(\beta_m) \|\nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\|_{0,\Gamma}^2 \\
&\qquad - \operatorname{Im}(\delta_m) \|\Delta_\Gamma(\mathbf{u}_m \cdot \mathbf{n})\|_{0,\Gamma}^2 = \operatorname{Im} \int_{\Omega_\mathsf{s}} \mathbf{f}_m \cdot \bar{\mathbf{u}}_m\; \mathrm{d}\mathbf{x}\,.
\end{aligned}$$

Since the sequence of functions $\{\operatorname{Im}(\mathcal{J}_m)\}$ is bounded ($\operatorname{Im}(\mathcal{J}_m) \to -\omega c \rho_\mathsf{f}$ a.e. on Γ as $m \to \infty$) and since $\operatorname{Im} \beta_m \to 0$ and $\delta_m \to 0$ (see (6.5a)) as $m \to \infty$, one deduces from the last equality and with the help of conditions (6.5b)-(6.5c) the following uniform bound

$$\|\mathbf{u}_m \cdot \mathbf{n}\|_{0,\Gamma} \leqslant C_1\,. \tag{6.7}$$

Since the tensor $\underline{\underline{\epsilon}}(\mathbf{u})$ is symmetric, thanks to the assumptions 2.2 *(i)-(iii)* together with the Korn inequality, we infer : there exists constants $C, c > 0$ such that for all $\mathbf{u} \in \mathbf{H}^1(\Omega_\mathsf{s})$

$$\int_{\Omega_\mathsf{s}} \underline{\underline{C}}\; \underline{\underline{\epsilon}}(\mathbf{u}) : \underline{\underline{\epsilon}}(\bar{\mathbf{u}}) \mathrm{d}\mathbf{x} \geqslant \alpha C \|\mathbf{u}\|_{1,\Omega_\mathsf{s}}^2 - \alpha c \|\mathbf{u}\|_{0,\Omega_\mathsf{s}}^2\,. \tag{6.8}$$

Then taking the real part of the equality (6.6) when $\mathbf{v} = \mathbf{u}_m$, there holds

$$\int_{\Omega_s} \underline{\underline{\sigma}}(\mathbf{u}_m) : \underline{\underline{\epsilon}}(\overline{\mathbf{u}_m})\,\mathrm{d}\mathbf{x} - \omega^2\rho\|\mathbf{u}_m\|^2_{0,\Omega_s} + \int_\Gamma \mathrm{Re}(\mathcal{J}_m)|\mathbf{u}_m\cdot\mathbf{n}|^2\,\mathrm{d}\sigma$$
$$- \int_\Gamma \nabla_\Gamma(\mathbf{u}_m\cdot\mathbf{n})\cdot\nabla_\Gamma(\mathrm{Re}\,\beta_m\,\bar{\mathbf{u}}_m\cdot\mathbf{n})\,\mathrm{d}\sigma - \gamma_m\int_\Gamma(\mathcal{H}\mathbb{I}-\mathcal{R})|\nabla_\Gamma(\mathbf{u}_m\cdot\mathbf{n})|^2\,\mathrm{d}\sigma$$
$$= -\,\mathrm{Re}\int_{\Omega_s}\mathbf{f}_m\cdot\bar{\mathbf{u}}_m\,\mathrm{d}\mathbf{x}\ ,$$

and since the sequences of functions $\{\mathrm{Re}\,\mathcal{J}_m\}$, $\{\mathrm{Re}\,\beta_m\}$, $\{\nabla_\Gamma(\mathrm{Re}\,\beta_m)\}$ and the sequence $\{\gamma_m\}$ are bounded, one obtains with the help of conditions (6.5a)-(6.5b)-(6.5c) together with the previous inequalities (6.7)-(6.8), a uniform bound in $\mathbf{H}^1(\Omega_s)$ (and thus in the space $\mathbf{V}^2$ using the condition (6.5b) again) for the sequence $\{\mathbf{u}_m\}$:

$$\|\mathbf{u}_m\|_{1,\Omega_s} + \|\nabla_\Gamma(\mathbf{u}_m\cdot\mathbf{n})\|_{0,\Gamma} + \|\Delta_\Gamma(\mathbf{u}_m\cdot\mathbf{n})\|_{0,\Gamma} \leqslant C_2\ . \tag{6.9}$$

Another consequence of (6.9) is that the sequence $\{\mathbf{u}_m\cdot\mathbf{n}\}$ is bounded in $\mathrm{H}^2(\Gamma)$:

$$\|\mathbf{u}_m\cdot\mathbf{n}\|_{2,\Gamma} \leqslant C\ . \tag{6.10}$$

6.1.2. Limit of the sequence and conclusion. The domain Ω_s being bounded, the embedding of $\mathbf{H}^1(\Omega_s)$ in $\mathbf{L}^2(\Omega_s)$ is compact. Hence as a consequence of (6.9), using the Rellich Lemma we can extract a subsequence of $\{\mathbf{u}_m\}$ (still denoted by $\{\mathbf{u}_m\}$) which is converging in $\mathbf{L}^2(\Omega_s)$. By the Banach-Alaoglu theorem, one can assume that the sequence $\{\underline{\underline{\nabla}}\mathbf{u}_m\}$ is weakly converging in $\mathsf{L}^2(\Omega_s)$ and the sequence $\{\Delta_\Gamma(\mathbf{u}_m\cdot\mathbf{n})\}$ is weakly converging in $\mathrm{L}^2(\Gamma)$. As a consequence of (6.10) and since the embedding of $\mathrm{H}^2(\Gamma)$ in $\mathrm{H}^1(\Gamma)$ is compact, up to the extraction of a subsequence we can assume that the sequence $\{\mathbf{u}_m\cdot\mathbf{n}\}$ is strongly converging in $\mathrm{H}^1(\Gamma)$: We deduce that there is $\mathbf{u}\in\mathbf{L}^2(\Omega_s)$ such that

$$\begin{cases} \underline{\underline{\epsilon}}(\mathbf{u}_m) \rightharpoonup \underline{\underline{\epsilon}}(\mathbf{u}) & \text{in } \mathsf{L}^2(\Omega_s) \\ \Delta_\Gamma(\mathbf{u}_m\cdot\mathbf{n}) \rightharpoonup \Delta_\Gamma(\mathbf{u}\cdot\mathbf{n}) & \text{in } \mathrm{L}^2(\Gamma) \\ \mathbf{u}_m \to \mathbf{u} & \text{in } \mathbf{L}^2(\Omega_s) \\ \mathbf{u}_m\cdot\mathbf{n} \to \mathbf{u}\cdot\mathbf{n} & \text{in } \mathrm{H}^1(\Gamma)\ . \end{cases} \tag{6.11}$$

Using Assumption 2.3, we are going to prove that $\mathbf{u} = 0$, which will contradict (6.5b), and finally prove estimate (5.18b). Let $\mathbf{v}\in\mathbf{V}^2$ be a test function in (6.6)

$$\int_{\Omega_s}\left(\underline{\underline{\sigma}}(\mathbf{u}_m) : \underline{\underline{\epsilon}}(\bar{\mathbf{v}}) - \omega^2\rho\mathbf{u}_m\cdot\bar{\mathbf{v}}\right)\,\mathrm{d}\mathbf{x} + \int_\Gamma \mathcal{J}_m\mathbf{u}_m\cdot\mathbf{n}\bar{\mathbf{v}}\cdot\mathbf{n}\,\mathrm{d}\sigma \tag{6.12}$$
$$- \int_\Gamma \nabla_\Gamma(\mathbf{u}_m\cdot\mathbf{n})\cdot\nabla_\Gamma(\beta_m\bar{\mathbf{v}}\cdot\mathbf{n})\,\mathrm{d}\sigma - \gamma_m\int_\Gamma(\mathcal{H}\mathbb{I}-\mathcal{R})\nabla_\Gamma(\mathbf{u}_m\cdot\mathbf{n})\cdot\nabla_\Gamma(\bar{\mathbf{v}}\cdot\mathbf{n})\,\mathrm{d}\sigma$$
$$+ \delta_m\int_\Gamma \Delta_\Gamma(\mathbf{u}_m\cdot\mathbf{n})\Delta_\Gamma(\bar{\mathbf{v}}\cdot\mathbf{n})\,\mathrm{d}\sigma = -\int_{\Omega_s}\mathbf{f}_m\cdot\bar{\mathbf{v}}\,\mathrm{d}\mathbf{x}\ .$$

Since $\mathcal{J}_m\to -i\omega c\rho_f$, $\beta_m\to 0$ and $\nabla_\Gamma\beta_m\to 0$ a.e. on Γ and since $\gamma_m\to 0$ and $\delta_m\to 0$ as $m\to\infty$, taking limits as $m\to\infty$ one deduces from the previous equalities (6.11)-(6.12) that $\mathbf{u}\in\mathbf{V}^2$ satisfies for all $\mathbf{v}\in\mathbf{V}^2$

$$a_s(\mathbf{u},\mathbf{v}) - i\omega c\rho_f\int_\Gamma \mathbf{u}\cdot\mathbf{n}\ \bar{\mathbf{v}}\cdot\mathbf{n}\,\mathrm{d}\sigma = 0\ . \tag{6.13}$$

Taking the imaginary part of the previous equality when $\mathbf{v} = \mathbf{u}$, one deduces $\mathbf{u} \cdot \mathbf{n} = 0$ on Γ. Then integrating by parts we find that $\mathbf{u}$ satisfies the problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}) + \omega^2 \rho \mathbf{u} = 0 & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}) = 0 \quad \text{and} \quad \mathbf{u} \cdot \mathbf{n} = 0 & \text{on} \quad \Gamma \,. \end{cases}$$

According to Assumption 2.3, we infer

$$\mathbf{u} = 0 \quad \text{in} \quad \Omega_{\mathsf{s}} \,.$$

Thus according to (6.11), there holds

$$\begin{cases} \mathbf{u}_m \to 0 & \text{in} \quad \mathbf{L}^2(\Omega_{\mathsf{s}}) \\ \nabla_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \to 0 & \text{in} \quad \mathrm{L}^2(\Gamma) \\ \Delta_\Gamma(\mathbf{u}_m \cdot \mathbf{n}) \to 0 & \text{in} \quad \mathrm{L}^2(\Gamma) \,, \end{cases} \tag{6.14}$$

which contradicts (6.5b) and ends the proof of Lemma 5.4.

6.2. Proof of error estimates . In this section we prove the Theorem 3.4. Since the problem (3.1) is of Fredholm type, it is sufficient to prove that any solution $\mathbf{u}_\varepsilon^k$ of (3.1) satisfies the error estimate (3.5)

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_{\mathsf{s}}} \leqslant C\varepsilon^{k+1} \,.$$

We prove hereafter the estimate (3.5) in two steps (Sec. 6.2.1 and Sec. 6.2.2).

6.2.1. Step A. The first step consists in deriving an expansion of $\mathbf{u}_\varepsilon^k$ and to show that the truncated expansions of $\mathbf{u}_\varepsilon^k$ and $\mathbf{u}_\varepsilon$ coincide up to the order ε^k:

$$\mathbf{u}_\varepsilon = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \mathbf{r}_\varepsilon^k \,, \tag{6.15}$$

$$\mathbf{u}_\varepsilon^k = \mathbf{u}_0 + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots + \varepsilon^k \mathbf{u}_k + \tilde{\mathbf{r}}_\varepsilon^k \,. \tag{6.16}$$

Hereafter, we justify the expansion (6.16). By construction, $\mathbf{u}_\varepsilon^k$ admits an expansion

$$\mathbf{u}_\varepsilon^k = \mathbf{v}_0 + \varepsilon \mathbf{v}_1 + \varepsilon^2 \mathbf{v}_2 + \cdots + \varepsilon^k \mathbf{v}_k + \tilde{\mathbf{r}}_\varepsilon^k$$

where each term $\mathbf{v}_n$, for $0 \leqslant n \leqslant k$, satisfies the problem (5.4) as well as the term $\mathbf{u}_n$. Using the spectral Assumption 2.3, we infer that for all $0 \leqslant n \leqslant k$, $\mathbf{v}_n = \mathbf{u}_n$ in Ω_{s}, and the expansion (6.16) holds.

Hence,

$$\|\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon^k\|_{1,\Omega_{\mathsf{s}}} = \|\mathbf{r}_\varepsilon^k - \tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_{\mathsf{s}}} \,.$$

The estimate of the remainder $\mathbf{r}_\varepsilon^k$ is already proved in Thm 5.2 (Sec. 5.4) : $\|\mathbf{r}_\varepsilon^k\|_{1,\Omega_{\mathsf{s}}} \leqslant C\varepsilon^{k+1}$. In the next step, we prove estimates for the remainder $\tilde{\mathbf{r}}_\varepsilon^k$.

6.2.2. Step B. According to (6.16), the remainder $\tilde{\mathbf{r}}_\varepsilon^k$ satisfies the elastic equation in Ω_{s}. We apply the operator $\mathbf{T} + \mathbf{F}_{k,\varepsilon}$ (where $\mathbf{F}_{k,\varepsilon}(\mathbf{u}) := \mathsf{F}_{k,\varepsilon}(\mathbf{u} \cdot \mathbf{n})\mathbf{n}$) to the remainder $\tilde{\mathbf{r}}_\varepsilon^k$. Then, we prove hereafter that

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^k) + \mathsf{F}_{k,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^k \cdot \mathbf{n})\mathbf{n} = \mathcal{O}(\varepsilon^{k+1}) \quad \text{on} \quad \Gamma \,.$$

Since $\mathbf{u}_\varepsilon^0 = \mathbf{u}_0$, then $\tilde{\mathbf{r}}_\varepsilon^0 = 0$. Relying on the construction of equivalent conditions detailed in section 5.3, there holds

$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^1) + \mathsf{F}_{1,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^1 \cdot \mathbf{n})\mathbf{n} = \varepsilon^2 i\omega c\rho_{\mathsf{f}} P_1(D)(\mathbf{u}_1 \cdot \mathbf{n})\, \mathbf{n} \quad \text{on} \quad \Gamma$$
$$\mathbf{T}(\tilde{\mathbf{r}}_\varepsilon^2) + \mathsf{F}_{2,\varepsilon}(\tilde{\mathbf{r}}_\varepsilon^2 \cdot \mathbf{n})\mathbf{n} = \varepsilon^3 i\omega c\rho_{\mathsf{f}} \left(P_1(D)(\mathbf{u}_2 \cdot \mathbf{n}) + P_2(D)\left((\varepsilon\mathbf{u}_2 + \mathbf{u}_1) \cdot \mathbf{n}\right)\right) \mathbf{n}\ .$$

Then, according to estimates (5.17b) one deduces the uniform estimate

$$\|\tilde{\mathbf{r}}_\varepsilon^k\|_{1,\Omega_{\mathsf{s}}} \leqslant C\varepsilon^{k+1}\ ,$$

which ends the proof of Theorem 3.4.

7. Conclusion. In this paper, a new equivalent conditions has been proposed to approximate a thin layer of water with a Robin boundary condition on top of it. The asymptotic model has been derived up to the third order thanks to a two scale asymptotic expansion. The stability of the new condition and the convergence of the asymptotic model have been proven. The equivalent condition has been designed in order to be easily implemented in Finite Element codes, in particular in those based on Discontinuous Galerkin methods. Thus, the next of this work will be the performance analysis of the equivalent condition on numerical simulations. The next issue to tackle is the case where the layer of fluid, which usually models the ocean, is coupled with the atmosphere. In such a case, a first solution consists in replacing the ocean by an Equivalent Boundary Condition. However, since the atmosphere can be considered as infinite, it is usually modeled thanks to a high order Absorbing Boundary Condition (ABC). Hence, a very promising perspective of the work is the construction of Equivalent Boundary Condition in the case where a high order ABC is imposed on top of the water. Finally, the extension of the proposed EBC to transient problems is far from trivial and should be the object of a furture work.

Appendix A. First terms of the multiscale expansion.

The aim of this appendix is to detail the calculus of the first terms $\mathfrak{p}_0, \mathbf{u}_0, \mathfrak{p}_1, \mathbf{u}_1, \mathfrak{p}_2, \mathbf{u}_2$ (see Section 5.2) of the multiscale expansion (5.1)-(5.2).

In the case $n = 0$, one obtains from (5.5)

$$\mathfrak{p}_0 = \mathfrak{p}_0(y_\alpha)\ ,$$

where $\mathfrak{p}_0(y_\alpha)$ has to be determined. Then, from (5.4) we deduce that $\mathbf{u}_0$ solves the problem

$$\text{(A.1)} \qquad \left\{ \begin{array}{ll} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_0) = -\mathfrak{p}_0(y_\alpha)\mathbf{n} & \text{on} \quad \Gamma\ . \end{array} \right.$$

At step $n = 1$, according to (5.5) since $\mathfrak{p}_0 = \mathfrak{p}_0(y_\alpha)$ there holds

$$\text{(A.2)} \qquad \left\{ \begin{array}{lll} \partial_3^2 \mathfrak{p}_1 = 0 & \text{for} & Y_3 \in (0,1) \\ \partial_3 \mathfrak{p}_1 = \rho_{\mathsf{f}} \omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{for} & Y_3 = 0 \\ \partial_3 \mathfrak{p}_1 = i\kappa \mathfrak{p}_0 & \text{for} & Y_3 = 1\ . \end{array} \right.$$

Since the right-hand side of the first equation in (A.2) is zero, the function $\partial_3\mathfrak{p}_1(= \partial_3\mathfrak{p}_1(y_\alpha)) = \rho_{\mathsf{f}}\omega^2\mathbf{u}_0 \cdot \mathbf{n}$ is independent of the variable Y_3. Hence, there exists two functions a_1, b_1 defined on Γ such that $\mathfrak{p}_1(y_\alpha, Y_3) = a_1(y_\alpha)Y_3 + b_1(y_\alpha)$. Using the last two equations in the system (A.2), the function a_1 satisfies

$$\left\{ \begin{array}{ll} a_1(y_\alpha) = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{on} \quad \Gamma \\ a_1(y_\alpha) = i\kappa \mathfrak{p}_0(y_\alpha)\ . & \end{array} \right.$$

We infer that $\mathfrak{p}_1$ writes as follow

$$\mathfrak{p}_1(y_\alpha, Y_3) = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma} Y_3 + b_1(y_\alpha) \ ,$$

where $b_1(y_\alpha) = \mathfrak{p}_1(y_\alpha, 0)$ has to be determined. Using the above expression of $\mathfrak{p}_0$ we can explicit the boundary condition in (A.1)

$$\mathbf{T}(\mathbf{u}_0) - i\omega c \rho_{\mathsf{f}} \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = 0 \quad \text{on} \quad \Gamma \ .$$

Hence $\mathbf{u}_0$ solves the problem (5.6) and $\mathfrak{p}_0$ satisfies (5.7). According to (5.4), $\mathbf{u}_1$ solves

$$\text{(A.3)} \qquad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_1) = -\mathfrak{p}_1(y_\alpha, 0)\mathbf{n} & \text{on} \quad \Gamma \ . \end{cases}$$

In the problem (A.3), the right-hand side term $\mathfrak{p}_1(y_\alpha, 0) = b_1(y_\alpha)$ has to be explicited. At step $n = 2$, according to (5.5) we find

$$\text{(A.4)} \qquad \begin{cases} \partial_3^2 \mathfrak{p}_2 = -2\mathcal{H}\partial_3 \mathfrak{p}_1(y_\alpha) - \Delta_\Gamma \mathfrak{p}_0(y_\alpha) - \kappa^2 \mathfrak{p}_0(y_\alpha) & \text{for} \quad Y_3 \in (0,1) \\ \partial_3 \mathfrak{p}_2 = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_1 \cdot \mathbf{n} & \text{for} \quad Y_3 = 0 \\ \partial_3 \mathfrak{p}_2 = i\kappa \mathfrak{p}_1(y_\alpha, 1) & \text{for} \quad Y_3 = 1 \ . \end{cases}$$

Since the right-hand side of the first equation in (A.4) does not depend on Y_3, there exists functions a_2, b_2 and c_2 such that $\mathfrak{p}_2(y_\alpha, Y_3) = a_2(y_\alpha)Y_3^2 + b_2(y_\alpha)Y_3 + c_2(y_\alpha)$, where the couple of functions $(a_2,\, b_2)$ solves the system

$$\begin{cases} 2a_2(y_\alpha) = -2\mathcal{H}\partial_3 \mathfrak{p}_1(y_\alpha) - \Delta_\Gamma \mathfrak{p}_0(y_\alpha) - \kappa^2 \mathfrak{p}_0(y_\alpha) & \text{for} \quad Y_3 \in (0,1) \\ b_2(y_\alpha) = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_1 \cdot \mathbf{n} & \text{for} \quad Y_3 = 0 \\ 2a_2(y_\alpha) + b_2(y_\alpha) = i\kappa \left(\partial_3 \mathfrak{p}_1(y_\alpha) + b_1(y_\alpha)\right) & \text{for} \quad Y_3 = 1 \ . \end{cases}$$

One eliminates a_2 and b_2 in the previous system to explicit the function b_1:

$$b_1(y_\alpha) = (i\kappa)^{-1} \left(-2\mathcal{H}\partial_3 \mathfrak{p}_1(y_\alpha) - \Delta_\Gamma \mathfrak{p}_0(y_\alpha) - \kappa^2 \mathfrak{p}_0(y_\alpha) + \rho_{\mathsf{f}}\omega^2 \mathbf{u}_1 \cdot \mathbf{n}\right) - \partial_3 \mathfrak{p}_1(y_\alpha) \ ,$$

Since $\partial_3 \mathfrak{p}_1(y_\alpha) = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma}$ and $\mathfrak{p}_0(y_\alpha)$ is given by (5.7), one deduces

$$b_1(y_\alpha) = (i\kappa)^{-1}\rho_{\mathsf{f}}\omega^2 \left(-2\mathcal{H} - (i\kappa)^{-1}\Delta_\Gamma\right)(\mathbf{u}_0 \cdot \mathbf{n})_{|\Gamma} + (i\kappa)^{-1}\rho_{\mathsf{f}}\omega^2 \mathbf{u}_1 \cdot \mathbf{n}_{|\Gamma} \ .$$

Finally we can explicit the boundary condition in (A.3):

$$\mathbf{T}(\mathbf{u}_1) - i\omega c \rho_{\mathsf{f}} \mathbf{u}_1 \cdot \mathbf{n}\,\mathbf{n} = (i\kappa)^{-1}\rho_{\mathsf{f}}\omega^2 \left(2\mathcal{H} + (i\kappa)^{-1}\Delta_\Gamma\right)(\mathbf{u}_0 \cdot \mathbf{n})_{|\Gamma}\mathbf{n} \quad \text{on} \quad \Gamma \ .$$

Hence $\mathbf{u}_1$ solves the boundary value problem (5.8) and $\mathfrak{p}_1$ satisfies (5.9).

Similar and tedious calculi lead to the expressions of $\mathbf{u}_2$ (5.10) and $\mathfrak{p}_2$ (5.11). According to (5.4), $\mathbf{u}_2$ solves

$$\text{(A.5)} \qquad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_2) + \omega^2 \rho \mathbf{u}_2 = 0 & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_2) = -\mathfrak{p}_2(y_\alpha, 0)\mathbf{n} & \text{on} \quad \Gamma \ . \end{cases}$$

In the problem (A.5), the right-hand side term $\mathfrak{p}_2(y_\alpha, 0) = c_2(y_\alpha)$ has to be explicited. At step $n = 3$, according to (5.5) one deduces

$$\text{(A.6)} \qquad \begin{cases} \partial_3^2 \mathfrak{p}_3 = -\left(\mathrm{L}^1 \mathfrak{p}_2 + \mathrm{L}^2 \mathfrak{p}_1 + \mathrm{L}^3 \mathfrak{p}_0\right)(y_\alpha, Y_3) & \text{for} \quad Y_3 \in (0,1) \\ \partial_3 \mathfrak{p}_3 = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_2 \cdot \mathbf{n}_{|\Gamma} & \text{for} \quad Y_3 = 0 \\ \partial_3 \mathfrak{p}_3 = i\kappa \mathfrak{p}_2(y_\alpha, 1) & \text{for} \quad Y_3 = 1 \ . \end{cases}$$

Since the right-hand side of the first equation in (A.6) is polynomial of degree 1 with respect to Y_3, there exists functions (a_3, b_3, c_3, d_3) such that $\mathfrak{p}_3(y_\alpha, Y_3) = a_3(y_\alpha)Y_3^3 + b_3(y_\alpha)Y_3^2 + c_3(y_\alpha)Y_3 + d_3(y_\alpha)$, where functions $(a_3,\, b_3,\, c_3)$ solve the system

$$\begin{cases} 6a_3(y_\alpha)Y_3 + 2b_3(y_\alpha) = -\left(\mathrm{L}^1\mathfrak{p}_2 + \mathrm{L}^2\mathfrak{p}_1 + \mathrm{L}^3\mathfrak{p}_0\right)(y_\alpha, Y_3) & \text{for} \quad Y_3 \in (0,1) \\ c_3(y_\alpha) = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_2 \cdot \mathbf{n} & \text{for} \quad Y_3 = 0 \\ (3a_3 + 2b_3 + c_3)(y_\alpha) = i\kappa\,(a_2 + b_2 + c_2)\,(y_\alpha) & \text{for} \quad Y_3 = 1\ . \end{cases}$$

One eliminates functions a_3, b_3 and c_3 in the previous system to explicit the function c_2. Tedious calculi lead to

$$c_2(y_\alpha) = -i\omega c\rho_{\mathsf{f}}\left(\mathbf{u}_2 \cdot \mathbf{n} + P_1(D)(\mathbf{u}_1 \cdot \mathbf{n}) + P_2(D)(\mathbf{u}_0 \cdot \mathbf{n})\right)\ .$$

Hence $\mathbf{u}_2$ solves the boundary value problem (5.10) and $\mathfrak{p}_2$ satisfies (5.11).

Appendix B. Equivalent Conditions - The case of a thin layer with a variable thickness.

In this section, we consider the transmission problem (2.3) set in a smooth bounded domain $\Omega^\varepsilon \subset \mathbb{R}^2$ made of a solid, elastic object occupying a subdomain Ω_{s} entirely immersed in a fluid region occupying the subdomain $\Omega_{\mathsf{f}}^\varepsilon$, which is a thin layer with a variable thickness along the interface Γ

$$\Omega_{\mathsf{f}}^\varepsilon = \{\mathbf{x} = \mathbf{x}(t) + sf(t)\mathbf{n}(t) \in \mathbb{R}^2 \;|\quad \mathbf{x}(t) \in \Gamma, \quad s \in (0,\varepsilon)\}\ .$$

Here t is an arc-length coordinate on Γ and f denotes a smooth function such that $f(t) \neq 0$. In this framework it is also possible to mimic the diffraction problem in the fluid region $\Omega_{\mathsf{f}}^\varepsilon$ with equivalent conditions set on Γ.

B.1. Statement of equivalent conditions. For $k = 0, 1$ one derives an equivalent condition (of order $k+1$) set on Γ and which is satisfied by $\mathbf{u}_\varepsilon^k$ which solves the following boundary value problem

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_\varepsilon^k) + \omega^2\rho\mathbf{u}_\varepsilon^k = \mathbf{f} & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_\varepsilon^k) + \mathsf{B}_{k,\varepsilon}(\mathbf{u}_\varepsilon^k \cdot \mathbf{n})\,\mathbf{n} = 0 & \text{on} \quad \Gamma\ . \end{cases}$$

Here $\mathsf{B}_{k,\varepsilon}$ is a surfacic differential operator acting on functions defined on Γ and $\mathbf{f}$ is a data.

Order 1.

$$\mathbf{T}(\mathbf{u}_0) - i\omega c\rho_f \mathbf{u}_0 \cdot \mathbf{n}\,\mathbf{n} = 0 \quad \text{on} \quad \Gamma \quad (\mathbf{u}_0 = \mathbf{u}_\varepsilon^0) \tag{B.1}$$

Order 2.

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c\rho_f\left(1 + \varepsilon\left(-f(t)\mathrm{c}(t) + i\kappa^{-1}[g(t)\partial_t + |f(t)|\partial_t^2]\right)\right)(\mathbf{u}_\varepsilon^1 \cdot \mathbf{n})\,\mathbf{n} = 0 \tag{B.2}$$

Here g is the function defined on Γ as

$$g(t) = \left(2 - f^2(t)\right)\operatorname{sgn}(f(t))\,f'(t)\ . \tag{B.3}$$

The formal derivation of these ECs is presented in Section B.4.

REMARK B.1. *When $f(t) = 1$, there holds $g(t) = 0$ and we recover the EC of order 2 for a thin layer with a uniform thickness ε in the bidimensional case (compare with (3.3))*

$$\mathbf{T}(\mathbf{u}_\varepsilon^1) - i\omega c\rho_f\left(1 + \varepsilon\left(-\mathrm{c}(t) + i\kappa^{-1}\partial_t^2\right)\right)(\mathbf{u}_\varepsilon^1 \cdot \mathbf{n})\,\mathbf{n} = 0$$

B.2. Geometrical tools - Notations. We introduce a "*system of coordinates*" (t,s) in $\Omega_{\mathsf{f}}^{\varepsilon}$. Here, t is an *arc-length coordinate* on the curve Γ : $\Gamma = \{\mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^2 \mid \quad t \in [0,L)\}$ (L is the length of the curve Γ) and $s \in (0,\varepsilon)$ is a coordinate such that $sf(t)$ represents the distance to the point $\mathbf{x}(t)$ on the curve Γ and f is a smooth L-periodic function defined on the torus $\mathbb{R}/L\mathbb{Z}$ which depends on the arc-length t such that $f(t) \neq 0$. Then, there exists a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the thin layer $\Omega_{\mathsf{f}}^{\varepsilon}$ can be parameterized with coordinates (t,s) :

$$\Omega_{\mathsf{f}}^{\varepsilon} = \{\mathbf{x} = \mathbf{x}(t) + sf(t)\mathbf{n}(t) \in \mathbb{R}^2 \mid \quad \mathbf{x}(t) \in \Gamma, \quad s \in (0,\varepsilon)\} \ .$$

Here, $\mathbf{n}(t) = \mathbf{n}\left(\mathbf{x}(t)\right)$ denotes the normal vector on Γ at the point $\mathbf{x}(t)$. Let $(g_{ij})_{i,j=1,2}$ be the Euclidean metric of the layer $\Omega_{\mathsf{f}}^{\varepsilon}$ defined through the local coordinates (t,s):

$$g_{11} =< \partial_t \Phi, \partial_t \Phi > , g_{22} =< \partial_s \Phi, \partial_s \Phi > , g_{12} = g_{21} =< \partial_t \Phi, \partial_s \Phi >$$

where $\Phi(t,s) = \mathbf{x}(t) + sf(t)\mathbf{n}(t)$ and $< \cdot,\cdot >$ is the Euclidean scalar product in $\mathbb{R}^2$.

One defines the unit tangent vector τ and the scalar curvature $\mathrm{c}(t)$ of Γ in $\mathbf{x}(t)$ as $\partial_t\mathbf{x}(t) = \tau$ and $\partial_t\mathbf{n}(t) = \mathrm{c}(t)\tau$. Then there holds

$$g_{11}(s,t) = \left(1 + sf(t)\mathrm{c}(t)\right)^2 + \left(sf'(t)\right)^2 \ ,$$
$$g_{22}(s,t) = f(t)^2 \quad \text{and} \quad g_{12}(s,t) = sf(t)f'(t) \ .$$

Remark B.2. *We denote by* g *the determinant of the metric* (g_{ij}) *:* g *is a polynomial function of degree* 2 *with respect to* s *:*

$$\mathsf{g}(s,t) = f^2(t)\left(1 + 2f(t)\mathrm{c}(t)s + f(t)^2\mathrm{c}(t)^2 s^2\right) \ .$$

There holds $\sqrt{\mathsf{g}(s,t)} = |f(t)|\left(1 + sf(t)\mathrm{c}(t)\right)$. *We assume that* $\varepsilon \in (0,\varepsilon_0)$ *where* $\varepsilon_0 < \frac{1}{\|f\mathrm{c}\|_\infty}$. *Then* Φ *is a* $\mathcal{C}^1$*-diffeomorphism from* $\mathbb{R}/L\mathbb{Z} \times (0,\varepsilon)$ *onto its image since* $\mathsf{g}(s,t) \neq 0$.

For any function p defined in $\Omega_{\mathsf{f}}^{\varepsilon}$, we denote $\mathfrak{p}$ the function defined in the "scaled domain" $\Omega_{\mathfrak{f}} = \Gamma \times (0,1)$ such that :

$$\mathsf{p}(\mathbf{x}) = \mathfrak{p}(t,S) \ , \quad (t, S = \frac{s}{\varepsilon}) \in [0,L) \times (0,1) \ .$$

B.3. Formal asymptotic expansion. We can exhibit formal series expansions in powers of ε for the elastic displacement $\mathbf{u}_\varepsilon$ and for the acoustic pressure p_ε :

$$\mathbf{u}_\varepsilon(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \varepsilon\mathbf{u}_1(\mathbf{x}) + \varepsilon^2\mathbf{u}_2(\mathbf{x}) + \cdots ,$$
$$\mathsf{p}_\varepsilon(\mathbf{x}) = \mathsf{p}_0(\mathbf{x};\varepsilon) + \varepsilon\mathsf{p}_1(\mathbf{x};\varepsilon) + \varepsilon^2\mathsf{p}_2(\mathbf{x};\varepsilon) + \cdots , \quad \mathsf{p}_j(\mathbf{x};\varepsilon) = \mathfrak{p}_j(t,\frac{s}{\varepsilon}) \, .$$

The term $\mathfrak{p}_j$ is a "profile" defined on $\Gamma \times (0,1)$. The formal calculus concerning the problem are presented in Sec. B.4 and the first asymptotics are given in Sec. B.4.1.

B.3.1. Expansion of the Helmholtz operator. We write the Laplace operator set in the layer through the metric $(g_{ij})_{i,j=1,2}$:

$$\Delta_{t,s} = \frac{1}{\sqrt{\mathsf{g}}}\{\partial_s\left(\frac{1}{\sqrt{\mathsf{g}}}(g_{11}\partial_s - g_{12}\partial_t)\right) + \partial_t\left(\frac{1}{\sqrt{\mathsf{g}}}(g_{22}\partial_t - g_{12}\partial_s)\right)\}$$

We make the scaling $S = \varepsilon^{-1}s \in (0,1)$ into the local coordinates. It maps $[0,L) \times (0,\varepsilon)$ onto $[0,L) \times (0,1)$. The small parameter ε does not appear anymore in the geometry but in the equations (4.9) written through the expression of the Helmholtz operator into power of ε in the thin layer :

$$\Delta+\kappa^2\mathrm{Id}=\varepsilon^{-2}\left[f(t)^{-2}\partial_S^2+\varepsilon f(t)^{-1}\mathrm{c}(t)\partial_S+\sum_{n=2}^{N-1}\varepsilon^n A_n+\varepsilon^N\mathrm{R}_\varepsilon^N\right]$$

for all $N\in\mathbb{N}^*$. We denote by A_1 the operator $A_1=f(t)^{-1}\mathrm{c}(t)\partial_S$. Here, the operator $A_n=A_n(t,S,\partial_t,\partial_S)$ have smooth coefficients in t, and polynomial in S. The operator R_ε^N has smooth coefficients in t, and S, and bounded in ε, [9]. There holds

$$A_2=\kappa^2\mathbb{I}+S^2g_1\partial_S^2+Sg_2\partial_S+Sg_3\partial_t\partial_S+g_4\partial_t+\partial_t^2\ .$$

Here the functions g_i are defined as

$$g_1=(f')^2f^{-2}\ ,\quad g_2=2(f')^2f^{-2}-f^{-1}f''-\mathrm{sgn}(f)\mathrm{c}^2\ ,$$
$$g_3=-2f'f^{-1}\quad\text{and}\quad g_4=(f^{-1}-f)f'$$

Expansion of the normal derivative. We write the normal derivative set on $\Gamma\cup\Gamma^\varepsilon$ through the metric $(g_{ij})_{i,j=1,2}$:

$$\partial_{\mathbf{n}}\circ\Phi|_{s\in\{0,1\}}=\frac{1}{\sqrt{\mathsf{g}}}\left(\sqrt{g_{11}}\partial_s-\frac{g_{12}}{\sqrt{g_{11}}}\partial_t\right)|_{s\in\{0,1\}}$$

After the scaling $S=\varepsilon^{-1}s$, the normal derivative expands in power series of ε as

$$\partial_{\mathbf{n}}=\varepsilon^{-1}|f(t)|^{-1}\partial_S\quad\text{on}\quad\Gamma$$

$$\partial_{\mathbf{n}}=\varepsilon^{-1}|f(t)|^{-1}\partial_S+\sum_{n=1}^{N-1}\varepsilon^nB_n+\varepsilon^N\mathrm{Q}_\varepsilon^N\quad\text{on}\quad\Gamma^\varepsilon$$

for all $N\in\mathbb{N}^*$. Here $B_1=B_1(t,\partial_t,\partial_S)=\left(\frac{|f(t)|^{-1}}{2}f'(t)^2\partial_S-\mathrm{sgn}(f)f'(t)\partial_t\right)$, the operator $B_n=B_n(t,\partial_t,\partial_S)$ have smooth coefficients in t and the operator Q_ε^N has smooth coefficients in t and bounded in ε.

B.4. Formal derivations, first terms and construction of equivalent conditions. One makes the scaling $s\mapsto S=\varepsilon^{-1}s$ in the transmission problem (2.3), it writes :

$$\text{(B.4)}\qquad\begin{cases}\varepsilon^{-2}\left(f(t)^{-2}\partial_S^2+\displaystyle\sum_{l\geqslant1}\varepsilon^lA_l\right)\mathfrak{p}_\varepsilon=0 & \text{in}\quad[0,L)\times(0,1)\\ \varepsilon^{-1}|f(t)|^{-1}\partial_S\mathfrak{p}_\varepsilon=\rho_{\mathsf{f}}\omega^2\mathbf{u}_\varepsilon\cdot\mathbf{n} & \text{on}\quad[0,L)\times\{0\}\\ \left(\varepsilon^{-1}|f(t)|^{-1}\partial_S+\displaystyle\sum_{l\geqslant1}\varepsilon^lB_l\right)\mathfrak{p}_\varepsilon-i\kappa\mathfrak{p}_\varepsilon=0 & \text{on}\quad[0,L)\times\{1\}\\ \nabla\cdot\underline{\underline{\sigma}}(\mathbf{u}_\varepsilon)+\omega^2\rho\mathbf{u}_\varepsilon=\mathbf{f}\delta_0^n & \text{in}\quad\Omega_{\mathsf{s}}\\ \mathbf{T}(\mathbf{u}_\varepsilon)=-\mathfrak{p}_\varepsilon\mathbf{n} & \text{on}\quad\Gamma\ .\end{cases}$$

Inserting the Ansatz (5.1)-(5.2) in equations (B.4), we get the following two families of problems coupled by their boundary conditions on Γ (i.e. when $S = 0$):

$$\text{(B.5)}\qquad \begin{cases} \partial_S^2 \mathfrak{p}_n = -f^2(t) \displaystyle\sum_{l+p=n, l\geqslant 1} A_l \mathfrak{p}_p & \text{for} \quad S \in (0,1) \\ \partial_S \mathfrak{p}_n = \rho_{\mathsf{f}} \omega^2 |f(t)| \mathbf{u}_{n-1} \cdot \mathbf{n} & \text{for} \quad S = 0 \\ \partial_S \mathfrak{p}_n = i\kappa |f(t)| \mathfrak{p}_{n-1} + |f(t)| \displaystyle\sum_{l+p=n-1, l\geqslant 1} B_l \mathfrak{p}_p & \text{for} \quad S = 1 \end{cases}$$

$$\text{(B.6)}\qquad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_n) + \omega^2 \rho \mathbf{u}_n = \mathbf{f} \delta_0^n & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_n) = -\mathfrak{p}_n \mathbf{n} & \text{on} \quad \Gamma \ . \end{cases}$$

We explicit the first terms $(\mathbf{u}_0, \mathfrak{p}_0)$ and $(\mathbf{u}_1, \mathfrak{p}_1)$ in Section B.4.1 and we give details for the calculus in Section B.4.2.

B.4.1. First terms. In the case $n = 0$, $\mathbf{u}_0$ solves the problem

$$\text{(B.7)}\qquad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_0) - i\omega c \rho_{\mathsf{f}} \mathbf{u}_0 \cdot \mathbf{n}\, \mathbf{n} = 0 & \text{on} \quad \Gamma \ , \end{cases}$$

and then we obtain

$$\text{(B.8)}\qquad \mathfrak{p}_0 (= \mathfrak{p}_0(t)) = -i\omega c \rho_f \mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma} \ .$$

At the step $n = 1$, we find that $\mathbf{u}_1$ solves the boundary value problem
(B.9)

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 \\ \mathbf{T}(\mathbf{u}_1) - i\omega c \rho_{\mathsf{f}} \mathbf{u}_1 \cdot \mathbf{n}\, \mathbf{n} = i\omega c \rho_f \left(-f(t) \mathrm{c}(t) + i\kappa^{-1} [g(t) \partial_t + |f(t)| \partial_t^2] \right) (\mathbf{u}_0 \cdot \mathbf{n}) \mathbf{n} \end{cases}$$

and

$$\text{(B.10)}\qquad \mathfrak{p}_1(t,S) = \rho_{\mathsf{f}} \omega^2 \mathbf{u}_0 \cdot \mathbf{n}_{|\Gamma} S + b_1(t) \ ,$$

where

$$b_1(t) = -i\omega c \rho_f \mathbf{u}_1 \cdot \mathbf{n} + i\omega c \rho_f f(t) \mathrm{c}(t) \mathbf{u}_0 \cdot \mathbf{n} + c^2 \rho_f [g(t) \partial_t + |f(t)| \partial_t^2] \mathbf{u}_0 \cdot \mathbf{n} \ .$$

B.4.2. Formal calculus. In the case $n = 0$, we obtain from (B.5)

$$\mathfrak{p}_0 = \mathfrak{p}_0(t) \ ,$$

where $\mathfrak{p}_0(t)$ has to be determined. Then (B.6) yields $\mathbf{u}_0$ solves the problem

$$\text{(B.11)}\qquad \begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_0) + \omega^2 \rho \mathbf{u}_0 = \mathbf{f} & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_0) = -\mathfrak{p}_0(t) \mathbf{n} & \text{on} \quad \Gamma \ . \end{cases}$$

At step $n = 1$, according to (B.5) since $\mathfrak{p}_0 = \mathfrak{p}_0(t)$ there holds

$$\text{(B.12)}\qquad \begin{cases} \partial_S^2 \mathfrak{p}_1(t,S) = 0 \quad \text{for} \quad S \in (0,1) \\ \partial_S \mathfrak{p}_1(t,0) = |f(t)| \rho_{\mathsf{f}} \omega^2 \mathbf{u}_0 \cdot \mathbf{n} \\ \partial_S \mathfrak{p}_1(t,1) = |f(t)| i\kappa \mathfrak{p}_0(t) \ . \end{cases}$$

Since the right-hand side of the first equation in (B.12) is zero, the function $\partial_S \mathfrak{p}_1(= \partial_S \mathfrak{p}_1(t)) = \rho_{\mathsf{f}}\omega^2 \mathbf{u}_0 \cdot \mathbf{n} = |f(t)| i\kappa \mathfrak{p}_0(t)$ is independent of the variable S. Hence, there exists two functions a_1, b_1 defined on Γ such that $\mathfrak{p}_1(t,S) = a_1(t)S + b_1(t)$. Using the last two equations in the system (B.12), a_1 satisfies

$$\begin{cases} a_1(t) = |f(t)|\rho_{\mathsf{f}}\omega^2 \mathbf{u}_0 \cdot \mathbf{n} & \text{on} \quad \Gamma \\ a_1(t) = |f(t)| i\kappa \mathfrak{p}_0(t) \ . \end{cases}$$

We infer that $\mathfrak{p}_0$ satisfies (B.8) and $\mathfrak{p}_1$ writes as follow

$$\mathfrak{p}_1(t,S) = |f(t)|\rho_{\mathsf{f}}\omega^2 \mathbf{u}_0 \cdot \mathbf{n} S + b_1(t) \ ,$$

where $b_1(t) = \mathfrak{p}_1(t,0)$ has to be determined. Using the above expression of $\mathfrak{p}_0$ we can explicit the boundary condition in (B.11)

$$\mathbf{T}(\mathbf{u}_0) - i\omega c \rho_{\mathsf{f}} \mathbf{u}_0 \cdot \mathbf{n}\, \mathbf{n} = 0 \quad \text{on} \quad \Gamma \ . \tag{B.13}$$

Hence $\mathbf{u}_0$ solves the problem (B.7). According to (B.6), $\mathbf{u}_1$ solves

$$\begin{cases} \nabla \cdot \underline{\underline{\sigma}}(\mathbf{u}_1) + \omega^2 \rho \mathbf{u}_1 = 0 & \text{in} \quad \Omega_{\mathsf{s}} \\ \mathbf{T}(\mathbf{u}_1) = -\mathfrak{p}_1(t,0)\mathbf{n} & \text{on} \quad \Gamma \ . \end{cases} \tag{B.14}$$

In the problem (B.14), we have to explicit $\mathfrak{p}_1(t,0) = b_1(t)$. At step $n = 2$, according to (B.5) we find

$$\begin{cases} \partial_S^2 \mathfrak{p}_2 = f^2(t) \left(f^{-1}(t) \mathrm{c}(t) \partial_S \mathfrak{p}_1(t) + A_0 \mathfrak{p}_0(t) + \kappa^2 \mathfrak{p}_0(t) \right) & \text{for} \quad S \in (0,1) \\ \partial_S \mathfrak{p}_2 = |f(t)| \rho_{\mathsf{f}} \omega^2 \mathbf{u}_1 \cdot \mathbf{n} & \text{for} \quad S = 0 \\ \partial_S \mathfrak{p}_2 = |f(t)| \left(i\kappa \mathfrak{p}_1(t,1) - B_1 \mathfrak{p}_0(t) \right) & \text{for} \quad S = 1 \ . \end{cases} \tag{B.15}$$

Since the right-hand side of the first equation in (B.15) does not dependent on S, there exists functions a_2, b_2 and c_2 such that $\mathfrak{p}_2(t,S) = a_2(t)S^2 + b_2(t)S + c_2(t)$, where the couple of functions (a_2, b_2) solves the system

$$\begin{cases} 2a_2(t) = -f(t)\mathrm{c}(t)a_1(t) - (f - f^3)f'(t)\partial_t \mathfrak{p}_0(t) - f^2(t)\Delta_\Gamma \mathfrak{p}_0(t) - \kappa^2 \mathfrak{p}_0(t) \\ b_2(t) = |f(t)|\rho_{\mathsf{f}}\omega^2 \mathbf{u}_1 \cdot \mathbf{n} \\ 2a_2(t) + b_2(t) = |f(t)| \left(\mathrm{sgn}(f) f'(t) \partial_t \mathfrak{p}_0(t) + i\kappa \mathfrak{p}_1(t,1) \right) \end{cases} \tag{B.16}$$

We eliminate a_2 and b_2 in the system (B.16) to explicit the function b_1:

$$b_1(t) = -i\omega c \rho_f \mathbf{u}_1 \cdot \mathbf{n} + i\omega c \rho_f f(t)\mathrm{c}(t)\mathbf{u}_0 \cdot \mathbf{n} + c^2 \rho_f [g(t)\partial_t + |f(t)|\partial_t^2] \mathbf{u}_0 \cdot \mathbf{n} \ .$$

Remind that g is the function defined in (B.3). One deduces $\mathfrak{p}_1$ satisfies (B.10) and we can explicit the boundary condition in equation (B.14)

$$\mathbf{T}(\mathbf{u}_1) - i\omega c \rho_{\mathsf{f}} \mathbf{u}_1 \cdot \mathbf{n}\, \mathbf{n} = -i\omega c \rho_f f(t)\mathrm{c}(t)\mathbf{u}_0 \cdot \mathbf{n}\mathbf{n} - c^2 \rho_f [g(t)\partial_t + |f(t)|\partial_t^2] \mathbf{u}_0 \cdot \mathbf{n}\mathbf{n} \ . \tag{B.17}$$

Hence $\mathbf{u}_1$ solves the problem (B.9).

B.4.3. Construction of equivalent conditions. Following the method used in Section 5.3 one derives equivalent conditions (B.1) and (B.2) from boundary conditions in (B.7) and (B.9).

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