Poisson Access Networks with Shadowing: Modelling and Statistical Inference

Mokhtar Zahdi Alaya

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Poisson Access Networks with Shadowing: Modelling and Statistical Inference

Submitted by :

Mokhtar Zahdi ALAYA

Supervisor : **Dr. Bartłomiej BŁASZCZYSZYN**

Associate Supervisor : **Dr. Mohamed Kadhem KARRAY**

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"Begin at the beginning, and go on till you come to the end. Then, stop."

L. Carol, Alice's Adventures in Wonderland.

To my mother ***Sahara***.

Abstract

Since interference is the main performance-limiting factor in most wireless networks, it is crucial to characterize the interference statistics. The two main determinants of the interference are the network geometry (spatial distribution of concurrently transmitting nodes) and the path loss law (signal attenuation). In order to explain the above, the main purpose of this thesis is of study the path-loss with respect to the serving base stations (BS) and the interference factor, defined as the ratio of the sum of the path-gains from interfering BS to the path-gain from the serving BS. In this thesis, the locations of nodes are modeled as a random Poisson process. Regarding the signal propagation model, we consider a random shadowing that characterizes in a statistical manner the effect of various obstacles. We provide results on the probability distribution function of both the path-loss and the interference factors.

Keywords : Wireless cellular networks, Point Poisson process, Shadowing, Path-loss, Interference, Path-loss exponent.

Contents

Introduction

Stochastic geometry is the study of a random process whose outcomes are geometrical objects or spatial patterns, that is, random subsets on the plane or in higher dimensions; it is intrinsically related to the theory of point processes. It has applications to biology, astronomy and material sciences. Nowadays, it is also used in telecommunications which allows to capture the non-regular and variable geometry of the network and variability of radio channel conditions in probabilistic manner primarily offering various averaging methods.

Modeling of the attenuation of an electromagnetic wave as it propagates in space is a major component in the analysis and design of wireless system. This phenomenon, also called propagation loss, is caused by the decay of the signal power with the distance from the emitter, existing even in the free space propagations models, and due to various obstacles between emitters and receivers, trees, buildings, hills, etc., present in real network profiles. The propagation loss is typically modeled by the product of a deterministic function of the distance, which represents average path-loss on the given distance in the network and a random variable, called *shadowing*.

There are two key ingredients in the analysis of wireless cellular networks and thus their values can be considered as some important requirements to the study of QoS. The first one is the path-loss to the serving BS, which is the one received with the strongest signal and not necessarily the closet one. The second is the so called interference factor defined as the ratio of the sum of the path-gains from interfering BS to the path-gain from the serving BS.

The remaining part of this thesis is organized as follows. In the first chapter, we introduce the theoretical framework by presenting the so known tools in the stochastic geometry which are the point Poisson process and the so called Palm distributions. In the second chapter, we describe our model which is based on the consideration of a Poisson point process. Finally, we study some mathematical results of the distributions of the path-loss and the interference factors which are two ingredients in the analysis of the quality of service (QoS) for a wireless cellular networks.

Chapter 1

Theoretical Framework

The aim of this chapter is to introduce the basic tools and structures of some classical results of stochastic geometry and thus to lay the foundations of for much of this thesis. Our main reference for this chapter is [BB09, vol.I].

1.1 Point Processes

1.1.1 General Notation

A point process is a counting process that represents a random set of points in a space. The usual spaces are the real line, the plane, the multi-dimensional Euclidean space $\mathbb{R}^d$, or, more generally, a complete, separable metric space (a Polish space). Following the standard convention we will discuss point processes on a polish space $\mathbb{E}$. We let $\mathcal{B}(\mathbb{E})$ be the family of Borel sets of $\mathbb{E}$ and let $\widehat{\mathcal{B}}(\mathbb{E})$ denote the family of bounded Borel sets (a set is bounded if it is contained in a compact set). We refer to $\mathbb{E}$ simply as a space, and denote other spaces of this type by $\mathbb{E}'$, $\widetilde{\mathbb{E}}$, etc.

We begin with an informal description of a point process. A random set of points on $\mathbb{E}$ is a countable set of $\mathbb{E}$-valued random elements X_n such that only a finite number of the points are in any bounded set. Thus,

$$\Phi(B) = \sum_{n=1}^{\Phi(\mathbb{E})} \epsilon_{X_n}(B), \quad B \in \mathcal{B}(\mathbb{E})$$

denotes the number of points in B, where $\epsilon_X(B) = 1$ if $X \in B$ and 0 otherwise (a Dirac measure with unit mass at X). This counting measure Φ, as we will define below, is a point process in $\mathbb{E}$ with point locations X_n.

Note that Φ takes values in the set $\mathbb{M}$ of all measures ν on $(\mathbb{E}, \mathcal{B}(\mathbb{E}))$ that are locally finite

($\nu(B) < \infty$, for all $B \in \widehat{\mathcal{B}}(\mathbb{E})$). Each measure in $\mathbb{M}$ has the form

$$\nu(B) = \sum_{n=1}^{\nu(\mathbb{E})} \epsilon_{X_n}(B), \quad B \in \mathcal{B}(\mathbb{E}),$$

The set $\mathbb{M}$ is endowed with the σ-field $\mathcal{M}$ generated by the sets $\{\mathbb{M} \ni \nu \mapsto \nu(B)\}$ (in other words by the mappings $\{\mathbb{M} \ni \nu \mapsto \nu(B)\}$), for $B \in \mathcal{B}(\mathbb{E})$. We are now ready for a formal definition.

1.1.2 Definitions and Classical Results

Definition 1.1.1 *A point process on a space $\mathbb{E}$ is a measurable map Φ from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the space $(\mathbb{M}, \mathcal{M})$. The quantity $\Phi(B)$ is the number of points in the set $B \in \mathcal{B}(\mathbb{E})$. Hence,*

$$\Phi(B) = \sum_{n=1}^{\Phi(\mathbb{E})} \epsilon_{X_n}(B), \quad B \in \mathcal{B}(\mathbb{E}),$$

where X_n denotes the locations of the points of Φ.

One can think of Φ as a counting process of $\mathbb{E} = \mathbb{R}^d$ that is locally finite ($\Phi(B) < \infty$ *a.s.* for the bounded sets B). The probability distribution of the point process Φ i.e., $\mathbb{P}(\Phi \in \cdot)$ is determined by its finite-dimensional distributions

$$\mathbb{P}[\Phi(B_1) = n_1, ..., \Phi(B_n) = n_k], \;\; B_1, ..., B_k \;\in \widehat{\mathcal{B}}(\mathbb{E}).$$

In other words, two point processes Φ and Φ' in E are equals in distribution, denotes by $\Phi \overset{D}{=} \Phi'$, if their finite dimensional distributions are equals :

$$\mathbb{P}[\Phi(B_1) = n_1, ..., \Phi(B_n) = n_k] = \mathbb{P}[\Phi'(B_1) = n_1, ..., \Phi'(B_n) = n_k], \;\; B_1, ..., B_k \in \widehat{\mathcal{B}}(\mathbb{E}).$$

In construction a point process, it suffices to define the probabilities on sets B_i that generate $\mathcal{B}(\mathbb{E})$.

Definition 1.1.2 *One says that the point process Φ is simple if*

$$\mathbb{P}\Big(\Phi(\{x\}) = 0 \,or\, 1, \,\forall\, x\Big) = 1,$$

i.e., with probability 1, $\Phi = \sum_n \epsilon_{X_n}$, the points $\{X_n\}$ are pairwise different.

Definition 1.1.3 ***(Mean Measure)*** *The mean measure of the point process Φ is the measure M on $(\mathbb{E}, \mathcal{B}(\mathbb{E}))$ defined by*

$$M(B) := \boldsymbol{E}[\Phi(B)], \quad B \;\in\; \mathcal{B}(\mathbb{E}).$$

The mean measure M is also known as the first moment measure. Higher moment can be introduced in a similar way.

Note that $M(B)$ may be infinte, even if B is bounded. When $\mathbb{E} = \mathbb{R}^d$, the intensity measure is sometimes of the form $M(B) = \int_B \lambda(x)dx$, where $\lambda(x)$ is the rate of Φ at the location x and dx denotes the Lebesgue measure.

Theorem 1.1.1 ***(Campbell's Theorem)***
Let Φ be a point process and consider a measurable function $f : \mathbb{E} \to [0, \infty)$. Then

$$\int_{\mathbb{E}} f(x)\ \Phi(dx) \equiv \sum_{x \in \Phi} f(x)$$

is a random variable and

$$\boldsymbol{E}\Big[\int_{\mathbb{E}} f(x)\ \Phi(dx)\Big] = \int_{\mathbb{E}} f(x)\ M(dx), \tag{1.1}$$

where M is the intensity measure of Φ.

Proof. For $B \in \mathcal{B}(\mathbb{E})$,

$$\Phi(B) = \int_{\mathbb{E}} \mathbf{1}_B\ d\Phi,$$

this is a nonnegative measurable function, and

$$\mathbf{E}\Big[\int_{\mathbb{E}} \mathbf{1}_B\, d\Phi\Big] = \mathbf{E}[\Phi(B)] = M(B) = \int_{\mathbb{E}} \mathbf{1}_B\ dM.$$

Thus, the assertion holds for indicator functions of Borel sets and therefore also for linear combinations of such functions. By a standard argument of integration theory, it holds for nonnegative measurable functions. ■

Remark 1.1.1 *We have formulated Campbell's theorem only for nonnegative measurable functions, but it is clear that it holds also for $M-$integrable functions. The same remark refers to the subsequent relatives of Campbell's theorem, they will later tacitly be applied to integrable functions.*
For a simple point process Φ, Campbell's theorem can be written in the form

$$\boldsymbol{E}\Big[\sum_{x \in \Phi} f(x)\Big] = \int_{\mathbb{E}} f dM.$$

A Laplace transform is a tool for characterizing the distribution and the moments of a nonnegative random variable. This transform is also useful for establishing convergence in distribution of random variables. The analogous tool for point processes is Laplace functional.

Definition 1.1.4 ***(Laplace Functional of Point Process)*** *The Laplace functional of the point process Φ is defined by*

$$\mathcal{L}_\Phi(f) := \boldsymbol{E}\Big[\exp\Big(-\int_{\mathbb{E}} f(x)\,\Phi(dx)\Big)\Big], \tag{1.2}$$

where, $f : \mathbb{E} \to [0, \infty)$ is a measurable function.

The Laplace functional of a point process uniquely determines its distribution. Also Laplace functionals are often more convient to use than finite-dimentionnel distributions in deriving the distribution of a point process constructed as a function of random variables or point process. Our main focus hereafter will be on the Poisson point processes.

1.2 Poisson Point Processes

Poisson point processes can be introduced in quite general measurable spaces with additional structures. Here we restrict ourselves, as in the previous sections, to a Polish space $\mathbb{E}$. We first introduce the two characteristic properties of Poisson point processes.

Definition 1.2.1 *A point process Φ on a space $\mathbb{E}$ is Poisson with intensity measure Λ that is locally finite if the following conditions are satisfied.*

a) Φ has independent increments; i.e, the random variables $\Phi(B_1), ..., \Phi(B_n)$ are stochastically independent for pairwise disjoint $B_1, ..., B_n \in \widehat{\mathcal{B}}(\mathbb{E})$.

b) For each $B \in \widehat{\mathcal{B}}(\mathbb{E})$, the quantity $\Phi(B)$ is a Poisson random variable with mean $\Lambda(B)$, i.e.,

$$\mathbb{P}(\Phi(B) = k) = \frac{\Lambda(B)^k}{k!}\, e^{-\Lambda(B)}.$$

Note that $\Phi(B) = 0$ *a.s.* when $\Lambda(B) = 0$. Note that the number of points $\Phi(x)$ exactly at x has the poisson distribution with mean $\Lambda(x)$, so $\Phi(x) = 0$ a.s. when $\Lambda(x) = 0$. From the definition, it follows that the finite-dimensional distributions of a Poisson point process are uniquely determined by its intensity measure, and vice versa.

Theorem 1.2.1 ***(Poisson Laplace Functional)*** *For a Poisson point process on $\mathbb{E}$ with intensity measure Λ, and a measurable function $f : \mathbb{E} \to [0, \infty)$*

$$\mathcal{L}_\Phi(f) = exp\Big(-\int_E (1 - e^{-f(x)})\,\Lambda(dx)\Big). \tag{1.3}$$

Moreover, Φ is a simple point process if and only if Λ is diffuse.

Proof. It is sufficient to consider simple functions of the form

$$f = \sum_{i=1}^{n} c_i \mathbf{1}_{B_i},$$

where c_i are nonnegative and the B_i are disjoint Borelian subsets of $\mathbb{E}$. The Laplace functional of Φ at f is given by

$$\mathbf{E}\Big[\exp\Big(-\int_{\mathbb{E}} f(x)\,\Phi(dx)\Big)\Big] = \mathbf{E}\Big[\exp\Big(-\sum_{1}^{n} c_i\,\Phi(B_i)\Big)\Big],$$

by independence Assumption $a)$ it can be written as

$$\mathbf{E}\Big[\exp\Big(-\int_{\mathbb{E}} f(x)\,\Phi(dx)\Big)\Big] = \prod_{1}^{n} \mathbf{E}[\exp(-c_i\,\Phi(B_i))] = \prod_{1}^{n} \exp[-\Lambda(B_i)(1-e^{-c_i})]$$

because the distribution of $\Phi(B_i)$ is a Poisson according to $b)$, finally

$$\mathbf{E}\Big[\exp\Big(-\int_{\mathbb{E}} f(x)\,\Phi(dx)\Big)\Big] = \exp\Big[-\sum_{1}^{n}\Lambda(B_i)(1-e^{-c_i})\Big] = \exp\Big[-\int(1-e^{-f(x)})\,\Lambda(dx)\Big].$$

Now, if the measure Λ has a mass $\lambda(\{x\}) > 0$ on $x \in \mathbb{E}$, the variable $\Phi(\{x\})$ is Poisson with parameter $\Lambda(\{x\})$, in particular $\mathbb{P}(\Phi(\{x\}) = 2) > 0$, hence Φ cannot be simple. Conversely, if Λ is diffuse, let B a bounded element of $\mathcal{B}(\mathbb{E})$ then,

$$\begin{aligned}
\mathbb{P}\{\Phi \text{ is simple on } B\} &= \\
&= \sum_{n=2}^{\infty} \mathbb{P}\{\Phi(B) = n\}\mathbb{P}\{\forall \text{n points of } \Phi \text{ are distincts} \,|\Phi(B) = n\} \\
&= \sum_{n=2}^{\infty} e^{-\Lambda(B)} \frac{(\Lambda(B))^n}{n!} \frac{1}{(\Lambda(B))^n} \int \ldots \int_{B^n} 1\{x_j \text{ distincts}\}\Lambda(dx_1)\ldots\Lambda(dx_n) \\
&= 1.
\end{aligned}$$

Whence the desired result. ∎

As we will see, many point processes involving complex phenomena or systems can be represented by functions of Poisson point processes. In these settings a Poisson point process is typically the basic data that defines or initializes a system, and various characteristics of the system are deterministic or random function of the Poisson point process. A basic issue in this regard is; if the point locations of a Poisson point processes are mapped to some space by a deterministic or random mapping, then do the resulting new points also form a Poisson point process? This issue on a variety of contexts is the underlying theme for the next section.

1.2.1 Operations Preserving the Poisson Law

a) Superposition

Lemma 1.2.1 *Let $\Phi_1, ..., \Phi_n$ denote independent Poisson point processes on $\mathbb{E}$ with respective intensity measures $\Lambda_1, ..., \Lambda_n$. Then the sum (or superposition) $\Phi = \sum_{i=1}^{n} \Phi_i$ is a Poisson point process with intensity measure $\Lambda = \sum_{i=1}^{n} \Lambda_i$. This is also true for $n = \infty$ provided Λ is locally finite.*

Proof. It is sufficient to see that for $n \geq 1$, the Laplace functional of the point process $\Phi_1 + ... + \Phi_n$ at f is given by

$$\exp\Big[- \int (1 - e^{-f(x)})(\Lambda_1 + ... + \Lambda_n)(dx) \Big],$$

therefore, this point process is a Poisson with intensity measure $\Lambda_1 + ... + \Lambda_n$. ∎

b) Transformation of Space

Suppose Φ is a Poisson point process on $\mathbb{E}$ with intensity measure Λ. Consider a transformation of Φ in which its points in $\mathbb{E}$ are mapped to a space $\mathbb{E}'$ (possibly $\mathbb{E}$) by the rule that a point of Φ located at $x \in \mathbb{E}$ is mapped to the location $G(x) \in \mathbb{E}'$, where $G : \mathbb{E} \to \mathbb{E}'$ is a given map. We represent this transformation of Φ by the point process N on $\mathbb{E} \times \mathbb{E}'$ defined by

$$N(B \times B') \equiv \sum_{n} \mathbf{1}\Big((X_n, G(X_n)) \in B \times B'\Big) = \Phi(B \cap G^{-1}(B')), B \in \mathcal{B}(\mathbb{E}),\ B' \in \mathcal{B}(\mathbb{E}').$$

Keep in mind that $\sum_{n} = \sum_{n=1}^{\Phi(E)}$. The quantity $N(B \times B')$ denotes the number of points of Φ in $B \in \mathcal{B}(\mathbb{E})$ that are mapped into $B' \in \mathcal{B}(\mathbb{E}')$. Then the transformed points in the space $\mathbb{E}'$ are represented by the point process Φ' defined by

$$\Phi'(B) = N(\mathbb{E} \times B') = \sum_{n} \epsilon_{G(X_n)}(B) = \Phi(G^{-1}(B)),\ \ B \in \mathbb{E}'.$$

Lemma 1.2.2 *Under the preceding assumptions, the transformation process N is a Poisson with mean measure*

$$\boldsymbol{E}[N(B \times B')] = \Lambda(B \cap G^{-1}(B')),\ B \in \mathcal{B}(\mathbb{E}),\ B' \in \mathcal{B}(\mathbb{E}').$$

Hence, the process Φ' is a Poisson point process with intensity measure $\boldsymbol{E}[\Phi'(B')] = \Lambda(G^{-1}(B'))$, $B' \in \mathcal{B}(\mathbb{E}')$, provided this measure is finite for each compact B'.

Proof. We will show that N satisfies the two conditions in the definitions of a Poisson point process. Since Φ is a Poisson point process, $N(B \times B') = \Phi(B \cap G^{-1}(B'))$ has the Poisson distribution with mean $\Lambda(B \cap G^{-1}(B'))$. This intensity is finite for any B' when B is a compact. It remains to verify that N has independent increments. It suffices to show that $N(B_i \times B_i')$, $i = 1, ..., k$, are independent for disjoint $B_1, ..., B_k$ in $\mathcal{B}(\mathbb{E})$ and disjoint $B_1', ..., B_k'$ in $\mathcal{B}(\mathbb{E})$. This independence follows since $B_i \cap G^{-1}(B_i')$, $i = 1, ..., k$, are disjoint and Φ has independent increments. Thus, N has independent increments and hence is a Poisson point process.

Next, note that process $\Phi'(B') = N(E \times B')$ has independent increments since N does, and $\Phi'(B')$ has a Poisson distribution with $\mathbf{E}[\Phi'(B')] = \Lambda(G^{-1}(B'))$. Thus, Φ' is a Poisson process when $\Lambda(G^{-1}(B'))$ is finite for each compact B'. ■

We are now ready to consider random transformation of Poisson point processes.

c) Random Transformation of Points

The focus of this section is on a transformation of a Poisson point process on the space $\mathbb{E}$ in which each of its points is independently assigned a random mark on a space $\mathbb{K}$ depending only on the particular point location. The distribution of the marks will be determined by probability kernels.

A mark assigned to a point at $x \in \mathbb{E}$, will take value in a set $B \in \mathbb{K}$ according the probability kernel $p(x, B)$ from $\mathbb{E} \to \mathbb{K}$. Such a kernel is a function $p : \mathbb{E} \times \mathbb{K} \to [0, 1]$ such that $p(., B)$ is a measurable function on $\mathbb{E}$ and $p(x, .)$ is a probability measure on $\mathbb{K}$. Our interest will on modelling the initial points as well as the marks by a *marked point process* on $\mathbb{E} \times \mathbb{K}$. The formal definition is as follows.

Definition 1.2.2 *Let* $\Phi = \sum_n \epsilon_{X_n}$ *be a Poisson point process on* $\mathbb{E}$ *with intensity* Λ. *Let* $\widetilde{\Phi} = \sum_n \epsilon_{(X_n, S_n)}$ *be a point process on* $\mathbb{E} \times \mathbb{K}$ *such that*

$$\mathbb{P}(S_n \in K \big| \Phi) = p(X_n, K), \ K \subset \mathbb{K}, \ n \leq \Phi(\mathbb{E}),$$

where $p(x, K)$ *is a probability kernel from* $\mathbb{E}$ *to* $\mathbb{K}$. *The* S_n *are p-marks of the* X_n, *and the point process* $\widetilde{\Phi}$ *of the initial points and their marks is a p-marked Poisson process associated with* Φ.

The Laplace functional of the marked point process $\widetilde{\Phi}$ is related to that of Φ as follows. this relation is useful for deriving properties of $\widetilde{\Phi}$, when Φ has a tractable Laplace functional.

Proposition 1.2.1 *The Laplace functional of the marked point process $\widetilde{\Phi}$ associated with Φ is*

$$L_{\widetilde{\Phi}}(f) = \boldsymbol{E}\Big\{\exp\Big[\int_{\mathbb{E}} log[\int_{\mathbb{K}} e^{-f(x,s)}\, p(x,ds)]\, \Phi(dx)\Big].\Big\} \tag{1.4}$$

That is $L_{\widetilde{\Phi}}(f) = L_{\Phi}(h)$, where $h(x) = -log[\int_{\mathbb{K}} e^{-f(x,s)}\, p(x,ds)]$.

Proof. Conditioning on Φ and using the property that the S_n are conditionally independent given Φ, we have

$$\begin{aligned}
L_{\widetilde{\Phi}}(f) &= \mathbf{E}\Big\{\mathbf{E}\Big[e^{-\sum_n f(X_n,S_n)}\Big]\Big|\Phi\Big\} \\
&= \mathbf{E}\Big\{\prod_n \mathbf{E}\Big[e^{-f(X_n,S_n)}\Big]\Big\} \\
&= \mathbf{E}\Big\{\prod_n \int_K e^{-f(X_n,s)} p(X_n,ds)\Big\} \\
&= \mathbf{E}\Big\{\exp\Big[\sum_n log \int_K e^{-f(X_n,s)} p(X_n,ds)\Big]\Big\}.
\end{aligned}$$

Using the property $\sum_n g(X_n) = \int_{\mathbb{E}} g(x)\Phi(dx)$, for $g : \mathbb{E} \to \mathbb{R}$, the last expectation equals to the right side of (1.4), whence the desired result. ■

The following is the major result that random transformations of Poisson processes are also Poisson.

Theorem 1.2.2 ***(Displacement Theorem)*** *The point process $\widetilde{\Phi} = \sum_n \epsilon_{(X_n,S_n)}$ in Definition 1.2.2 is a Poisson point process on $\mathbb{E} \times \mathbb{K}$ with intensity measure $\Lambda_{\widetilde{\Phi}}$ defined by*

$$\Lambda_{\widetilde{\Phi}}(B \times K) = \int_B p(x,K)\Lambda(dx), \quad B \subset \mathbb{E}, K \subset \mathbb{K}. \tag{1.5}$$

Hence, the point process of marks values $\Phi' = \sum_n \epsilon_{S_n}$ is a Poisson point process on $\mathbb{K}$ with intensity measure $\Lambda'(K) = \int_{\mathbb{E}} p(x,K)\Lambda(dx), K \in \mathbb{K}$, provided this measure is locally finite.

Proof. From Proposition 1.2.1, we know that $L_{\widetilde{\Phi}}(f) = L_{\Phi}(h)$, where $L_{\Phi}(h)$ is a Poisson Laplace functional of the form given by the Theorem 1.2.1. Then

$$\begin{aligned}
L_{\widetilde{\Phi}}(f) &= \exp\Big[-\int_{\mathbb{E}}(1 - e^{-h(x)})\Lambda(dx)\Big] \\
&= \exp\Big[-\int_{\mathbb{E}\times K}(1 - e^{-f(x,s)})p(x,ds)\Lambda(dx)\Big].
\end{aligned}$$

But this is the Laplace functional of a Poisson point process with mean given by (1.5). This proves the first assertion of the theorem. The second assertion that $\Phi'(\cdot) = \widetilde{\Phi}(\mathbb{E} \times \cdot)$

is a Poisson process follows since it is the Poisson process $\widetilde{\Phi}$ on a subset of its space $\mathbb{E} \times \mathbb{K}$.
■

1.3 Palm Theory

In the study of a point process we are often interested in properties relating to a *typical point* of the process. This requires the calculation of conditional probabilities of events given that there is a point in the process at a specified location. It leads to the concept of the Palm distribution of the point process, and the related Campbell-Mecke formula. These tools allow us to define new characteristics of a point process, such as the nearest neighbor distance distribution function.One simple question about a point process Φ is: what is the probability distribution of the distance from a point of Φ to its nearest neighbour (the nearest other point of Φ).
If x is known to be a point of Φ, then the nearest neighbour distance is $R_x = dist(x, \Phi \setminus \{x\})$, and we seek the "conditional probability" $\mathbb{P}(R_x \leq r | x \in \Phi)$. The problem is that this is not a conditional probability in the elementary sense, because the event $\{x \in \Phi\}$ typically has probability zero. In this section, we consider point processes on $\mathbb{R}^d$ and introduce the Palm calculus, a powerful tool to describe conditional distributions of point processes. In the probability theory, it is possible under suitable assumptions to introduce conditional probabilities, which are not defined in an elementary way, by means of desintegration procedures. A somewhat similar procedure is possible for point processes and leads to the notion of Palm distribution P_x with respect to a given point $x \in \mathbb{R}^d$.

1.3.1 Palm Distributions in a General Setting

Definition 1.3.1 *For a point process on $\mathbb{R}^d$, define the reduced Campbell measure $C^!$ on $\mathbb{R}^d \times \mathbb{M}$ by*

$$C^!(B \times \Gamma) := \boldsymbol{E}\Big[\int_{\mathbb{R}^d} \boldsymbol{1}(x \in B)\boldsymbol{1}(\Phi \setminus \epsilon_x \in \Gamma)\Phi(dx)\Big]. \tag{1.6}$$

The Campbell measure C is defined by

$$C(B \times \Gamma) := \boldsymbol{E}\Big[\int_{\mathbb{R}^d} \boldsymbol{1}(x \in B)\boldsymbol{1}(\Phi \in \Gamma)\Phi(dx)\Big]. \tag{1.7}$$

The measure $C^!(B \times \Gamma)$ is a refinement of the mean measure M. It gives the expected number of points of Φ in B such that when removing a particular point from Φ. We have an extension of the Campbell theorem

$$\mathbf{E}\Big[\int_{\mathbb{R}^d} f(x,\Phi)\ \Phi(dx)\Big] = \int_{\mathbb{R}^d}\int_{\mathbb{M}} f(x,\eta)\ C(d(x,\eta)), \tag{1.8}$$

also

$$\mathbf{E}\Big[\int_{\mathbb{R}^d} f(x,\Phi\backslash\epsilon_x)\ \Phi(dx)\Big] = \int_{\mathbb{R}^d}\int_{\mathbb{M}} f(x,\eta)\ C^!(d(x,\eta)), \tag{1.9}$$

for a nonnegative measurable function $f : \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}$.

For each $\Gamma \in \mathbb{M}$, $C^!(\cdot \times \Gamma) << M(\cdot)$, so $C^!(\cdot \times \Gamma)$ is absolutely continuous with respect to M. Then by the Radon-Nikodym theorem, there exists a $M-$almost surely density $x \to P_x^!(\Gamma)$ so that

$$C^!(B \times \Gamma) = \int_B P_x^!\, M(dx),\ for\ all\ B \in \mathbb{R}^d. \tag{1.10}$$

The function $P_x^! = P_x^!(\Gamma)$ depends on Γ. Moreover, if $M(\cdot)$ is a locally finite measure, $P_x^!$ can be chosen as a probability distribution on $\mathbb{M}$ for each given x.

Definition 1.3.2 *Given a point process with a locally finite mean measure, the distribution $P_x^!(\cdot)$ is called the reduced Palm distribution of Φ given at a point x.*

Theorem 1.3.1 ***(Reduced Campbell-Little-Mecke Formula)*** *Let Φ be a point process on $\mathbb{R}^d$ with finite mean measure M and let $f : \mathbb{R}^d \times \mathbb{M} \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\boldsymbol{E}\Big[\int_{\mathbb{R}^d} f(x,\Phi\backslash\epsilon_x)\Phi(dx)\Big] = \int_{\mathbb{R}^d}\int_{\mathbb{M}} f(x,\eta)\, P_x(d\eta)\, M(dx). \tag{1.11}$$

The family $\{P_x : x \in \mathbb{R}^d\}$ is a disintegration of the Campbell measure C with respect to the mean mesure. It is plausible to interpret $\mathbb{P}^x$ as the distribution of Φ, given that x is a point in Φ. For simple point process Φ this intuitive meaning becomes clearer since we can interpret P_x as the conditional distribution $\mathbb{P}(\Phi \in \cdot | x \in \Phi)$.
For a Poisson point process Φ, the Palm distribution P_x is closely related to the distribution of Φ as showing in the following theorem.

Theorem 1.3.2 ***(Slivnyak's Theorem)*** *Let Φ be a Poisson point process on $\mathbb{R}^d$ with intensity measure Λ. For almost all $x \in \mathbb{R}^d$,*

$$P_x^!(\cdot) = \mathbb{P}_\Phi(\cdot) = \mathbb{P}(\Phi \in \cdot),$$

that is the reduced Palm distribution of the Poisson point process is equal to its original distribution.

Proof. The proof is based on a direct verification of the integral formula

$$C^!(B \times \Gamma) = \int_B \mathbb{P}(\Phi \in \Gamma) M(dx) = \Lambda(B)\mathbb{P}(\Phi \in \Gamma).$$

It is enough to prove this formula for all Γ of the form $\{\nu : \nu(A) = n\}$, for all A a Borel bounded and $n \geq 0$. For all Γ

$$C^!(B \times \Gamma) = \mathbf{E}\Big[\sum_{X_i \in B} \mathbf{1}\big(\Phi \backslash \epsilon_{X_i}(A) = n\big)\Big].$$

If $A \cap B = \varnothing$, then it yields

$$\mathbf{E}\Big[\sum_{X_i \in B} \mathbf{1}\big((\Phi \backslash \epsilon_{X_i}(A) = n\big)\Big] = \mathbf{E}[\Phi(B)\,\mathbf{1}(\Phi(A) = n)] = \Lambda(B)\mathbb{P}\{\Phi(A) = n\}.$$

Now, if $A \cap B \neq \varnothing$,

$$\begin{aligned}
&\mathbf{E}\Big[\sum_{X_i \in B} \mathbf{1}\big((\Phi \backslash \epsilon_{X_i}(A) = n)\big)\Big] \\
&= \mathbf{E}[\Phi(B \backslash A)\mathbf{1}(\Phi(A) = n)] + \mathbf{E}[\Phi(B \cap A)\mathbf{1}(\Phi(A) = n + 1)] \\
&= \Lambda(B \backslash A)\mathbb{P}\{\Phi(A) = n\} + \mathbf{E}[\Phi(B \cap A)\mathbf{1}(\Phi(A \backslash B) = n - \Phi(B \cap A) + 1)].
\end{aligned}$$

In the other hand we have,

$$\begin{aligned}
&\mathbf{E}[\Phi(B \cap A)\mathbf{1}(\Phi(A \backslash B) = n - \Phi(B \cap A) + 1)] \\
&= e^{-\Lambda(B \cap A)} \sum_{k=0}^{n+1} \Big(\frac{(\Lambda(B \cap A))^k}{k!} k e^{-\Lambda(A \backslash B)} \frac{(\Lambda(A \backslash B))^{n-k+1}}{(n_k + 1)!} \Big) \\
&= e^{-\Lambda(A)} \sum_{k=0}^{n+1} \Big(\frac{(\Lambda(B \cap A))^k}{(k-1)!} k e^{-\Lambda(A \backslash B)} \frac{(\Lambda(B \backslash A))^{n-k+1}}{(n - (k-1))!} \Big) \\
&= e^{-\Lambda(A)} \frac{(\Lambda(B \cap A))}{n!} \sum_{k=0}^{n} \Big(\frac{n!}{k!(n-k)!} (\Lambda(B \cap A))^k \Lambda(A \backslash B))^{n-k} \Big) \\
&= \Lambda(B \cap A) e^{-\Lambda(A)} \frac{(\Lambda(A))^n}{n!} \\
&= \Lambda(A \cap B)\mathbb{P}\{\Phi(A) = n\}.
\end{aligned}$$

■

Remark 1.3.1 *We remark that it is often to see $P_x(\cdot)$ and $P_x^!$ as the distributions of some point processes Φ_x and $\Phi_x^!$ called, respectively the Palm and the reduced Palm version of Φ. For Poisson point process one can take $\Phi_x^! = \Phi$ and $\Phi_x = \Phi + \epsilon_x$ for all x. Using this convention, wa can rewrite the reduced Campbell formula for Poisson point process, $\Phi = \{x_i\}_i$, as following*

$$\boldsymbol{E}\Big[\sum_{x_i \in \Phi} f(x_i, \Phi \backslash x_i)\Big] = \int_{\mathbb{R}^d} \boldsymbol{E}[f(x, \Phi)]\, M(dx).$$

We next deduce a characterization of Poisson point process by using the property expressed in Slivnyak's theorem.

Theorem 1.3.3 ***(Mecke's Theorem)*** *Let Φ be a point process with a mean measure M locally finite. Then Φ is the Poisson point process with intensity measure $\Lambda = M$ if and only if*

$$P_x^!(\cdot) = \mathbb{P}(\Phi \in \cdot).$$

Proof. Necessary: it is deduced from the result in Slivnyak's theorem.
Sufficiency: It is enough to prove that for any bounded B

$$\mathbb{P}\{\Phi(B) = n\} = \mathbb{P}\{\Phi(B) = 0\}\frac{(M(B))^n}{n!}. \tag{1.12}$$

Using the definition of the reduced Palm distribution with $\Gamma = \{\nu : \nu(B) = n\}$,

$$C^!(B \times \{\nu : \nu(B) = n\}) = E\Big[\sum_{x_i \in B} 1(\Phi(B) = n+1)\Big] = (n+1)\mathbb{P}\{\Phi(B) = n+1\}.$$

Taking account the assumption that $P_x^!(\Gamma) = \mathbb{P}\{\Phi \in \Gamma\}$, for all Γ it follows

$$C^!(B \times \Gamma) = \int_B \mathbb{P}(\Phi \in \Gamma) M(dx) = M(B)\mathbb{P}(\Phi \in \Gamma).$$

Thus

$$(n+1)\mathbb{P}\{\Phi(B) = n+1\} = M(B)\mathbb{P}\{\Phi(B) = n\},$$

from which (1.12) follows. ∎

1.3.2 Palm Distributions in the Stationary Case

The Palm distribution P_x of a point process Φ on $\mathbb{R}^d$, which we introduced in the previous section, is a probability measure on $(\mathbb{M}, \mathcal{M})$ where we can define a translation operator of vector $x \in \mathbb{R}^d$o r some others can call it a *shift* S_x given as following

$$S_x\mu(B) = \mu(B + x),$$

here $B + x := \{y + x \in \mathbb{R}^d : y \in B\}$.
Note that if $\mu = \sum_i \epsilon_{x_i}$ then $S_x\mu = \sum_i \epsilon_{(x_i - x)}$.

We now assume that the basic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is supplied with an additional structure. For each $x \in \mathbb{R}^d$, suppose $\theta_x : \Omega \to \Omega$ is a bijection such that the map $(t, w) \to \theta_x(w)$ is measurable and $\theta_s(\theta_t(w)) = \theta_{x+y}(w), w \in \Omega,\ x, y \in \mathbb{R}^d$. In particular

$\theta_0(w) = w$ and ${\theta_x}^{-1} = \theta_{-x}$.

$\theta \equiv \{\theta_x : x \in \mathbb{R}^d\}$ is called a *flow* on $(\Omega, \mathcal{F}, \mathbb{P})$ (parameterized by $\mathbb{R}^d$), and the quadruple $(\Omega, \mathcal{F}, \mathcal{P}, \theta)$is named a dynamical system.

Definition 1.3.3 *A point process Φ on $\mathbb{R}^d$ is called stationary if:*

- *the probability measure $\mathbb{P}$ on Ω is invariant under θ in the sense that $\mathbb{P}(\theta_x \in A) = \mathbb{P}(A), A \ \in \mathcal{F}, x \in \mathbb{R}^d$.*
- *Φ is adapted to the flow θ, i.e.,*

$$\Phi(\theta_x(w))(B) = S_x\Phi(w)(B) \quad for w \in \Omega, x \in \mathbb{R}^d, \ B \in B(\mathbb{R}^d).$$

Given a stationary point process Φ, its mean measure is a multiple of Lebesgue measure: $M(dx) = \lambda\, dx$. Obviously $\lambda = \mathbf{E}[\Phi(B)]$ for any set $B \in \mathbb{R}^d$ of lebesgue measure 1. One can define the *Campbell-Matthes measure* of the stationary point process Φ as the following measure on $\mathbb{R}^d \times M$:

$$C(B \times \Gamma) := \mathbf{E}\Big[\int_B \mathbf{1}(S_x(\Phi) \in \Gamma)\Phi(dx)\Big].$$

Definition 1.3.4 *For a stationary point process Φ, we call the constant λ described above the intensity parameter of Φ. For an arbitrary Borel set $A \in B(\mathbb{R}^d)$ with $|A| = 1$ and for $\Gamma \in \mathcal{M}$, let*

$$\mathbb{P}^o(\Gamma) = \frac{1}{\lambda}\boldsymbol{E}\Big[\int_{\mathbb{R}^d} \boldsymbol{1}(x \in A)\ \boldsymbol{1}(S_x(\Phi) \in \Gamma)\Phi(dx)\Big]. \tag{1.13}$$

The distribution $\mathbb{P}^o$ is called the Palm -Matthes distribution of the stationary point process Φ.

Let coming back to the notation of the shift operator $\Phi(\theta_x(w)) = S_x\Phi(w)$, one defines the Campbell-Matthes measure on $\mathbb{R}^d \times \Omega$, such that

$$C(B \times F) := \mathbf{E}\Big[\int_{\mathbb{R}^d}\int_{\Omega} \mathbf{1}(x \in B)\ \mathbf{1}(\theta_x w \in F)\Phi(dx)\Big],\ for\, all\, F \in \mathcal{F}.$$

Now, using this notation of the shift we can also state an equivalent to the Theorem 1.3.1 for a stationary point process.

Theorem 1.3.4 ***(Campbell-Little-Mecke-Matthes)***
Consider a stationary point process Φ on $\mathbb{R}^d$ with intensity $\lambda > 0$, and let $f : \mathbb{R}^d \times \Omega \to \mathbb{R}$ be a nonnegative measurable function. Then,

$$\boldsymbol{E}\Big[\int_{\mathbb{R}^d} f(x, \theta_x(w))\Phi(dx)\Big] = \lambda\int_{\mathbb{R}^d}\int_{\Omega} f(x, w)\mathbb{P}^o(dw)dx = \lambda\int_{\mathbb{R}^d} \boldsymbol{E}^o[f(x, \Phi)]dx, \tag{1.14}$$

where $\boldsymbol{E}^o$ is the expectation under the distribution $\mathbb{P}^o$.

The aim of the following lemma is to clarify the relationships between P^o and P_x which are two probability measures on $\mathbb{M}$.

Lemma 1.3.1 *For almost all $x \in \mathbb{R}^d$ and for all $\Gamma \in \mathcal{M}$,*

$$P_x \circ S_x^{-1}(\cdot) = \mathbb{P}^o_\Phi(\cdot).$$

Proof. let B a Borel bounded and $\Gamma \in \mathcal{M}$,

$$\begin{aligned} \lambda \int_B (P_x \circ S_x^{-1}(\Gamma) - \mathbb{P}^o(\Gamma))dx &= \lambda \int_{\mathbb{R}^d} \mathbf{1}(x \in B) \int_{\mathbb{M}} \mathbf{1}(S_x(\mu \in \Gamma) P_x(d\mu) \\ &\quad - \lambda \int_{\mathbb{R}^d} \mathbf{1}(x \in B) \int_\Omega \mathbf{1}(\Phi(w) \in \Gamma) \mathbb{P}^o(dw). \end{aligned}$$

Using the Campbell-Little-Mecke-Matthes Theorem, it follows

$$\begin{aligned} \lambda \int_B (P_x \circ S_x^{-1}(\Gamma) - \mathbb{P}^o(\Gamma))dx &= \mathbf{E}\Big[\int_{\mathbb{R}^d} \mathbf{1}(S_x\Phi \in \Gamma)\Phi(dx) \Big] - \mathbf{E}\Big[\int_{\mathbb{R}^d} \mathbf{1}(\Phi(\theta_x(w))\phi(dx) \Big] \\ &= 0. \end{aligned}$$

Whence the desired result. ■

Remark 1.3.2 *The distribution $\mathbb{P}^o$ is often interpreted as that of the point process "seen from a typical point" or seen from a "randomly chosen point" of Φ. It should not be surprising that in the case of a stationary point process we actually define one conditional distribution given a point at the origin o. This is motivated by the stationarity of the original distribution. Furthermore, the distribution P^x is then obtained as the image $t_x\mathbb{P}^o$ of $\mathbb{P}^o$ under the translation by the vector x. This means that the conditional distribution of points of Φ seen from the origin given Φ has a point there is exactly the same as the conditional distribution of points of Φ seen from an arbitrary location x given Φ has a point at x. Henceforth, note by Slivnyak Theorem 1.3.2 that for a stationary Poisson point process Φ, $\mathbb{P}^o$ corresponds to the law of $\Phi + \epsilon_o$ under the original distribution.*

Chapter 2

Model Description

2.1 Necessity of the Modelling

Optimisation of the network's architecture is one of the principal ways for telecommunication operators to increase the effectiveness of their system. The strategy is based on an economical analysis of statistical data and aims to find an architecture that best meets the future service demand. The current planning methods require the complete knowledge of the network and the traffic together with prediction for the future service demand. Therefore the corresponding optimisation programs use a very large number of parameters whose values are not all known exactly, and due to their specifity and complexity can not be applied to other networks. Moreover, the complete description does not give a transparent functional relation between the performance characteristics and the description of the network's topology. Therefore the development of reliable models possessing a few number of structuring parameters and taking into account spatial characteristics of the network opens new possibilities.

2.2 System Model

We assume that the base stations (BS) are located at the points of a stationary Poisson point process $\Phi := \{X_n, n \in \mathbb{N}\}$ of intensity λ BS per km^2 on the plane $\mathbb{R}^2$. The path-loss over a wireless link is modeled by distance component and a shadowing component. It is usually assumed that the distance part is deterministic while the shadowing part is random.

For a given BS $X \in \Phi$ and a given a location $y \in \mathbb{R}^2$ on the plane we denote by $L_X(y)$

and $P_X(y)$ respectively the propagation-loss and gain between BS X and the location y. In what following we will always assume that

$$\begin{aligned} P_X(y) &= \frac{S_X(y)}{l(|X-y|)} \qquad (2.1) \\ L_X(y) &= \frac{l(|X-y|)}{S_X(y)}. \qquad (2.2) \end{aligned}$$

where $l(.)$ is a non-decreasing, deterministic function of the distance between an emitter and a receiver, and $S_X(.)$ is a random shadowing field related to the BS X. In what follows we will always assume that the locations of BS $\{X_n, n \in N\} \equiv \Phi$ and their respective shadowing fields $S_X(.)$ form an independently marked version $\widetilde{\Phi} = (X, S_X(.))_{X\in\Phi}$ of the point process Φ. Regarding the distribution of the marks (shadowing fields) of this process, they are assumed to have the same distribution for all $y \in \mathbb{R}^2$.

For the deterministic path-loss function $l(.)$ the following particular model is often used and will be our default hypothesis in this thesis:

[H1] $\begin{cases} l(r) = (Kr)^\beta, \ \textit{where } K > 0 \textit{ and } \beta > 2 \textit{ are constants}, \\ \beta \textit{ is called the path-loss exponent (PLE).} \end{cases}$

[H2] $\begin{cases} \textit{For all } y, S_X(y) \textit{ is a log-normal random variablewhich can be expressed as } e^{m+\sigma Z}, \\ \textit{where } Z \textit{ is a standard Gaussian random varaible (with mean 0 and variance 1).} \end{cases}$

Note that if the shadowing is log-normal then the path-loss (at a given location) expressed in dB is a Gaussian random variable.

2.3 Path-gain Factor

In what follows we will assume that each given location $y \in \mathbb{R}^2$ is served by the BS $X_y^* \in \Phi$ with respect to which it has the highest propagation gain $P_{X_y^*}(y)$, so in other words, the strongest received signal, given all BS emit with the same power, i.e., such that

$$P_{X_y^*}(y) = \max_{n\in\mathbb{N}} \frac{S_{X_n}(y)}{l(|X_n - y|)}, \qquad (2.3)$$

Consequently we have,

$$P_{X_y^*}(y) \geq P_X(y), \ \forall X \ \in \Phi.$$

We notice that $P_{X_y^*}(y)$ is the propagation gain experienced by a user located at y with respect to its serving BS. Obviously it effects the quality of service of this user. In this context we will call *path-gain factor* of a user y and denote by $P_{X_y^*}(y)$. It depends on the location y but also on the path-gain conditions of this location with respect to all BS in the network.

2.4 Interference Modeling

In wireless networks, interference is one of the central elements in system design, since network performance is often limited by competition of users for common resources. For a given location $y \in \mathbb{R}^2$ the *interference factor* $f(y)$ is defined as

$$f(y) = f(y, \widetilde{\Phi}) = \sum_{X \in \Phi,\ X \neq X_y^*} \frac{P_X(y)}{P_{X_y^*}(y)}, \tag{2.4}$$

provided X_y^* is well defined. Note that , $f(y) = \widetilde{f}(y) - 1$ where $\widetilde{f}(y) = \sum_{X \in \Phi} \frac{P_X(y)}{P_{X_y^*}(y)}$. Without loss of generality, since the network is homogenous, the interference measure at the origin $f(o)$ is representative of the interference seen by all the other receiver nodes in the network. It is given by,

$$f \equiv f(o) = \sum_{X \in \Phi,\ X \neq X_o^*} \frac{P_X(o)}{P_{X_o^*}(o)} = \widetilde{f}(o) - 1 = \frac{1}{P_{X^*}} \sum_{X_n \in \Phi} \frac{S_{X_n}}{l(|X_n|)} - 1,$$

where $P_{X^*} = P_{X_o^*}(o)$.

The interference power seen by the receiver at the origin can be viewed as a shot noise process described as

$$I \equiv I(o) := \sum_{n \in \mathbb{N}} \frac{S_{X_n}}{l(|X_n|)}. \tag{2.5}$$

If we define

$$L \equiv L(o) := \min_{n \in \mathbb{N}} L_{X_n}(o), \tag{2.6}$$

then we can express the interference factor in terms of L and the shot noise I as following

$$f = I \times L - 1. \tag{2.7}$$

Study of the path-loss and interference factors, which are relatively simple objects, can give an important insight into more involved quality of service indicators. That is the goal of the next chapter.

Chapter 3

Mathematical Results

3.1 Analysis of the Path-gain Factor

The main goal of this section is to provide an explicit expression of the distributions of the path-gain factor P_{X^*} for an infinite Poisson point process modeling the locations of the BS.

Theorem 3.1.1 *Consider an infinite Poisson process Φ model of the BS, with arbitrary shadowing whose marginal distribution has finite moment of order $\frac{2}{\beta}$ and any deterministic path-loss function $0 < l(r) < \infty$. Then, the distribution of P_{X^*} has the following form*

$$\mathbb{P}\Big(P_{X^*} \leq r\Big) = \exp\left(- \lambda \int_{\mathbb{R}^2} \Big(1 - F_{S_X}\Big(rl(|X|)\Big)\Big) \, dX \right). \tag{3.1}$$

Proof. We will show two methods to prove this theorem, the first one is based on the Laplace functional of the Poisson point process (Theorem 1.2.1), whereas the second one on the Displacement theorem (Theorem 1.2.2). Let us now focus on the first point by calculating immediately the probability distribution function of P_{X^*}

$$\begin{aligned}
\mathbb{P}\Big(P_{X^*} \leq r\Big) &= \mathbb{P}\Big(\max_n \frac{S_{X_n}}{l(|X_n|)} \leq r\Big) \\
&= \mathbb{P}\Big(\forall n, \ \frac{S_n}{l(|X_n|)} \leq r\Big) \\
&= \mathbb{P}\Big(\forall n, \ S_n \leq rl(|X_n|)\Big).
\end{aligned}$$

Since the marks S_n are independently distributed, it yields

$$\begin{aligned}
\mathbb{P}\Big(P_{X^*} \leq r\Big) &= \mathbf{E}\Big[\mathbf{E}\Big[\prod_{X_n \in \ \Phi} \mathbf{1}\Big(S_n \leq rl(|X_n|)\Big) \mid \Phi\Big]\Big] \\
&= \mathbf{E}\Big[\prod_{X_n \in \ \Phi} \mathbf{E}\Big[\mathbf{1}\Big(S_n \leq rl(|X_n|)\Big) \mid \Phi\Big]\Big] \\
&= \mathbf{E}\Big[\prod_{X_n \in \ \Phi} F_S\Big(rl(|X_n|)\Big)\Big] \\
&= \mathbf{E}\Big[\exp \sum_{X_n \in \ \Phi} logF_S\Big(rl(|X_n|)\Big)\Big] \\
&= \mathcal{L}_\Phi\Bigg(- logF_S\Big(rl(|X|)\Big)\Bigg)
\end{aligned}$$

Using Theorem 1.2.1 we obtain the desired result (3.1). Now we will develop the second method. Let us consider the Poisson point process $\widetilde{\Phi}$ on $\mathbb{R}^2 \times \mathbb{R}_+$, with the intensity measure $\widetilde{\Lambda}(.)$. Namely we have,

$$\begin{cases}
\widetilde{\Phi} := \sum_{n \in \mathbb{N}} \epsilon_{(X_n, S_n)} \\
\widetilde{\Lambda}(d(x,s)) = \lambda F_S(ds)\, dx.
\end{cases}$$

Here, the couple (x, s) is a generic element. Therefore,

$$\begin{aligned}
\mathbb{P}\Big(P_X^* \leq r\Big) &= \mathbb{P}\Big(\{\forall n,\ S_{X_n} \leq r\ l(|X_n|)\}\Big) \\
&= \mathbb{P}\Bigg(\widetilde{\Phi}\Big(\{(X,S)\ :\ \frac{S}{l(|X|)} > r\}\Big) = 0\Bigg)
\end{aligned}$$

Let us consider the following set

$$C := \{(X,S)\ :\ \frac{S}{l(|X|)} > r\}.$$

Henceforth the cumulate distribution function of P_X^* verifies,

$$\begin{aligned}
\mathbb{P}\Big(P_X^* \leq r\Big) &= \mathbb{P}\Big(\widetilde{\Phi}(C) = 0\Big) \\
&= \exp(-\widetilde{\Lambda}(C)) \\
&= \exp\Big(- \lambda \int_{\mathbb{R}^3} \mathbf{1}\Big((x,s) \in C\Big) F_S(ds)\ dx\Big) \\
&= \exp\Big(- \lambda \int_{\mathbb{R}^2} \Big[\int_{\mathbb{R}_+} \mathbf{1}\Big(\frac{s}{l(|x|)} > r\Big) F_S(ds)\Big]\ dx\Big) \\
&= \exp\Big(- \lambda \int_{\mathbb{R}^2} \Big(1 - F_S\Big(r\ l(|x|)\Big)\Big) dx\Big),
\end{aligned}$$

which establishes (3.1). ■

Corollary 3.1.2 *Assume the hypothesis **[H1]** of the previous chapter, we have remarked that the distribution function of P_{X^*} depends only on the moment $\boldsymbol{E}[S^{\frac{2}{\beta}}]$ of the shadowing as such*

$$\mathbb{P}\Big(P_{X^*} \leq r\Big) = \exp\Big(- \frac{\pi\,\lambda}{K^2 r^{\frac{2}{\beta}}} \boldsymbol{E}[S^{\frac{2}{\beta}}]\Big). \tag{3.2}$$

Proof. From Theorem 3.1.1, we have $\mathbb{P}\Big(P_{X^*} \leq r\Big) = \exp\Big(-\lambda \int_{\mathbb{R}^2} \Big(1 - F_{S_x}\Big(rl(|x|)\Big)\Big)\, dx\Big)$. Taking $l(r) = (Kr)^{\beta}$, it follows

$$\begin{aligned}
\mathbb{P}\Big(P_{X^*} \leq r\Big) &= \exp\Big(- \lambda \int_{\mathbb{R}^2} \mathbb{P}\big(S > r(K|x|)^{\beta}\big)\, dx\Big) \\
&= \exp\Big(- 2\pi\,\lambda \int_0^{\infty} \rho \mathbb{P}\big(S > r(K\rho)^{\beta}\big) d\rho\Big) \\
&\overset{t := r(K\rho)^{\beta}}{=} \exp\Big(- \frac{\pi\,\lambda}{K^2 r^{\frac{2}{\beta}}} \int_0^{\infty} t^{\frac{2}{\beta}-1} \mathbb{P}(S > t) dt\Big) \\
&= \exp\Big(- \frac{\pi\,\lambda}{K^2 r^{\frac{2}{\beta}}} \mathbf{E}[S^{\frac{2}{\beta}}]\Big).
\end{aligned}$$

∎

This is the *Fréchet distribution* with shape $\frac{2}{\beta}$ and scale parameter $\Big(\frac{2\lambda\pi}{K^2\beta}\mathbf{E}[S^{\frac{2}{\beta}}]\Big)^{\frac{\beta}{2}}$.

Example 3.1.1 *Assume an infinite Poisson model Φ of BS locations satisfying the hypothesis **[H1]** and **[H2]** (log-normal shadowing). Using Corollary 3.1.2 we can get an explicit expression of the cumulate distribution function of the path-gain,*

$$\mathbb{P}\Big(P_{X^*} \leq r\Big) = \exp\Big(- \frac{\lambda\pi}{K^2 r^{\frac{2}{\beta}}}\; e^{\frac{2\sigma^2}{\beta^2} + \frac{2m}{\beta}}\Big). \tag{3.3}$$

3.2 Analysis of the Interference Factor

We will prove a general result. Let us consider point process $\Psi := \{\frac{l(|X_n|)}{S_{X_n}}, X_n \in \Phi\}$ on $\mathbb{R}^+$. As we see Ψ IS constructed from the Poisson point process Φ. The following lemma shows .

Lemma 3.2.1 *Assume **[H1]**, Ψ is a non-homogeneous Poisson point process on $\mathbb{R}^+$ with intensity measure given by*

$$\Lambda_{\Psi}([0,t]) = \frac{\pi\lambda t^{\frac{2}{\beta}}}{K^2} \boldsymbol{E}[S^{\frac{2}{\beta}}]. \tag{3.4}$$

Proof. By the displacement theorem Ψ is a Poisson point process on $\mathbb{R}^+$ of intensity measure given by

$$\begin{aligned}
\Lambda_\Psi([0,t]) &= \mathbf{E}[\Psi([0,t])] \\
&= \lambda \int_{\mathbb{R}^2} \mathbb{P}\Big((\frac{l(|x|)}{S}) < t \Big)\, dx \\
&= 2\pi\lambda \int_0^\infty \mathbb{P}(\frac{l(r)}{S} < t) r dr \\
&= \frac{\pi\lambda t^{\frac{2}{\beta}}}{K^2} \mathbf{E}[S^{\frac{2}{\beta}}].
\end{aligned}$$

Whence the desired result. ■

Remark 3.2.1 *Lemma 3.2.1 says that the distribution of any functional of Ψ does not depend on the distribution of the shadowing S but only on its moment $\boldsymbol{E}[S^{\frac{2}{\beta}}]$. Observe that the path-gain factor P_X^* and the interference factor $\widetilde{f}$ are examples of such functionals. Consequently, Lemma 3.2.1 confirms Corollary 3.1.2 and extends it to the interference factor f. This latter result will be useful on the proof of Proposition 3.3.1.*

Lemma 3.2.2 *Taking into account the previous hypothesis **[H1]**, the distribution of the interference factor f does not depend on the intensity λ of the Poisson point process Φ.*

Proof. Let us construct a new point process $\Phi' = \{Y_n, n \in \mathbb{N}\}$ by taking $X_n = \frac{Y_n}{\sqrt{\lambda}}$. Lemma 1.2.1 shows that Φ' is Poisson point process of intensity 1. Using the new expression of P_{X^*} is now

$$P_{X^*} = \max_n \frac{S_{X_n}}{l(|\frac{Y_n}{\sqrt{\lambda}}|)} = \lambda^{\frac{\beta}{2}} \max_n \frac{S_{X_n}}{l(|Y_n|)}.$$

The expression of the interference factor is given by

$$\begin{aligned}
f(o) &= \widetilde{f}(o) - 1 \\
&= \sum_{X\in\Phi} \frac{P_X}{P_{X^*}} - 1 \\
&= \sum_{Y\in\Phi'} \frac{P_{\frac{Y}{\sqrt{\lambda}}}}{P_{\frac{Y}{\sqrt{\lambda}}^*}} - 1 \\
&= \Big(\lambda^{\frac{\beta}{2}} \max_n \frac{S_{X_n}}{l(|Y_n|)}\Big)^{-1} \sum_{Y_n\in\Phi'} \lambda^{\frac{\beta}{2}} \frac{S_{X_n}}{l(|Y_n|)} - 1 \\
&= \Big(\max_n \frac{S_{X_n}}{l(|Y_n|)}\Big)^{-1} \sum_{Y_n\in\Phi'} \frac{S_{X_n}}{l(|Y_n|)} - 1.
\end{aligned}$$

Whence the desired result. ■

Now we derive a general expression for the mean interference in networks whose nodes are distributed as a stationary point process $\Phi = \{X_1, X_2, ...\} \subset \mathbb{R}^2$ of intensity λ.

Proposition 3.2.1 *Assume Poisson network with hypothesis **[H1]**. The interference factor $f(o)$ does not depend on the marginal distribution of shadowing field $S_X(\cdot)$ even not on its moments , provided $\boldsymbol{E}[S^{\frac{2}{\beta}}] < \infty$. Moreover, we have $\boldsymbol{E}[f(o)] = \frac{2}{\beta-2}$.*

Proof. By Lemma 3.2.2 the distribution of the interference factor f does not depend on the intensity λ of the Poisson point process Φ. Using this observation and taking into account that $\mathbf{E}[S^{\frac{2}{\beta}}] < \infty$, we can show that the distribution of interference factor does not depend, not only on the distribution of the shadowing, but also on the moment $\mathbf{E}[S^{\frac{2}{\beta}}]$ of the shadowing. To see this note that the intensity measure of the process Ψ is $\Lambda_\Psi([0,t]) = \frac{\pi\lambda t^{\frac{2}{\beta}}}{K^2}\mathbf{E}[S^{\frac{2}{\beta}}]$. We can now consider a new intensity $\lambda' = \frac{\lambda}{\mathbf{E}[S^{\frac{2}{\beta}}]}$ of Φ. Henceforth, the new intensity measure $\Lambda'_\Psi([0,t]) = \frac{\pi\lambda}{K^2}t^{\frac{2}{\beta}}$ of the interference factor f does not depends on the shadowing S.

The above observations show that in the infinite Poisson network the existence of the shadowing has no impact on the interference factor, thus that we can assume that $S \equiv 1$. Hence,

$$\begin{aligned}
\mathbf{E}[f(o)] &= \mathbf{E}[\widetilde{f}(o)] - 1 \\
&= E\Big[\frac{1}{P_{X^*}}\sum_{X_i\in\Phi}\frac{1}{l(|X_i|)}\Big] - 1 \\
&= \mathbf{E}\Big[\int_{\mathbb{R}^2}\frac{1}{l(|X|)}\frac{1}{P_{X^*}}\Phi(dX)\Big] - 1 \\
&= \mathbf{E}\Big[\int_{\mathbb{R}^2}\frac{\theta_X\circ\theta_{-X}(w)}{l(|X|)}\frac{1}{P_{X^*}(\theta_X\circ\theta_{-X}(w))}\Phi(dX)\Big] - 1 \\
&= \mathbf{E}\Big[\int_{\mathbb{R}^2}\frac{\theta_X\circ\theta_{-X}(w)}{l(|X|)}\frac{1}{l(\theta_X\circ\theta_{-X}(w))}\Phi(dX)\Big] - 1.
\end{aligned}$$

By using Slivnyak's theorem, we obtain

$$\mathbf{E}[\widetilde{f}(o)] = \int_{\mathbb{R}^2}\mathbf{E}^\circ\Big[\frac{\theta_X(w)}{l(|X|)}\frac{1}{l(\theta_{-X}(w))}\Big]\,dX - 1 = \int_{\mathbb{R}^2}\mathbf{E}^\circ\Big[\frac{1}{l(|X|)}\frac{1}{l(\theta_{-X}(w))}\Big]\,dX - 1.$$

Consequently, it follows

$$\begin{aligned}
\mathbf{E}[\widetilde{f}(o)] &= \int_{\mathbb{R}^2}\mathbf{E}\Big[\frac{1}{l(|X|)}\inf_{X_j\in\ \Phi+\delta_0} l(|X_j+X|)\Big]dX - 1 \\
&= \int_{\mathbb{R}^2}E\Big[\frac{1}{l(|X|)}\min\Big(l(|Y|),\inf_{X_j\in\ \Phi} l(|X_j+X|)\Big)\Big]dX - 1 \\
&= \int_{\mathbb{R}^2}E\Big[\min\Big(1, l(|Y|)\inf_{X_j\in\ \Phi} l(|X_j+X|)\Big)\Big]dX - 1 \\
&= \int_{\mathbb{R}^2}\int_0^1\mathbb{P}\Big(l(|Y|)\inf_{X_j\in\ \Phi} l(|X_j+X|) > t\Big)dt\ dX - 1 \\
&= \int_{\mathbb{R}^2}\int_0^1\mathbb{P}\Big(\inf_{X_j\in\ \Phi} l(|X_j+X|) > tl(|X|)\Big)dt\ dX - 1
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2}\int_0^1 \exp\Big(-\int_{\mathbb{R}^2}\Big[1-\mathbf{1}\Big(\frac{l(|Y|)}{t\,l(|X|)}\Big)\Big]dY\Big)dt\;dX-1\\
&= \int_{\mathbb{R}^2}\int_0^1 \exp\Big(-\pi\Big(l^{-1}(t\,l(|Y|))\Big)^2\Big)\;dt\;dX-1\\
&= 2\pi\int_0^{+\infty} u\;du\int_0^1 dt\exp\Big(-(l^{-1}(t\,l(u)))^2\Big)-1\\
&= 2\pi\int_0^{+\infty} u\;du\int_0^1 dt\exp(-\pi t^{\frac{2}{\beta}}u^2)-1\\
&= \pi\int_0^{+\infty} dv\int_0^1 dt\exp(-\pi t^{\frac{2}{\beta}}v^{\frac{2}{\beta}}))-1\\
&= \pi\int_0^1 dt\frac{1}{\pi t^{\frac{2}{\beta}}}-1\\
&= \int_0^1 dt\frac{1}{t^{\frac{2}{\beta}}}-1\\
&= \frac{\beta}{\beta-2}-1\\
&= \frac{2}{\beta-2}.
\end{aligned}
$$

■

Now we focus on the characterization of the distribution function of the interference factor f by calculating its Laplace function.

Lemma 3.2.3 *The Laplace transform of the shot noise* $I=\sum_{n\in N}\frac{S_n}{l(|X_n|)}$ *is given by*

$$\mathcal{L}_I = exp\Big(-2\pi\lambda\int_0^{\infty}\Big(1-\mathcal{L}_S(\frac{t}{l(r)}\Big)rdr\Big),$$

where $\mathcal{L}_S(t)=\boldsymbol{E}[e^{-tS}]$.

Proof. Using Campbell theorem, it follows that

$$
\begin{aligned}
\mathcal{L}_I(t) &= \mathbf{E}[-tI] = \exp\Big(-\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\Big(1-e^{-t\frac{s}{l(|x|)}}\Big)F_S(d\,s)dx\Big)\\
&= \exp\Big(-\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\Big(1-e^{-t\frac{s}{l(|x|)}}\Big)F_S(d\rho)\,dx\Big)\\
&= \exp\Big(-2\pi\lambda\int_{\mathbb{R}^2}\int_{\mathbb{R}^+}\Big(1-e^{-t\frac{s}{l(|x|)}}\Big)F_S(d\rho)\,r\,dr\Big)\\
&= \exp\Big(-2\pi\lambda\int_{\mathbb{R}^+}\Big(1-\mathcal{L}_S(\frac{t}{l(r)})\Big)r\,dr\Big).
\end{aligned}
$$

which is the desired expression. ■

Corollary 3.2.1 *Assume the hypothesis **[H1]** then the Laplace functional* $\mathcal{L}_I(t)$ *of the shot noise* I *verifies,*

$$\mathcal{L}_I(t) = \exp\Big(-\frac{2\pi\lambda}{\beta K^2}\Gamma(-\frac{2}{\beta})\boldsymbol{E}[S^{\frac{2}{\beta}}]\,t^{\frac{2}{\beta}}\Big). \tag{3.5}$$

Proof. The proof is based on the exponential formula of the Poisson point process Ψ defined in the beginning of this section. $I = \sum_{n\in N} \frac{1}{L_{X_n}} = \int \varphi(x)\Psi(dx)$ where $\varphi(x) = \frac{1}{x}$. By the exponential formula, it yields

$$\begin{aligned}
\mathbf{E}[e^{-tI}] &= \exp\Big[\int_{\mathbb{R}+} \Big(e^{t\varphi(s)} - 1\Big)\Lambda_\Psi(ds)\Big] \\
&= \exp\Big[\int_{\mathbb{R}+} \Big(e^{-\frac{t}{s}} - 1\Big)\frac{\lambda\pi\mathbf{E}[S^{\frac{2}{\beta}}]}{K^2}\frac{2}{\beta}s^{\frac{2}{\beta}-1}ds\Big] \\
&= \exp\Big[\frac{2\pi\lambda}{K^2\beta}\mathbf{E}[S^{\frac{2}{\beta}}]\int_{\mathbb{R}+}(e^{-\frac{t}{s}} - 1)\, ds\Big] \\
&= \exp\Big(-\frac{2\pi\lambda}{\beta K^2}\Gamma(-\frac{2}{\beta})\mathbf{E}[S^{\frac{2}{\beta}}]\, t^{\frac{2}{\beta}}\Big).
\end{aligned}$$

■

The next results are more characterized the distribution of the interference factor and showed the dependence of this latter with the path-gain. They are inspired by [Ka11] and [Ng11].

Proposition 3.2.2 ***(Karray 2011)*** *Assume **[H1]**, the Laplace functional of the interference factor f is given by*

$$\boldsymbol{E}[e^{-zf}] = \frac{1}{e^{-z} + z^{\frac{2}{\beta}}\Big(\Gamma(1-\frac{2}{\beta}) - \Gamma(1-\frac{2}{\beta}, z)\Big)}. \tag{3.6}$$

where $\Gamma(a) = \int_0^\infty t^{a-1}e^{-a}dt$ is the gamma function and $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-a}dt$ is the upper incomplete gamma function.

Proof. By Proposition 3.2.1, we can take $S \equiv 1$. We have $f = I \times L - 1$ where L was given by (2.6) thus the Laplace transform of f is

$$\mathbf{E}[e^{-z\,f}] \;=\; \int_0^\infty \mathbf{E}[e^{-z\,f}\big|L = s]\,\mathbb{P}_L(ds).$$

where $\mathbb{P}_L$ is the distribution of L. Moreover, we can easily calculate this probability as following.

$$\begin{aligned}
&\mathbb{P}(L \geq t) = \mathbb{P}(L_{X_n}(0) \geq t, \forall n \in N) = \mathbb{P}(\Psi([0,t]) = 0) \\
&= \exp\Big(-\Lambda_\Psi([0,t])\Big) = \exp\Big(-\frac{\pi\lambda t^{\frac{2}{\beta}}}{K^2}\Big) = \exp\Big(-a\, t^{\frac{2}{\beta}}\Big),
\end{aligned}$$

where $a := \frac{\pi\lambda}{K^2}$.

Returning to the calculus of the Laplace transform of f it follows,
$\mathbf{E}[e^{-zf}] = \int_0^\infty \underbrace{\mathbf{E}[e^{-z(Is-1)}\big|L = s]}_{:=\mathbf{E}[e^{-zsI'}]}\mathbb{P}_L(ds)$, where I' is an additive shot noise $I' = \sum_{n\in N}\frac{1}{\widetilde{L}_{X_n}}$,

and $\{\widetilde{L}_{X_n}, n \in \mathbb{N}\}$ is a restriction of the Poisson point process Ψ on the interval $]t, \infty)$, having the same intensity measure of Ψ on $]t, \infty)$. On the one hand, by Campbell and Laplace functional theorems, we have also

$$\begin{aligned}\mathbf{E}[e^{-zsI'}] &= exp\Big[\int_s^{\infty}(e^{-\frac{zs}{t}}-1)a\frac{2}{\beta}t^{\frac{2}{\beta}-1}dt\Big] \\ &= exp\Big[-a\frac{2}{\beta}\int_s^{\infty}(1-e^{-\frac{zs}{t}})\,a\,t^{\frac{2}{\beta}-1}\,dt\Big]\end{aligned}$$

Now we aim to calculate the following integral $\int_s^{\infty}(1-e^{-\frac{zs}{t}})\,a\,t^{\frac{2}{\beta}-1}dt$ by making a change of variables, namely, $u := \frac{zs}{t}$. Therefore, it yields

$$\begin{aligned}\int_s^{\infty}(1-e^{-\frac{zs}{t}})at^{\frac{2}{\beta}-1}dt &= \int_0^{z}(1-e^{-u})(\frac{zs}{u})^{\frac{2}{\beta}-1}\frac{zs}{u^2}du \\ &= (zs)^{\frac{2}{\beta}}\Bigg\{\Big[(1-e^{-u})\frac{u^{-\frac{2}{\beta}}}{-\frac{2}{\beta}}\Big]_0^z-\int_0^z e^{-u}\,\frac{u^{-\frac{2}{\beta}}}{-\frac{2}{\beta}}du\Bigg\} \\ &= s^{\frac{2}{\beta}}\frac{\beta}{2}\Big(-1+e^{-z}+z^{\frac{2}{\beta}}\Big(\Gamma(1-\frac{2}{\beta})-\Gamma(1-\frac{2}{\beta},z)\Big)\Big)\end{aligned}$$

Let us consider the following function which we denote

$$h(z) = e^{-z}+z^{\frac{2}{\beta}}\Big(\Gamma(1-\frac{2}{\beta})-\Gamma(1-\frac{2}{\beta},z)\Big),$$

then we obtain

$$\begin{aligned}\mathbf{E}[e^{-zf}] &= \int_0^{\infty}exp\Big(-a\,(h(z)-1)\,s^{\frac{2}{\beta}}\Big)\frac{2}{\beta}\,a\,s^{\frac{2}{\beta}-1}\,e^{-as^{\frac{2}{\beta}}}\,ds \\ &= \frac{2}{\beta}\,a\int_0^{\infty}s^{\frac{2}{\beta}-1}e^{-ah(z)s^{\frac{2}{\beta}}}ds \\ &= \frac{1}{h(z)}\int_0^{\infty}a\,h(z)\,\frac{2}{\beta}\,s^{\frac{2}{\beta}-1}\,e^{-a\,h(z)\,s^{\frac{2}{\beta}}}ds \\ &= \frac{1}{h(z)}\Big[-e^{-a\,h(z)\,s^{\frac{2}{\beta}}}\Big]_0^{\infty} = \frac{1}{h(z)}.\end{aligned}$$

Finally, we get our desired result $\mathbf{E}[e^{-zf}] = \frac{1}{e^{-z}+z^{\frac{2}{\beta}}\left(\Gamma\left(1-\frac{2}{\beta}\right)-\Gamma\left(1-\frac{2}{\beta},z\right)\right)}$. ■

Remark 3.2.2 *We can verify that the mean of the interference factor f ,which was calculated in Proposition 3.2.1, can be deduced from its Laplace transform. Indeed, $h'(z) = \frac{2}{\beta}\int_0^1 e^{-zv}v^{-\frac{2}{\beta}}dv$, for $z = 0$, $h'(0) = \frac{2}{\beta-2}$. It follows that $\boldsymbol{E}[f] = \frac{2}{\beta-2}$*

3.3 Joint Distribution Path-gain Interference Factors

Proposition 3.3.1 ***(Karray 2011)*** *Assume **[H1]**, the joint distribution of the path-gain interference factors is given by*

$$\boldsymbol{E}\Big[\boldsymbol{1}\{P_{X^*}\le u\}\,e^{-zI}\Big] = \exp\Big(-\frac{2\pi\lambda}{\beta\,K^2}\boldsymbol{E}[S^{\frac{2}{\beta}}]\,z^{\frac{2}{\beta}}\Big[\Gamma(-\frac{2}{\beta})+\Gamma(-\frac{2}{\beta},u\,z)\Big]\Big). \qquad (3.7)$$

Proof.

$$\begin{aligned}
\mathbf{E}\Big[\mathbf{1}\{P \le u\}\, e^{-zI}\Big] &= \mathbf{E}\Big[\prod_n 1\{P_{X_n} \le u\}\, e^{-zI}\Big] \\
&= \mathbf{E}\Big[exp \sum_n \big\{log\big(\mathbf{1}\{P_{X_n} \le u\}\big) - zP_{X_n}\big\}\Big] \\
&= \exp\Big\{ - \lambda \int_{\mathbb{R}^2} (1 - E\big[\mathbf{1}\{\frac{S}{l(|x|)} \le u\}\, e^{-z\frac{S}{l(|x|)}}\big])\Big\} \\
&= \mathcal{L}_I(z)\, \exp\Big\{ - \lambda \int_{\mathbb{R}^2} \mathbf{E}\Big[\mathbf{1}\{\frac{S}{l(|x|)} > u\}\, e^{-z\frac{S}{l(|x|)}}\, dx\Big]\Big\} \\
&= \mathcal{L}_I(z)\, \exp\Big\{ - 2\pi\lambda \int_{\mathbb{R}^+} \mathbf{E}\Big[\mathbf{1}\{\frac{S}{(Kr)^\beta} > u\}\, e^{-z\frac{S}{l(r)}}\Big])rdr\Big\} \\
&= \mathcal{L}_I(z)\, \exp\Big\{ - 2\pi\lambda \mathbf{E}\Big[\int_0^{\frac{1}{K}(\frac{S}{u})^{\frac{1}{\beta}}} e^{-z\frac{S}{(Kr)^\beta}}\, r\, dr\Big]\Big\} \\
&\overset{v:=\frac{z\,S}{(K\,r)^\beta}}{=} \mathcal{L}_I(z)\, \exp\Big\{ - 2\pi\lambda \mathbf{E}[S^{\frac{2}{\beta}}] \int_{u\,z}^\infty \frac{e^{-v}}{v^{\frac{2}{\beta}+1}}\, \frac{z^{\frac{2}{\beta}}}{\beta K^2}\, dv\Big\} \\
&= \mathcal{L}_I(z)\, exp\Big\{ - \frac{2\pi\lambda}{\beta\, K^2} \mathbf{E}[S^{\frac{2}{\beta}}]\, z^{\frac{2}{\beta}} \int_{uz}^\infty \frac{e^{-v}}{v^{\frac{2}{\beta}+1}}\, dv\Big\}
\end{aligned}$$

Using corollary 3.3.1 we obtain

$$\begin{aligned}
\mathbf{E}\Big[\mathbf{1}\{P_{X^*} \le u\}\, e^{-zI}\Big] &= exp\Big\{ - \frac{2\pi\lambda}{\beta\, K^2} \mathbf{E}[S^{\frac{2}{\beta}}]\, z^{\frac{2}{\beta}} \Big(\Gamma(-\frac{2}{\beta}) + \int_{u\,z}^\infty \frac{e^{-v}}{v^{\frac{2}{\beta}+1}}\, dv\Big) \\
&= \exp\Big\{ - \frac{2\pi\lambda}{\beta\, K^2} E[S^{\frac{2}{\beta}}]\, z^{\frac{2}{\beta}} \Big(\Gamma(-\frac{2}{\beta}) + \Gamma(-\frac{2}{\beta}, uz)\Big)\Big\}.
\end{aligned}$$

■

Corollary 3.3.1 *The path-loss and the interference factors are not independent random variables.*

Proof. Let us calculate

$$\mathbf{E}\Big[\mathbf{1}\big\{L \ge u\big\}\, e^{-zf}\Big] = \int_u^\infty \mathbf{E}[e^{-zf}\big|L = s]\, \mathbb{P}_L(ds) = \frac{1}{h(z)}\Big(1 - e^{-a\,u^{\frac{2}{\beta}}\, h(z)}\Big).$$

We observe that, $\mathbf{E}\Big[\mathbf{1}\big\{L \ge u\big\}\, e^{-zf}\Big] \ne P(L \ge u)\, \mathbf{E}[e^{-zf}]$. Whence the path-loss and the interferecne factor are not independent random variables. ■

3.4 Path-loss Exponent Estimation

In wireless channels, the path loss exponent β has a strong impact on the quality of the links, and hence, it needs to be accurately estimated for the efficient design and operation of wireless networks. In this section we propose a statistical technique for the estimation of β from the measurements of P_{X^*}. Consider our model with the hypothesis ***[H1]***, from equation (3.2) we have the cumulate distribution function of the path-gain factor P_{X^*} as following,

$$\mathbb{P}\Big(P_{X^*} \leq r\Big) = \exp\Big(-\frac{\pi\,\lambda}{K^2 r^{\frac{2}{\beta}}}\mathbf{E}[S^{\frac{2}{\beta}}]\Big).$$

Thus

$$-\log\Big[\mathbb{P}\Big(P_{X^*} \leq r\Big)\Big] = \frac{\pi\,\lambda}{K^2 r^{\frac{2}{\beta}}}\mathbf{E}[S^{\frac{2}{\beta}}].$$

Then it follows,

$$\begin{aligned}\log\Big(-\log\Big[\mathbb{P}\Big(P_{X^*} \leq r\Big)\Big]\Big) &= \log\Big(\frac{\pi\,\lambda}{K^2 r^{\frac{2}{\beta}}}\mathbf{E}[S^{\frac{2}{\beta}}]\Big)\\ &= \log(\frac{\pi\lambda}{K^2}) - \frac{2}{\beta}\log(r) + \log\Big(\mathbf{E}[S^{\frac{2}{\beta}}]\Big).\end{aligned}$$

Therefore,

$$\log\Big(-\log\Big[\mathbb{P}\Big(P_{X^*} \leq r\Big)\Big]\Big) = \log(\frac{\pi\lambda}{K^2}) - \frac{2}{\beta}\log(r) + \log\Big(\mathbf{E}[S^{\frac{2}{\beta}}]\Big).$$

Making the change of the variable $t := \log(r)$, which corresponds to the expression of P_{X^*} in dB, it yields

$$\log\Big(-\log\Big[\mathbb{P}\Big(P_{X^*} \leq e^t\Big)\Big]\Big) = -\frac{2}{\beta}\,t + \Big[\log(\frac{\pi\lambda}{K^2}) + \log\Big(\mathbf{E}[S^{\frac{2}{\beta}}]\Big)\Big].$$

Hence,

$$\log\Big(-\log\Big[\mathbb{P}\Big(P_{X^*} \leq e^t\Big)\Big]\Big) = A\,t + B$$

where,

$$\begin{cases} A = -\frac{2}{\beta} \\ B = \log(\frac{\pi\lambda}{K^2}) + \log\Big(\mathbf{E}[S^{\frac{2}{\beta}}]\Big). \end{cases}$$

Finally, we can use linear regression model (see [SL03]) to estimate the path-loss exponent β from the constant A.

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