Fast computation of shifted Popov forms of polynomial matrices via systems of modular polynomial equations

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ABSTRACT

We give a Las Vegas algorithm which computes the shifted Popov form of an $m \times m$ nonsingular polynomial matrix of degree d in expected $\widetilde{\mathcal{O}}(m^\omega d)$ field operations, where ω is the exponent of matrix multiplication and $\widetilde{\mathcal{O}}(\cdot)$ indicates that logarithmic factors are omitted. This is the first algorithm in $\widetilde{\mathcal{O}}(m^\omega d)$ for shifted row reduction with arbitrary shifts.

Using partial linearization, we reduce the problem to the case $d \leqslant \lceil \sigma/m \rceil$ where σ is the generic determinant bound, with σ/m bounded from above by both the average row degree and the average column degree of the matrix. The cost above becomes $\widetilde{\mathcal{O}}(m^\omega \lceil \sigma/m \rceil)$, improving upon the cost of the fastest previously known algorithm for row reduction, which is deterministic.

Our algorithm first builds a system of modular equations whose solution set is the row space of the input matrix, and then finds the basis in shifted Popov form of this set. We give a deterministic algorithm for this second step supporting arbitrary moduli in $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ field operations, where m is the number of unknowns and σ is the sum of the degrees of the moduli. This extends previous results with the same cost bound in the specific cases of order basis computation and M-Padé approximation, in which the moduli are products of known linear factors.

Keywords

Shifted Popov form; polynomial matrices; row reduction; Hermite form; system of modular equations.

1. INTRODUCTION

In this paper, we consider two problems of linear algebra over the ring $\mathbb{K}[X]$ of univariate polynomials, for some field $\mathbb{K}$: computing the shifted Popov form of a matrix, and solving systems of modular equations.

1.1 Shifted Popov form

A polynomial matrix $\mathbf{P}$ is row reduced [22, Section 6.3.2] if its rows have some type of minimal degree (we give precise definitions below). Besides, if $\mathbf{P}$ satisfies an additional normalization property, then it is said to be in Popov form [22, Section 6.7.2]. Given a matrix $\mathbf{A}$, the efficient computation of a (row) reduced form of $\mathbf{A}$ and of the Popov form of $\mathbf{A}$ has received a lot of attention recently [14, 28, 16].

In many applications one rather considers the degrees of the rows of $\mathbf{P}$ shifted by some integers which specify degree weights on the columns of $\mathbf{P}$, for example in list-decoding algorithms [2, 7], robust Private Information Retrieval [12], and more generally in polynomial versions of the Coppersmith method [9, 10]. A well-known specific shifted Popov form is the Hermite form; there has been recent progress on its fast computation [17, 15, 35]. The case of an arbitrary shift has been studied in [6].

For a *shift* $\mathbf{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n$, the $\mathbf{s}$-*degree* of $\mathbf{p} = [p_1, \ldots, p_n] \in \mathbb{K}[X]^{1\times n}$ is $\max_{1 \leqslant j \leqslant n}(\deg(p_j) + s_j)$; the $\mathbf{s}$-*row degree* of $\mathbf{P} \in \mathbb{K}[X]^{m \times n}$ is $\mathrm{rdeg}_{\mathbf{s}}(\mathbf{P}) = (d_1, \ldots, d_m)$ with d_i the $\mathbf{s}$-degree of the i-th row of $\mathbf{P}$. Then, the $\mathbf{s}$-*leading matrix* of $\mathbf{P} = [p_{i,j}]_{ij}$ is the matrix $\mathrm{lm}_{\mathbf{s}}(\mathbf{P}) \in \mathbb{K}^{m\times n}$ whose entry (i, j) is the coefficient of degree $d_i - s_j$ of $p_{i,j}$.

Now, we assume that $m \leqslant n$ and $\mathbf{P}$ has full rank. Then, $\mathbf{P}$ is said to be $\mathbf{s}$-*reduced* [22, 6] if $\mathrm{lm}_{\mathbf{s}}(\mathbf{P})$ has full rank. For a full rank $\mathbf{A} \in \mathbb{K}[X]^{m\times n}$, an $\mathbf{s}$-*reduced form of* $\mathbf{A}$ is an $\mathbf{s}$-reduced matrix $\mathbf{P}$ whose row space is the same as that of $\mathbf{A}$; by row space we mean the $\mathbb{K}[X]$-module generated by the rows of the matrix. Equivalently, $\mathbf{P}$ is left-unimodularly equivalent to $\mathbf{A}$ and the tuple $\mathrm{rdeg}_{\mathbf{s}}(\mathbf{P})$ sorted in nondecreasing order is lexicographically minimal among the $\mathbf{s}$-row degrees of all matrices left-unimodularly equivalent to $\mathbf{A}$.

Specific $\mathbf{s}$-reduced matrices are those in $\mathbf{s}$-*Popov form* [22, 5, 6], as defined below. One interesting property is that the $\mathbf{s}$-Popov form is canonical: there is a unique $\mathbf{s}$-reduced form of $\mathbf{A}$ which is in $\mathbf{s}$-Popov form, called the $\mathbf{s}$-*Popov form of* $\mathbf{A}$.

Definition 1.1 (Pivot). *Let $\mathbf{p} = [p_j]_j \in \mathbb{K}[X]^{1\times n}$ be nonzero and let $\mathbf{s} \in \mathbb{Z}^n$. The* $\mathbf{s}$-pivot index *of $\mathbf{p}$ is the largest index j such that $\mathrm{rdeg}_{\mathbf{s}}(\mathbf{p}) = \deg(p_j) + s_j$. Then we call p_j and $\deg(p_j)$ the* $\mathbf{s}$-pivot entry *and the* $\mathbf{s}$-pivot degree *of $\mathbf{p}$.*

We remark that adding a constant to the entries of $\mathbf{s}$ does not change the notion of $\mathbf{s}$-pivot. For example, we will sometimes assume $\min(\mathbf{s}) = 0$ without loss of generality.

Definition 1.2 (Shifted Popov form). *Let $m \leqslant n$, let $\mathbf{P} \in \mathbb{K}[X]^{m\times n}$ be full rank, and let $\mathbf{s} \in \mathbb{Z}^n$. Then, $\mathbf{P}$ is said to be in* $\mathbf{s}$-Popov form *if the $\mathbf{s}$-pivot indices of its rows are strictly increasing, the corresponding $\mathbf{s}$-pivot entries are monic, and in each column of $\mathbf{P}$ which contains a pivot the nonpivot entries have degree less than the pivot entry.*

In this case, the $\mathbf{s}$-pivot degree *of $\mathbf{P}$ is $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_m) \in \mathbb{N}^m$, with δ_i the $\mathbf{s}$-pivot degree of the i-th row of $\mathbf{P}$.*

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Here, although we will encounter Popov forms of rectangular matrices in intermediate nullspace computations, our main focus is on computing shifted Popov forms of *square nonsingular matrices*. For the general case, studied in [6], a fast solution would require further developments. A square matrix in $\mathbf{s}$-Popov form has its $\mathbf{s}$-pivot entries on the diagonal, and its $\mathbf{s}$-pivot degree is the tuple of degrees of its diagonal entries and coincides with its column degree.

> Problem 1 (Shifted Popov normal form).
> Input: *the base field $\mathbb{K}$, a nonsingular matrix $\mathbf{A} \in \mathbb{K}[X]^{m\times m}$, a shift $\mathbf{s} \in \mathbb{Z}^m$.*
> Output: *the $\mathbf{s}$-Popov form of $\mathbf{A}$.*

Two well-known specific cases are the Popov form [27, 22] for the *uniform* shift $\mathbf{s} = \mathbf{0}$, and the Hermite form [19, 22] for the shift $\mathbf{h} = (0, \delta, 2\delta, \ldots, (m-1)\delta) \in \mathbb{N}^m$ with $\delta = m \deg(\mathbf{A})$ [6, Lemma 2.6]. For a broader perspective on shifted reduced forms, we refer the reader to [6].

For such problems involving $m \times m$ matrices of degree d, one often wishes to obtain a cost bound similar to that of polynomial matrix multiplication in the same dimensions: $\widetilde{\mathcal{O}}(m^\omega d)$ operations in $\mathbb{K}$. Here, ω is so that we can multiply $m\times m$ matrices over a commutative ring in $\mathcal{O}(m^\omega)$ operations in that ring, the best known bound being $\omega < 2.38$ [11, 25]. For example, one can compute $\mathbf{0}$-reduced [14, 16], $\mathbf{0}$-Popov [28], and Hermite [15, 35] forms of $m\times m$ nonsingular matrices of degree d in $\widetilde{\mathcal{O}}(m^\omega d)$ field operations.

Nevertheless, d may be significantly larger than the average degree of the entries of the matrix, in which case the cost $\widetilde{\mathcal{O}}(m^\omega d)$ seems unsatisfactory. Recently, for the computation of order bases [30, 34], nullspace bases [36], interpolation bases [20, 21], and matrix inversion [37], fast algorithms do take into account some types of average degrees of the matrices rather than their degree. Here, in particular, we achieve a similar improvement for the computation of shifted Popov forms of a matrix.

Given $\mathbf{A} = [a_{i,j}]_{ij} \in \mathbb{K}[X]^{m\times m}$, we denote by $\sigma(\mathbf{A})$ the *generic bound for* $\deg(\det(\mathbf{A}))$ [16, Section 6], that is,

$$\sigma(\mathbf{A}) = \max_{\pi\in S_m} \sum_{1\leqslant i\leqslant m} \overline{\deg}(a_{i,\pi_i}) \tag{1}$$

where S_m is the set of permutations of $\{1, \ldots, m\}$, and $\overline{\deg}(p)$ is defined over $\mathbb{K}[X]$ as $\overline{\deg}(0) = 0$ and $\overline{\deg}(p) = \deg(p)$ for $p \neq 0$. We have $\deg(\det(\mathbf{A})) \leqslant \sigma(\mathbf{A}) \leqslant m \deg(\mathbf{A})$, and $\sigma(\mathbf{A}) \leqslant \min(|\mathrm{rdeg}(\mathbf{A})|, |\mathrm{cdeg}(\mathbf{A})|)$ with $|\mathrm{rdeg}(\mathbf{A})|$ and $|\mathrm{cdeg}(\mathbf{A})|$ the sums of the row and column degrees of $\mathbf{A}$. We note that $\sigma(\mathbf{A})$ can be substantially smaller than $|\mathrm{rdeg}(\mathbf{A})|$ and $|\mathrm{cdeg}(\mathbf{A})|$, for example if $\mathbf{A}$ has one row and one column of uniformly large degree and other entries of low degree.

Theorem 1.3. *There is a Las Vegas randomized algorithm which solves Problem 1 in expected $\widetilde{\mathcal{O}}(m^\omega \lceil \sigma(\mathbf{A})/m \rceil) \subseteq \widetilde{\mathcal{O}}(m^\omega \deg(\mathbf{A}))$ field operations.*

The ceiling function indicates that the cost is $\widetilde{\mathcal{O}}(m^\omega)$ when $\sigma(\mathbf{A})$ is small compared to m, in which case $\mathbf{A}$ has mostly constant entries. Here we are mainly interested in the case $m \in \mathcal{O}(\sigma(\mathbf{A}))$: the cost bound may be written $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma(\mathbf{A}))$ and is both in $\widetilde{\mathcal{O}}(m^{\omega-1}|\mathrm{rdeg}(\mathbf{A})|)$ and $\widetilde{\mathcal{O}}(m^{\omega-1}|\mathrm{cdeg}(\mathbf{A})|)$.

Previous work on fast algorithms related to Problem 1 is summarized in Table 1. The fastest known algorithm for the

Ref.	Problem	Cost bound	
[18]	Hermite form	$\widetilde{\mathcal{O}}(m^4 d)$	
[31]	Hermite form	$\widetilde{\mathcal{O}}(m^{\omega+1} d)$	
[33]	Popov & Hermite forms	$\widetilde{\mathcal{O}}(m^{\omega+1} d + (md)^\omega)$	
[1, 2]	weak Popov form	$\widetilde{\mathcal{O}}(m^{\omega+1} d)$	
[26]	Popov & Hermite forms	$\mathcal{O}(m^3 d^2)$	
[14]	$\mathbf{0}$-reduction	$\widetilde{\mathcal{O}}(m^\omega d)$	$\star$
[28]	Popov form of $\mathbf{0}$-reduced	$\widetilde{\mathcal{O}}(m^\omega d)$	
[17]	Hermite form	$\widetilde{\mathcal{O}}(m^\omega d)$	$\star$
[16]	$\mathbf{0}$-reduction	$\widetilde{\mathcal{O}}(m^\omega d)$	
[35]	Hermite form	$\widetilde{\mathcal{O}}(m^\omega d)$	
[16]+[28]	$\mathbf{s}$-Popov form for any $\mathbf{s}$	$\widetilde{\mathcal{O}}(m^\omega (d+\mu))$	
Here	$\mathbf{s}$-Popov form for any $\mathbf{s}$	$\widetilde{\mathcal{O}}(m^\omega \lceil \sigma(\mathbf{A})/m \rceil)$	$\star$

Table 1: Fast algorithms for shifted reduction problems ($d = \deg(\mathbf{A})$; $\star$ = probabilistic; $\mu = \max(\mathbf{s}) - \min(\mathbf{s})$)**.**

$\mathbf{0}$-Popov form is deterministic and has cost $\widetilde{\mathcal{O}}(m^\omega d)$ with $d = \deg(\mathbf{A})$; it first computes a $\mathbf{0}$-reduced form of $\mathbf{A}$ [16], and then its $\mathbf{0}$-Popov form via normalization [28]. Obtaining the Hermite form in $\widetilde{\mathcal{O}}(m^\omega d)$ was first achieved by a probabilistic algorithm in [15], and then deterministically in [35].

For an arbitrary $\mathbf{s}$, the algorithm in [6] is fraction-free and uses a number of operations that is, depending on $\mathbf{s}$, at least quintic in m and quadratic in $\deg(\mathbf{A})$.

When $\mathbf{s}$ is not uniform there is a folklore solution based on the fact that $\mathbf{Q}$ is in $\mathbf{s}$-Popov form if and only if $\mathbf{QD}$ is in $\mathbf{0}$-Popov form, with $\mathbf{D} = \mathrm{diag}(X^{s_1}, \ldots, X^{s_m})$ and assuming $\mathbf{s} \geqslant 0$. Then, this solution computes the $\mathbf{0}$-Popov form $\mathbf{P}$ of $\mathbf{AD}$ using [16, 28] and returns $\mathbf{PD}^{-1}$. This approach uses $\widetilde{\mathcal{O}}(m^\omega(d+\mu))$ operations where $\mu = \max(\mathbf{s}) - \min(\mathbf{s})$, which is not satisfactory when μ is large. For example, its cost for computing the Hermite form is $\widetilde{\mathcal{O}}(m^{\omega+2}d)$. This is the worst case since one can assume without loss of generality that $\mu \in \mathcal{O}(m \deg(\det(\mathbf{A}))) \subseteq \mathcal{O}(m^2 d)$ [21, Appendix A].

Here we obtain, to the best of our knowledge, the best known cost bound $\widetilde{\mathcal{O}}(m^\omega \lceil \sigma(\mathbf{A})/m \rceil) \subseteq \widetilde{\mathcal{O}}(m^\omega d)$ for an arbitrary shift $\mathbf{s}$. This removes the dependency in μ, which means in some cases a speedup by a factor m^2. Besides, this is also an improvement for both specific cases $\mathbf{s} = \mathbf{0}$ and $\mathbf{s} = \mathbf{h}$ when $\mathbf{A}$ has unbalanced degrees.

One of the main difficulties in row reduction algorithms is to control the size of the manipulated matrices, that is, the number of coefficients from $\mathbb{K}$ needed for their dense representation. A major issue when dealing with arbitrary shifts is that the size of an $\mathbf{s}$-reduced form of $\mathbf{A}$ may be beyond our target cost. This is a further motivation for focusing on the computation of the $\mathbf{s}$-Popov form of $\mathbf{A}$: by definition, the sum of its column degrees is $\deg(\det(\mathbf{A}))$, and therefore its size is at most $m^2 + m\deg(\det(\mathbf{A}))$, independently of $\mathbf{s}$.

Consider for example $\mathbf{A} = \left[\begin{smallmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 \end{smallmatrix}\right]$ for any $\mathbf{0}$-reduced $\mathbf{B}_1$ and $\mathbf{B}_2$ in $\mathbb{K}[X]^{m\times m}$. Then, taking $\mathbf{s} = (0, \ldots, 0, d, \ldots, d)$ with $d > 0$, $\left[\begin{smallmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{C} & \mathbf{B}_2 \end{smallmatrix}\right]$ is an $\mathbf{s}$-reduced form of $\mathbf{A}$ for any $\mathbf{C} \in \mathbb{K}[X]^{m\times m}$ with $\deg(\mathbf{C}) \leqslant d$; for some $\mathbf{C}$ it has size $\Theta(m^2 d)$, with d arbitrary large independently of $\deg(\mathbf{A})$.

Furthermore, the size of the unimodular transformation leading from $\mathbf{A}$ to $\mathbf{P}$ may be beyond the target cost, which is why fast algorithms for $\mathbf{0}$-reduction and Hermite form do not directly perform unimodular transformations on $\mathbf{A}$ to reduce the degrees of its entries. Instead, they proceed in two steps: first, they work on $\mathbf{A}$ to find some equations which describe its row space, and then they find a basis of solutions to these equations in $\mathbf{0}$-reduced form or Hermite form. We will follow a similar two-step strategy for an arbitrary shift.

It seems that some new ingredient is needed, since for both $\mathbf{s} = \mathbf{0}$ and $\mathbf{s} = \mathbf{h}$ the fastest algorithms use shift-specific properties at some point of the process: namely, the facts that a $\mathbf{0}$-reduced form of $\mathbf{A}$ has degree at most $\deg(\mathbf{A})$ and that the Hermite form of $\mathbf{A}$ is triangular.

As in [17], we first compute the Smith form $\mathbf{S}$ of $\mathbf{A}$ and partial information on a right unimodular transformation $\mathbf{V}$; this is where the probabilistic aspect comes from. This gives a description of the row space of $\mathbf{A}$ as the set of row vectors $\mathbf{p} \in \mathbb{K}[X]^{1\times m}$ such that $\mathbf{pV} = \mathbf{qS}$ for some $\mathbf{q} \in \mathbb{K}[X]^{1\times m}$. Since $\mathbf{S}$ is diagonal, this can be seen as a system of modular equations: the second step is the fast computation of a basis of solutions in $\mathbf{s}$-Popov form, which is our new ingredient.

1.2 Systems of modular equations

Hereafter, $\mathbb{K}[X]_{\neq 0}$ denotes the set of nonzero polynomials. We fix some moduli $\mathfrak{M} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_n) \in \mathbb{K}[X]^n_{\neq 0}$, and for $\mathbf{A}, \mathbf{B} \in \mathbb{K}[X]^{m\times n}$ we write $\mathbf{A} = \mathbf{B} \bmod \mathfrak{M}$ if there exists $\mathbf{Q} \in \mathbb{K}[X]^{m\times n}$ such that $\mathbf{A} = \mathbf{B} + \mathbf{Q}\,\mathrm{diag}(\mathfrak{M})$. Given $\mathbf{F} \in \mathbb{K}[X]^{m\times n}$ specifying the equations, we call *solution for* $(\mathfrak{M}, \mathbf{F})$ any $\mathbf{p} \in \mathbb{K}[X]^{1\times m}$ such that $\mathbf{pF} = 0 \bmod \mathfrak{M}$.

The set of all such $\mathbf{p}$ is a $\mathbb{K}[X]$-submodule of $\mathbb{K}[X]^{1\times m}$ which contains $\mathrm{lcm}(\mathfrak{m}_1, \ldots, \mathfrak{m}_n)\mathbb{K}[X]^{1\times m}$, and is thus free of rank m [24, p. 146]. Then, we represent any basis of this module as the rows of a matrix $\mathbf{P} \in \mathbb{K}[X]^{m\times m}$, called a *solution basis for* $(\mathfrak{M}, \mathbf{F})$. Here, for example for the application to Problem 1, we are interested in such bases that are $\mathbf{s}$-reduced, in which case $\mathbf{P}$ is said to be an $\mathbf{s}$-*minimal solution basis for* $(\mathfrak{M}, \mathbf{F})$. The unique such basis which is in $\mathbf{s}$-Popov form is called the $\mathbf{s}$-*Popov solution basis for* $(\mathfrak{M}, \mathbf{F})$.

Problem 2 (Minimal solution basis).
Input: *the base field $\mathbb{K}$, moduli $\mathfrak{M} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_n) \in \mathbb{K}[X]^n_{\neq 0}$, a matrix $\mathbf{F} \in \mathbb{K}[X]^{m\times n}$ such that $\deg(\mathbf{F}_{*,j}) < \deg(\mathfrak{m}_j)$, a shift $\mathbf{s} \in \mathbb{Z}^m$.*
Output: *an $\mathbf{s}$-minimal solution basis for $(\mathfrak{M}, \mathbf{F})$.*

Well-known specific cases of this problem are *Hermite-Padé approximation* with a single equation modulo some power of X, and *M-Padé approximation* [3, 32] with moduli that are products of known linear factors. Moreover, an $\mathbf{s}$-*order basis for* $\mathbf{F}$ *and* $(\sigma_1, \ldots, \sigma_n)$ [34] is an $\mathbf{s}$-minimal solution basis for $(\mathfrak{M}, \mathbf{F})$ with $\mathfrak{M} = (X^{\sigma_1}, \ldots, X^{\sigma_n})$.

An overview of fast algorithms for Problem 2 is given in Table 2. For M-Padé approximation, and thus in particular for order basis computation, there is an algorithm to compute the $\mathbf{s}$-Popov solution basis using $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ operations, with $\sigma = \deg(\mathfrak{m}_1) + \cdots + \deg(\mathfrak{m}_n)$ [21]. Here, for $n \in \mathcal{O}(m)$, we extend this result to arbitrary moduli.

Theorem 1.4. *Assuming $n \in \mathcal{O}(m)$, there is a deterministic algorithm which solves Problem 2 using $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ field operations, with $\sigma = \deg(\mathfrak{m}_1) + \cdots + \deg(\mathfrak{m}_n)$, and returns the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M}, \mathbf{F})$.*

We note that Problem 2 is a minimal interpolation basis problem [5, 20] when the so-called *multiplication matrix* $\mathbf{M}$ is block diagonal with companion blocks. Indeed, $\mathbf{p}$ is a solution for $(\mathfrak{M}, \mathbf{F})$ if and only if $\mathbf{p}$ is an *interpolant for* $(\mathbf{E}, \mathbf{M})$ [20, Definition 1.1], where $\mathbf{E} \in \mathbb{K}^{m\times\sigma}$ is the concatenation of the coefficient vectors of the columns of $\mathbf{F}$ and $\mathbf{M} \in \mathbb{K}^{\sigma\times\sigma}$ is $\mathrm{diag}(\mathbf{M}_1, \ldots, \mathbf{M}_n)$ with $\mathbf{M}_j$ the companion matrix associated with $\mathfrak{m}_j$. In this context, the multiplication $\mathbf{p} \cdot \mathbf{E}$ defined by $\mathbf{M}$ as in [5, 20] precisely corresponds to $\mathbf{pF} \bmod \mathfrak{M}$.

In particular, Theorem 1.4 follows from [20, Theorem 1.4] when $\sigma \in \mathcal{O}(m)$. If some of the moduli have small degree, we use this result for base cases of our recursive algorithm.

Ref.	Cost bound	Moduli	Particularities
[3, 32]	$\mathcal{O}(m^2\sigma^2)$	split	
[4]	$\mathcal{O}(m\sigma^2)$	$\mathfrak{m}_j = X^{\sigma/n}$	partial basis
[4]	$\widetilde{\mathcal{O}}(m^\omega\sigma)$	$\mathfrak{m}_j = X^{\sigma/n}$	
[14]	$\widetilde{\mathcal{O}}(m^\omega\sigma/n)$	$\mathfrak{m}_j = X^{\sigma/n}$	
[30]	$\widetilde{\mathcal{O}}(m^\omega\lceil\sigma/m\rceil)$	$\mathfrak{m}_j = X^{\sigma/n}$	partial basis, $\lvert\mathbf{s}\rvert \leqslant \sigma$
[34]	$\widetilde{\mathcal{O}}(m^\omega\lceil\sigma/m\rceil)$	$\mathfrak{m}_j = X^{\sigma/n}$	$\lvert\mathbf{s}\rvert \leqslant \sigma$
[8]	$\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$, probabilistic	any	returns a single small degree solution
[20]	$\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$	split	$\lvert\mathbf{s}\rvert \leqslant \sigma$
[20]	$\widetilde{\mathcal{O}}(m\sigma^{\omega-1})$	any	$\mathbf{s}$-Popov, $\sigma \in \mathcal{O}(m)$
[21]	$\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$	split	$\mathbf{s}$-Popov
Here	$\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$	any	$\mathbf{s}$-Popov

Table 2: Fast algorithms for Problem 2 ($n \in \mathcal{O}(m)$; *partial basis* = returns small degree rows of an $\mathbf{s}$-minimal solution basis; *split* = product of known linear factors)**.**

In the case of M-Padé approximation, knowing the moduli as products of linear factors leads to rewriting the problem as a minimal interpolation basis computation with $\mathbf{M}$ in Jordan form [5, 20]. Since $\mathbf{M}$ is upper triangular, one can then rely on recurrence relations to solve the problem iteratively [3, 32, 4, 5]. The fast algorithms in [4, 14, 34, 20, 21], beyond the techniques used to achieve efficiency, are essentially divide-and-conquer versions of this iterative solution and are thus based on the same recurrence relations.

However, for arbitrary moduli the matrix $\mathbf{M}$ is not triangular and there is no such recurrence in general. Then, a natural idea is to relate solution bases to nullspace bases: Problem 2 asks to find $\mathbf{P}$ such that there is some quotient $\mathbf{Q}$ with $[\mathbf{P}|\mathbf{Q}]\mathbf{N} = \mathbf{0}$ for $\mathbf{N} = [\mathbf{F}^\mathsf{T}| -\mathrm{diag}(\mathfrak{M})]^\mathsf{T}$. More precisely, $[\mathbf{P}|\mathbf{Q}]$ can be obtained as a $\mathbf{u}$-minimal nullspace basis of $\mathbf{N}$ for the shift $\mathbf{u} = (\mathbf{s} - \min(\mathbf{s}), \mathbf{0}) \in \mathbb{N}^{m+n}$.

Using recent ingredients from [17, 21] outlined in the next paragraphs, the main remaining difficulty is to deal with this nullspace problem when $n = 1$. Here, we give a $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ algorithm to solve it using its specific properties: $\mathbf{N}$ is the column $[\mathbf{F}^\mathsf{T}|\mathfrak{m}_1]^\mathsf{T}$ with $\deg(\mathbf{F}) < \deg(\mathfrak{m}_1) = \sigma$, and the last entry of $\mathbf{u}$ is $\min(\mathbf{u})$. First, when $\max(\mathbf{u}) \in \mathcal{O}(\sigma)$ we show that $[\mathbf{P}|\mathbf{Q}]$ can be efficiently obtained as a submatrix of the $\mathbf{u}$-Popov order basis for $\mathbf{N}$ and order $\mathcal{O}(\sigma)$. Then, when $\max(\mathbf{u})$ is large compared to σ and assuming $\mathbf{u}$ is sorted non-decreasingly, $\mathbf{P}$ has a lower block triangular shape. We show how this shape can be revealed, along with the $\mathbf{s}$-pivot degree of $\mathbf{P}$, using a divide-and-conquer approach which splits $\mathbf{u}$ into two shifts of amplitude about $\max(\mathbf{u})/2$.

Then, for $n \geqslant 1$ we use a divide-and-conquer approach on n which is classical in such contexts: two solution bases $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ are computed recursively in shifted Popov form and are multiplied together to obtain the $\mathbf{s}$-minimal solution basis $\mathbf{P}^{(2)}\mathbf{P}^{(1)}$ for $(\mathfrak{M}, \mathbf{F})$. However this product is usually not in $\mathbf{s}$-Popov form and may have size beyond our target cost. Thus, as in [21], instead of computing $\mathbf{P}^{(2)}\mathbf{P}^{(1)}$, we use $\mathbf{P}^{(2)}$ and $\mathbf{P}^{(1)}$ to deduce the $\mathbf{s}$-pivot degree of $\mathbf{P}$.

In both recursions above, we focus on finding the $\mathbf{s}$-pivot degree of $\mathbf{P}$. Using ideas and results from [17, 21], we show that this knowledge about the degrees in $\mathbf{P}$ allows us to complete the computation of $\mathbf{P}$ within the target cost.

2. FAST COMPUTATION OF THE SHIFTED POPOV SOLUTION BASIS

Hereafter, we call $\mathbf{s}$-*minimal degree of* $(\mathfrak{M},\mathbf{F})$ the $\mathbf{s}$-pivot degree $\boldsymbol{\delta}$ of the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M},\mathbf{F})$; $\boldsymbol{\delta}$ coincides with the column degree of this basis. A central result for the cost analysis is that $|\boldsymbol{\delta}| = \delta_1 + \cdots + \delta_m$ is at most $\sigma = \deg(\mathfrak{m}_1)+\cdots+\deg(\mathfrak{m}_n)$. This is classical for M-Padé approximation [32, Theorem 4.1] and holds for minimal interpolation bases in general (see for example [20, Lemma 7.17]).

2.1 Solution bases from nullspace bases and fast algorithm for known minimal degree

This subsection summarizes and slightly extends results from [17, Section 3]. We first show that the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M},\mathbf{F})$ is the principal $m \times m$ submatrix of the $\mathbf{u}$-Popov nullspace basis of $[\mathbf{F}^\top | \operatorname{diag}(\mathfrak{M})]^\top$ for some $\mathbf{u} \in \mathbb{Z}^{m+n}$.

Lemma 2.1. *Let $\mathfrak{M} = (\mathfrak{m}_1,\ldots,\mathfrak{m}_n) \in \mathbb{K}[X]^n_{\neq 0}$, $\mathbf{s} \in \mathbb{Z}^m$, $\mathbf{F} \in \mathbb{K}[X]^{m\times n}$ with $\deg(\mathbf{F}_{*,j}) < \deg(\mathfrak{m}_j)$, $\mathbf{P} \in \mathbb{K}[X]^{m\times m}$, and $\mathbf{w} \in \mathbb{Z}^n$ be such that $\max(\mathbf{w}) \leqslant \min(\mathbf{s})$. Then, $\mathbf{P}$ is the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M},\mathbf{F})$ if and only if $[\mathbf{P}|\mathbf{Q}]$ is the $\mathbf{u}$-Popov nullspace basis of $[\mathbf{F}^\top | \operatorname{diag}(\mathfrak{M})]^\top$ for some $\mathbf{Q} \in \mathbb{K}[X]^{m\times n}$ and $\mathbf{u} = (\mathbf{s},\mathbf{w}) \in \mathbb{Z}^{m+n}$. In this case, $\deg(\mathbf{Q}) < \deg(\mathbf{P})$ and $[\mathbf{P}|\mathbf{Q}]$ has $\mathbf{s}$-pivot index $(1,2,\ldots,m)$.*

Proof. Let $\mathbf{N} = [\mathbf{F}^\top | \operatorname{diag}(\mathfrak{M})]^\top$. It is easily verified that $\mathbf{P}$ is a solution basis for $(\mathfrak{M},\mathbf{F})$ if and only if there is some $\mathbf{Q} \in \mathbb{K}[X]^{m\times n}$ such that $[\mathbf{P}|\mathbf{Q}]$ is a nullspace basis of $\mathbf{N}$.

Now, having $\deg(\mathbf{F}_{*,j}) < \deg(\mathfrak{m}_j)$ implies that any $[\mathbf{p}|\mathbf{q}] \in \mathbb{K}[X]^{1\times(m+n)}$ in the nullspace of $\mathbf{N}$ satisfies $\deg(\mathbf{q}) < \deg(\mathbf{p})$, and since $\max(\mathbf{w}) \leqslant \min(\mathbf{s})$ we get $\operatorname{rdeg}_{\mathbf{w}}(\mathbf{q}) < \operatorname{rdeg}_{\mathbf{s}}(\mathbf{p})$. In particular, for any matrix $[\mathbf{P}|\mathbf{Q}] \in \mathbb{K}[X]^{m\times(m+n)}$ such that $[\mathbf{P}|\mathbf{Q}]\mathbf{N} = 0$, we have $\operatorname{lm}_{\mathbf{u}}([\mathbf{P}|\mathbf{Q}]) = [\operatorname{lm}_{\mathbf{s}}(\mathbf{P})|\mathbf{0}]$. This implies that $\mathbf{P}$ is in $\mathbf{s}$-Popov form if and only if $[\mathbf{P}|\mathbf{Q}]$ is in $\mathbf{u}$-Popov form with $\mathbf{s}$-pivot index $(1,\ldots,m)$. $\square$

We now show that, when we have *a priori* knowledge about the $\mathbf{s}$-pivot entries of a $\mathbf{s}$-Popov nullspace basis, it can be computed efficiently via an $\mathbf{s}$-Popov order basis.

Lemma 2.2. *Let $\mathbf{s} \in \mathbb{Z}^{m+n}$ and let $\mathbf{N} \in \mathbb{K}[X]^{(m+n)\times n}$ be of full rank. Let $\mathbf{B} \in \mathbb{K}[X]^{m\times(m+n)}$ be the $\mathbf{s}$-Popov nullspace basis for $\mathbf{N}$, $(\pi_1,\ldots,\pi_m)$ be its $\mathbf{s}$-pivot index, $(\delta_1,\ldots,\delta_m)$ be its $\mathbf{s}$-pivot degree, and $\delta \geqslant \deg(\mathbf{B})$ be a degree bound. Then, let $\mathbf{u} = (u_1,\ldots,u_{m+n}) \in \mathbb{Z}^{m+n}_{\leqslant 0}$ with*

$$u_j = \begin{cases} -\delta - 1 & \text{if } j \notin \{\pi_1,\ldots,\pi_m\}, \\ -\delta_i & \text{if } j = \pi_i. \end{cases}$$

Writing $(\sigma_1,\ldots,\sigma_n)$ for the column degree of $\mathbf{N}$, let $\tau_j = \sigma_j + \delta + 1$ for $1 \leqslant j \leqslant n$ and let $\mathbf{A}$ be the $\mathbf{u}$-Popov order basis for $\mathbf{N}$ and $(\tau_1,\ldots,\tau_n)$. Then, $\mathbf{B}$ is the submatrix of $\mathbf{A}$ formed by its rows at indices $\{\pi_1,\ldots,\pi_m\}$.

Proof. First, $\mathbf{B}$ is in $\mathbf{u}$-Popov form with $\operatorname{rdeg}_{\mathbf{u}}(\mathbf{B}) = \mathbf{0}$. Define $\mathbf{C} \in \mathbb{K}[X]^{(m+n)\times(m+n)}$ whose i-th row is $\mathbf{B}_{j,*}$ if $i = \pi_j$ and $\mathbf{A}_{i,*}$ if $i \notin \{\pi_1,\ldots,\pi_m\}$: we want to prove $\mathbf{C} = \mathbf{A}$.

Let $\mathbf{p} = [p_j]_j \in \mathbb{K}[X]^{1\times(m+n)}$ be a row of $\mathbf{A}$, and assume $\operatorname{rdeg}_{\mathbf{u}}(\mathbf{p}) < 0$. This means $\deg(p_j) < -u_j$ for all j, so that $\deg(\mathbf{p}) < \max(-\mathbf{u}) = \delta + 1$. Then, for all $1 \leqslant j \leqslant n$ we have $\deg(\mathbf{p}\mathbf{N}_{*,j}) < \sigma_j + \delta + 1 = \tau_j$, and from $\mathbf{p}\mathbf{N}_{*,j} = 0 \bmod X^{\tau_j}$ we obtain $\mathbf{p}\mathbf{N}_{*,j} = 0$, which is absurd by minimality of $\mathbf{B}$. As a result, $\operatorname{rdeg}_{\mathbf{u}}(\mathbf{A}) \geqslant \mathbf{0} = \operatorname{rdeg}_{\mathbf{u}}(\mathbf{B})$ componentwise.

Besides, $\mathbf{C}\mathbf{F} = 0 \bmod (X^{\tau_1},\ldots,X^{\tau_n})$ and since $\mathbf{C}$ has its $\mathbf{u}$-pivot entries on the diagonal, it is $\mathbf{u}$-reduced: by minimality of $\mathbf{A}$, we obtain $\operatorname{rdeg}_{\mathbf{u}}(\mathbf{A}) = \operatorname{rdeg}_{\mathbf{u}}(\mathbf{C})$. Then, it is easily verified that $\mathbf{C}$ is in $\mathbf{u}$-Popov form, hence $\mathbf{C} = \mathbf{A}$. $\square$

In particular, computing the $\mathbf{s}$-Popov nullspace basis $\mathbf{B}$, when its $\mathbf{s}$-pivot index, its $\mathbf{s}$-pivot degree, and $\delta \geqslant \deg(\mathbf{B})$ are known, can be done in $\widetilde{\mathcal{O}}(m^{\omega-1}(\sigma + n\delta))$ with $\sigma = \sigma_1 + \cdots + \sigma_n$ using the order basis algorithm in [21].

As for Problem 2, with Lemma 2.1 this gives an algorithm for computing $\mathbf{P}$ *and* the quotients $\mathbf{Q} = -\mathbf{P}\mathbf{F}/\operatorname{diag}(\mathfrak{M})$ when we know *a priori* the $\mathbf{s}$-minimal degree $\boldsymbol{\delta}$ of $(\mathfrak{M},\mathbf{F})$. Here, we would choose $\delta = \max(\boldsymbol{\delta}) \geqslant \deg([\mathbf{P}|\mathbf{Q}])$: in some cases $\delta = \Theta(\sigma)$ and this has cost bound $\widetilde{\mathcal{O}}(m^{\omega-1}(\sigma + n\sigma))$, which exceeds our target $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$. An issue is that $\mathbf{Q}$ has size $\mathcal{O}(mn\sigma)$ when $\mathbf{P}$ has columns of large degree; yet here we are not interested in $\mathbf{Q}$. This can be solved using partial linearization to expand the columns of large degree in $\mathbf{P}$ into more columns of smaller degree as in the next result, which holds in general for interpolation bases [21, Lemma 4.2].

Lemma 2.3. *Let $\mathfrak{M} \in \mathbb{K}[X]^n_{\neq 0}$ with entries having degrees $(\sigma_1,\ldots,\sigma_n)$. Let $\mathbf{F} \in \mathbb{K}[X]^{m\times n}$ and $\mathbf{s} \in \mathbb{Z}^m$. Furthermore, let $\boldsymbol{\delta} = (\delta_1,\ldots,\delta_m)$ denote the $\mathbf{s}$-minimal degree of $(\mathfrak{M},\mathbf{F})$.*

Writing $\sigma = \sigma_1 + \cdots + \sigma_n$, let $\delta = \lceil \sigma/m \rceil \geqslant 1$, and for $i \in \{1,\ldots,m\}$ write $\delta_i = (\alpha_i - 1)\delta + \beta_i$ with $\alpha_i \geqslant 1$ and $0 \leqslant \beta_i < \delta$, and let $\widetilde{m} = \alpha_1 + \cdots + \alpha_m$. Define $\widetilde{\boldsymbol{\delta}} \in \mathbb{N}^{\widetilde{m}}$ as

$$\widetilde{\boldsymbol{\delta}} = (\underbrace{\delta,\ldots,\delta,\beta_1}_{\alpha_1},\ldots,\underbrace{\delta,\ldots,\delta,\beta_m}_{\alpha_m}) \qquad (2)$$

and the expansion-compression matrix $\mathcal{E} \in \mathbb{K}[X]^{\widetilde{m}\times m}$ as

$$\mathcal{E} = \begin{bmatrix} 1 & & \\ X^{\delta} & & \\ \vdots & & \\ X^{(\alpha_1-1)\delta} & & \\ & \ddots & \\ & & 1 \\ & & X^{\delta} \\ & & \vdots \\ & & X^{(\alpha_m-1)\delta} \end{bmatrix}. \qquad (3)$$

Let $\mathbf{d} = -\widetilde{\boldsymbol{\delta}} \in \mathbb{Z}^{\widetilde{m}}$ and $\mathbf{P} \in \mathbb{K}[X]^{\widetilde{m}\times\widetilde{m}}$ be the $\mathbf{d}$-Popov solution basis for $(\mathfrak{M}, \mathcal{E}\mathbf{F} \bmod \mathfrak{M})$. Then, $\mathbf{P}$ has $\mathbf{d}$-pivot degree $\widetilde{\boldsymbol{\delta}}$ and the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M},\mathbf{F})$ is the submatrix of $\mathbf{P}\mathcal{E}$ formed by its rows at indices $\{\alpha_1+\cdots+\alpha_i, 1 \leqslant i \leqslant m\}$.

This leads to Algorithm 1, which solves Problem 2 efficiently when the $\mathbf{s}$-minimal degree $\boldsymbol{\delta}$ is known *a priori*.

Algorithm 1 (KnownDegPolModSys).
Input: polynomials $\mathfrak{M} = (\mathfrak{m}_1,\ldots,\mathfrak{m}_n) \in \mathbb{K}[X]^n_{\neq 0}$, a matrix $\mathbf{F} \in \mathbb{K}[X]^{m\times n}$ with $\deg(\mathbf{F}_{*,j}) < \deg(\mathfrak{m}_j)$, a shift $\mathbf{s} \in \mathbb{Z}^m$, $\boldsymbol{\delta} = (\delta_1,\ldots,\delta_m)$ the $\mathbf{s}$-minimal degree of $(\mathfrak{M},\mathbf{F})$.
Output: the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M},\mathbf{F})$.

1. $\delta \leftarrow \lceil (\deg(\mathfrak{m}_1) + \cdots + \deg(\mathfrak{m}_n))/m \rceil$, $\alpha_i \leftarrow \lfloor \delta_i/\delta \rfloor + 1$ for $1 \leqslant i \leqslant m$, $\widetilde{m} \leftarrow \alpha_1 + \cdots + \alpha_m$, $\widetilde{\boldsymbol{\delta}}$ as in (2), $\mathcal{E}$ as in (3), $\widetilde{\mathbf{F}} \leftarrow \mathcal{E}\mathbf{F} \bmod \mathfrak{M}$
2. $\mathbf{u} \leftarrow (-\widetilde{\boldsymbol{\delta}}, -\delta-1,\ldots,-\delta-1) \in \mathbb{Z}^{\widetilde{m}+n}$, $\boldsymbol{\tau} \leftarrow (\deg(\mathfrak{m}_j) + \delta + 1)_{1 \leqslant j \leqslant n}$
3. $\widetilde{\mathbf{P}} \leftarrow$ the $\mathbf{u}$-Popov order basis for $[\widetilde{\mathbf{F}}^\top | \operatorname{diag}(\mathfrak{M})]^\top$ and $\boldsymbol{\tau}$, $\mathbf{P} \leftarrow$ the principal $\widetilde{m} \times \widetilde{m}$ submatrix of $\widetilde{\mathbf{P}}$
4. Return the submatrix of $\mathbf{P}\mathcal{E}$ formed by the rows at indices $\alpha_1 + \cdots + \alpha_i$ for $1 \leqslant i \leqslant m$

Proposition 2.4. *Algorithm* KnownDegPolModSys *is correct. Writing $\sigma = \deg(\mathfrak{m}_1) + \cdots + \deg(\mathfrak{m}_n)$ and assuming $\sigma \geqslant m \geqslant n$, it uses $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ operations in $\mathbb{K}$.*

Proof. By Lemmas 2.3 and 2.1, since $\min(-\widetilde{\boldsymbol{\delta}}) > -\delta - 1$ and $\mathbf{u} = (-\widetilde{\boldsymbol{\delta}}, -\delta-1,\ldots,-\delta-1)$, the $-\boldsymbol{\delta}$-Popov solution basis for $(\mathfrak{M}, \widetilde{\mathbf{F}})$ is the principal $\widetilde{m} \times \widetilde{m}$ submatrix of the $\mathbf{u}$-Popov nullspace basis $\mathbf{B}$ for $[\widetilde{\mathbf{F}}^\mathsf{T} | \operatorname{diag}(\mathfrak{M})]^\mathsf{T}$, and $\mathbf{B}$ has $\mathbf{u}$-pivot index $\{1, \ldots, \widetilde{m}\}$, $\mathbf{u}$-pivot degree $\widetilde{\boldsymbol{\delta}}$, and $\deg(\mathbf{B}) \leqslant \delta$. Then, by Lemma 2.2, $\mathbf{B}$ is formed by the first $\widetilde{m}$ rows of $\widetilde{\mathbf{P}}$ at Step **3**, hence $\mathbf{P}$ is the $\mathbf{d}$-Popov solution basis for $(\mathfrak{M}, \mathbf{F})$. The correctness then follows from Lemma 2.3.

Since $|\boldsymbol{\delta}| \leqslant \sigma$, $\mathcal{E}$ has $\widetilde{m} \leqslant 2m$ rows and $\mathcal{E}\mathbf{F} \bmod \mathfrak{M}$ can be computed in $\widetilde{\mathcal{O}}(m\sigma)$ operations using fast polynomial division [13]. The cost bound of Step **3** follows from [21, Theorem 1.4] since $\tau_1 + \cdots + \tau_n = \sigma + n(1 + \lceil \sigma/m \rceil) \in \mathcal{O}(\sigma)$. $\square$

2.2 The case of one equation

We now present our main new ingredients, focusing on the case $n = 1$. First, we show that when the shift $\mathbf{s}$ has a small *amplitude* $\operatorname{amp}(\mathbf{s}) = \max(\mathbf{s}) - \min(\mathbf{s})$, one can solve Problem 2 via an order basis computation at small order.

Lemma 2.5. *Let $\mathfrak{m} \in \mathbb{K}[X]_{\neq 0}$, $\mathbf{s} \in \mathbb{Z}^m$, and $\mathbf{F} \in \mathbb{K}[X]^{m \times 1}$ with $\deg(\mathbf{F}) < \deg(\mathfrak{m}) = \sigma$. Then, for any $\tau \geqslant \operatorname{amp}(\mathbf{s}) + 2\sigma$, the $\mathbf{s}$-Popov solution basis for $(\mathfrak{m}, \mathbf{F})$ is the principal $m \times m$ submatrix of the $\mathbf{u}$-Popov order basis for $[\mathbf{F}^\mathsf{T} | \mathfrak{m}]^\mathsf{T}$ and τ, with $\mathbf{u} = (\mathbf{s}, \min(\mathbf{s})) \in \mathbb{Z}^{m+1}$.*

Proof. Let $\mathbf{A} = \left[\begin{smallmatrix} \mathbf{P} & \mathbf{q} \\ \mathbf{p} & q \end{smallmatrix}\right]$ denote the $\mathbf{u}$-Popov order basis for $[\mathbf{F}^\mathsf{T} | \mathfrak{m}]^\mathsf{T}$ and τ, where $\mathbf{P} \in \mathbb{K}[X]^{m \times m}$ and $q \in \mathbb{K}[X]$. Consider $\mathbf{B} = [\bar{\mathbf{P}} | \bar{\mathbf{q}}]$ the $\mathbf{u}$-Popov nullspace basis of $[\mathbf{F}^\mathsf{T} | \mathfrak{m}]^\mathsf{T}$: thanks to Lemma 2.1, it is enough to prove that $\mathbf{B} = [\mathbf{P} | \mathbf{q}]$.

First, we have $\operatorname{rdeg}(\mathbf{p}) \leqslant \deg(q)$ by choice of $\mathbf{u}$, so that $q\mathfrak{m} \neq 0$ implies $\deg(\mathbf{p}\mathbf{F} + q\mathfrak{m}) = \deg(q) + \sigma$. Since $\mathbf{p}\mathbf{F} + q\mathfrak{m} = 0 \bmod X^\tau$, this gives $\deg(q) + \sigma \geqslant \tau$. This also shows that the $\mathbf{u}$-pivot entries of $\mathbf{B}$ are located in $\bar{\mathbf{P}}$.

Then, since the sum of the $\mathbf{u}$-pivot degrees of $\mathbf{A}$ is at most τ, the sum of the $\mathbf{s}$-pivot degrees of $\mathbf{P}$ is at most σ; with $[\mathbf{P} | \mathbf{q}]$ in $\mathbf{u}$-Popov form, this gives $\deg(\mathbf{q}) < \sigma + \operatorname{amp}(\mathbf{s}) \leqslant \tau - \sigma$. We obtain $\deg(\mathbf{P}\mathbf{F} + \mathbf{q}\mathfrak{m}) < \tau$, so that $\mathbf{P}\mathbf{F} + \mathbf{q}\mathfrak{m} = 0$. Thus, the minimality of $\mathbf{B}$ and $\mathbf{A}$ gives the conclusion. $\square$

When $\operatorname{amp}(\mathbf{s}) \in \mathcal{O}(\sigma)$, this gives a fast solution to our problem. In what follows, we present a divide-and-conquer approach on $\operatorname{amp}(\mathbf{s})$, with base case $\operatorname{amp}(\mathbf{s}) \in \mathcal{O}(\sigma)$.

We first give an overview, assuming $\mathbf{s}$ is non-decreasing. A key ingredient is that when $\operatorname{amp}(\mathbf{s})$ is large compared to σ, then $\mathbf{P}$ has a lower block triangular shape, since it is in $\mathbf{s}$-Popov form with sum of $\mathbf{s}$-pivot degrees $|\boldsymbol{\delta}| \leqslant \sigma$. Typically, if $s_{i+1} - s_i \geqslant \sigma$ for some i then $\mathbf{P} = \left[\begin{smallmatrix} \mathbf{P}^{(1)} & \mathbf{0} \\ * & \mathbf{P}^{(2)} \end{smallmatrix}\right]$ with $\mathbf{P}^{(1)} \in \mathbb{K}[X]^{i \times i}$. Even though the block sizes are unknown in general, we show that they can be revealed efficiently along with $\boldsymbol{\delta}$ by a divide-and-conquer algorithm, as follows.

First, we use a recursive call with the first j entries $\mathbf{s}^{(0)}$ of $\mathbf{s}$ and $\mathbf{F}^{(0)}$ of $\mathbf{F}$, where j is such that $\operatorname{amp}(\mathbf{s}^{(0)})$ is about half of $\operatorname{amp}(\mathbf{s})$. This reveals the first $i \leqslant j$ entries $\boldsymbol{\delta}^{(1)}$ of $\boldsymbol{\delta}$ and the first i rows $[\mathbf{P}^{(1)} | \mathbf{0}]$ of $\mathbf{P}$, with $\mathbf{P}^{(1)} \in \mathbb{K}[X]^{i \times i}$. A central point is that $\operatorname{amp}(\mathbf{s}^{(2)})$ is about half of $\operatorname{amp}(\mathbf{s})$ as well, where $\mathbf{s}^{(2)}$ is the tail of $\mathbf{s}$ starting at the entry $i + 1$.

Then, knowing the degrees $\boldsymbol{\delta}^{(1)}$ allows us to set up an order basis computation that yields a *residual*, that is, a column $\mathbf{G} \in \mathbb{K}[X]^{(m-i) \times 1}$ and a modulus $\mathfrak{n}$ such that we can continue the computation of $\mathbf{P}$ using a second recursive call, which consists in computing the $\mathbf{s}^{(2)}$-Popov solution basis for $(\mathfrak{n}, \mathbf{G})$. From these two calls we obtain $\boldsymbol{\delta}$, and then we recover $\mathbf{P}$ using Algorithm 1.

Now we present the details. We fix $\mathbf{F} \in \mathbb{K}[X]^{m \times 1}$, $\mathfrak{m} \in \mathbb{K}[X]_{\neq 0}$ with $\sigma = \deg(\mathfrak{m}) > \deg(\mathbf{F})$, $\mathbf{s} \in \mathbb{Z}^m$, $\mathbf{P}$ the $\mathbf{s}$-Popov solution basis for $(\mathfrak{m}, \mathbf{F})$, and $\boldsymbol{\delta}$ its $\mathbf{s}$-pivot degree. In what follows, $\boldsymbol{\pi}^{\mathbf{s}} = (\pi_1, \ldots, \pi_m)$ is any permutation of $\{1, \ldots, m\}$ such that $(s_{\pi_1}, \ldots, s_{\pi_m})$ is non-decreasing.

Then, for $\mathbf{t} = (t_1, \ldots, t_m) \in \mathbb{Z}^m$ we write $\mathbf{t}_{[i:j]}$ for the subtuple of $\mathbf{t}$ formed by its entries at indices $\{\pi_i, \ldots, \pi_j\}$, and for a matrix $\mathbf{M} \in \mathbb{K}[X]^{m \times m}$ we write $\mathbf{M}_{[i:j,k:l]}$ for the submatrix of $\mathbf{M}$ formed by its rows at indices $\{\pi_i, \pi_{i+1}, \ldots, \pi_j\}$ and columns at indices $\{\pi_k, \pi_{k+1}, \ldots, \pi_l\}$. The main ideas in this subsection can be understood by focusing on the case of a non-decreasing $\mathbf{s}$, taking $\pi_i = i$ for all i: then we have $\mathbf{t}_{[i:j]} = (t_i, t_{i+1}, \ldots, t_j)$ and $\mathbf{M}_{[i:j,k:l]} = (\mathbf{M}_{u,v})_{i \leqslant u \leqslant j, k \leqslant v \leqslant l}$.

We now introduce the notion of splitting index, which will help us to locate zero blocks in $\mathbf{P}$.

Definition 2.6 (Splitting index). *Let $\mathbf{d} \in \mathbb{N}^m$, $\mathbf{t} \in \mathbb{Z}^m$ and $\boldsymbol{\pi}^{\mathbf{t}} = (\mu_i)_i$. Then, $i \in \{1, \ldots, m-1\}$ is a* splitting index *for $(\mathbf{d}, \mathbf{t})$ if $d_{\mu_j} + t_{\mu_j} - t_{\mu_{i+1}} < 0$ for all $j \in \{1, \ldots, i\}$.*

In particular, if i is a splitting index for $(\boldsymbol{\delta}, \mathbf{s})$, then we have $[\mathbf{P}_{[:i,:i]} | \mathbf{P}_{[:i,i+1:]}] = [\mathbf{P}_{[:i,:i]} | \mathbf{0}]$. Our algorithm first looks for such a splitting index, and then uses $\mathbf{P}_{[:i,i+1:]} = \mathbf{0}$ to split the problem into two subproblems of dimensions i and $m - i$.

To find a splitting index, we rely on the following property: if $(\mathbf{d}, \mathbf{t})$ does not admit a splitting index, then $|\mathbf{d}| > \operatorname{amp}(\mathbf{t})$. This allows us to partition $\mathbf{s}$ into ℓ subtuples which all contain a splitting index, as follows.

Given $\alpha \in \mathbb{Z}_{>0}$ we let $\ell = 1 + \lfloor \operatorname{amp}(\mathbf{s})/\alpha \rfloor$ and we consider the subtuples $\mathbf{s}_1, \ldots, \mathbf{s}_\ell$ of $\mathbf{s}$ where $\mathbf{s}_k$ consists of the entries of $\mathbf{s}$ in $\{\min(\mathbf{s}) + (k-1)\alpha, \ldots, \min(\mathbf{s}) + k\alpha - 1\}$; this gives a subroutine $\textsc{Partition}(\mathbf{s}, \alpha) = (\mathbf{s}_1, \ldots, \mathbf{s}_\ell)$. Now we take $\alpha \geqslant 2\sigma$ and we assume $s_{\pi_{i+1}} - s_{\pi_i} \leqslant \sigma$ for $1 \leqslant i < m$ without loss of generality [21, Appendix A]. Then, for $1 \leqslant k < \ell$, since $|\boldsymbol{\delta}| \leqslant \sigma$ and $\operatorname{amp}(\mathbf{t}) \geqslant \sigma$ with $\mathbf{t} = (\mathbf{s}_k, \min(\mathbf{s}_{k+1}))$, by the above remark $\mathbf{s}_k$ contains a splitting index for $(\boldsymbol{\delta}, \mathbf{s})$.

Still, we do not know in advance which entries of $\mathbf{s}_k$ correspond to splitting indices for $(\boldsymbol{\delta}, \mathbf{s})$. Thus we recursively compute the $\mathbf{s}$-Popov solution basis $\mathbf{P}^{(0)}$ for $\mathbf{s}_1, \ldots, \mathbf{s}_{\ell/2}$, and we are now going to prove that this gives us a splitting index which divides the computation into two subproblems, the first of which has been already solved by computing $\mathbf{P}^{(0)}$.

Lemma 2.7. *Let $j \in \{2, \ldots, m\}$, $\mathbf{s}^{(0)} = \mathbf{s}_{[:j]}$, $\mathbf{P}^{(0)}$ be the $\mathbf{s}^{(0)}$-Popov solution basis for $(\mathfrak{m}, \mathbf{F}_{[:j]})$, and $\boldsymbol{\delta}^{(0)}$ be its $\mathbf{s}^{(0)}$-pivot degree. Suppose that there is a splitting index $i \leqslant j$ for $(\boldsymbol{\delta}^{(0)}, \mathbf{s}^{(0)})$. Let $\mathbf{P}^{(1)} \in \mathbb{K}[X]^{i \times i}$ be the $\mathbf{s}^{(1)}$-Popov solution basis for $(\mathfrak{m}, \mathbf{F}_{[:i]})$ with $\mathbf{s}^{(1)} = \mathbf{s}_{[:i]}$, and let $\boldsymbol{\delta}^{(1)}$ be its $\mathbf{s}^{(1)}$-pivot degree. Then i is a splitting index for $(\boldsymbol{\delta}, \mathbf{s})$ and $\mathbf{P}_{[:i,:i]} = \mathbf{P}^{(1)} = \mathbf{P}^{(0)}_{[:i,:i]}$, hence $\boldsymbol{\delta}_{[:i]} = \boldsymbol{\delta}^{(1)} = \boldsymbol{\delta}^{(0)}_{[:i]}$ (where $\mathbf{P}^{(0)}$ and $\boldsymbol{\delta}^{(0)}$ are indexed by $\{\pi_1, \ldots, \pi_j\}$ sorted increasingly).*

Proof. Since i is a splitting index for $(\boldsymbol{\delta}^{(0)}, \mathbf{s}^{(0)})$ we have $[\mathbf{P}^{(0)}_{[:i,:i]} | \mathbf{P}^{(0)}_{[:i,i+1:]}] = [\mathbf{Q} | \mathbf{0}]$ for some $\mathbf{Q} \in \mathbb{K}[X]^{i \times i}$. Now, for any $\mathbf{B} \in \mathbb{K}[X]^{m \times m}$ with $[\mathbf{B}_{[:i,:i]} | \mathbf{B}_{[:i,i+1:]}] = [\mathbf{P}^{(1)} | \mathbf{0}]$, $\mathbf{B}_{[:i,:]}$ is in $\mathbf{s}$-Popov form with its rows being solutions for $(\mathfrak{M}, \mathbf{F})$. Then, by minimality of $\mathbf{P}$, $\mathbf{P}_{[:i,:]}$ has $\mathbf{s}$-pivot degree at most $\boldsymbol{\delta}^{(1)}$ componentwise, so that i is also a splitting index for $(\boldsymbol{\delta}, \mathbf{s})$, and in particular $[\mathbf{P}_{[:i,:i]} | \mathbf{P}_{[:i,i+1:]}] = [\mathbf{R} | \mathbf{0}]$ for some $\mathbf{R} \in \mathbb{K}[X]^{i \times i}$. It remains to prove that $\mathbf{Q} = \mathbf{R} = \mathbf{P}^{(1)}$.

Since $\mathbf{R}\mathbf{F}_{[:i]} = 0 \bmod \mathfrak{m}$ and $\mathbf{R} = \mathbf{P}_{[:i,:i]}$ is in $\mathbf{s}^{(1)}$-Popov form, proving that all solutions $\mathbf{p} \in \mathbb{K}[X]^{1 \times i}$ for $(\mathfrak{m}, \mathbf{F}_{[:i]})$ are in the row space of $\mathbf{R}$ is enough to obtain $\mathbf{R} = \mathbf{P}^{(1)}$. Since $\mathbf{q} \in \mathbb{K}[X]^{1 \times m}$ defined by $[\mathbf{q}_{[:i]} | \mathbf{q}_{[i+1:]}] = [\mathbf{p} | \mathbf{0}]$ is a solution for $(\mathfrak{m}, \mathbf{F})$, $\mathbf{q} = \boldsymbol{\lambda}\mathbf{P}$ for some $\boldsymbol{\lambda} \in \mathbb{K}[X]^{1 \times m}$. Now $\mathbf{P}$ is nonsingular, thus $\mathbf{P}_{[:i,i+1:]} = \mathbf{0}$ implies that $[\boldsymbol{\lambda}_{[:i]} | \boldsymbol{\lambda}_{[i+1:]}] =$

$[\boldsymbol{\mu}|\mathbf{0}]$ with $\boldsymbol{\mu} \in \mathbb{K}[X]^{1\times i}$, hence $\mathbf{p} = \mathbf{q}_{[:i]} = \boldsymbol{\lambda}_{[:i]}\mathbf{P}_{[:i,:i]} + \boldsymbol{\lambda}_{[i+1:]}\mathbf{P}_{[i+1:,:i]} = \boldsymbol{\mu}\mathbf{Q}$. Similar arguments give $\mathbf{Q} = \mathbf{P}^{(1)}$. $\square$

The next two lemmas show that knowing $\boldsymbol{\delta}^{(1)}$, which is $\boldsymbol{\delta}_{[:i]}$, allows us to compute a so-called *residual* $(\mathfrak{n}, \mathbf{G})$ from which we can complete the computation of $\boldsymbol{\delta}$ and $\mathbf{P}$.

LEMMA 2.8. *Let* $\mathbf{s}^{(2)} = \mathbf{s}_{[i+1:]}$, $\mathbf{d} = -\boldsymbol{\delta}^{(1)} + \min(\mathbf{s}^{(2)}) - 2\sigma \in \mathbb{Z}^i$, $\mathbf{v} \in \mathbb{Z}^m$ *be such that* $[\mathbf{v}_{[:i]}|\mathbf{v}_{[i+1:]}] = [\mathbf{d}|\mathbf{s}^{(2)}]$, *and* $\mathbf{u} = (\mathbf{v}, \min(\mathbf{d})) \in \mathbb{Z}^{m+1}$. *Let* $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{smallmatrix}\right]$ *be the* $\mathbf{u}$*-Popov order basis for* $[\mathbf{F}^\mathsf{T}|\mathfrak{m}]^\mathsf{T}$ *and* 2σ, *where* $\mathbf{A} \in \mathbb{K}[X]^{m\times m}$ *and* $q \in \mathbb{K}[X]$. *Then we have* $\deg(q) \geqslant \sigma$, $\mathbf{A}_{[:i,i+1:]} = \mathbf{0}$, $\mathbf{p}_{[i+1:]} = \mathbf{0}$, *and* $[\mathbf{A}_{[:i,:i]}|\mathbf{q}_{[:i]}] = [\mathbf{P}^{(1)}|\mathbf{q}^{(1)}]$ *with* $\mathbf{q}^{(1)} = -\mathbf{P}^{(1)}\mathbf{F}_{[:i]}/\mathfrak{m}$.

PROOF. Since $\mathbf{u} = (\mathbf{v}, \min(\mathbf{v}))$ we have $\deg(\mathbf{p}) \leqslant \deg(q)$, and since $\deg(\mathbf{F}) < \deg(\mathfrak{m})$ the degree of $\mathbf{pF}+q\mathfrak{m}$ is $\deg(q)+\sigma$; then $\mathbf{pF}+q\mathfrak{m} = 0 \bmod X^{2\sigma}$ implies $\deg(q)+\sigma \geqslant 2\sigma$. Now, since $\mathbf{A}$ is in $\mathbf{v}$-Popov form and $\deg(\mathbf{A}) \leqslant 2\sigma - \deg(q) < 2\sigma$, from $\min(\mathbf{s}^{(2)}) \geqslant \max(\mathbf{d})+2\sigma$ we get $\mathbf{A}_{[:i,i+1:]} = \mathbf{0}$. Besides, $\mathbf{p}_{[i+1:]} = \mathbf{0}$ since either $\deg(q) < 2\sigma$ and then $\min(\mathbf{s}^{(2)}) > \min(\mathbf{d})+\deg(q)$, or $\mathbf{A}$ is the identity matrix and then $\mathbf{p} = \mathbf{0}$.

Furthermore, by Lemma 2.1 $[\mathbf{P}^{(1)}|\mathbf{q}^{(1)}]$ is the $(\mathbf{d}, \min(\mathbf{d}))$-Popov nullspace basis for $[\mathbf{F}_{[:i]}^\mathsf{T}|\mathfrak{m}]^\mathsf{T}$, with $(\mathbf{d}, \min(\mathbf{d}))$-pivot index $\{1, \ldots, i\}$, $(\mathbf{d}, \min(\mathbf{d}))$-pivot degree $\boldsymbol{\delta}^{(1)}$ and degree at most $\max(\boldsymbol{\delta}^{(1)})$. Then, as in the proof of Lemma 2.2, one can show that $[\mathbf{A}_{[:i,:i]}|\mathbf{q}_{[:i]}] = [\mathbf{P}^{(1)}|\mathbf{q}^{(1)}]$. $\square$

Thus, up to row and column permutations this order basis is $\left[\begin{smallmatrix} \mathbf{P}^{(1)} & \mathbf{0} & \mathbf{q}^{(1)} \\ * & \mathbf{P}^{(2)} & * \\ * & \mathbf{0} & q \end{smallmatrix}\right]$ with $\mathbf{P}^{(2)} = \mathbf{A}_{[i+1:,i+1:]} \in \mathbb{K}[X]^{(m-i)\times(m-i)}$ in $\mathbf{s}^{(2)}$-Popov form; let $\boldsymbol{\delta}^{(2)}$ denote its $\mathbf{s}^{(2)}$-pivot degree.

LEMMA 2.9. *Let* $\mathfrak{n} = X^{-2\sigma}(\mathbf{p}_{[i+1:]}\mathbf{F}_{[i+1:]} + q\mathfrak{m}) \in \mathbb{K}[X]$ *and* $\mathbf{G} = X^{-2\sigma}(\mathbf{A}_{[i+1:,:]}\mathbf{F} + \mathbf{q}_{[i+1:]}\mathfrak{m}) \in \mathbb{K}[X]^{(m-i)\times 1}$. *Then,* $\deg(\mathbf{G}) < \deg(\mathfrak{n}) \leqslant \sigma - |\boldsymbol{\delta}^{(1)}| - |\boldsymbol{\delta}^{(2)}|$. *Let* $\mathbf{P}^{(3)}$ *be the* $\mathbf{t}$*-Popov solution basis for* $(\mathfrak{n}, \mathbf{G})$ *with* $\mathbf{t} = \mathrm{rdeg}_{\mathbf{s}^{(2)}}(\mathbf{P}^{(2)})$ *and* $\boldsymbol{\delta}^{(3)}$ *be its* $\mathbf{t}$*-pivot degree. Then,* $(\boldsymbol{\delta}_{[:i]}, \boldsymbol{\delta}_{[i+1:]}) = (\boldsymbol{\delta}^{(1)}, \boldsymbol{\delta}^{(2)} + \boldsymbol{\delta}^{(3)})$.

PROOF. The sum $|\boldsymbol{\delta}^{(1)}| + |\boldsymbol{\delta}^{(2)}| + \deg(q)$ of the $\mathbf{u}$-pivot degrees of $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{smallmatrix}\right]$ is at most the order 2σ. Thus, we have $\deg(\mathfrak{n}) = \deg(q) - \sigma \leqslant \sigma - |\boldsymbol{\delta}^{(1)}| - |\boldsymbol{\delta}^{(2)}|$, $\deg(\mathbf{A}_{[i+1:,:i]}) < |\boldsymbol{\delta}^{(1)}| \leqslant \sigma$, $\deg(\mathbf{A}_{[i+1:,i+1:]}) \leqslant |\boldsymbol{\delta}^{(2)}| \leqslant \sigma$, and $\deg(\mathbf{q}_{[i+1:]}) < \deg(q)$. This implies $\deg(\mathbf{G}) < \deg(q) - \sigma = \deg(\mathfrak{n})$.

Let $\mathbf{q}^{(3)} = -\mathbf{P}^{(3)}\mathbf{G}/\mathfrak{n}$ and $t = \mathrm{rdeg}_{\mathbf{u}}([\mathbf{p}|q]) = \deg(q) + \min(\mathbf{d}) \leqslant \min(\mathbf{s}^{(2)}) \leqslant \min(\mathbf{t})$. By Lemma 2.1, $[\mathbf{P}^{(3)}|\mathbf{q}^{(3)}]$ is the $(\mathbf{t}, t)$-Popov nullspace basis for $[\mathbf{G}^\mathsf{T}|\mathfrak{n}]^\mathsf{T}$. Defining $\mathbf{B} \in \mathbb{K}[X]^{m\times m}$ and $\mathbf{c} \in \mathbb{K}[X]^{m\times 1}$ by $\left[\begin{smallmatrix} \mathbf{B}_{[:i,:i]} & \mathbf{B}_{[:i,i+1:]} & \mathbf{c}_{[:i]} \\ \mathbf{B}_{[i+1:,:i]} & \mathbf{B}_{[i+1:,i+1:]} & \mathbf{c}_{[i+1:]} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^{(3)} & \mathbf{q}^{(3)} \end{smallmatrix}\right]$, then $\left[\begin{smallmatrix} \mathbf{B} & \mathbf{c} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{smallmatrix}\right]$ is a $\mathbf{u}$-minimal nullspace basis of $[\mathbf{F}^\mathsf{T}|\mathfrak{m}]^\mathsf{T}$ [36, Theorem 3.9]. Thus Lemma 2.1 implies that $\bar{\mathbf{P}} = \left[\begin{smallmatrix} \mathbf{B} & \mathbf{c} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{A} \\ \mathbf{p} \end{smallmatrix}\right]$ is a $\mathbf{v}$-minimal solution basis for $(\mathfrak{m}, \mathbf{F})$.

It is easily checked that $\mathbf{P}$ is in $\mathbf{v}$-Popov form, so that the $\mathbf{v}$-Popov form of $\bar{\mathbf{P}}$ is $\mathbf{P}$ and its $\mathbf{v}$-pivot degree is $\boldsymbol{\delta}$. Besides $\left[\begin{smallmatrix} \bar{\mathbf{P}}_{[:i,:i]} & \bar{\mathbf{P}}_{[:i,i+1:]} \\ \bar{\mathbf{P}}_{[i+1:,:i]} & \bar{\mathbf{P}}_{[i+1:,i+1:]} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{P}^{(1)} & \mathbf{0} \\ \mathbf{P}^{(3)}\mathbf{A}_{2,1}+\mathbf{q}^{(3)}\mathbf{A}_{3,1} & \mathbf{P}^{(3)}\mathbf{P}^{(2)} \end{smallmatrix}\right]$, so that $(\boldsymbol{\delta}_{[:i]}, \boldsymbol{\delta}_{[i+1:]}) = (\boldsymbol{\delta}^{(1)}, \boldsymbol{\delta}^{(2)} + \boldsymbol{\delta}^{(3)})$ [21, Section 3]. $\square$

This results in Algorithm 2. It takes as input α which dictates the amplitude of the subtuples that partition $\mathbf{s}$; as mentioned above, the initial call can be made with $\alpha = 2\sigma$.

PROPOSITION 2.10. *Algorithm* POLMODSYSONE *is correct and uses* $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ *operations in* $\mathbb{K}$.

PROOF. The correctness follows from the results in this subsection. By [21, Theorem 1.4], each leaf of the recursion at Step **1.a** in dimension m uses $\widetilde{\mathcal{O}}(m^{\omega-1}\alpha)$ operations.

ALGORITHM 2 (POLMODSYSONE).

Input: a polynomial $\mathfrak{m} \in \mathbb{K}[X]_{\neq 0}$ of degree σ, a column $\mathbf{F} \in \mathbb{K}[X]^{m\times 1}$ with $\deg(\mathbf{F}) < \deg(\mathfrak{m})$, a shift $\mathbf{s} \in \mathbb{Z}^m$, a parameter $\alpha \in \mathbb{Z}_{>0}$ with $\alpha \geqslant 2\sigma$.

Output: the $\mathbf{s}$-Popov solution basis for $(\mathfrak{m}, \mathbf{F})$ and the $\mathbf{s}$-minimal degree $\boldsymbol{\delta}$ of $(\mathfrak{m}, \mathbf{F})$.

1. If $\mathrm{amp}(\mathbf{s}) \leqslant 2\alpha$:

- **a.** $\mathbf{A} \leftarrow$ the $(\mathbf{s}, \min(\mathbf{s}))$-Popov order basis for $[\mathbf{F}^\mathsf{T}|\mathfrak{m}]^\mathsf{T}$ and $2\alpha + 2\sigma$; return the principal $m \times m$ submatrix of $\mathbf{A}$ and the degrees of its diagonal entries

2. Else: /* $\ell = 1 + \lfloor \mathrm{amp}(\mathbf{s})/\alpha \rfloor \geqslant 3$ */

- **a.** $(\mathbf{s}_1, \ldots, \mathbf{s}_\ell) \leftarrow$ PARTITION$(\mathbf{s}, \alpha)$,
  $j \leftarrow$ sum of the lengths of $\mathbf{s}_1, \ldots, \mathbf{s}_{\lceil \ell/2 \rceil}$, $\mathbf{s}^{(0)} \leftarrow \mathbf{s}_{[:j]}$,
  $(\mathbf{P}^{(0)}, \boldsymbol{\delta}^{(0)}) \leftarrow$ POLMODSYSONE$(\mathfrak{m}, \mathbf{F}_{[:j]}, \mathbf{s}^{(0)}, \alpha)$
- **b.** $i \leftarrow$ the largest splitting index for $(\boldsymbol{\delta}^{(0)}, \mathbf{s}^{(0)})$, $\boldsymbol{\delta}^{(1)} \leftarrow \boldsymbol{\delta}^{(0)}_{[:i]}$, $\mathbf{s}^{(2)} \leftarrow \mathbf{s}_{[i+1:]}$, $\mathbf{d} = -\boldsymbol{\delta}^{(1)} + \min(\mathbf{s}^{(2)}) - 2\sigma$, $\mathbf{v} \in \mathbb{Z}^m$ with $[\mathbf{v}_{[:i]}|\mathbf{v}_{[i+1:]}] \leftarrow [\mathbf{d}|\mathbf{s}^{(2)}]$, $\mathbf{u} = (\mathbf{v}, \min(\mathbf{d}))$
- **c.** $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{q} \\ \mathbf{p} & q \end{smallmatrix}\right] \leftarrow \mathbf{u}$-Popov order basis for $[\mathbf{F}^\mathsf{T}|\mathfrak{m}]^\mathsf{T}$ and 2σ,
  $\boldsymbol{\delta}^{(2)} \leftarrow$ the $\mathbf{s}^{(2)}$-pivot degree of $\mathbf{A}_{[i+1:,i+1:]}$
  $\mathbf{G} \leftarrow X^{-2\sigma}(\mathbf{A}_{[i+1:,:]}\mathbf{F} + \mathbf{q}_{[i+1:]}\mathfrak{m})$,
  $\mathfrak{n} \leftarrow X^{-2\sigma}(\mathbf{p}_{[i+1:]}\mathbf{F}_{[i+1:]} + q\mathfrak{m})$.
- **d.** $\mathbf{t} \leftarrow \mathbf{s}^{(2)} + \boldsymbol{\delta}^{(2)} = \mathrm{rdeg}_{\mathbf{s}^{(2)}}(\mathbf{A}_{[i+1:,i+1:]})$,
  $(\mathbf{P}^{(3)}, \boldsymbol{\delta}^{(3)}) \leftarrow$ POLMODSYSONE$(\mathfrak{n}, \mathbf{G}, \mathbf{t}, \alpha)$
- **e.** $\boldsymbol{\delta} \in \mathbb{N}^m$ with $(\boldsymbol{\delta}_{[:i]}, \boldsymbol{\delta}_{[i+1:]}) \leftarrow (\boldsymbol{\delta}^{(1)}, \boldsymbol{\delta}^{(2)} + \boldsymbol{\delta}^{(3)})$
  $\mathbf{P} \leftarrow$ KNOWNDEGPOLMODSYS$(\mathfrak{m}, \mathbf{F}, \mathbf{s}, \boldsymbol{\delta})$
- **f.** Return $(\mathbf{P}, \boldsymbol{\delta})$

Running the algorithm with initial input $\alpha = 2\sigma$, the recursive tree has depth $\mathcal{O}(\log(\ell)) = \mathcal{O}(\log(1 + \mathrm{amp}(\mathbf{s})/2\sigma))$, with $\mathrm{amp}(\mathbf{s})/2\sigma \in \mathcal{O}(m^2)$ [21, Appendix A]. All recursive calls are for a modulus of degree $\sigma < \alpha$. The order basis computation at Step **2.c** uses $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ operations; the computation of $\mathbf{G}$ and $\mathfrak{n}$ at Step **2.c** can be done in time $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ using partial linearization as in Lemma 2.11 below; Step **2.e** uses $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ operations by Proposition 2.4.

On a given level of the tree, the sum of the dimensions of the column vector in input of each sub-problem is in $\mathcal{O}(m)$. Since $a^{\omega-1} + b^{\omega-1} \leqslant (a+b)^{\omega-1}$ for all $a, b > 0$, each level of the tree uses a total of $\widetilde{\mathcal{O}}(m^{\omega-1}\alpha)$ operations. $\square$

2.3 Fast divide-and-conquer algorithm

Now that we have an efficient algorithm for $n = 1$, our main algorithm uses a divide-and-conquer approach on n. Similarly to [21, Algorithm 1], from the two bases obtained recursively we first deduce the $\mathbf{s}$-minimal degree $\boldsymbol{\delta}$, and then we use this knowledge to compute $\mathbf{P}$ with Algorithm 1. When $\sigma = \deg(\mathfrak{m}_1) + \cdots + \deg(\mathfrak{m}_n) \in \mathcal{O}(m)$, we rely on the algorithm LINEARIZATIONMIB in [20, Algorithm 9].

The computation of the so-called *residual* at Step **3.c** can be done efficiently using partial linearization, as follows.

LEMMA 2.11. *Let* $\mathfrak{M} = (\mathfrak{m}_j)_j \in \mathbb{K}[X]^n_{\neq 0}$, $\mathbf{P} \in \mathbb{K}[X]^{m\times m}$, $\mathbf{F} \in \mathbb{K}[X]^{m\times n}$ *with* $m \geqslant n$ *and* $\deg(\mathbf{F}_{*,j}) < \sigma_j = \deg(\mathfrak{m}_j)$, *and let* $\sigma \geqslant m$ *such that* $\sigma \geqslant \sigma_1 + \cdots + \sigma_n$ *and* $|\mathrm{cdeg}(\mathbf{P})| \leqslant \sigma$. *Then* $\mathbf{PF} \bmod \mathfrak{M}$ *can be computed in* $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ *operations.*

PROOF. Using notation from Lemma 2.3, we let $\widetilde{\mathbf{P}} \in \mathbb{K}[X]^{m\times\widetilde{m}}$ such that $\mathbf{P} = \widetilde{\mathbf{P}}\mathcal{E}$ and $\deg(\widetilde{\mathbf{P}}) < \lceil |\mathrm{cdeg}(\mathbf{P})|/m \rceil$. As above, $\widetilde{\mathbf{F}} = \mathcal{E}\mathbf{F} \bmod \mathfrak{M}$ can be computed in time $\widetilde{\mathcal{O}}(m\sigma)$. Here we want to compute $\mathbf{PF} \bmod \mathfrak{M} = \widetilde{\mathbf{P}}\,\widetilde{\mathbf{F}} \bmod \mathfrak{M}$.

Algorithm 3 (PolModSys).

Input: polynomials $\mathfrak{M} = (\mathfrak{m}_1, \ldots, \mathfrak{m}_n) \in \mathbb{K}[X]^n_{\neq 0}$, a matrix $\mathbf{F} \in \mathbb{K}[X]^{m \times n}$ with $\deg(\mathbf{F}_{*,j}) < \deg(\mathfrak{m}_j)$, a shift $\mathbf{s} \in \mathbb{Z}^m$.

Output: the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M}, \mathbf{F})$ and the $\mathbf{s}$-minimal degree $\boldsymbol{\delta}$ of $(\mathfrak{M}, \mathbf{F})$.

1. If $\sigma = \deg(\mathfrak{m}_1) + \cdots + \deg(\mathfrak{m}_n) \leqslant m$:

 a. Build $\mathbf{E} \in \mathbb{K}^{m \times \sigma}$ and $\mathbf{M} \in \mathbb{K}^{\sigma \times \sigma}$ as in Section 1.2

 b. Return LinearizationMIB$(\mathbf{E}, \mathbf{M}, \mathbf{s}, 2^{\lceil \log_2(\sigma) \rceil})$

2. Else if $n = 1$: Return PolModSysOne$(\mathfrak{m}_1, \mathbf{F}, \mathbf{s}, 2\sigma)$

3. Else:

 a. $\mathfrak{M}^{(1)}, \mathbf{F}^{(1)} \leftarrow (\mathfrak{m}_1, \ldots, \mathfrak{m}_{\lfloor n/2 \rfloor}),\ \mathbf{F}_{*,1\ldots\lfloor n/2 \rfloor}$
 $\mathfrak{M}^{(2)}, \mathbf{F}^{(2)} \leftarrow (\mathfrak{m}_{\lfloor n/2 \rfloor+1}, \ldots, \mathfrak{m}_n),\ \mathbf{F}_{*,\lfloor n/2 \rfloor+1\ldots n}$

 b. $\mathbf{P}^{(1)}, \boldsymbol{\delta}^{(1)} \leftarrow$ PolModSys$(\mathfrak{M}^{(1)}, \mathbf{F}^{(1)}, \mathbf{s})$

 c. $\mathbf{R} \leftarrow \mathbf{P}^{(1)}\mathbf{F}^{(2)} \bmod \mathfrak{M}^{(2)}$

 d. $\mathbf{P}^{(2)}, \boldsymbol{\delta}^{(2)} \leftarrow$ PolModSys$(\mathfrak{M}^{(2)}, \mathbf{R}, \mathrm{rdeg}_{\mathbf{s}}(\mathbf{P}^{(1)}))$

 e. $\mathbf{P} \leftarrow$ KnownDegPolModSys$(\mathfrak{M}, \mathbf{F}, \mathbf{s}, \boldsymbol{\delta}^{(1)} + \boldsymbol{\delta}^{(2)})$

 f. Return $(\mathbf{P}, \boldsymbol{\delta}^{(1)} + \boldsymbol{\delta}^{(2)})$

We have $\deg(\widetilde{\mathbf{P}}) \leqslant \lceil \sigma/m \rceil \leqslant 2\sigma/m$. Since $|\mathrm{cdeg}(\widetilde{\mathbf{F}})| < \sigma$ and $n \leqslant m \leqslant \widetilde{m} \leqslant 2m$, $\widetilde{\mathbf{F}}$ can be partially linearized into $\mathcal{O}(m)$ columns of degree $\mathcal{O}(\sigma/m)$. Then, $\widetilde{\mathbf{P}}\,\widetilde{\mathbf{F}}$ is computed in $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ operations. The j-th column of $\widetilde{\mathbf{P}}\,\widetilde{\mathbf{F}}$ has $\widetilde{m} \leqslant 2m$ rows and degree less than $\sigma_j + 2\sigma/m$: it can be reduced modulo $\mathfrak{m}_j$ in $\widetilde{\mathcal{O}}(\sigma + m\sigma_j)$ operations [13, Chapter 9]; summing over $1 \leqslant j \leqslant n$ with $n \leqslant m$, this is in $\widetilde{\mathcal{O}}(m\sigma)$. $\square$

Proof of Theorem 1.4. The correctness and the cost $\widetilde{\mathcal{O}}(m^{\omega-1}\sigma)$ for Steps **1** and **2** of Algorithm 3 follow from [20, Theorem 1.4] and Proposition 2.10. With the costs of Steps **3.c** and **3.e** given in Proposition 2.4 and Lemma 2.11, we obtain the announced cost bound.

Now, using notation in Step **3**, suppose $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ are the $\mathbf{s}$- and $\mathrm{rdeg}_{\mathbf{s}}(\mathbf{P}^{(1)})$-Popov solution bases for $(\mathfrak{M}^{(1)}, \mathbf{F}^{(1)})$ and $(\mathfrak{M}^{(2)}, \mathbf{R})$. Then $\mathbf{P}^{(2)}\mathbf{P}^{(1)}$ is a solution basis for $(\mathfrak{M}, \mathbf{F})$: if $\mathbf{p}$ is a solution for $(\mathfrak{M}, \mathbf{F})$, it is one for $(\mathfrak{M}^{(1)}, \mathbf{F}^{(1)})$ and thus $\mathbf{p} = \boldsymbol{\lambda}\mathbf{P}^{(1)}$ for some $\boldsymbol{\lambda}$, and it is one for $(\mathfrak{M}^{(2)}, \mathbf{F}^{(2)})$ so that $\mathbf{p}\mathbf{F}^{(2)} = \boldsymbol{\lambda}\mathbf{P}^{(1)}\mathbf{F}^{(2)} = \boldsymbol{\lambda}\mathbf{R} = \mathbf{0} \bmod \mathfrak{M}^{(2)}$ and thus $\boldsymbol{\lambda} = \boldsymbol{\mu}\mathbf{P}^{(2)}$ for some $\boldsymbol{\mu}$; then $\mathbf{p} = \boldsymbol{\mu}\mathbf{P}^{(2)}\mathbf{P}^{(1)}$.

Then $\mathbf{P}^{(2)}\mathbf{P}^{(1)}$ is an $\mathbf{s}$-minimal solution basis for $(\mathfrak{M}, \mathbf{F})$ and its $\mathbf{s}$-Popov form has $\mathbf{s}$-pivot degree $\boldsymbol{\delta}^{(1)} + \boldsymbol{\delta}^{(2)}$ [21, Section 3]. The correctness follows from Proposition 2.4. $\square$

3. FAST COMPUTATION OF THE SHIFTED POPOV FORM OF A MATRIX

3.1 Fast shifted Popov form algorithm

Our fast method for computing the $\mathbf{s}$-Popov form of a nonsingular $\mathbf{A} \in \mathbb{K}[X]^{m \times m}$ uses two steps, as follows.

1. Compute the Smith form of $\mathbf{A}$, giving the moduli $\mathfrak{M}$, and a corresponding right unimodular transformation, giving the equations $\mathbf{F}$, so that $\mathbf{A}$ is a solution basis for $(\mathfrak{M}, \mathbf{F})$.

2. Find the $\mathbf{s}$-Popov solution basis for $(\mathfrak{M}, \mathbf{F})$.

We first show the correctness of this approach.

Lemma 3.1. *Let $\mathbf{A} \in \mathbb{K}[X]^{m \times m}$ be nonsingular and $\mathbf{S} = \mathbf{U}\mathbf{A}\mathbf{V}$ be the Smith form of $\mathbf{A}$ with $\mathbf{U}$ and $\mathbf{V}$ unimodular. Let $\mathfrak{M} \in \mathbb{K}[X]^m_{\neq 0}$ and $\mathbf{F} \in \mathbb{K}[X]^{m \times m}$ be such that $\mathbf{S} = \mathrm{diag}(\mathfrak{M})$ and $\mathbf{F} = \mathbf{V} \bmod \mathfrak{M}$. Then $\mathbf{A}$ is a solution basis for $(\mathfrak{M}, \mathbf{F})$.*

Proof. Let $\mathbf{p} \in \mathbb{K}[X]^{1 \times m}$. If $\mathbf{p}$ is in the row space of $\mathbf{A}$ then $\mathbf{p}$ is a solution for $(\mathfrak{M}, \mathbf{F})$ since $\mathbf{A}\mathbf{V} = \mathbf{U}^{-1}\mathbf{S}$ with $\mathbf{U}^{-1}$ over $\mathbb{K}[X]$. Now if $\mathbf{p}\mathbf{F} = 0 \bmod \mathfrak{M}$, then $\mathbf{p}\mathbf{V} = \mathbf{q}\mathbf{S}$ for some $\mathbf{q}$ and $\mathbf{p} = \mathbf{q}\mathbf{U}\mathbf{A}$ is in the row space of $\mathbf{A}$. $\square$

Concerning the cost of Step **1**, such $\mathfrak{M}$ and $\mathbf{F}$ can be obtained in expected $\widetilde{\mathcal{O}}(m^\omega \deg(\mathbf{A}))$ operations, by computing

1.a $\mathbf{R}$ a row reduced form of $\mathbf{A}$ [16, Theorem 18],

1.b $\mathrm{diag}(\mathfrak{M})$ the Smith form of $\mathbf{R}$ [29, Algorithm 12],

1.c $(*, \mathbf{F})$ a reduced Smith transform for $\mathbf{R}$ [15, Figure 3.2];

as in [15, Figure 6.1], Steps **1.b** and **1.c** should be performed in conjunction with the preconditioning techniques detailed in [23]. One may take for $\mathfrak{M}$ only the nontrivial Smith factors, and for $\mathbf{F}$ only the nonzero columns of the transform.

The product of the moduli in $\mathfrak{M}$ is $\det(\mathbf{A})$ so that the sum of their degrees is $\deg(\det(\mathbf{A}))$. Then, according to Theorem 1.4, Step **2** of the algorithm outlined above costs $\widetilde{\mathcal{O}}(m^{\omega-1}\deg(\det(\mathbf{A})))$ operations. Thus this algorithm solves Problem 1 in expected $\widetilde{\mathcal{O}}(m^\omega \deg(\mathbf{A}))$ field operations.

3.2 Reducing to almost uniform degrees

In this subsection, we use the partial linearization techniques from [16, Section 6] to prove the following result.

Proposition 3.2. *Let $\mathbf{A} \in \mathbb{K}[X]^{m \times m}$ be nonsingular and let $\mathbf{s} \in \mathbb{Z}^m$. With no field operation, one can build a nonsingular $\widetilde{\mathbf{A}} \in \mathbb{K}[X]^{\widetilde{m} \times \widetilde{m}}$ and a shift $\mathbf{u} \in \mathbb{Z}^{\widetilde{m}}$ such that $\widetilde{m} \leqslant 3m$, $\deg(\widetilde{\mathbf{A}}) \leqslant \lceil \sigma(\mathbf{A})/m \rceil$, and the $\mathbf{s}$-Popov form of $\mathbf{A}$ is the principal $m \times m$ submatrix of the $\mathbf{u}$-Popov form of $\widetilde{\mathbf{A}}$.*

With the algorithm in the previous subsection, this implies Theorem 1.3. In the specific case of Hermite form computation, for which there is a deterministic algorithm with cost bound $\widetilde{\mathcal{O}}(m^\omega \deg(\mathbf{A}))$ [35], one can verify that this leads to a *deterministic* algorithm using $\widetilde{\mathcal{O}}(m^\omega \lceil \sigma(\mathbf{A})/m \rceil)$ operations. (However, for $\mathbf{s} = \mathbf{0}$ this does not give a $\widetilde{\mathcal{O}}(m^\omega \lceil \sigma(\mathbf{A})/m \rceil)$ *deterministic* algorithm for the Popov form using [16, 28], since the corresponding $\mathbf{u}$ is $(\mathbf{0}, t, \ldots, t)$ with $t \geqslant \deg(\mathbf{A})$.)

Definition 3.3 (Column partial linearization). *Let $\mathbf{A} \in \mathbb{K}[X]^{m \times m}$ and $\boldsymbol{\delta} = (\delta_i)_i \in \mathbb{N}^m$. Then let $\delta = 1 + \lfloor (\delta_1 + \cdots + \delta_m)/m \rfloor$, let $\alpha_i \geqslant 1$ and $0 \leqslant \beta_i < \delta$ be such that $\delta_i = (\alpha_i - 1)\delta + \beta_i$ for $1 \leqslant i \leqslant m$, let $\widetilde{m} = \alpha_1 + \cdots + \alpha_m$, and let $\mathcal{E} = [\mathbf{I} | \mathbf{E}^\mathsf{T}]^\mathsf{T} \in \mathbb{K}[X]^{\widetilde{m} \times m}$ be the expansion-compression matrix with $\mathbf{I}$ the identity matrix and*

$$\mathbf{E} = \begin{bmatrix} X^\delta & & \\ \vdots & & \\ X^{(\alpha_1-1)\delta} & & \\ & \ddots & \\ & & X^\delta \\ & & \vdots \\ & & X^{(\alpha_m-1)\delta} \end{bmatrix}. \tag{4}$$

The column partial linearization $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A}) \in \mathbb{K}[X]^{\widetilde{m} \times \widetilde{m}}$ of $\mathbf{A}$ *is defined as follows:*

- *the first m rows of $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$ form the unique matrix $\widetilde{\mathbf{A}} \in \mathbb{K}[X]^{m \times \widetilde{m}}$ such that $\mathbf{A} = \widetilde{\mathbf{A}}\mathcal{E}$ and $\widetilde{\mathbf{A}}$ has all columns of degree less than δ except possibly those at indices $m + (\alpha_1 - 1) + \cdots + (\alpha_i - 1)$ for $1 \leqslant i \leqslant m$,*
- *for $1 \leqslant i \leqslant m$, the row $m + (\alpha_1 - 1) + \cdots + (\alpha_{i-1} - 1) + 1$ of $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$ is $[0, \ldots, 0, -X^\delta, 0, \ldots, 0, 1, 0, \ldots, 0]$ where $-X^\delta$ is at index i and 1 is on the diagonal,*

- *for* $1 \leqslant i \leqslant m$ *and* $2 \leqslant j \leqslant \alpha_i-1$, *the row* $m+(\alpha_1-1)+\cdots+(\alpha_{i-1}-1)+j$ *of* $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$ *is* $[0,\ldots,0,-X^{\delta},1,0,\ldots,0]$ *where* 1 *is on the diagonal.*

Defining the *row partial linearization* $\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A})$ of $\mathbf{A}$ similarly, both linearizations are related by $\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A}) = \mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}}$.

Now we show that for a well-chosen $\mathbf{u}$, one can directly read the $\mathbf{s}$-Popov form of $\mathbf{A}$ as a submatrix of the $\mathbf{u}$-Popov form of $\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A})$ (resp. $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$).

Lemma 3.4. *Let* $\mathbf{A} \in \mathbb{K}[X]^{m\times m}$ *be nonsingular,* $\mathbf{s} \in \mathbb{Z}^m$, $\mathbf{P}$ *be the* $\mathbf{s}$*-Popov form of* $\mathbf{A}$, *and* $\boldsymbol{\delta} \in \mathbb{N}^m$. *We have that:*

(i) if $\widetilde{m}$ *is the dimension of* $\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A})$ *and* $\mathbf{u} = (\mathbf{s}, t, \ldots, t)$ *is in* $\mathbb{Z}^{\widetilde{m}}$ *with* $t \geqslant \max(\mathbf{s}) + \deg(\mathbf{P})$, *then the* $\mathbf{u}$*-Popov form of* $\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A})$ *is* $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ * & \mathbf{I} \end{bmatrix}$;

(ii) if $\widetilde{m}$ *is the dimension of* $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$, $\mathbf{E}$ *is as in* (4), *and* $\mathbf{u} = (\mathbf{s}, \mathbf{t}) \in \mathbb{Z}^{\widetilde{m}}$ *for any* $\mathbf{t} \in \mathbb{Z}^{\widetilde{m}-m}$, *then the* $\mathbf{u}$*-Popov form of* $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A}) \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E} & \mathbf{I} \end{bmatrix}$ *is* $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$;

(iii) if $\widetilde{m}$ *is the dimension of* $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$ *and* $\mathbf{u} = (\mathbf{s}, t, \ldots, t)$ *is in* $\mathbb{Z}^{\widetilde{m}}$ *with* $t \geqslant \max(\mathbf{s}) + \deg(\mathbf{P})$, *then the* $\mathbf{u}$*-Popov form of* $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$ *is* $\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ * & \mathbf{I} \end{bmatrix}$.

Proof. *(i)* $\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A})$ is left-unimodularly equivalent to $\left[\begin{smallmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{smallmatrix}\right]$ for some $\mathbf{B} \in \mathbb{K}[X]^{(\widetilde{m}-m)\times m}$ [16, Theorem 10 (i)]. Then, let $\mathbf{R}$ be the remainder of $\mathbf{B}$ modulo $\mathbf{P}$, that is, the unique matrix in $\mathbb{K}[X]^{(\widetilde{m}-m)\times m}$ which has column degree bounded by the column degree of $\mathbf{P}$ componentwise and such that $\mathbf{R} = \mathbf{B} + \mathbf{QP}$ for some matrix $\mathbf{Q}$ (see for example [22, Theorem 6.3-15], noting that $\mathbf{P}$ is $\mathbf{0}$-column reduced).

Let $\mathbf{W}$ denote the unimodular matrix such that $\mathbf{P} = \mathbf{WA}$. Then, $\left[\begin{smallmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{QW} & \mathbf{I} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{B} & \mathbf{I} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{smallmatrix}\right]$ is left-unimodularly equivalent to $\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A})$. Besides, since $\deg(\mathbf{R}) < \deg(\mathbf{P})$, we have that $\left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{smallmatrix}\right]$ is in $\mathbf{u}$-Popov form by choice of t.

(ii) The matrix $\left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{smallmatrix}\right]$ is obviously in $\mathbf{u}$-Popov form: it remains to prove that it is left-unimodularly equivalent to $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})\left[\begin{smallmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E} & \mathbf{I} \end{smallmatrix}\right]$. Let $\mathbf{T}$ denote the trailing principal submatrix $\mathbf{T} = \mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})_{m+1\ldots\widetilde{m},m+1\ldots\widetilde{m}}$, and let $\mathbf{W}$ be the unimodular matrix such that $\mathbf{WP} = \mathbf{A}$. Then, $\mathbf{T}$ is unit lower triangular, thus unimodular, and by construction of $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$, for some matrix $\mathbf{B}$ we have $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})\left[\begin{smallmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{E} & \mathbf{I} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{T} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{W} & \mathbf{B} \\ \mathbf{0} & \mathbf{T} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{smallmatrix}\right]$.

(iii) From *(ii)*, $\mathcal{L}^{\mathrm{c}}_{\boldsymbol{\delta}}(\mathbf{A})$ is left-unimodularly equivalent to $\left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{smallmatrix}\right]\left[\begin{smallmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{E} & \mathbf{I} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ -\mathbf{E} & \mathbf{I} \end{smallmatrix}\right]$. Using arguments in the proof of *(i)* above, by choice of t the $\mathbf{u}$-Popov form of $\left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ -\mathbf{E} & \mathbf{I} \end{smallmatrix}\right]$ is $\left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{R} & \mathbf{I} \end{smallmatrix}\right]$ with $\mathbf{R}$ the remainder of $-\mathbf{E}$ modulo $\mathbf{P}$. $\square$

In the usual case where $\deg(\mathbf{P})$ is not known *a priori*, one may choose t using the inequality $\deg(\mathbf{P}) \leqslant \deg(\det(\mathbf{P})) = \deg(\det(\mathbf{A})) \leqslant m\deg(\mathbf{A})$.

This result implies Proposition 3.2 thanks to the following remark from [16]. Let π_1, π_2 be permutation matrices such that $\mathbf{B} = \pi_1\mathbf{A}\pi_2 = [b_{i,j}]_{ij}$ satisfies $\deg(b_{i,i}) \geqslant \deg(b_{j,k})$ for all $j,k \geqslant i$ and $1 \leqslant i \leqslant m$. Defining $\mathbf{d} = (d_i)_i \in \mathbb{N}^m$ by $d_i = \overline{\deg}(b_{i,i}) = \begin{cases} \deg(b_{i,i}) & \text{if } b_{i,i} \neq 0 \\ 0 & \text{otherwise} \end{cases}$, we have $d_1+\cdots+d_m \leqslant \sigma(\mathbf{A})$ by definition of $\sigma(\mathbf{A})$ in (1). Let $\boldsymbol{\delta} = \pi_1^{-1}\mathbf{d}$, where $\mathbf{d}$ is seen as a column vector, and $\boldsymbol{\gamma} = \operatorname{cdeg}(\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A}))$. Then the matrix $\widetilde{\mathbf{A}} = \mathcal{L}^{\mathrm{c}}_{\boldsymbol{\gamma}}(\mathcal{L}^{\mathrm{r}}_{\boldsymbol{\delta}}(\mathbf{A}))$ is $\widetilde{m}\times\widetilde{m}$ with $\widetilde{m} < 3m$, and we have $\deg(\widetilde{\mathbf{A}}) \leqslant \lceil\sigma(\mathbf{A})/m\rceil$ [16, Corollary 3]. Lemma 3.4 further shows that the $\mathbf{s}$-Popov form of $\mathbf{A}$ is the principal $m\times m$ submatrix of the $\mathbf{u}$-Popov form of $\widetilde{\mathbf{A}}$, for the shift $\mathbf{u} = (\mathbf{s}, t, \ldots, t) \in \mathbb{Z}^{\widetilde{m}}$ with $t = \max(\mathbf{s}) + m\deg(\mathbf{A})$.

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