Stability Analysis for Time-Varying Systems with Delay using Linear Lyapunov Functionals and a Positive Systems Approach

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Abstract—**We prove stability of time-varying systems with delays, using linear Lyapunov functionals and positive systems, and we provide robustness of the stability with respect to multiplicative uncertainty in the vector fields. We allow cases where the delay may be unknown, and where the vector fields defining the systems are not necessarily bounded. We illustrate our work using a chain of integrators and other examples.**

I. Introduction

This paper continues our search for new ways to prove stability properties for time-varying nonlinear systems with delays. Our search is motivated by physical phenomena governing many applications that lead to nonlinear systems with delay [2], [3], [8]. However, these systems are beyond the scope of classical methods, e.g., frequency domain and linear matrix inequality techniques. Therefore, despite their importance, relatively few contributions are devoted to time-varying systems with delays (but see [9] for a polytopic approach to time-varying delays which does not cover the results we give here).

Moreover, the existing stability results are largely limited to systems where the delays only occur in the control. Prediction is very useful for systems with arbitrarily long input delays [2], [3], [14]. However, prediction may not always apply when the delays occur in the vector fields defining the system, which is the situation we are concerned with here (but see [1] for predictive Lyapunov function based methods under delays in the input and in the vector fields of the system). Also, it may not always be easy to find Lyapunov functions that can be transformed into the Lyapunov-Krasovskii functionals that are often used to prove stability properties for delay systems.

The work [16] uses a different approach to prove stability of delayed neutral systems, based on nonnegative and cooperative systems. Here, we use analogous tools for time-varying systems. We use three key ingredients. The first involves operators with integral terms that produce systems with distributed delays, interconnected with integral equations. Then, we prove that all solutions of the interconnections are components of solutions of nonnegative systems. Finally, we use linear Lyapunov-Krasovskii functionals for the higher dimensional systems, which ensure global asymptotic stability for the original systems. Our Lyapunov functional design owes a great deal to the functionals in [11] for time invariant systems; see also [6], [10].

We use decompositions of functions involving their cooperative parts, as was done in [5], [6] for interval observers or time invariant systems. This technique shares similarities with the internal positive representation from [4], [7]. Two advantages of our new technique are that (a) its assumptions are relatively simple and (b) it establishes exponential stability for systems that do not seem to be covered by other techniques. We establish our first theorem in Section III, under a bound on the delay. However, the result requires that certain vector fields satisfy suitable bounds, which may not always hold in practice.

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Therefore, we also present an approach in Section V for systems with unbounded vector fields and arbitrary unknown constant delays, under a condition related to the signs of the components of the vector fields of the system. Since we establish global asymptotic stability using stabilizing time-varying terms with delay, our results contrast with [22], [23], which use comparison systems and non-smooth time invariant vector Lyapunov functions to prove ultimate boundedness and local results (using the stabilizing effect of time invariant terms). See also [20], which proves local stability under delays, and global stability under homogeneity conditions that we do not require here.

Compared with treatments of more elementary systems (such as undelayed linear time invariant systems, where the stability analysis often entails solving a linear matrix inequality), our assumptions and analysis are more complicated. However, we believe that this is the price to pay to obtain the very general results we obtain here for nonlinear delay systems with state dependent coefficient matrices that may contain uncertainties. Our examples in Section VI illustrate the value added by our results and the checkability of our assumptions, and include a key chain of integrators where our allowable delay bound is larger than bounds that were available in the literature.

II. Definitions and Notation

In what follows, the dimensions are arbitrary. For any matrix $M \in \mathbb{R}^{p\times q}$, let m_{ij} denote its entry in row i and column j for all i and j. The $k\times n$ matrix in which each entry is 0 is denoted by 0. The usual Euclidean norm of vectors, and the induced norm of matrices, are denoted by $|\cdot|$. All inequalities must be understood to hold for each entry of the corresponding matrices, i.e., given any matrices A and B of the same size, we write $A \le B$ to mean that $a_{ij} \le b_{ij}$ for all i and j. A square matrix is said to be *cooperative* or *Metzler* provided all of its off-diagonal entries are nonnegative. A matrix $M \in \mathbb{R}^{r\times s}$ is said to be *nonnegative* (resp., *positive*) provided every entry m_{ij} of M satisfies $m_{ij} \ge 0$ (resp., > 0). For simplicity, we always take the initial times for the trajectories of our systems to be $t_0 = 0$. We let M^+ denote the matrix whose position (i,j) entry is $\max\{0, m_{ij}\}$ for all i and j, and $M^- = M^+ - M$. We let $M^s = M^+ + M^-$, so M^s is obtained by taking the absolute values of all entries of M.

Let C^1 denote the set of all continuously differentiable functions, whose domains and ranges will be clear from the context. Given any constant $\tau > 0$, we let $C([-\tau,0],\mathbb{R}^n)$ denote the set of all continuous $\mathbb{R}^n$-valued functions defined on $[-\tau,0]$. We often abbreviate this set as $C_{\rm in}$, and we call it the set of all *initial functions*. A system is said to be *positive* for a class of initial functions $\mathcal{S}_0$ provided for each positive valued initial function in $\mathcal{S}_0$, the unique solution stays positive for all $t \ge 0$. For any continuous function $\varphi : [-\tau,\infty) \to \mathbb{R}^n$ and all $t \ge 0$, we define φ_t by $\varphi_t(m) = \varphi(t+m)$ for all $m \in [-\tau, 0]$, i.e., $\varphi_t \in C_{\rm in}$ is the translation operator.

III. Statement of First Result and Discussion

We first provide a stability analysis for time-varying systems

$$\dot{x}(t) = A_1(t,x_t)x(t) + A_2(t,x_t)x(t-\tau)\ , \qquad (1)$$

where x is valued in $\mathbb{R}^n$, $\tau > 0$ is a constant delay, the initial functions are in $C_{\rm in}$, and A_1 and A_2 are locally Lipschitz (but see Remark 4 for generalizations with uncertainty). This includes the key case of linear time-varying systems of the form $\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau)$ and nonlinear systems with linearizations of that form.

Set $B_1^a(t,\phi,\psi) = A_1(t,\phi) + A_2(t+\tau,\psi)$, $B_2^a(t,\psi) = -A_2(t+\tau,\psi)$, and $B_3^a(t,\phi,m,\psi) = B_1^a(t,\phi,\psi)B_2^a(m,\psi)$ for all $t \geq 0$, $m \geq 0$, and ϕ and ψ in $C_{\rm in}$. Fix matrix valued functions $\bar{B}_1^a$ and $\underline{B}_1^a$ such that $B_1^a = \bar{B}_1^a - \underline{B}_1^a$, $\bar{B}_1^a$ is everywhere Metzler, and $\underline{B}_1^a$ is everywhere nonnegative. For instance, take $\bar{B}_1^a = B_1^a + \mathcal{G}$ and $\underline{B}_1^a = \mathcal{G}$ for a positive matrix $\mathcal{G}$ with large enough entries. Assume:

Assumption 1: All of the entries of the matrix A_1 in (1) are bounded functions. Also, there are constants $\mathfrak{c}_j > 0$ for $j = 1,2,3,4,6$ and a C^1 function $p : [0,\infty) \to \mathbb{R}^n$ satisfying

$$\begin{array}{l} \dot{p}(t)^\top + p(t)^\top B^b(t) \leq -\mathfrak{c}_1 p(t)^\top, \\ \mathfrak{c}_2(1\ ...\ 1)^\top \leq p(t) \leq \mathfrak{c}_3(1\ ...\ 1)^\top, \\ p^\top(t)\sup_{\psi\in C_{\rm in}}(B_2^a(t,\psi))^s \leq \mathfrak{c}_4 p^\top(t), \text{ and} \\ p^\top(t)\sup_{\phi\in C_{\rm in},\psi\in C_{\rm in},m\geq 0}(B_3^a(t,\phi,m,\psi))^s \leq \mathfrak{c}_6 p^\top(t) \end{array} \tag{2}$$

for all $t \geq 0$, where B^b is a function such that $\bar{B}_1^a(t,\phi,\psi) + \underline{B}_1^a(t,\phi,\psi) \leq B^b(t)$ for all $\phi \in C_{\rm in}$, $\psi \in C_{\rm in}$, and $t \geq 0$. □

Setting $\mathfrak{c}_5 = \mathfrak{c}_3\mathfrak{c}_4/\mathfrak{c}_2$, we also assume the following:

Assumption 2: The delay $\tau \geq 0$ satisfies

$$\left(\frac{\mathfrak{c}_6}{\mathfrak{c}_1} + \mathfrak{c}_5\right)\tau < 1 \tag{3}$$

where the $\mathfrak{c}_i$'s are as above. □

We can then prove the following:

Theorem 1: If (1) satisfies Assumptions 1-2, then (1) is uniformly globally exponentially stable to 0 for all initial functions in $C_{\rm in}$. □

Remark 1: If $\bar{B}_1^a + \underline{B}_1^a$ is bounded from above by a constant Hurwitz matrix, then there is a constant vector $p > 0$ such that the first inequality in (2) holds for all $t \geq 0$ [11]. When B_1^a is bounded from above by a suitable matrix $B^m(t)$, we can sometimes change coordinates to transform $\dot{X} = B_1^m(t)X$ into an autonomous positive system [17], which may facilitate checking Assumption 1. Without extra assumptions, it is unclear how to drop the requirements in the last two inequalities in (2), which are analogous to the requirements in [16, Assumption A2, Eq. (4)] for the case of neutral systems. □

Remark 2: Assumption 2 bounds τ, but does not require that τ be known. One cannot expect stability without assuming that τ is small enough, even if A_1 is independent of x_t, since Assumptions 1-2 do not imply that $\dot{X} = A_1(t)X$ is asymptotically stable. A key difference between [16] and Theorem 1 is that the potential stabilizing effect of the delayed term $A_2(t,x_t)x(t-\tau)$ is not taken into account in [16], but it is in Theorem 1; see Section VI. □

Remark 3: Conditions (2) hold if

$$\begin{array}{l} \dot{p}(t)^\top + p(t)^\top\left(\bar{B}_1^a(t,\phi,\psi) + \underline{B}_1^a(t,\phi,\psi)\right) \leq -\mathfrak{c}_1 p(t)^\top, \\ \mathfrak{c}_2(1\ ...\ 1)^\top \leq p(t) \leq \mathfrak{c}_3(1\ ...\ 1)^\top, \\ p^\top(t)(B_2^a(t,\psi))^s \leq \mathfrak{c}_4 p^\top(t), \text{ and} \\ p^\top(t)(B_3^a(t,\phi,m,\psi))^s \leq \mathfrak{c}_6 p^\top(t) \end{array} \tag{4}$$

hold for all $t \geq 0$ and all ϕ and ψ in $C_{\rm in}$, since then we can satisfy (2) by taking $B^b(t) = \sup_{\phi\in C_{\rm in},\psi\in C_{\rm in}}\left(\bar{B}_1^a(t,\phi,\psi) + \underline{B}_1^a(t,\phi,\psi)\right)$. Any conservativeness of Assumptions 1-2 mostly stems from the fact that B_1^a can be Hurwitz while $\bar{B}_1^a + \underline{B}_1^a$ is not Hurwitz [17]. To broaden the class of systems for which Assumptions 1-2 apply, one can use changes of coordinates (like those in [18]) that transform a Hurwitz matrix into a Hurwitz and Metzler matrix. □

IV. Proof of Theorem 1

Step 1: Obtaining a Comparison System

Since Assumption 2 ensures that A_1 and A_2 are bounded, all trajectories of (1) are defined over $[-\tau,\infty)$ [12]. Therefore, by specifying one trajectory $x(t)$, we can define $B_1(t) = B_1^a(t,x_t,x_{t+\tau})$, $\bar{B}_1(t) = \bar{B}_1^a(t,x_t,x_{t+\tau})$, $\underline{B}_1(t) = \underline{B}_1^a(t,x_t,x_{t+\tau})$, $B_2(t) = B_2^a(t,x_{t+\tau})$, and $B_3(t,m) = B_3^a(t,x_t,m,x_{m+\tau})$. For each $t \geq 0$, set

$$\xi(t) = x(t) + \int_{t-\tau}^t A_2(m+\tau,x_{m+\tau})x(m)\mathrm{d}m\ . \tag{5}$$

Since $B_1(t)B_2(m) = B_3(t,m)$ for all t and m, we then have

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & B_1(t)\xi(t) + \int_{t-\tau}^t B_3(t,m)x(m)\mathrm{d}m \\ x(t) & = & \xi(t) + \int_{t-\tau}^t B_2(m)x(m)\mathrm{d}m\ . \end{array}\right. \tag{6}$$

Hence, all solutions of (1) converge to 0 if all solutions of (6) with initial conditions in $(\phi_\xi,\phi_x) \in C_{\rm in}$ satisfying the matching condition

$$\phi_x(0) = \phi_\xi(0) + \int_{-\tau}^0 B_2(m)\phi_x(m)\mathrm{d}m \tag{7}$$

are defined over $[-\tau,\infty)$ and converge exponentially to 0. Note that the trajectory $x(t)$ enters explicitly in (6), as well as through our formulas for the functions B_i that appear in (6), and the functions B_i are defined along a fixed choice of $x(t)$. Nevertheless, since the constants $\mathfrak{c}_i$ in Assumptions 1-2 are independent of the trajectory, and since the constants in our final exponential stability estimate will be independent of $x(t)$, we will still be able to establish our exponential stability estimate for all solutions $(\xi(t),x(t))$ of (6) that satisfy the matching condition (7), which in particular will give the exponential stability result for the trajectory $x(t)$ we used to define the B_i's.

One can prove that all the solutions of (6) for all initial conditions $(\phi_\xi,\phi_x) \in C_{\rm in}$ satisfying the matching condition (7) are continuous and uniquely defined over $[-\tau,\infty)$ by noticing that they satisfy

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & B_1(t)\xi(t) + \int_{t-\tau}^t B_3(t,m)x(m)\mathrm{d}m \\ \dot{x}(t) & = & B_1(t)\xi(t) + \int_{t-\tau}^t B_3(t,m)x(m)\mathrm{d}m \\ & & +\,B_2(t)x(t) - B_2(t-\tau)x(t-\tau) \end{array}\right. \tag{8}$$

for all $t \geq 0$. It is an interconnection of a system with a distributed delay. To analyze the stability of (8), we first write $B_1(t) = \bar{B}_1(t) - \underline{B}_1(t)$, $B_2(m) = B_2^+(m) - B_2^-(m)$, and $B_3(t,m) = B_3^+(t,m) - B_3^-(t,m)$, and observe that (6) is equivalent to

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & \bar{B}_1(t)\xi(t) - \underline{B}_1(t)\xi(t) \\ & & +\int_{t-\tau}^t (B_3^+(t,m) - B_3^-(t,m))x(m)\mathrm{d}m \\ x(t) & = & \xi(t) + \int_{t-\tau}^t (B_2^+(m) - B_2^-(m))x(m)\mathrm{d}m. \end{array}\right. \tag{9}$$

We analyze the trajectories of the equivalent system (9).

Step 2: Analyzing the Comparison System (9)

To analyze the stability of (9), we combine (i) our novel linear Lyapunov-Krasovskii functional approach and (ii) an approach from [19] that doubles the dimension of the system. Consider the system

$$\left\{\begin{array}{rcl} \dot{\xi}(t) & = & \bar{B}_1(t)\xi(t) + \underline{B}_1(t)\Psi(t) \\ & & +\int_{t-\tau}^t B_3^+(t,m)x(m)\mathrm{d}m \\ & & +\int_{t-\tau}^t B_3^-(t,m)Z(m)\mathrm{d}m \\ x(t) & = & \xi(t) + \int_{t-\tau}^t B_2^+(m)x(m)\mathrm{d}m \\ & & +\int_{t-\tau}^t B_2^-(m)Z(m)\mathrm{d}m \\ \dot{\Psi}(t) & = & \bar{B}_1(t)\Psi(t) + \underline{B}_1(t)\xi(t) \\ & & +\int_{t-\tau}^t B_3^+(t,m)Z(m)\mathrm{d}m \\ & & +\int_{t-\tau}^t B_3^-(t,m)x(m)\mathrm{d}m \\ Z(t) & = & \Psi(t) + \int_{t-\tau}^t B_2^+(m)Z(m)\mathrm{d}m \\ & & +\int_{t-\tau}^t B_2^-(m)x(m)\mathrm{d}m. \end{array}\right. \tag{10}$$

One can easily check that $(\xi,x,-\xi,-x)$ is a solution of (10), if (ξ,x) is a solution of (9). Hence, if all solutions of (10) satisfying the matching condition

$$\begin{array}{l} x(0) = \xi(0) + \int_{-\tau}^0 B_2^+(m)x(m)\mathrm{d}m + \int_{-\tau}^0 B_2^-(m)Z(m)\mathrm{d}m \\ Z(0) = \Psi(0) + \int_{-\tau}^0 B_2^+(m)Z(m)\mathrm{d}m + \int_{-\tau}^0 B_2^-(m)x(m)\mathrm{d}m \end{array} \tag{11}$$

are continuous on $[-\tau,\infty)$ and converge exponentially to 0, then all solutions of (6) that satisfy (7) converge exponentially to 0. Note that we are not asserting that (9) and (10) are equivalent, since positivity of (10) will not imply positivity of (9). However, the way we embed solutions of (9) as components of solutions of (10) will ensure that our stability result for (10) implies the desired stability for (9).

Arguing as we did when we studied the existence of the solutions of (6), one can prove that all solutions of (10) satisfying (11) are continuous and uniquely defined over $[-\tau,\infty)$. Also, we prove in the appendix that (10) is positive for the class $\mathcal{S}_0$ of all initial functions satisfying (11). Moreover, it is linear. Hence, it is globally exponentially stable if it is globally exponentially stable on only the positive orthant. To see why, let $\mathcal{X}$ be any solution of (10) with any nonzero initial condition $\phi_{\mathcal{X}}=(\phi_\xi,\phi_x,\phi_\Psi,\phi_Z)$ satisfying (11). Then we can find a positive valued solution $\mathcal{X}_a$ of (10) and a negative valued solution $\mathcal{X}_b$ of (10) (both satisfying the matching condition) such that the corresponding initial functions $\phi_{\mathcal{X}_a}$ and $\phi_{\mathcal{X}_b}$ satisfy

$$\phi_{\mathcal{X}_b}(t)<\phi_{\mathcal{X}}(t)<\phi_{\mathcal{X}_a}(t)\ \text{ for all } t\in[-\tau,0]. \tag{12}$$

To see why such $\mathcal{X}_a$ and $\mathcal{X}_b$ exist, it suffices to find a negative valued $\phi_{\mathcal{X}_b}:[-\tau,0]\to\mathbb{R}^{4n}$ and a positive valued $\phi_{\mathcal{X}_a}:[-\tau,0]\to\mathbb{R}^{4n}$ satisfying (12), both satisfying the matching condition, since then the positivity of $\mathcal{X}_a$ and the negativity of $\mathcal{X}_b$ follow from our proof of the positivity of (10) in the appendix. To find $\phi_{\mathcal{X}_b}$ and $\phi_{\mathcal{X}_a}$, let $L_{i,j}$ be the corresponding entries of B_2^s, so $B_2^s(m)=[L_{i,j}(m)]$ for all m. Since B_2^s is bounded, there is a constant $H>0$ such that

$$\max_{1\le i\le n}\int_{-\tau}^{0}\sum_{j=1}^{n}L_{i,j}(m)e^{Hm}\mathrm{d}m\ \le\ \tfrac{1}{2}\ . \tag{13}$$

Let δ_i be the integral in (13), so $\delta_i\in[0,1/2]$ for all $i\in\{1,2,\dots,n\}$. Set $E_*(t)=(e^{Ht},e^{Ht},\dots,e^{Ht})^\top\in\mathbb{R}^n$ and $\Delta_*=(1-\delta_1,1-\delta_2,\dots,1-\delta_n)^\top\in\mathbb{R}^n$. Then the positive valued function $\phi_d(t)=(\phi_{d,\xi},\phi_{d,x},\phi_{d,\Psi},\phi_{d,Z})^\top(t)=\big(\Delta_*,E_*(t),\Delta_*,E_*(t)\big)^\top$ satisfies (11). Set $\bar\phi=2\max_{t\in[-\tau,0]}|\phi_{\mathcal{X}}(t)|$ and $\phi_0=\min\{1-\max_i\delta_i,e^{-H\tau}\}$. Then $\phi_{\mathcal{X}_a}(t)=(\bar\phi/\phi_0)\phi_d(t)$ is positive valued and $\phi_{\mathcal{X}_b}(t)=-(\bar\phi/\phi_0)\phi_d(t)$ is negative valued, and (12) holds.

Next, assume that (10) satisfies the exponential stability property on the positive orthant. Since positivity and linearity of (10) give positivity of $\mathcal{X}_a(t)-\mathcal{X}(t)$ and $\mathcal{X}(t)-\mathcal{X}_b(t)$ for all $t\ge0$, we get $\lim_{t\to\infty}(\mathcal{X}_a(t)-\mathcal{X}(t))=\lim_{t\to\infty}(\mathcal{X}(t)-\mathcal{X}_b(t))=0$, so $\lim_{t\to\infty}(\mathcal{X}_a(t)-\mathcal{X}_b(t))=0$, where the limits are exponential convergence. (The fact that the coefficient matrices in (10) depend on the state values x_t does not matter, since the positivity proof does not use any information about the specific dependencies of the coefficient matrices on x_t.) Since $\mathcal{X}=(\mathcal{X}-\mathcal{X}_b)+(\mathcal{X}_b-\mathcal{X}_a)+\mathcal{X}_a$ is a sum of three terms that exponentially converge to 0, and $|\phi_{\mathcal{X}_a}|_{[-\tau,0]}=|\phi_{\mathcal{X}_b}|_{[-\tau,0]}=2|\phi_{\mathcal{X}}|_{[-\tau,0]}|\phi_d|_{[-\tau,0]}/\phi_0$ where $|\cdot|_{[-\tau,0]}$ is the sup norm, it follows that $\mathcal{X}(t)$ converges to 0 exponentially. Hence, we next study positive solutions of (10) satisfying (11).

Step 3: Exponential Stability of Positive Solutions (ξ,x,Ψ,Z) *of (10)*

Set $c=x+Z$ and $\gamma=\xi+\Psi$. Then (10) and the decompositions $B_i^s=B_i^++B_i^-$ for $i=2,3$ and $B_1^*=\bar B_1+\underline{B}_1$ give

$$\left\{\begin{array}{rcl}\dot\gamma(t)&=&B_1^*(t)\gamma(t)+\int_{t-\tau}^{t}B_3^s(t,m)c(m)\mathrm{d}m\\ c(t)&=&\gamma(t)+\int_{t-\tau}^{t}B_2^s(m)c(m)\mathrm{d}m.\end{array}\right. \tag{14}$$

We use the linear function $V_1:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ defined by $V_1(t,\gamma)=p(t)^\top\gamma$, where p is from Assumption 1 for the choice of $x(t)$ that we used to define the B_i's at the start of the proof. Along all positive trajectories of (14), we get the following for all $t\ge0$:

$$\begin{array}{rcl}\dot V_1(t)&=&\left[\dot p(t)^\top+p(t)^\top B_1^*(t)\right]\gamma(t)\\ &&+\int_{t-\tau}^{t}p(t)^\top B_3^s(t,m)c(m)\mathrm{d}m.\end{array} \tag{15}$$

Assumption 1 gives $p^\top(t)B_3^s(t,m)c(m)\le\mathfrak{c}_6p^\top(t)c(m)$ and $p^\top(t)B_2^s(m)c(m)\le\mathfrak{c}_4p^\top(t)c(m)\le\mathfrak{c}_5p^\top(m)c(m)$, since c is positive valued. Setting $v(t)=V_1(t,\gamma(t))$ gives

$$\begin{array}{l}\dot v(t)\ \le\ -\mathfrak{c}_1v(t)+\mathfrak{c}_6\int_{t-\tau}^{t}p(t)^\top c(m)\mathrm{d}m\quad\text{and}\\ p(t)^\top c(t)\ \le\ v(t)+\mathfrak{c}_5\int_{t-\tau}^{t}p^\top(m)c(m)\mathrm{d}m\end{array} \tag{16}$$

hold for all $t\ge0$, where we combined (14) and (15) and used the positivity of the solution (ξ,x,Ψ,Z). We next prove:

Claim 1: There are constants $g\in(\tau,1/\mathfrak{c}_5)$ and $h>0$ such that

$$hg-\mathfrak{c}_1<0\ \text{ and }\ \mathfrak{c}_6+h(g\mathfrak{c}_5-1)<0 \tag{17}$$

hold, where $\mathfrak{c}_1$, $\mathfrak{c}_5$, and $\mathfrak{c}_6$ are from Assumptions 1-2. □

Proof of Claim 1. From (3), we get $\tau<1/\mathfrak{c}_5$, so we can choose a constant $g\in(\tau,1/\mathfrak{c}_5)$. We rewrite our objectives (17) as

$$\frac{\mathfrak{c}_6}{1-g\mathfrak{c}_5}<h<\frac{\mathfrak{c}_1}{g}. \tag{18}$$

There exists $h>0$ such that (18) holds if and only if there is $g\in(\tau,1/\mathfrak{c}_5)$ such that $(1-g\mathfrak{c}_5)/\mathfrak{c}_6>g/\mathfrak{c}_1$, i.e., $1>(\mathfrak{c}_6/\mathfrak{c}_1+\mathfrak{c}_5)g$. By Assumption 2, $1>(\mathfrak{c}_6/\mathfrak{c}_1+\mathfrak{c}_5)\tau$. Therefore there exists $g\in(\tau,1/\mathfrak{c}_5)$ such that $1>(\mathfrak{c}_6/\mathfrak{c}_1+\mathfrak{c}_5)\,g$, which proves the claim. □

We use g and h from Claim 1 to define V_2 and V_3 by

$$V_2(t,c_t)=\int_{t-\tau}^{t}(g-t+\ell)p^\top(\ell)c(\ell)\mathrm{d}\ell\ \text{ and} \tag{19}$$

$$V_3(t,\gamma(t),c_t)=V_1(t,\gamma(t))+hV_2(t,c_t). \tag{20}$$

By the second inequality in (16), the time derivative of $V_2(t,c_t)$ along all componentwise positive solutions of (14) satisfies

$$\begin{array}{rcl}\dot V_2&=&-\int_{t-\tau}^{t}p^\top(\ell)c(\ell)\mathrm{d}\ell+gp^\top(t)c(t)\\ &&-(g-\tau)p^\top(t-\tau)c(t-\tau)\\ &\le&(g\mathfrak{c}_5-1)\int_{t-\tau}^{t}p^\top(\ell)c(\ell)\mathrm{d}\ell+gv(t)\end{array} \tag{21}$$

for all $t\ge0$, since the fact that $g\ge\tau$ lets us drop the term $-(g-\tau)p(t-\tau)^\top c(t-\tau)$. Along all positive valued solutions of (10),

$$\begin{array}{r}g\int_{t-\tau}^{t}p^\top(\ell)c(\ell)\mathrm{d}\ell\ \ge\ V_2(t,c_t)\ \ge\ (g-\tau)\int_{t-\tau}^{t}p^\top(\ell)c(\ell)\mathrm{d}\ell\\ \ge\ (g-\tau)\mathfrak{c}_2\int_{t-\tau}^{t}\sum_{i=1}^{n}c_i(\ell)\mathrm{d}\ell\end{array} \tag{22}$$

for all $t\ge0$, by our lower bound on $p(t)$ from Assumption 1.

Along all trajectories of (14), we can combine (16) and (21) to get

$$\begin{array}{rcl}\dot V_3&\le&-\mathfrak{c}_1v(t)+\mathfrak{c}_6\int_{t-\tau}^{t}p^\top(m)c(m)\mathrm{d}m\\ &&+h(g\mathfrak{c}_5-1)\int_{t-\tau}^{t}p^\top(\ell)c(\ell)\mathrm{d}\ell+hgv(t)\\ &=&(hg-\mathfrak{c}_1)\,v(t)\\ &&+\,[\mathfrak{c}_6+h(g\mathfrak{c}_5-1)]\int_{t-\tau}^{t}p^\top(m)c(m)\mathrm{d}m\end{array} \tag{23}$$

for all $t\ge0$. Then we can use the upper bound on V_2 from (22) and (17) to show that the constant $\mathfrak{c}_7=\min\{\mathfrak{c}_1-hg,(h(1-g\mathfrak{c}_5)-\mathfrak{c}_6)/(gh)\}$ is such that $\dot V_3(t)\le-\mathfrak{c}_7V_3(t,\gamma(t),c_t)$ for all $t\ge0$. For all $t_1\ge t_2\ge0$, we can integrate the preceding inequality to get

$$\begin{array}{l}V_1(t_1,\gamma(t_1))+hV_2(t_1,c_{t_1})\\ \le\exp(-\mathfrak{c}_7(t_1-t_2))\left[V_1(t_2,\gamma(t_2))+hV_2(t_2,c_{t_2})\right]\ ,\end{array} \tag{24}$$

by our formula for V_3 in (20). Since (22) holds for all $t\ge0$ along all positive trajectories of (10), our choice of $\mathfrak{c}_2$ in Assumption 1 gives

$$\begin{array}{l}\mathfrak{c}_2\sum_{i=1}^{n}\gamma_i(t_1)+h(g-\tau)\mathfrak{c}_2\sum_{i=1}^{n}\int_{t_1-\tau}^{t_1}c_i(\ell)\mathrm{d}\ell\le\\ e^{-\mathfrak{c}_7(t_1-t_2)}\big[V_1(t_2,\gamma(t_2))+hV_2(t_2,c_{t_2})\big].\end{array} \tag{25}$$

We conclude from (25) that

$$\gamma(t)+\int_{t-\tau}^{t}c(m)\mathrm{d}m\to0\ \text{ exponentially.} \tag{26}$$

It follows that $c(t)\to0$ exponentially, since (14) and (26) provide a constant matrix $\tilde c>0$ and constants $\mathfrak{c}_8>0$ and $\mathfrak{c}_9>0$ such that $|c(t)|\ \le\ |\gamma(t)+\tilde c\int_{t-\tau}^{t}c(m)\mathrm{d}m|\ \le\ \mathfrak{c}_9\exp(-\mathfrak{c}_8t)(|\gamma(0)|+$

$\tau|c|_{[-\tau,0]}) \leq \mathfrak{c}_9\exp(-\mathfrak{c}_8 t)(|c(0)| + (1 + |\tilde{c}|)\tau|c|_{[-\tau,0]})$. Since $(\xi(t), x(t), \Psi(t), Z(t)) > 0$, $c(t) = x(t) + Z(t)$, and $\gamma(t) = \xi(t) + \Psi(t)$ hold for all $t \geq 0$, it follows from (26) that all positive solutions of (10)-(11) converge exponentially to zero. Hence, our argument at the end of Step 2 implies that (6) is uniformly globally exponentially stable to 0. This proves Theorem 1.

Remark 4: Often, the exact values of the entries of the matrices A_1 and A_2 in (1) are uncertain, but their signs are known. We represent this case using multiplicative uncertainties on their entries, i.e.,

$$\dot{x}(t) = (\delta A_1)(t, x_t)x(t) + (\delta A_2)(t, x_t)x(t-\tau) \tag{27}$$

where the (i,j) entry of $(\delta A_p)(t, x_t)$ is $\delta_{pij}(t, x_t)a_{pij}(t, x_t)$ for all i and j in $\{1, 2, \ldots, n\}$ and the matrices $A_p = [a_{pij}]$ are known for $p = 1, 2$. To prove our robustness property, we assume that each $\delta_{pij} : [0, \infty) \times C_{\rm in} \to [\underline{\delta}, \bar{\delta}]$ is continuous but unknown, but that their constant lower and upper bounds $\underline{\delta} > 0$ and $\bar{\delta} > 0$ are known. We maintain our Assumptions 1-2 exactly as before, except we replace the first inequality in (2) by the requirement that $\dot{p}(t)^\top + p(t)^\top(\overline{\mathcal{L}} + \underline{\mathcal{L}})(t, x_t, x_{t+\tau}) \leq -\mathfrak{c}_1 p(t)^\top$, where $\mathcal{L}(t, x_t, x_{t+\tau}) = \delta A_1(t, x_t) + \delta A_2(t+\tau, x_{t+\tau})$, and where $\overline{\mathcal{L}}$ and $\underline{\mathcal{L}}$ are any matrices such that $\overline{\mathcal{L}}$ is Metzler, $\underline{\mathcal{L}} \geq 0$, and $\mathcal{L} = \overline{\mathcal{L}} - \underline{\mathcal{L}}$ on the domain of $\mathcal{L}$; we replace the last inequality in (2) by $p^\top(t)((A_1(t,\phi))^s + (A_2(t+\tau,\psi))^s)B_2^a(m,\psi))^s \leq \mathfrak{c}_6 p^\top(t)$; and we replace (3) by the inequality

$$\bar{\delta}\left(\bar{\delta}\tfrac{\mathfrak{c}_6}{\mathfrak{c}_1} + \mathfrak{c}_5\right)\tau < 1. \tag{28}$$

Under these new assumptions, we can show that (27) is uniformly globally exponentially stable to zero. The proof is similar to the proof of Theorem 1, except one replaces A_1 and A_2 by (δA_1) and (δA_2) respectively throughout the earlier proof. The $\mathfrak{c}_6$ is multiplied by $\bar{\delta}^2$ in (28), because both matrices B_1^a and B_2^a in the formula for $B_3^a(t, \phi, m, \psi)$ have their entries multiplied by the uncertainties. □

V. Another Result

A. Statement of Result

We next present an alternative result for systems of the form

$$\dot{x}(t) = M(t, x_t)x(t) + P(t-\tau, x_{t-\tau})x(t-\tau) \tag{29}$$

under the following assumptions:

Assumption 3: For all $t \geq 0$ and all functions ϕ in $C_{\rm in}$, the matrix $M(t,\phi) \in \mathbb{R}^{n\times n}$ is Metzler and $P(t,\phi) \in \mathbb{R}^{n\times n}$ is nonnegative. Also, M and P are locally Lipschitz in ϕ and continuous in t. □

Assumption 4: There exist a constant positive vector $v \in \mathbb{R}^n$ and a locally Lipschitz function $c : C_{\rm in} \to (0, \infty)$ such that $v^\top(M(t,\phi) + P(t,\phi)) \leq -c(\phi)v^\top$ holds for all $t \geq 0$ and $\phi \in C_{\rm in}$. □

We can then prove:

Theorem 2: Fix any constant $\tau > 0$. If (29) satisfies Assumptions 3-4, then (29) is uniformly globally asymptotically stable to 0. □

Remark 5: Theorem 2 differs from Theorem 1 because P is nonnegative valued, so $P(t, x_{t-\tau})x(t-\tau)$ never has a stabilizing effect. We do not require bounds or growth conditions on M or P, or a uniform positive lower bound on c or a bound on τ. Since the off diagonal entries of $M + P$ are nonnegative valued and $v > 0$, Assumption 4 requires that all main diagonal entries of M be negative valued. While Theorem 1 shows global exponential stability, Theorem 2 shows the weaker global asymptotic stability condition. □

B. Proof of Theorem 2

Consider any solution $x(t)$ of (27). Fix a value $T > 0$ such that $x(t)$ is defined on $[-\tau, T)$. We also set $M_\tau(t) = M(t, x_t)$, $P_\tau(t) = P(t, x_t)$, and $c_\tau(t) = c(x_t)$. Then for all $t \in [0, T)$, we have

$$\dot{x}(t) = M_\tau(t)x(t) + P_\tau(t-\tau)x(t-\tau). \tag{30}$$

Let $x_L : [-\tau, 0] \to (0, \infty)$ be defined componentwise by $x_{L,i}(t) = |x_i(t)| + \max_{\ell\in[-\tau,0]}|x(\ell)|$ for $i = 1, 2, \ldots, n$, assuming without loss of generality that x_0 is nonzero. Then $-x_L(t) < x(t) < x_L(t)$ for all $t \in [-\tau, 0]$ and $\max_{\ell\in[-\tau,0]}|x(\ell)| \leq \max_{\ell\in[-\tau,0]}|x_L(\ell)| \leq 2\sqrt{n}\max_{\ell\in[-\tau,0]}|x(\ell)|$. We extend x_L to $[-\tau, T)$, by solving

$$\dot{x}_L(t) = M_\tau(t)x_L(t) + P_\tau(t-\tau)x_L(t-\tau) \tag{31}$$

on $[0, T)$. For all $t \in [0, T)$, the matrices $M_\tau(t)$ and $P_\tau(t)$ are Metzler and nonnegative valued, respectively. It follows that

$$-x_L(t) \leq x(t) \leq x_L(t) \tag{32}$$

and $-x_L(t) \leq 0 \leq x_L(t)$ hold for all $t \in [-\tau, T)$. The bounds (32) follow by showing that $x_L(t) - x(t)$ and $x(t) + x_L(t)$ have positive valued initial functions and are solutions of $\dot{y}(t) = M_\tau(t)y(t) + P_\tau(t-\tau)y(t-\tau)$ and so are nonnegative valued on $[-\tau, T)$; see the appendix below for analogous arguments. (As in the proof of Theorem 1, the fact that the coefficient matrices depend on the state does not matter, since the analysis does not use any information about the specific dependencies of the coefficient matrices on x_t.)

Next note that for all $t \in [0, T)$, we have $v^\top\dot{x}_L(t) = v^\top M_\tau(t)x_L(t) + v^\top P_\tau(t-\tau)x_L(t-\tau)$. Hence, the function

$$a(t) = v^\top x_L(t) + \int_{t-\tau}^{t} v^\top P_\tau(m)x_L(m)\mathrm{d}m$$

satisfies

$$\dot{a}(t) = v^\top(M_\tau(t) + P_\tau(t))x_L(t) \leq -c_\tau(t)v^\top x_L(t) \tag{33}$$

for all $t \in [0, T)$, by Assumption 4. Since a is nonnegative valued and $c_\tau(t)v^\top x_L(t) \geq 0$ for all $t \in [0, T)$, it follows from (33) that $a(t)$ is bounded over $[0, T)$. Hence, our formula for $a(t)$ and the nonnegative valuedness of $v^\top P_\tau(m)x_L(m)$ imply that $v^\top x_L(t)$ and so also $x_L(t)$ are bounded on $[0, T)$. It follows from (32) that $x(t)$ is bounded on $[0, T)$, so the finite escape phenomenon cannot occur, so each trajectory of (29) is defined over $[-\tau, \infty)$. Also, $x_L(t)$ is bounded, since $a(t) \leq a(0)$ for all $t \geq 0$. Hence, (32) ensures that each trajectory $x(t)$ is bounded. Also, (32) hold for all $t \geq -\tau$, and (30) and (31) hold for all $t \geq 0$.

Since x_L is bounded and positive valued and c_τ is positive valued, we can find a constant $c_* > 0$ such that $c_* v^\top P_\tau(t)x_L(t) \leq c_\tau(t)v^\top x_L(t)$ and $\inf_{t\geq 0} c_\tau(t) = \inf_{t\geq 0} c(x_t) > c_*$, for all $t \geq 0$. Therefore, we can use (33) to find a constant $c_{**} > 0$ such that the time derivative of

$$a^\sharp(t) = a(t) + \tfrac{c_*}{2\tau}\textstyle\int_{t-\tau}^{t}\int_{s}^{t} v^\top x_L(\ell)\mathrm{d}\ell\mathrm{d}s$$

satisfies

$$\begin{aligned} \dot{a}^\sharp(t) &\leq -\tfrac{1}{2}c_\tau(t)v^\top x_L(t) - \tfrac{c_*}{2\tau}\textstyle\int_{t-\tau}^{t} v^\top x_L(\ell)\mathrm{d}\ell \\ &\leq -c_{**}a^\sharp(t) \end{aligned} \tag{34}$$

for all $t \geq 0$, because

$$\tfrac{d}{dt}\textstyle\int_{t-\tau}^{t}\int_{s}^{t} v^\top x_L(\ell)\mathrm{d}\ell\mathrm{d}s = \tau v^\top x_L(t) - \int_{t-\tau}^{t} v^\top x_L(\ell)\mathrm{d}\ell$$

holds for all $t \geq 0$. The exponential decay estimate (34) provides a uniform global asymptotic stability estimate for x_L, i.e., a function $\beta \in \mathcal{KL}$ [15] such that $|x(t)| \leq |x_L(t)| \leq \beta(\max_{\ell\in[-\tau,0]}|x_L(\ell)|, t)$ for all $t \geq 0$, which gives the conclusion, because $\max_{\ell\in[-\tau,0]}|x_L(\ell)| \leq 2\sqrt{n}\max_{\ell\in[-\tau,0]}|x(\ell)|$.

VI. Illustrations

A. System with a stabilizing term without delay

We first consider the one-dimensional system

$$\dot{x}(t) = l_1\cos^2(t)x(t) + l_2\sin(t)x(t-\tau)\ , \tag{35}$$

where $\tau \geq 0$, $l_1 \in \mathbb{R}$, and $l_2 \in \mathbb{R}$ are constants. We use Theorem 1 to find conditions on τ, l_1 and l_2 that ensure that (35) is exponentially stable. Using the above notation with the dependencies on ϕ and ψ omitted, we choose $B_1^a(t) = \bar{B}_1^a(t) = l_1 \cos^2(t) + l_2 \sin(t+\tau)$, $\underline{B}_1^a(t) = 0$, $B_2^a(t) = -l_2 \sin(t+\tau)$, and $B_3^a(t,m) = -l_2 \sin(m+\tau)(l_1 \cos^2(t) + l_2 \sin(t+\tau))$. Let us determine conditions ensuring that there are a C^1 positive valued function $p : \mathbb{R} \to \mathbb{R}$ and a constant $\mathfrak{c}_1 > 0$ such that $\dot{p}(t) + p(t)B_1^a(t) = -\mathfrak{c}_1 p(t)$, for all $t \geq 0$, which is equivalent to $\dot{p}(t) = -[\mathfrak{c}_1 + l_1 \cos^2(t) + l_2 \sin(t+\tau)]p(t)$. Assuming that $l_1 < 0$ and picking $\mathfrak{c}_1 = -0.5 l_1$ gives $\dot{p}(t) = [-0.5 l_1 \cos(2t) - l_2 \sin(t+\tau)]\, p(t)$, by the double angle formula for cosine. Thus, we can choose $p(t) = \exp(-(l_1/4)\sin(2t) + l_2 \cos(t+\tau))$. We deduce that Assumptions 1 and 2 hold with

$$\begin{array}{l} \mathfrak{c}_2 = \exp(-|l_1|/4 - |l_2|),\ \mathfrak{c}_3 = 1/\mathfrak{c}_2, \\ \mathfrak{c}_5 = |l_2| \exp(|l_1|/2 + 2|l_2|),\ \mathfrak{c}_6 = (|l_1| + |l_2|)|l_2|, \end{array} \tag{36}$$

and all $\tau > 0$ such that $(\mathfrak{c}_6/\mathfrak{c}_1 + \mathfrak{c}_5)\,\tau < 1$, which is equivalent to

$$|l_2| \left(\tfrac{2}{|l_1|}(|l_1| + |l_2|) + e^{0.5|l_1| + 2|l_2|} \right) \tau \ < \ 1 \, . \tag{37}$$

By Theorem 1, we conclude that the system (35) is uniformly globally exponentially stable to 0, provided $l_1 < 0$ and (37) hold.

B. System with a stabilizing term with delay

We next illustrate Theorem 1 using the chain of integrators

$$\dot{\xi}_1(t) = v_1(t-\tau),\ \ \dot{\xi}_2(t) = v_2(t-\tau),\ \ \dot{\xi}_3(t) = v_1(t)\xi_2(t) \tag{38}$$

for any constant delay τ that satisfies

$$\tau \in \left(0, \tfrac{1}{3+2\sqrt{e}}\right] . \tag{39}$$

In [15, Section 6.2], we solved a tracking problem for (38) for the reference trajectory $(-\cos(t), 0, 0)^\top$ when $\tau = 0$, by building a strict Lyapunov function, but there is no clear analog of this earlier construction under our condition (39). Here we solve the problem of globally asymptotically tracking the trajectory $(\sin(t), 0, 0)^\top$.

Fix a constant a_0 such that

$$a_0 \in \left(0, \tfrac{1}{4\pi}\right) \tag{40}$$

and set $\gamma_1(t) = \xi_1(t) - \sin(t)$ and $v_1(t) = \cos(t+\tau) - a_0 \arctan(\gamma_1(t))$. This gives

$$\left\{ \begin{array}{l} \dot{\gamma}_1(t) = -a_0 \arctan(\gamma_1(t-\tau)),\ \ \dot{\xi}_2(t) = v_2(t-\tau), \\ \dot{\xi}_3(t) = v_1(t)\xi_2(t). \end{array} \right. \tag{41}$$

Since the origin of $\dot{\gamma}_1(t) = -a_0 \arctan(\gamma_1(t-\tau))$ is globally asymptotically stable to zero, the tracking dynamics (41) will be globally asymptotically stable to zero if the origin of

$$\dot{\xi}_2(t) = v_2(t-\tau),\ \ \dot{\xi}_3(t) = v_1(t)\xi_2(t) \tag{42}$$

is globally exponentially stable (GES) to 0 with a GES estimate that is independent of γ_1.

To show this GES property for (42), fix any γ_1 satisfying $\dot{\gamma}_1(t) = -a_0 \arctan(\gamma_1(t-\tau))$ for all $t \geq 0$, and apply backstepping. For each $t \geq 0$, we set $\mathcal{G}(t) = \cos(t+\tau) - a_0 \arctan(\gamma_1(t-\tau))$, so

$$\begin{array}{l} v_1(t+\tau)\mathcal{G}(t+\tau) = -a_0 \cos(t+2\tau) \arctan(\gamma_1(t)) \\ -\, a_0 \arctan(\gamma_1(t+\tau)) \cos(t+2\tau) \\ +a_0^2 \arctan(\gamma_1(t+\tau)) \arctan(\gamma_1(t)) + \cos^2(t+2\tau). \end{array} \tag{43}$$

Using $\gamma_2(t) = \xi_2(t) + \mathcal{G}(t)\xi_3(t-\tau)$, (42) becomes

$$\begin{array}{rcl} \dot{\gamma}_2(t) &=& v_2(t-\tau) + \Big[\mathcal{G}(t) v_1(t-\tau)\xi_2(t-\tau) \\ && -\left(\sin(t+\tau) - \frac{a_0^2 \arctan(\gamma_1(t-2\tau))}{1+\gamma_1^2(t-\tau)}\right)\xi_3(t-\tau)\Big] \\ \dot{\xi}_3(t) &=& v_1(t)(\gamma_2(t) - \mathcal{G}(t)\xi_3(t-\tau)). \end{array} \tag{44}$$

Choose $v_2(t-\tau) = -\gamma_2(t-\tau) - \mathcal{R}(t, \gamma_1(t-\tau), \gamma_1(t-2\tau), \xi_2(t-\tau), \xi_3(t-\tau))$, where $\mathcal{R}$ is the quantity in brackets in (44). This gives

$$\left\{ \begin{array}{rcl} \dot{x}_1(t) &=& v_1(t)[x_2(t) - \mathcal{G}(t)x_1(t-\tau)] \\ \dot{x}_2(t) &=& -x_2(t-\tau)\ , \end{array} \right. \tag{45}$$

where $x_1 = \xi_3$ and $x_2 = \gamma_2$. Then Theorem 1 can be used to study (45). Using the notation from the proof of Theorem 1, we choose

$$\begin{array}{l} B_1(t) = \left[\begin{array}{cc} -v_1(t+\tau)\mathcal{G}(t+\tau) & v_1(t) \\ 0 & -1 \end{array} \right] \text{ and} \\ B_1^*(t) = \left[\begin{array}{cc} -v_1(t+\tau)\mathcal{G}(t+\tau) & |v_1(t)| \\ 0 & -1 \end{array} \right], \end{array} \tag{46}$$

by defining $B_1^* = \bar{B}_1 + \underline{B}_1$ as before, and taking the upper right entries of $\bar{B}_1(t)$ and $\underline{B}_1(t)$ to be $\max\{0, v_1(t)\}$ and $\max\{0, v_1(t)\} - v_1(t)$ for all $t \geq 0$, respectively. Choose $p(t) = (\exp(0.25\sin(2t+4\tau)), 2(1+\pi a_0/2)e^{1/4})^\top$. Then since $a_0 \in (0, 1/(4\pi))$, condition (43) gives $-v_1(t+\tau)\mathcal{G}(t+\tau) \leq -\cos^2(t+2\tau) + 17\pi a_0/16$ for all $t \geq 0$. Therefore, the double angle formula for cosine gives

$$\begin{array}{l} \dot{p}(t)^\top + p(t)^\top B_1^*(t) \\ = \left[\begin{array}{l} \left(\frac{1}{2}\cos(2t+4\tau) - v_1(t+\tau)\mathcal{G}(t+\tau)\right) e^{\frac{1}{4}\sin(2t+4\tau)} \\ |v_1(t)| e^{\frac{1}{4}\sin(2t+4\tau)} - 2(1+\pi a_0/2)e^{1/4} \end{array} \right]^\top \\ \leq -\left[\left(\frac{1}{2} - \frac{17}{16}\pi a_0\right) e^{\frac{1}{4}\sin(2t+4\tau)} \quad (1+\pi a_0/2)e^{\frac{1}{4}} \right] \\ \leq -\left(\frac{1}{2} - \frac{17}{16}\pi a_0\right) p(t)^\top . \end{array}$$

Recalling our delay bound (39) and the second inequality in (40), it is now easy to check that (42) satisfies our Assumptions 1-2 with $\mathfrak{c}_1 = (1/2) - 17\pi a_0/16$, $\mathfrak{c}_2 = e^{-1/4}$, $\mathfrak{c}_3 = 2(1+\pi a_0/2)e^{1/4}$, $\mathfrak{c}_5 = 2e^{1/2}(1+\pi a_0/2)^3$, and $\mathfrak{c}_6 = (3/2)(1+\pi a_0/2)$ when (39) holds and $a_0 > 0$ is small enough. Hence, Theorem 1 applies. Combined with the GAS property of the γ_1 subsystem of (41), we conclude that the tracking dynamics (41) is UGAS to zero, as claimed. See Fig. 1 for a simulation. We are not aware of any other technique that makes it possible to prove GES of (45) to 0 under our delay bound (39).

C. System that is nonlinear in the state

We next illustrate Theorem 2 using the one-dimensional system

$$\dot{x}(t) = -x(t)e^{-x(t)} + 0.5x(t-\tau)e^{-x(t-\tau)} \, . \tag{47}$$

Omitting the time dependence in the coefficients, we can apply Theorem 2 with $M(x_t) = -e^{-x(t)}$ and $P(x_{t-\tau}) = 0.5e^{-x(t-\tau)}$, since then $M(\phi) + P(\phi) = -0.5e^{-\phi}$ holds for all $\phi \in C_{\text{in}}$ and Assumption 4 holds with $c(\phi) = \frac{1}{2}e^{-\phi}$ and $v = 1$. This example does not seem to be covered by earlier results, such as the Razumikhin Theorem or results based on Lyapunov-Krasovskii functionals.

VII. Conclusions

Stabilizing time-varying nonlinear systems with delays is challenging and beyond the scope of standard frequency domain and linear matrix inequality methods. The state-of-the-art results were largely limited to systems with input delays or required Lyapunov-Krasovskii functionals that may not always be easy to find [21]. Here, we used a very different approach, by expressing the original system as a coupling of (a) an integral equation and (b) a differential equation with a distributed delay. Another novel feature is our viewing the system solutions as solutions of a higher order system, and using positive systems to reduce the stabilization problem to a study of positive solutions of the higher order system. This improved on the work [16] for neutral systems, which did not take the potentially stabilizing effect of the delayed term into account. The positivity helped us prove exponential stability, using linear Lyapunov-Krasovskii functionals. We illustrated our work in a chain of integrators, where we provided

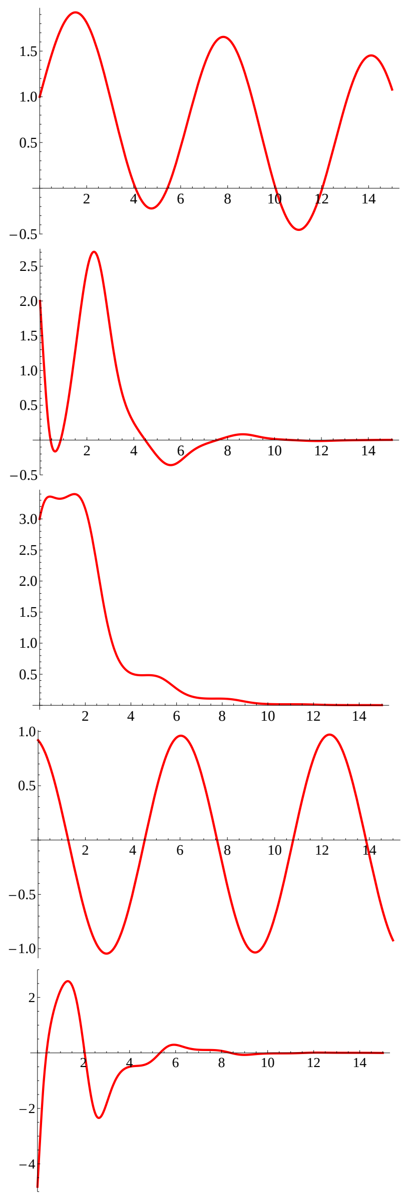

Fig. 1. Simulations of Chain of Integrators (38) Tracking $(\sin(t), 0, 0)^\top$ with our Closed Loop Control, Initial Function $\xi_0 = (1, 2, 3)^\top$, Delay $\tau = 1/6$ and $a_0 = 1/(4\pi)$. Top to Bottom: $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$, $v_1(t)$, and $v_2(t)$.

a larger upper bound on the allowable delays than was available in existing results. We plan to generalize our work to hyperbolic PDEs and difference equations [13].

Appendix: Positiveness of the System (10)

Let $(\phi_\xi, \phi_x, \phi_\Psi, \phi_Z) \in C_{\rm in}$ be any positive valued initial condition satisfying (11). We prove that the solution of (10) with $(\phi_\xi, \phi_x, \phi_\Psi, \phi_Z) \in C_{\rm in}$ as the initial function is positive for all $t \in [-\tau, \infty)$. Throughout the sequel, let $\bar{B}_{1ij}$ denote the (i,j) entry of $\bar{B}_1$ for all i and j. We prove the positivity of (10) by contradiction.

Case 1: Suppose that there were $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $\xi_i(t_c) = 0$. Since $\bar{B}_1(t)$ is Metzler and $\underline{B}_1(t)$, $B_3^+(t,m)$, and $B_3^-(t,m)$ are nonnegative valued, it follows from (10) that $\dot{\xi}_i(t) \geq \bar{B}_{1ii}(t)\xi_i(t)$ for all $t \in [0, t_c]$. By integrating this inequality, we get $\xi_i(t_c) \geq \exp(\int_0^{t_c} \bar{B}_{1ii}(m){\rm d}m)\xi_i(0) > 0$. This yields a contradiction.

Case 2: Suppose that there is $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $x_i(t_c) = 0$. Since $B_2^+(m)$ and $B_2^-(m)$ are nonnegative valued, it follows from the structure of the x subdynamics of (10) that $x_i(t_c) \geq \xi_i(t_c) \geq 0$. Hence, $\xi_i(t_c) = 0$. From Case 1, we again have a contradiction.

Case 3: Suppose that there is $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $\Psi_i(t_c) = 0$. Arguing as in Case 1 with ξ_i replaced by Ψ_i, we can conclude.

Case 4: Suppose that there is $t_c > 0$ and $i \in \{1, ..., n\}$ such that $(\xi(t), x(t), \Psi(t), Z(t)) > 0$ for all $t \in [-\tau, t_c)$ and $Z_i(t_c) = 0$. Arguing as in Case 2, we can conclude from Case 3.

This concludes the proof of the positiveness of the system (10).

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