On the real-time calculation of the forward kinematics of suspended cable-driven parallel robots

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Abstract—*This paper addresses the problem of the forward kinematics of cable-driven parallel robots (CDPR) having elastic or non elastic cables under the assumption that the cables are submitted to small changes in their lengths compared to an initial situation at which the pose is known, a situation that is typical of a real-time forward kinematics used for control purposes. We show that at the initial point it is necessary to have not only the pose but also the* cable configuration*, i.e. having the knowledge of which cables are under tension and that the result of the forward kinematics should be both the pose and the cable configuration after the change in cable lengths. We exhibit a first algorithm that allows one to determine all possible final poses and cable configurations. Then a second algorithm is proposed to determine all possible cable configurations also during the coiling process which is an important point as the CDPR state, including the cable tensions, may drastically change at this point. The last algorithm assumes a model of the coiling process and is able to determine the unique final pose and cable configuration. These algorithms provide safer real-time forward kinematics which will improve the CDPR control*

Keywords: clear, cable-driven parallel robots,forward kinematics,cable configurations, real-time

I. Introduction

Cable-driven parallel robot (CDPR) are robots whose platform are connected to the ground by a set of cables that can be uncoiled or coiled. The study of CDPR has started about 30 years ago with the pioneering work of Albus [1] and Landsberger [2] but there has been recently a renewed interest in such a robot, both from a theoretical and application viewpoint. The kinematics analysis of CDPR is much more complex than the one of parallel robot with rigid legs as static equilibrium has to be taken into account [3], [4], [5], [6] and is still an open issue especially as not all cables of a robot with m cables may be under tension [7], [8], [9], [10], [11] and that only stable solutions have to be determined [12]. Numerous applications of CDPRs have been mentioned e.g. large scale maintenance studied in the European project Cablebot [13], rescue robots [14], [15], large telescope [16], rehabilitation [17], [18] and transfer robot for elderly people [19] to name a few.

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The proprioceptive measurement on such a robot is usually the cable lengths as other physical quantities such as orientation of the cables or their tensions are difficult to measure. The kinematic analysis of such robot is drastically influenced by the cable model that is used. For example a cable may be supposed mass-less and non elastic, mass-less but elastic or deformable and elastic. In this paper we consider only mass-less cables and a specific class of CDPR: *suspended CDPR*. In this class there is no cable that can exert a downward force that is larger than its own weight (figure 1).

Before going on we will introduce some notation. The output point of the coiling mechanism for cable i will be denoted A_i while this cable is attached at point B_i on the platform. We define an absolute frame $(O, \mathbf{x}, \mathbf{y}, \mathbf{z})$ and we assume that the coordinates of A_i in this frame are known. In the same manner we define a mobile frame $(C, \mathbf{x_r}, \mathbf{y_r}, \mathbf{z_r})$ that is attached to the platform (figure 1). Without lack of generality C will be assumed to be the center of mass of the platform with coordinates (x_c, y_c, z_c). We assume that the coordinates of B_i in the mobile frame are known. Our

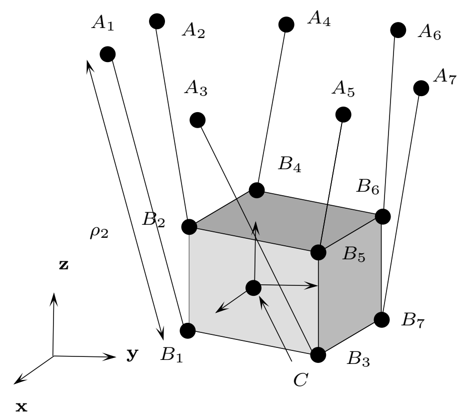

Fig. 1. A suspended CDPR

cable model assume that the cable profile is directed along the line going through the points A, B as soon as the cable is under tension. For non elastic cables the length of a cable will be denoted ρ while for elastic cable ρ will denote its real length while l^0 will denote its length at rest.

A major problem with CDPR is that they have usually several solutions to the forward kinematic (FK) problem (i.e. being given the cable lengths determine the pose of the platform). This is usually the case for parallel robots but the situation is even worse for CDPR and is still yet not fully understood. A major point is that each solution of the FK leads to a specific set of cable tensions together to a specific pose. As determining the pose and the tensions are key factors for safety and control it is essential to be able to determine the current state of the robot. This problem may stated as follows: being given a state of the robot $(\boldsymbol{\rho}, \mathbf{X})$ for non elastic cables or $(\mathbf{l^0}, \mathbf{X})$ for elastic cables, where $\mathbf{X}$ denotes the pose of the platform, determine the pose $\mathbf{X_1}$ of the robot when the components of $\boldsymbol{\rho}$ $(\mathbf{l^0})$ are changed to $\boldsymbol{\rho}+\boldsymbol{\Delta\rho}$ $(\mathbf{l^0} + \boldsymbol{\Delta}\mathbf{l^0})$, where $\boldsymbol{\Delta\rho}(\boldsymbol{l^0})$ is a vector of "small" amount of changes in the cable lengths that occurs after a "small" amount of time Δt. This is typically the problem we have to solve for the real-time calculation of the FK: after the calculation of the current pose based on the cable lengths measurement a control law will calculate a command for the actuators that will not be changed during the sampling time of the controller (we do not consider here other control schemes that do not use the measurements of the cable lengths, for example the one using the visual observation of the direction of the cables [20]).

For robot with rigid legs we have exhibited an algorithm that first check if $\Delta\rho$ is small enough so that there is an unique $\mathbf{X_1}$ that can be reached starting from $\mathbf{X}$ and if this is the case is able to compute $\mathbf{X_1}$ with an arbitrary accuracy [21].

Unfortunately we will see that such scheme will not work for CDPR. But first we may summarize the known result about the FK of CDPR as follow:

- for mass-less and not elastic cables (case 1) for a CDPR with m cables the FK always amounts to solve a *set* of square systems of equations. Indeed we cannot assume that in the current pose all m cables are under tension and to find all FK solutions we have to consider all possible combinations of cables under tension. Note that if $m > 6$ there cannot be more than 6 cables under tension simultaneously. Still for a given number of cables the minimal FK equations will be square with 6 equations for 6 cables and $m+6$ for $m < 6$ cables (6 parameters for the pose and m cable tensions). But even with this complexity all the solutions found by solving the different sets of equations may not all be valid. Indeed for a given solution $\mathbf{X_s}$ we have to examine first the tensions of the cables that must all be positive. Then we have to consider the lengths of the cables that are supposed to be slack: for each of them this length must be greater than the distance between the A, B points for the pose $\mathbf{X_s}$. Hence this differs from the FK of parallel robots with rigid legs where a single square system has to be solved and all solutions are valid.
- for mass-less but linearly elastic cables (case 2) all cables may be under tension even if $m > 6$ but it may also happen that we have slack cables. The tension τ in the cable is written as $\tau = k(\rho - l_0)$ where k is the linear stiffness, ρ is the real length of the cable and l_0 its length at rest that are supposed to be known. Hence the minimal FK has $m + 6$ unknowns (6 parameters for the pose and $m\rho$) and $m + 6$ equations (6 from the statics equilibrium and m equations that state that the distance between A, B should be ρ). Still one or more of the cables may be slack (i.e. $\rho < l_0$) and as in case 1 we have thus to consider all combinations of cables under tension
- for deformable and elastic cables (case 3) the FK amounts to solve a single square system of equations that has usually several solutions [22].

In case 3 the algorithm presented in [21] may be applied to solve the real-time FK but cannot for case 1 and 2. Indeed at $\mathbf{X}$ we know which cables are under tension and which one are slack (if any). Unfortunately at $\mathbf{X_1}$ the set of cables under tension may be different from the one at $\mathbf{X}$ and therefore the system of equations of which $\mathbf{X_1}$ is a solution is different from the one at $\mathbf{X}$.

This paper addresses the following topics for CDPR in case 1 and 2:

- determine if a change (or more than one) in the set of cables under tension may occur when changing the cable lengths by $\boldsymbol{\Delta\rho}(\boldsymbol{l^0})$
- if yes determine the new set(s) of cable under tension
- determine the pose $\mathbf{X_1}$

II. Solving the real-time FK

A. The statics equations

A suspended CDPR is usually submitted only to gravity (and possibly to small disturbances that we will neglect). The wrench applied on the platform will be denoted $\mathcal{F}$ and the equations relating this wrench to the tension $\boldsymbol{\tau}$ in the cables are:

$$\mathcal{F} = \mathbf{J^{-T}}\boldsymbol{\tau} \tag{1}$$

where $\mathbf{J^{-T}}$ is the $6 \times m$ jacobian matrix (where m is the number of cable under tension) whose i-th column is

$$((\frac{\mathbf{A_iB_i}}{\rho_i} \quad \frac{\mathbf{CB_i} \wedge \mathbf{A_iB_i}}{\rho_i}))$$

B. The cable configuration concept

A cable will be denoted *dominant* if it exerts an force on the platform and *non dominant* if it is slack. A *cables configuration* (CC) for a CDPR with m cables is a set of m booleans whose i-th member is set to 1 if the i-th cable is dominant and set to 0 otherwise. This concept is quite important because at a given pose the CDPR may have different cable configurations. For example we have considered a specific robot with 8 cables that has to move along an horizontal circle while keeping constant the orientation of its platform. Figure 2 shows the circular arcs for which the various CC with 6 cables are valid (the arcs have been offsetted to make them visible). It may be noticed that this trajectory cannot be fully completed with the same CC and

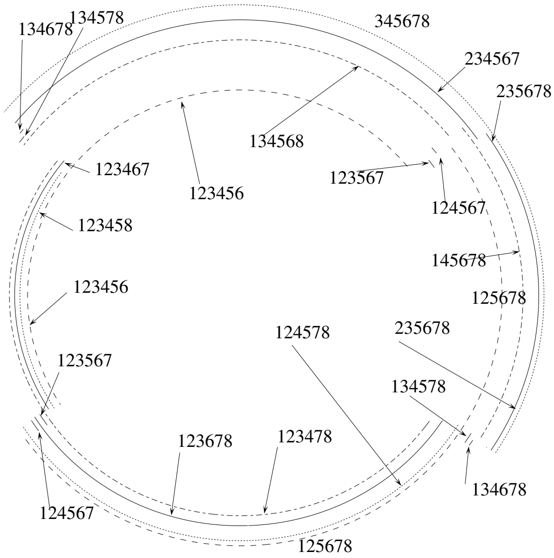

Fig. 2. The possible CC on a circular trajectory. The arc have the same radius but have been enlarged in order to show the different CC

that there may be up to 4 different CC's for a pose. The cable configuration allow to determine what are the cable tensions (which differ from one CC to another one) but also the performances of the CDPR at this pose. For example it is well known that an error $\boldsymbol{\Delta\rho}$ on the cable lengths $\boldsymbol{\rho}$ leads to an error in the platform positioning $\Delta\mathbf{X}$ such that

$$\Delta\mathbf{X} = \mathbf{J}\boldsymbol{\Delta\rho}$$

As $\mathbf{J}$ is dependent upon the CC so will be the positioning accuracy.

C. First algorithm

This algorithm will simply try to determine all the valid CC to which may lead the change from $\boldsymbol{\rho}$ to $\boldsymbol{\rho} + \boldsymbol{\Delta\rho}$ i.e. to solve the FK for each CC. For non elastic cables and a CDPR with a total of m cables, n of which are under tension we have the following equations and inequalities under the assumption that the numbering of the cables is such that the cables from 1 to n are under tension:

$$||\mathbf{A_iB_i}||^2 = (\rho_i + \Delta\rho_i)^2 \; i \in [1, n] \quad (2)$$
$$||\mathbf{A_iB_i}||^2 - (\rho_i + \Delta\rho_i)^2 \leq 0 \; i \in [n+1, m] \quad (3)$$
$$\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_n\} \;\; \tau_i > 0 \;\; \forall i \in [1, n]$$
$$\mathcal{F} = \mathbf{J^{-T}}\boldsymbol{\tau} \quad (4)$$

the statics equations being used in the FK solving only if $n < 6$ (if $n = 6$ only the equations (2) are used and the statics equations are used only a posteriori to eliminate solutions leading to at least one negative τ_i). To model the pose of the platform we use a parametrization that depends upon the number n:

- if $4 \geq n \leq 6$: we use as parameters the coordinates of 4 of the B says the coordinates of $\mathbf{OB_1}, \ldots, \mathbf{OB_4}$ that are called the *reference points*. It follows that we have $\forall j > 4 \;\; \mathbf{OB_j} = \sum_{m=1}^{m=4} \alpha_m \mathbf{OB_m}$ where the α_m are known constants. At the pose $\mathbf{X}$ the coordinates of the reference points B_j^s are known.
- if $n = 3$: we choose B_1, B_2, B_3 as references points and we have $\forall m > 3 \;\; \mathbf{B_1B_m} = \beta_1\mathbf{B1B2} + \beta_2\mathbf{B1B3} + \beta_3(\mathbf{B_1B2} \times \mathbf{B_1B_3})$ where the β are known constants.
- if $n = 2$: C, B_1, B_2 all lie in the same vertical plane that includes A_1, A_2 and the location of C may be deduced from the location of B_1, B_2 with 2 possible solutions (one below the line B_1B_2, one over it). Hence we choose the coordinates of B_1, B_2 as unknowns
- if $n = 1$: A_1, B_1, C lie on the same vertical line with C having two possible locations (over or below B_1). The z coordinates of B_1 is chosen as unknown

The motivation for using this parametrization will be explained later on. We will not consider the case $n \leq 2$ that can be trivially solved. Now let us look at the equation

$$\mathcal{F} = \mathbf{J^{-T}}\boldsymbol{\tau} \quad \boldsymbol{\tau} = \{\tau_1, \ldots, \tau_n\}$$

that is used for the FK solving if $n < 6$ with the components of $\boldsymbol{\tau}$ as unknowns. At the opposite of the pose parameters there is no a-priori rules that allows one to establish bounds for the elements of $\boldsymbol{\tau}$. However we may extract from these equations a linear system in $\boldsymbol{\tau}$ of size $n \times n$. After solving this system we may report the result in the remaining $6 - n$ equations. Hence we always end up with a system constituted of:

- n equations from the linear system
- 6 equations (n from (2) plus $6 - n$ remaining after the solving of the linear system)
- $m - n$ inequalities (3)

In terms of unknowns we have

- the pose parameters (12 if $4 \geq n \leq 6$, 9 if $n = 3$)
- the n elements of $\boldsymbol{\tau}$

As we may see we have always more unknowns than equations. However the parametrization always allow us to end up with a square system by adding equations stating that the distance between the reference points B_i, B_j is a known constant. We will no more mention this additional equations in the remaining of the paper but they are required to get a square FK system.

For elastic cables the change in cable lengths affect $\mathbf{l^0}$ which will become $\boldsymbol{l^0} + \boldsymbol{\Delta l^0}$. The equations are:

$$||\mathbf{A_iB_i}||^2 = \rho_i^2 \quad (5)$$
$$\mathcal{F} = \mathbf{J^{-T}}\boldsymbol{\tau} \quad \boldsymbol{\tau} = \{k(\rho_i - l_i^0 - \Delta l_i^0)\} \quad (6)$$

which are valid for all $i \in [1, n]$. We have also the inequalities

$$l_i^0 + \Delta l_i^0 - \rho_i \leq 0 \;\; \forall i \in [1, n] \quad (7)$$
$$||\mathbf{A_iB_i}||^2 - l_i^0 - \Delta l_i^0 \leq 0 \;\; \forall i \in [n+1, m] \quad (8)$$

but here we have to consider all combination with $n \leq m$. In terms of unknowns we have

- the pose parameters (12 if $4 \geq n \leq m$, 9 if $n = 3$)
- the ρ_i with $i \in [1, n]$

For this first algorithm we will assume that we may bound the pose parameters e.g. by assigning a maximal velocity to the platform from which we will deduce a cartesian box that will include the final position of the reference points. Under that assumption we have bounded variables (the pose parameters) and unbounded unknowns (the τ_i for the non elastic case and the ρ_i for the elastic case). The presence of bounded variables lead us to propose an interval analysis solving method (that requests bounded unknowns) and we will see later on that the presence of unbounded unknowns is not a problem, We will now describe our generic FK solving method.

C.1 Solving with interval analysis

Interval analysis allows to calculate exactly (i.e. with an arbitrary accuracy) all solutions of a system of equations that lie within a bounded region, called the *search space*. Without going into the details (that may be found in [23], [24], [25], [26]) the solving principle is first based on the *interval evaluation* of the equations: being given intervals for the unknowns $\mathbf{W}$ (which define a *box* in the unknowns space) and a function of these unknowns $f(\mathbf{W})$ the interval evaluation of f is an interval $[U, V]$ that is guaranteed to satisfy that for all vectors $\mathbf{W}$ whose components all lie in the corresponding intervals we have $U \leq f(\mathbf{W}) \leq V$. There are several methods for computing such an interval evaluation, all having the drawback that U may be underestimated (i.e. the minimum of f over the intervals is larger than U) and/or V may be overestimated (i.e. the maximum of f over the intervals is larger than V). However the differences between U, V and the minimum, maximum decrease with the size of the input intervals. Such an overestimation occurs when there are several occurrences of the same variable in f. A typical example of overestimation is to consider $f = x - x$ when $x \in [-1, 1]$ as $f([-1, 1]) = [-1, 1] - [-1, 1] = [-2, 2]$ that indeed include the solution 0 but with a large overestimation.

Clearly if $U > 0$ or $V < 0$, then f cannot cancel for any point in the box. The second component of an interval analysis algorithm is the *branch and bound* scheme. In this scheme we have a list $\mathcal{L}$ of box(es) which has, at the start of the algorithm, a single element, the search space and an index i initialized to 1. The algorithm look at the i-th box in the list and calculate the interval evaluation of each equation of the system for this box. If for each of these evaluations we have $U < 0$ and $V > 0$, then we bisect the box in two by selecting one of the unknowns and splitting its current interval at the mid-point. This process creates two new boxes that are stored at the end of $\mathcal{L}$ and the index i is incremented. If $U > 0$ or $V < 0$ then the index i is incremented. After each bisection the size of the box decreases so that we may use the third tool of interval analysis which is the Kantorovitch theorem. It states that if some conditions, that may be calculated with interval analysis, are fulfilled, then the box includes a single solution of the system and that this solution may be obtained by using the Newton-Raphson scheme with as initial guess the center of the box (see section V). If this theorem is fulfilled for a given box we have determined a solution of the system and the index i is incremented. The algorithm completes when the index i is larger than the number of elements in $\mathcal{L}$. Such an algorithm cannot miss a solution and will usually provide all the solutions in the search space unless the numerical accuracy is not high enough (in this case it is necessary to extend the floating point arithmetic and numerous packages allow to do it).

This principle may be extended to deal with inequality. For example if the problem is to check if $f(\mathbf{W}) \leq 0$ a box will be deleted from the list $\mathcal{L}$ if $U > 0$ and the inequalities will always be satisfied for any point of the box if $V \leq 0$.

For non elastic cables an objection to the use of interval analysis for $n < 6$ is that we don't have bounds for the components of $\boldsymbol{\tau}$. However they are not necessary. Indeed let us consider the linear system extracted from the statics equations. As the matrix $\mathbf{J}^{-\mathbf{T}}$ is pose dependent and as the pose parameters are intervals we have a so-called *linear interval system*. Interval analysis allows one to solve such system i.e. provide ranges for the components of $\boldsymbol{\tau}$ that include all the solutions of all scalar linear system included in the linear interval system. Without going into the details this solving may fail for a given box especially because the linear interval system includes one or several singular system, in which case the box will be bisected and as we may assume that the robot is not in a singular pose a sufficient number of bisections will always guarantee that the interval linear system may be solved.

For elastic cables we will distinguish two cases:

- $n \leq 6$: we use the interval solving for the linear system that may be extracted from (6) to bound the unknowns ρ_i together with the interval evaluation of (5) that provide another mean to evaluate these bounds
- $n > 6$: we use the interval evaluation of (5) to determine bounds for the ρ_i

C.2 Solving the FK

As seen previously the unknowns of our FK problems are just the pose parameters whose intervals are described by a box. We now describe the processing of a given box of the list $\mathcal{L}$ for non elastic cables:

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1. if $n < 6$
   (a) solve the linear interval system
   (b) if the solving fails bisect the box
   (c) if one component of $\boldsymbol{\tau}$ has a negative upper bound moves to the next box in the list
2. interval evaluation of the 6 equations (2) and the one remaining from (1). If one of this evaluation has a strictly negative upper bound or a strictly positive lower bound moves to the next box in the list

3. apply Kantorovitch theorem with $\mathbf{X_0}$ being the center of the box:
(a) if this test succeed calculate the solution and substitute the box by this solution
4. interval evaluation of the inequalities (3) for the non dominant cable(s)
(a) if the lower bound of one evaluation is strictly positive moves to the next box in the list
5. if the box has been reduced to a solution, then store the solution and moves to the next box in the list
6. bisect the box

For elastic cables the processing of a given box of the list $\mathcal{L}$ is:

1. if $n \leq 6$
(a) solve the linear interval system in ρ_i derived from (6)
(b) if the solving fails bisect the box
(c) if the upper bound of one ρ_i is lower than $l_i^0 + \Delta l_i^0$ moves to the next box in the list
(d) interval evaluation of the equations (5) and of the $m-n$ remaining equations of (6). If one of this evaluation has a strictly negative upper bound or a strictly positive lower bound moves to the next box in the list
2. if $n > 6$ determine bounds for the ρ_i by the interval evaluation of (5)
(a) if the upper bound of one ρ_i is lower than $l_i^0 + \Delta l_i^0$ moves to the next box in the list
(b) if the lower bound of one ρ_i is lower than $l_i^0 + \Delta l_i^0$ set this bound to $l_i^0 + \Delta l_i^0$
(c) interval evaluation of the 6 equations (6). If one of this evaluation has a strictly negative upper bound or a strictly positive lower bound moves to the next box in the list
3. apply Kantorovitch theorem on the full system with $\mathbf{X_0}$ being the center of the box:
(a) if this test succeeds calculate the solution and substitute the box by this solution
4. interval evaluation of the inequalities (7, 8)
(a) if the lower bound of one evaluation is strictly positive moves to the next box in the list
5. if the box has been reduced to a solution, then store the solution and moves to the next box in the list
6. bisect the box

Note that if $n > 6$ we may possible refine the interval for the ρ_i. Indeed (6) may be seen as a linear system in 6 arbitrary ρ_i. The solving of this system will provide bounds for these ρ_i that may used to refine the bounds determined by using (5). The specific parametrization of the platform pose that we have used may be explained here. Indeed all the equations involved in the FK are algebraic and of second order in terms of the unknowns. This implies that the Hessian matrix of the system will be a constant matrix whose norm can thus be computed beforehand. As this norm is used in the Kantorovich theorem having a constant norm allows one to speed up the Kantorovitch test.

Using this process we are able to find all the solutions of all FK problems under the assumption that there is no drastic change in the pose of the parameters during the coiling. If there is a single solution we may have determined the pose of the platform together with its CC after the change in the cable lengths. If we have multiple solutions we cannot determine the current pose and CC.

We will now describe a safe real-time FK method that takes into account possible CC change during the coiling process.

D. Second algorithm

In our previous algorithm we have assumed no drastic change in the platform pose. But unfortunately this assumption may not always hold. For example figure 3 shows changes of CC for infinitesimal changes of cable lengths that leads to a significant change in the platform pose. On the first image the robot has 6 cables under tension and its pose correspond to one solution of its FK. On the second image the CDPR moves suddenly to a CC with 4 cables under tension but one which is mechanically unstable which explain why (third image) it moves to a new pose with 6 cables under tension but on another kinematic branch of the FK than the one from which it started, thereby exhibiting a very different orientation.

Such situation cannot be detected with our first algorithm and requires to consider not only the final cable lengths but how these lengths change with respect to time. We will now consider the CC under the assumption that the cables lengths may have any value in their ranges $[\boldsymbol{\rho}, \boldsymbol{\rho} + \boldsymbol{\Delta\rho}]$ for non elastic cables or that their lengths at rest may have any value in $[\boldsymbol{l^0}, \boldsymbol{l^0} + \boldsymbol{\Delta l^0}]$ for elastic cables. Our purpose will be to determine what are the possible CC of the CDPR under that assumption. We still use the assumption that the pose of the platform may be bounded and we will check what are the possible CC in the neighborhood of $\mathbf{X}$. As the motion of the platform are continuous we will look at specific time at which the initial CC may co-exist with another CC. For non elastic cables such a situation may occur if

1. the tension in a dominant cable (or in several cables) part of the initial CC goes to 0
2. the length of a non dominant cable (or of several cables) not part of the initial CC becomes exactly the distance between the A, B

For elastic cables the situation may occur if

1. the real length ρ of one (or several) cable is exactly its length at rest

We will assume that the initial CC is such that 6 cables are under tension for non elastic cables and that all cables are under tension for elastic cables. We consider the full scale FK equations of the CDPR i.e. equations (2, 4) for non elastic cables and (5, 6) for elastic cables. For fixed values of $\boldsymbol{\rho}$ or $\boldsymbol{l^0}$ we have a square system of equations but here we assume that $\boldsymbol{\rho}, \boldsymbol{l^0}$ may have interval values i.e. that these constraints describe a family of square systems of equations. Note however that the Kantorovitch theorem (section V)

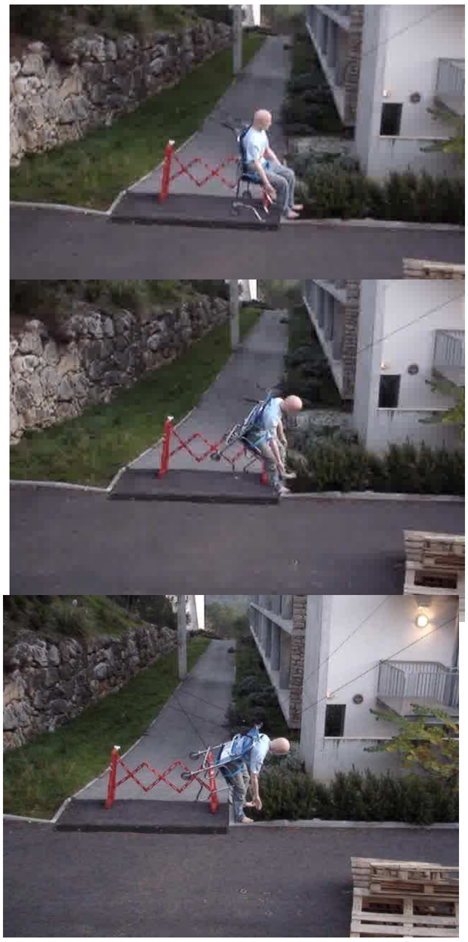

Fig. 3. An experiment showing changes of cable configuration for an infinitesimal change in the cable lengths leading to very different poses

may still be used. Indeed in spite of ρ, l^0 having interval values:

- the Jacobian matrix of the system at $\mathbf{X_0}$ is still a scalar matrix
- $\mathbf{F}(\mathbf{X_0})$ will have now interval values, leading to intervals for $\Gamma_0\mathbf{F}(\mathbf{X_0})$ but whose norm will still be bounded

In consequence the interval values of $\boldsymbol{\rho}, \boldsymbol{l^0}$ will prohibit us to calculate the solutions but if the Kantorovitch conditions are fulfilled, then we are sure that there will be a single solution for any system in the family.

We will now consider separately the non elastic and elastic cables cases.

D.1 Non elastic cables

We will consider as unknowns the pose parameters, the $\boldsymbol{\rho}$ and the $\boldsymbol{\tau}$ of the dominant cables and hence all the unknowns are bounded (the bounds for the τ being obtained with the method described in the first algorithm).

We will consider in sequence all combinations of cases where

1. one or several cables is (are) such that $\tau_j = 0$
2. the length of a cable (or of several cables) not part of the initial CC becomes exactly the distance between the A, B
3. the length of a cable (or of several cables) part of the initial CC becomes larger than the distance between the A, B

For the first case we will set $k\ \tau_j = 0$ into equations (4) and use as unknowns for an interval analysis algorithm the pose parameters, the $\boldsymbol{\rho}$ and the remaining components of $\boldsymbol{\tau}$ but only the pose parameters and the $\boldsymbol{\rho}$ will be bisected. Note that as the $\boldsymbol{\rho}$ have interval values we have a family of square system of equations. The processing of a given box of the algorithm is as follows:

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1. for all cables in the initial CC interval evaluate the equation $||\mathbf{A_i B_i}||^2 - \rho_i^2$. If one of the interval evaluation does not include 0 move to the next box
2. interval evaluate (3) for the cable not part of the CC. If one of these inequality has a positive lower bound move to the next box
3. extract an interval linear system of size $6 - k$ from (4) and solve it in the $6 - k$ elements of $\boldsymbol{\tau}$:
   - if the solving fails bisect the box and move to the next box in the list
   - if one of the τ_j has a negative upper bound move to the next box
4. plug the $\boldsymbol{\tau}$ in the remaining k equations from (4), interval evaluate them and if one of the interval evaluation does not include 0 move to the next box
5. if one of the τ_j has a negative lower bound then bisect the box and move to the next box in the list
6. we check if the Kantorovitch conditions hold
   - if no we bisect the box and move to the next box in the list
   - if yes we have determined that a new solution CC may be reached by the CDPR

In the second and third case we will assume that any number k, with $k \in [1, 6]$, of cables may be such that $||\mathbf{A_i B_i}||^2 = \rho_i^2$. The full FK system will still be square. We will use as unknowns for an interval analysis algorithm the pose parameters and the $k\rho$.

For a given box we use the following steps:

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1. for all k cables interval evaluate the equation $||\mathbf{A_i B_i}||^2 - \rho_i^2$. If one of the interval evaluation does not include 0 move to the next box
2. interval evaluate (3) for the $m - k$ cables not part of the k cables. If one of these inequalities has a positive lower bound move to the next box
3. solve the interval linear system of size k extracted from (4) in the τ:
   - if the solving fails we bisect the box and move to the next box in the list
   - if one of the τ has a negative upper bound, then move to the next box
4. if $k < 6$ plug the τ in the remaining $6 - k$ equations from (4), interval evaluate them and if one of the interval evaluation does not include 0, then move to the next box

5. check if the Kantorovitch conditions hold
- if no we bisect the box and move to the next box in the list
- if yes we have determined that a new solution CC may be reached by the CDPR

If solution CC's are found then there will be set of ρ's for which the CDPR may be submitted to a change of CC.

D.2 Elastic cables

We will consider as unknowns the pose parameters, the components of $\boldsymbol{\rho}$ and the components of $\mathbf{l^0}$ and hence all the unknowns are bounded (the bounds for the components of the $\boldsymbol{\rho}$ being obtained with the method described in the first algorithm).

We will consider in sequence all combinations of cases where one or several cables is (are) such that $\rho_j = l_j^0$ and use an interval analysis algorithm in each case after having removed from the equations (5,6) the equation(s) involving cable(s) j. As we remove as many equations as unknowns we still end up with a square system. For a given box of the algorithm:

- we interval evaluate the equations of the new system. If one interval evaluation does not include 0 we move to the next box
- if for a cable k the upper bound of $||\mathbf{A_k B_k}||$ is lower than the lower bound of l_k^0, then we move to the next box
- we check if the Kantorovitch conditions hold
  - if no we bisect the box
  - if yes we have determined that a new CC may be reached by the CDPR

After completing the algorithm for all cases we will have determined all possible CC of the CDPR during the cable lengths change that are different from the initial CC. If there is no such CC then the initial CC will be kept during the cable lengths changes and the possible final pose(s) will be determined using the first algorithm. If more than one solution is found, then we cannot solve properly the FK problem.

E. Real time FK with a coiling model

The first FK algorithm is just able to determine if a CC different from the initial one may be valid at the end of the cable motion and thus is not completely safe. The second FK algorithm will detect if a change of CC may occur during the cable motion and is therefore fully safe from the CC view point. However both algorithms are not fully safe for finding the pose at the end of the cable motion because they both assume small motion of the platform and still may provide not a single solution. Furthermore from the CC viewpoint the second algorithm is "worst case": although a change in CC may occur the real coiling process may be such that this change will not occur. To take into account the real coiling we will assume a model for this process.

E.1 Coiling model

A typical coiling model will provide the $\boldsymbol{\rho}$ or $\boldsymbol{l^0}$ as functions of time. Being given the state $\mathcal{S}$ of the actuator at time t_0 the cable length at time $t > t_0$ will be obtained as:

$$\rho(t) = G(\mathcal{S}, t) \tag{9}$$

For example assume that an electrical motor is used to turn the drum of a winch. If at time t_0 the velocity of the motor is V_0 and a control law imposes a desired velocity V_c, then the velocity of the motor is

$$V(t) = V_c + (V_0 - V_c)e^{-\frac{t-t_0}{U}}$$

where U is a known constant. If θ_0 is the rotation angle of the motor at time t_0, then the angle at time t is obtained as:

$$\theta(t) = V_c t - (V_0 - V_c)\, U \mathrm{e}^{-\frac{t-t_0}{U}} + (V0 - V_c)U - V_c t_0 + \theta_0$$

If ρ_0 is the cable length at time t_0, then the cable length at time t will be:

$$\rho(t) = \rho_0 + K(\theta(t) - \theta_0)$$

where K is a constant that depends upon the reduction gear of the motor and the drum radius.

E.2 FK for non elastic cables

We will assume that at time t_0 the CDPR is in the initial CC at pose $\mathbf{X}$ with cable lengths $\boldsymbol{\rho_0}$ and that we are interested in determining the robot pose and CC at time $t_0 + \Delta T$. If n is the number of cables under tension at time t_0 and if we assume the CDPR will stay in the same CC the governing equations of the system are:

$$||\mathbf{A_i B_i}||^2 = \rho_i(t) \; i \in [1, n] \tag{10}$$

$$||\mathbf{A_i B_i}||^2 - \rho_i(t)^2 \leq 0 \; i \in [n+1, m] \tag{11}$$

$$\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_n\} \;\; \tau_i > 0 \; \forall i \in [1, n]$$

$$\mathcal{F} = \mathbf{J^{-T}} \boldsymbol{\tau} \tag{12}$$

Let us choose a time increment $\Delta t < \Delta T$. The coiling model allows us to determine an interval value for each of the ρ_i that will include all possible values of ρ_i for any time in the range $[t_0 + \Delta t]$. We then apply Kantorovitch theorem on the 6 equations (10) if $n = 6$ or on the equations (10,12) if $n < 6$. If the Kantorovitch conditions do no hold we divide Δt by 2 and repeat the process. If they hold we are able to calculate the robot pose at time $t_0 + \Delta t$ together with the cable tensions. If the cable tensions are all positive and the inequalities (11) are verified we have a new starting pose and we repeat the process from this point until the time reaches $t_0 + \Delta T$. But it may perfectly happen that in the time interval $[t_0, t_0 + \Delta t]$ there is a time such one of the inequalities (11) are not satisfied or one of the cable tensions becomes negative. To determine if one such configuration exists we will consider the two following problems:

- for each cable j not in the CC find a time t_1 in $[t_0, t_0 + \Delta t]$ such that equations (10) are verified together with $||\mathbf{A_j B_j}||^2 = \rho_j(t)$. For each j we add an equation and the unknown t_1 and therefore we have a square system of equations. Interval analysis is used to solve this problem in t_1
- for each cable j in the CC find a time t_1 in $[t_0, t_0 + \Delta t]$ such that the solving of equations (12) leads to $\tau_j = 0$. For this purpose we set $\tau_j = 0$ and we solve the system (10,12) which is still square as we have removed the unknown τ_j but have added the unknown t_1.

Let define $\mathcal{T} = \{t_1^1, t_1^2, \ldots t_1^p\}$ be the set of times t_1 ordered by increasing value. At any of the time in this set two different CC coexist: the initial one CC_0 and a new one CC_1. We consider each of the time t_1 in the set by increasing value and consider the situation at time $t_1^j + \epsilon$ where ϵ is a small quantity such that $t_1^j + \epsilon < t_1^{j+1}$. At this time the CC of the CDPR may be either CC_0 or CC_1. In the first case at time t_1^j CC_0 and CC_1 were equivalent but the coiling moves the CDPR back to CC_0. In the later case the CDPR has moved to the new CC CC_1: a cable configuration change has occurred. We solve the FK system for both cable configurations and as the CDPR must be in an unique CC only one of the two FK systems has a valid solution (i.e. verifying the inequalities (11) and having positive τ). If the valid CC is CC_0 we move to the time t_1^{j+1} and repeat the process until either we have exhausted the time of $\mathcal{T}$ or we have found a time t_1 at which a change of CC occurs. In the former case we have determined the pose at time $t_0 + \Delta t$ together with the CDPR CC and in the later case we have determined the pose at time $t_1^j + \epsilon$ together with the CC at this pose. Hence in both cases we have a new starting state for the robot with known pose and CC and we may repeat the algorithm until the time reaches $t_0 + \Delta T$.

E.3 FK for elastic cables

Basically the same algorithm than for non elastic cables may be applied. The only difference is that a change in CC will occur only if for a cable j we have at some time t_1 $\rho_j = l_j^0(t_1)$. This constraint adds an equation to the FK system but t_1 is an additional unknown so that we still have a square system. Solving this system for all j leads to a set of time $\mathcal{T}$ that is used to determine if a CC change occurs with the same strategy than for non elastic cables.

F. Implementation

The three presented algorithm have been implemented in C++. All interval calculations are performed using the PROFIL/BIAS package [27] while the interval analysis solving algorithms are based on our interval analysis library `ALIAS` [28]. The `Maple` interface of this library allows us to produce part of the C++ code automatically (e.g. the code for the equations, the Jacobian and Hessian matrices). With this implementation all numerical round-off errors are taken into account. However we have noticed that in some case the required accuracy from completing the FK algorithms may be higher than the extended floating point accuracy. Currently we deal with this problem by using specific `Maple` procedures (for example a Newton scheme that allows one to calculate the roots of a system with an arbitrary number of digits). Clearly this is not compatible with a real-time algorithm but extended accuracy package such as MPFR [29] may be used for these special cases.

III. Examples

We have considered a CDPR with 8 cables. The coordinates of the A, B points are provided in tables I,II and are derived from the robot presented in [30].

	A_1	A_2	A_3	A_4
x	-7.175120	-7.315910	-7.302850	-7.160980
y	-5.243980	-5.102960	5.235980	5.372810
z	5.462460	5.472220	5.476150	5.485390
	A_5	A_6	A_7	A_8
x	7.182060	7.323310	7.301560	7.161290
y	5.347600	5.205840	-5.132550	-5.269460
z	5.488300	5.499030	5.489000	5.497070

TABLE I. Coordinates of the A points (in meter)

	B_1	B_2	B_3	B_4
x	0.503210	-0.509740	-0.503210	0.496070
y	-0.492830	0.350900	-0.269900	0.355620
z	0.000000	0.997530	0.000000	0.999540
	B_5	B_6	B_7	B_8
x	-0.503210	0.499640	0.502090	-0.504540
y	0.492830	-0.340280	0.274900	-0.346290
z	0.000000	0.999180	-0.000620	0.997520

TABLE II. Coordinates of the B points in the mobile frame (in meter)

We have used the third algorithm to fully simulate the motion of the robot along a horizontal circle centered at (0,0,2) with radius 1 with an orientation such that the vectors of the reference and mobile frame were identical. The velocity of the robot was set to 0.1m/s. It was supposed that the starting point of the trajectory was (1,0,2) with the configuration 345678. On this trajectory 11 changes of cable configurations were detected: 2 with only 5 cables under tension (34678, 13478 that occurred both twice on the trajectory) and 9 with 6 cables under tension (345678, 234567, 134678, 134568, 125678, 124578, 123678, 123478, 123456). The CC with only 5 cables under tension occur only for a small amount of time, typically less than 1 ms. Figure 4 shows the tension of cable 3 during the trajectory while figure 5 shows the tension of cable 8. In both figures we can see the large influence of CC changes on the cable tensions.

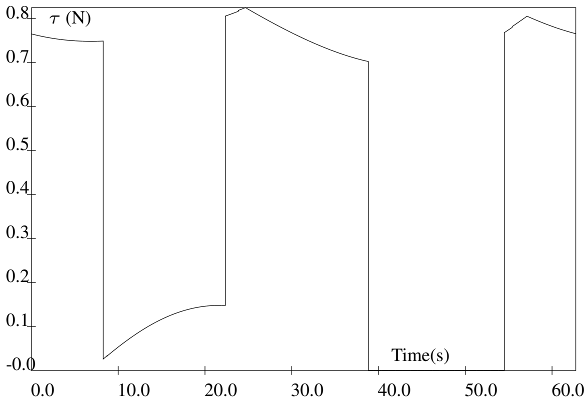

Fig. 4. The tension in cable 3 along the circular trajectory

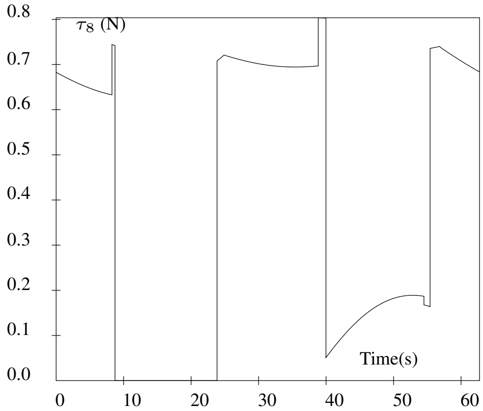

Fig. 5. The tension in cable 8 along the circular trajectory

We have timed the first and second algorithms for a maximal changes of 2mm in the ρ, l_0 on this trajectory and the computation time is usually less than 1ms which is satisfactory from a real-time point of view.

IV. Conclusions

This paper has addressed the real-time forward kinematic problem of CDPR i..e determining the pose of the robot after a small change in its cable lengths, under the assumption that the cables have no mass but may or not have elasticity. This problem is more complex than the forward kinematics of parallel robots with rigid legs as the inverse kinematics equations that are used to solve the forward kinematics depends upon the cable configuration. Hence a forward kinematic solver should take into account the point that the cable configuration at the final pose may be different from the one at the initial pose, thereby leading to different equations and consequently to a different final pose. We have proposed 3 algorithms for that purpose. one compute all possible pose and cable configurations after completion of the change in the cable lengths. The second one performs the same job but also compute changes in cable configurations during the coiling. The last algorithm assumes a model for the coiling process and determine the unique pose and cable configuration at the end of the coiling. All three algorithms are basically real time in the sense that they complete within a sampling period of the controller. The results of the FK solver show that drastic changes in the cable tensions may occur due to CC changes. As classical simulation tools will ignore these changes they may underestimate positioning errors and cable tensions.

An interesting problem will be to determine if the third algorithm is able to deal with the sequence presented in figure 3: during the large motion that leads from the CC with 4 cables under tension to a new stable CC with 6 cables under tension there is most probably no solution to the FK equations that respect the mechanical equilibrium. In that case it will be necessary to take into account the dynamics of the CDPR.

The presented algorithm do not deal with cables having significant mass and submitted to sagging. However in that case there is a single set of FK equations and the method described in [21] may be used after an adaptation to the sagging equations with either a simplified model [30], a mixed model [31] or a full model [22]. Surprisingly the real-time inverse kinematics may be more difficult for CDPR with sagging cables as our preliminary experiments have shown, but that has to be confirmed, that in some cases this problem may not have an exact solution, in which case it will be necessary to determine a solution that is the "closest" in some sense to the desired pose.

Another extension that has to be considered is toward non suspended CDPR. Here the downward pulling cables may be controlled in tension to ensure that all cables are under tension using strategies that have been exposed in numerous works [32], [33], [34], [35] but we believe that still there will be poses in the workspace for which cables may become slack. Finally it may also be necesseray to incorporate interference detection in the real-time FK [36], [37], [38].

V. Annex: Kantorovitch theorem

Assume that we have a square system of n equations $\mathbf{F}$ in the n unknowns $x_1, \ldots, x_n$:

$$\mathbf{F} = \{F_i(x_1, \ldots, x_n) = 0, i \in [1, n]\}$$

Let $\mathbf{X_0} = \{x_1^0, \ldots, x_n^0\}$ be a vector of specific values for the unknowns and define a ball U centered at $\mathbf{X_0}$ with radius B_0. We denote $||\mathbf{G}||$ the norm of a vector or a matrix $\mathbf{G}$ of dimension p or $p \times p$. This norm may be arbitrary but we will use here $\mathrm{Max}(|G_i|)\forall i \in [1, p]$ for vectors and $\mathrm{Max}(\sum_{j=1}^{j=p} |G_{ij}|)\forall i \in [1, p]$. Let's assume that the following conditions hold:

1. the Jacobian matrix of the system has an inverse Γ_0 at $\mathbf{X_0}$ such that $||\Gamma_0|| \leq A_0$
2. $||\Gamma_0 \mathbf{F}(\mathbf{X_0})|| \leq 2B_0$
3. $\sum_{k=1}^{n} |\frac{\partial^2 F_i(\mathbf{X}}{\partial x_j \partial x_k}| \leq C$ for $i, j = 1, \ldots, n$ and $\forall \mathbf{X} \in U$

then if $2nA_0B_0C \leq 1$ there is an unique solution of $\mathbf{F}$ in U and the Newton method used with $\mathbf{X_0}$ as initial estimate of the solution will converge toward this solution.

Note that the last condition may be verified with interval analysis as soon as the unknowns are all bounded.

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