File dissemination in dynamic graphs: The case of independent and correlated links in series

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In this paper we investigate the traversal time of a file across N communication links subject to stochastic changes in the sending rate of each link. Each link's sending rate is modeled by a finite-state Markov process. Two cases, one where links evolve independently of one another (N mutually independent Markov processes), and the second where their behaviors are dependent (these N Markov processes are not mutually independent) are considered. A particular instance where the above is encountered in ad hoc delay/tolerant networks where links are subject to intermittent unavailability.

CCS Concepts: • **Networks → Network performance modeling;**

General Terms: Design, Algorithms, Performance

Additional Key Words and Phrases: Dynamic communication path; Traversal time; Markov process; Laplace-Stietljes transform; Pearson correlation coefficient; Stochastic bound.

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1. INTRODUCTION

In this paper we investigate the traversal time of a file across N communication links (or edges) subject to stochastic changes. Throughout the paper the state of each link is modeled by a finite-state Markov process to account for the link's sending rate over time. We will address two cases, one where links are independent of each other and a second where they exhibit local dependencies.

The situation depicted above is encountered in many environments that include mobile ad hoc networks, vehicular networks, and disruption/delay tolerant networks (DTNs), where links are typically intermittently unavailable. Specifically, the state of a link at any time t (i.e., absent, present, high rate, low rate) may be random, as well as possibly depend on its state prior to t. A simple model is the two state on-off Markov process. Apart from mobile networks, vehicular networks, and DTNs, dynamic links are observed in naturally occurring in linear network topologies such as military convoys, underwater networks, traffic light networks, and linear sensor networks, e.g., for monitoring bridges. Links in these networks exhibit varying levels of intermittence due to a variety of factors such as relative mobility of vehicles, significant signal propagation loss due to rugged terrain, or multi-path fading and shadowing.

The time spent traversing a path with such intermittent connectivity includes both the time spent crossing the links along the path and the time spent waiting in node buffers for the links to appear. Unlike the case of a static path where each link is always available for use (and thus the path's traversal time is the sum of the queueing delays and the delays to cross the constituent links),

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computing statistics (e.g., distribution, average) of the path (or the end-to-end blue) traversal time along a dynamic path is not easy. This is the problem we study in this paper.

Throughout the paper we focus on the time to send a single file across a path composed of an arbitrary number of links. The file is sent according to the store-and-forward technique across each link (the first bit of the file cannot be transmitted across a link before the entire file is ready for transmission across that link). We focus on the transmission of only one file. We do not model queueing delays explicitly, however they can be accounted for in the distribution of the time needed to transmit the file across a link. File path traversal times are then sums of propagation delays and transmission times across the links, the latter including times during which links are not available for transmission (their transmission rate is null).

We make two sets of contributions (referred below to as S1 and S2) depending on whether or not the links behave independently of each other. In both cases the state of each link is modeled as a finite-state Markov process, which gives the available transmission rate of the link at any time.

S1: When links behave *independently* of each other and are initially in an arbitrary state, we derive (Section 3.3) a recursive scheme for the calculation of the joint Laplace-Stieltjes Transform (LST) of the joint traversal times across each link (when links are initially in steady-state it is easy to find this LST - see Section 3.2). Specializing this result to on-off links, we show that the expected path traversal time can be computed in $O(n^2)$ operations, with n the number of links, and we exhibit lower and upper stochastic bounds on the expected path traversal time (Section 3.4). Last, the case where transmission times are random is treated in Sections 3.5 and 3.6. Section 3.5 focuses on exponentially distributed transmission times for general Markov link models and Section 3.6 allows for generally distributed transmission times but for the case of on-off links.

S2: When links are described by on-off processes, adjacent links are characterized by a particular *local dependence structure*, and links are initially in steady-state, we obtain the LST of the path traversal time and the expected path traversal time in explicit form. The resulting expressions are presented as functions of the Neyman-Pearson correlation coefficient, which characterizes the amount of linear dependence between adjacent links (Section 4).

Besides the sections mentioned above, related works are discussed in Section 2, various notation, definitions and assumptions are given in Section 3.1. The paper ends with numerical results in Section 5 and concluding remarks in Section 6.

Notation: For any mapping $s \to f(s)$, $f'(a)$ denotes the derivative of $f(s)$ w.r.t. s evaluated at $s = a$. For any number $a \in [0,1]$ we denote $\bar{a} = 1 - a$. The $|J| \times |J|$ diagonal matrix with diagonal elements x_j, $j \in J$, is denoted by $\mathrm{diag}(x_j, j \in J)$. Here, $|J|$ denotes the cardinality of the set J. Last, $\mathbf{v}^T$ is the transpose of the row vector $\mathbf{v}$.

2. RELATED WORK

Our work can be cast into the framework of weighted graphs with stochastic weights, where in our case weights are the link traversal times. [Hall 1986] is the first paper to consider stochastic time-dependent travel times in a network. The author shows that for such networks classical shortest path algorithms (e.g. Dijkstra algorithm) do not necessarily find the best route between two nodes and proposes an adaptive decision rule to find the optimal route. More recently, a general framework for finding the optimal routing policy for stochastic time-dependent weights has been proposed in [Song and Chabini 2006]. A literature review on optimal routing for graphs with both deterministic and stochastic time-dependent weights can be found in the latter paper.

Time-varying graphs [Ferreira 2004] are useful in the study of communication networks with intermittent connectivity such as delay tolerant networks [Jain et al. 2004]. Markov random graphs have been proposed in [Clementi et al. 2007], studied in the context of random walks in [Figueiredo et al. 2012], and used to derive asymptotic scaling laws for flooding times in such networks [Clementi et al. 2008]. The closest work to ours is [Nain et al. 2013], which computes the expected traversal time between a source-destination pair across a linear network where all links are described by independent and identical discrete time two state on-off models under the assumptions that service requires either zero or one unit of time. Our work considers a continuous time model where links are described by finite state, non-identical and possibly correlated Markov chains with a richer set of transmission models.

Last, our correlated link model based on local interactions is an example of an interacting particle system on a line. Typically, studies of such systems concern themselves with the existence and characterization of the stationary distribution. They are in general not concerned with metrics such as traversal times. See [Liggett 1985] for a background of this area and other examples. An exception is [Buchholz and Felko 2015] where link traversal times are modeled by correlated random variables with phase-type distributions (if the traversal time of link u ends in phase x it will begin at link v in phase y with probability $P_{u,v}(x, y)$ if the path (u, v) is followed). This Markovian setting is different from our setting which need not be Markovian and also does not account for different transmission models.

3. INDEPENDENT LINKS IN SERIES

3.1. Notation, definition, model

Consider N links in series, labeled $1, 2, \ldots, N$. Unless stated otherwise, the following n is an arbitrary integer in $\{1, \ldots, N\}$. Let $X_n(t) \in E_n$ denote the state of link n at time $t \geq 0$, where E_n is a countable and finite set, and let $X(t) = (X_1(t), \ldots, X_N(t)) \in E := E_1 \times \cdots \times E_N$ be the state of all links at time t. Link states may be associated with different transmission rates to account for channel physical conditions, as discussed later on.

In the following, $\mathbb{P}_x$ and $\mathbb{E}_x$ will denote the probability and the expectation operator, respectively, given that $X(0) = x := (x_1, \ldots, x_N) \in E$.

Throughout the paper we assume that, for each n, $\mathcal{X}_n := \{X_n(t), t \geq 0\}$ is an irreducible Markov process on E_n with infinitesimal generator $\mathbf{Q}_n = [q_n(i, j)]_{i,j \in E_n}$. Let $\boldsymbol{\pi}_n(t) := (\mathbb{P}(X_n(t) = j), j \in E_n)$ for $t \geq 0$.

We define by D_n the *traversal time* across link n so that $T_n := \sum_{i=1}^{n} D_i$ is the traversal time across links $1, \ldots, n$, also called hereafter the path traversal time. The traversal time across a link is the time that elapses between the instant where the file is ready for transmission and the instant where the last bit of the file is transmitted. The traversal time across a link is the sum of the transmission time (which may account for queueing), the link propagation delay, and the total time during which the link was unavailable during the file transmission.

Define $Y_n := X_n(T_{n-1}) \in E_n$, the state of link n at the time T_{n-1} when the file becomes available for transmission across that link (by convention $T_0 = 0$).

For further use in this section, we introduce the following assumptions:

Assumption 1: *The Markov processes $\mathcal{X}_1, \ldots, \mathcal{X}_N$ are mutually independent.*

Assumption 2: $\mathbb{P}(D_n < x \,|\, \{(Y_j, D_j), j = 1, \ldots, n-1\}, Y_n = i) = \mathbb{P}(D_n < x \,|\, Y_n = i)$ *for all $x > 0$, $i \in E_n$, $n = 1, \ldots, N$.*

Assumption 2 states that, conditioned on the state of the link at the time the file becomes ready for transmission, the link traversal time does not depend on the "past history" of the file on the path (i.e. traversal times across previous links and state of these links). This assumption will hold, for instance, when links behave independently of each other – which will be assumed throughout Section 3 thanks either to Assumption 1 in Sections 3.2-3.3 or to the more restrictive Assumption 3 in Section 3.4 – and when the conditional link transmission times across links $1, \ldots, N$ form mutually independent rvs.

Our first objective is to calculate

$$G_{N,x}(s_1, \ldots, s_N) := \mathbb{E}_x[e^{-(s_1 D_1 + \cdots + s_N D_N)}] \quad (s_i \geq 0, i = 1, \ldots, N) \tag{1}$$

the LST of the joint link traversal times given that the links are initially in state $x \in E$. We will see that this calculation requires the knowledge of

$$F_{n,j}(s) := \mathbb{E}_x[e^{-sD_n} | Y_n = j] \quad (s \geq 0, j \in E_n) \tag{2}$$

the LST of the traversal time across link n given that the link is in state j when the file becomes available for transmission across the link. Last, introduce the conditional expectation

$$\gamma_{n,j} = \mathbb{E}_x[D_n | Y_n = j] = -F'_{n,j}(0) \quad (j \in E_n). \tag{3}$$

When both Assumptions 1 and 2 hold the dependence on the initial state x can be dropped in both definitions (2) and (3) as D_n does not depend on x when the state of link n is known at time T_{n-1}.

Quantities $F_{n,j}(\cdot)$ are calculated in Section 3.5 and in Section 3.6 for some cases of interest.

3.2. Stationary initial link state distribution

When the Markov processes $\mathcal{X}_1, \ldots, \mathcal{X}_N$ are in steady-state at time $t = 0$, with $(\pi^\star_n(j), j \in E_n)$ the stationary distribution vector of $\mathcal{X}_n$, rvs $D_1, \ldots, D_N$ are mutually independent under Assumptions 1 and 2 as shown in Appendix A, so that the unconditional LST of the joint link traversal times is given by

$$\mathbb{E}\left[e^{-(s_1 D_1 + \cdots + s_N D_N)}\right] = \prod_{n=1}^{N} \sum_{j \in E_n} \pi^\star_n(j) F_{n,j}(s_n) \tag{4}$$

under Assumptions 1 and 2. In particular,

$$\mathbb{E}[T_N] = \sum_{n=1}^{N} \sum_{j \in E_n} \pi^\star_n(j) \gamma_{n,j}. \tag{5}$$

3.3. Arbitrary initial link state distribution

In this section, we do not assume that the Markov processes $\mathcal{X}_1, \ldots, \mathcal{X}_N$ are in steady-steady at time $t = 0$. This introduces a dependence between the rvs $D_1, \ldots, D_N$ since, for instance, D_n depends on the state of link n when the file gets ready to be transmitted across that link (given by $X(T_{n-1})$), a quantity that itself depends on both the initial state of link n and $D_1 + \cdots + D_{n-1}$, the traversal time across links $1, \ldots, n-1$.

Assumption 3: *There exist constants $\{c_n(i, j, k), \lambda_n(k)\}_{i,j,k \in E_n}$ such that $(i, j \in E_n,\ t > 0)$*

$$\mathbb{P}(X_n(t) = j | X_n(0) = i) = \sum_{k \in E_n} c_n(i, j, k)\, e^{-\lambda_n(k) t}. \tag{6}$$

Assumption 3 is made for the sake of mathematical tractability (see derivation of (10) in the proof of Proposition 3.1). Thanks to the (Kolmogorov) result $\boldsymbol{\pi}_n(t) = \boldsymbol{\pi}_n(0) e^{\mathbf{Q}_n t}$ $(t \geq 0)$, it is well known that (6) holds when $\mathbf{Q}_n$ is diagonalizable. In this case, the constants $\{-\lambda_n(k)\}_{k \in E_n}$ in (6) are the eigenvalues of the matrix $\mathbf{Q}_n$ and the constants $\{c_n(i, j, k)\}_{i,j,k \in E_n}$ relate to the eigenvectors of $\mathbf{Q}_n$ (see Remark 3.3).

A particular case when $\mathbf{Q}_n$ is diagonalizable is when the Markov process $\mathcal{X}_n$ is reversible [Keilson 1965, p. 116]. This includes the two-state Markov process, which is investigated in Section 3.4. Being reversible for each n does not help in the calculation of $G_{N,x}(s_1, \ldots, s_N)$ (even when these Markov processes are mutually independent) beyond the fact that (6) holds, a fact that we exploit in Section 3.4 when we investigate two-state Markov processes.

Proposition 3.1 below provides a recursive scheme for calculating $G_{n,x}(s_1, \ldots, s_n)$ in terms of the mappings $\{F_{n,j}(\cdot)\}_{n,j}$ introduced in (2).

Proposition 3.1 (LST of joint traversal times).
Under Assumptions 1, 2, and 3

$$G_{1,x}(s_1) = F_{1,x_1}(s_1) \tag{7}$$

and, for $n = 2, \ldots, N$,

$$G_{n,x}(s_1, \ldots, s_n) = \sum_{j,k \in E_n} c_n(x_n, j, k) F_{n,j}(s_n) G_{n-1,x}(s_1 + \lambda_n(k), \ldots, s_{n-1} + \lambda_n(k)) \tag{8}$$

for any $x = (x_1, \ldots, x_N) \in E$.

Proof. Fix $x = (x_1, \ldots, x_N) \in E$. Eq. (7) directly follows from definitions (1) and (2) thanks to $T_0 = 0$. Now let $n \geq 2$. With the shorthand $\mathcal{A}_n((y_i, d_i)) := \{(Y_i = y_i, D_i = d_i), i = 1, \ldots, n-1\}$, we find that

$$\begin{aligned}
G_{n,x}(s_1, \ldots, s_n) = &\sum_{\substack{y_i \in E_i \\ i=1,\ldots,n-1}} \sum_{j \in E_n} \int_{d_1=0}^{\infty} \cdots \int_{d_{n-1}=0}^{\infty} e^{-\sum_{i=1}^{n-1} s_i d_i} \\
&\times \mathbb{E}_x\Big[e^{-s_n D_n} \,|\, \mathcal{A}_n((y_i, d_i)), X_n(D_1 + \cdots + D_{n-1}) = j\Big] \\
&\times \mathbb{P}_x(X_n(d_1 + \cdots + d_{n-1}) = j \,|\, \mathcal{A}_n((y_i, d_i)))\, d\mathbb{P}_x(Y_i = y_i, D_i < d_i, i = 1, \ldots, n-1) \\
= &\sum_{\substack{y_i \in E_i \\ i=1,\ldots,n-1}} \sum_{j \in E_n} \int_{d_1=0}^{\infty} \cdots \int_{d_{n-1}=0}^{\infty} e^{-\sum_{i=1}^{n-1} s_i d_i} F_{n,j}(s_n)\, \mathbb{P}_x(X_n(d_1 + \cdots + d_{n-1}) = j) \qquad (9) \\
&\times d\mathbb{P}_x(Y_i = y_i, D_i < d_i, i = 1, \ldots, n-1) \\
= &\sum_{j \in E_n} \int_{d_1=0}^{\infty} \cdots \int_{d_{n-1}=0}^{\infty} e^{-\sum_{i=1}^{n-1} s_i d_i} F_{n,j}(s_n)\, \mathbb{P}_x(X_n(d_1 + \cdots + d_{n-1}) = j)\, d\mathbb{P}_x(D_i < d_i, i = 1, \ldots, n-1) \\
= &\int_{d_1=0}^{\infty} \cdots \int_{d_{n-1}=0}^{\infty} \sum_{j,k \in E_n} F_{n,j}(s_n)\, c_n(x_n, j, k) e^{-\sum_{i=1}^{n-1} (s_i + \lambda_n(k)) d_i} d\mathbb{P}_x(D_i < d_i, i = 1, \ldots, n-1) \qquad (10) \\
= &\sum_{j,k \in E_n} F_{n,j}(s_n)\, c_n(x_n, j, k) G_{n-1,x}(s_1 + \lambda_n(k), \ldots, s_{n-1} + \lambda_n(k))
\end{aligned}$$

where (10) is a consequence of Assumption 3.

To establish (9) we have used the fact that (i) $\mathbb{E}_x\Big[e^{-s_n D_n} \,|\, \mathcal{A}_n((y_i, d_i)), X_n(D_1 + \cdots + D_{n-1}) = j\Big] = F_{n,j}(s_n)$ since $X_n(D_1 + \cdots + D_{n-1}) = Y_n$ from the definition of Y_n along with the fact that conditioning on the event $\{\mathcal{A}_n((y_i, d_i)), Y_n = j\}$ the rv D_n only depends on $\{Y_n = j\}$ the latter holding from Assumption 2, and (ii) $X_n(d_1 + \cdots + d_{n-1})$ does not depend on $\mathcal{A}_n((y_i\, d_i))$ thanks again to Assumption 1. This concludes the proof. □

Any statistics of the random vector $(D_1, \ldots, D_N)$ can be obtained from (7)-(8). As an illustration, consider

$$T_n(x) := \mathbb{E}_x[T_n] \tag{11}$$

the expected traversal time across links $1, \ldots, n$ given that the system is in state x at time $t = 0$. From the relation

$$T_n(x) = -g'_{n,x}(0) \tag{12}$$

with $g_{n,x}(s) := G_{n,x}(s, \ldots, s)$ and Proposition 3.1 we immediately get the following result:

Corollary 3.2 (Expected path traversal times).

Under Assumptions 1, 2, and 3

$$T_1(x) = -F'_{1,x_1}(0) \tag{13}$$

$$T_n(x) = \sum_{j,k\in E_n} c_n(x_n, j, k)\left[\gamma_{n,j}\, g_{n-1,x}(\lambda_n(k)) - g'_{n-1,x}(\lambda_n(k))\right] \tag{14}$$

for any $x = (x_1, \ldots, x_N) \in E$, $s \geq 0$, *where*

$$g_{1,x}(s) = F_{1,x_1}(s) \tag{15}$$

$$g_{n,x}(s) = \sum_{j,k\in E_n} c_n(x_n, j, k) F_{n,j}(s) g_{n-1,x}(s + \lambda_n(k))$$

$$\begin{aligned} g'_{1,x}(s) &= F'_{1,x_1}(s) \\ g'_{n,x}(s) &= \sum_{j,k\in E_n} c_n(x_n, j, k)\left[F'_{n,j}(s) g_{n-1,x}(s + \lambda_n(k)) + F_{n,j}(s) g'_{n-1,x}(s + \lambda_n(k))\right] \end{aligned} \tag{16}$$

for $n = 2, \ldots, N$.

In the next section we address the case where $\mathcal{X}_1, \ldots, \mathcal{X}_N$ are identical and mutually independent two-state Markov processes. We will show that $T_N(x)$ can be calculated in $O(N^2)$ operations from Corollary 3.2.

Remark 3.3 (*Links between* $\{c_n(i, j, k)\}_{i,j,k}$ *in (6) and eigenvectors of* $\mathbf{Q}_n$ *when diagonalizable*). When $\mathbf{Q}_n$ is diagonalizable then $\mathbf{Q}_n = \mathbf{S}_n\mathbf{D}_n\mathbf{S}_n^{-1}$ with $\mathbf{D}_n = \mathrm{diag}(-\lambda_n(i), i \in E_n)$ the diagonal matrix whose diagonal elements are the eigenvalues of $\mathbf{Q}_n$. The jth column of the (similarity) matrix $\mathbf{S}_n = [S_n(i, j)]_{i,j}$ is the eigenvector of $\mathbf{Q}_n$ associated with the eigenvalue $-\lambda_n(j)$. In this notation, the relation $\boldsymbol{\pi}_n(t) = \boldsymbol{\pi}_n(0)e^{\mathbf{Q}_n t}$ yields

$$\mathbb{P}(X_n(t) = j) = \sum_{i\in E_n} \mathbb{P}(X_n(0) = i) \sum_{k\in E_n} S_n(i, k) S^*_n(k, j) e^{-\lambda_n(k)t} \tag{17}$$

with $S^*_n(i, j))$ the (i, j)-entry of the matrix $\mathbf{S}_n^{-1}$. Multiplying both sides of (6) by $\mathbb{P}(X_n(0) = i)$ and summing up for all values of $i \in E_n$ shows by identification with the r.h.s. of (17) that $c_n(i, j, k) = S_n(i, k)S^*_n(k, j)$, which makes explicit the relationship between $c_n(i, j, k)$ and the eigenvectors of $\mathbf{Q}_n$.

3.4. The case of on-off links

We specialize the framework of Section 3.1 to the important case where links behave as independent two-state (on-off) Markov processes.

Furthermore, all links are assumed to be identical, namely they have the same state space $E_n = \{0, 1\}$ and infinitesimal generator

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \tag{18}$$

with $\lambda > 0$ (resp. $\mu > 0$) the transition rate from state 0 (resp. state 1) to state 1 (resp. state 0). In this case $F_{n,i}(s)$ and $\gamma_{n,i}$ do not depend on n and will be denoted by $F_i(s)$ and γ_i, respectively, for $i \in \{0, 1\}$. Let $\beta := \mu + \lambda$.

The above typically models the situation where the link is down (off) in state 0 and cannot transmit anything while it is up (on) in state 1 and ready to transmit at the same rate at every link.

Assumption 4: *Links behave as independent and identical two-state Markov processes with state-space* $\{0, 1\}$, *and no transmission takes place in state* 0.

Define $P_{i,j}(t) = \mathbb{P}(X_n(t) = j \,|\, X_n(0) = i)$ for $(i, j) \in \{0, 1\}^2$, where we recall that $X_n(t)$ is the state of link n at time t. Note that the l.h.s. of this definition does not depend on n since all links are identical.

The matrix $\mathbf{Q}$ is diagonalizable since it has two distinct eigenvalues, given by 0 and $-\beta$, so that (6) holds. More specifically

$$P_{1,0}(t) = \pi_0(1 - e^{-\beta t}), \quad P_{0,0}(t) = \pi_0 + \pi_1 e^{-\beta t} \tag{19}$$

$$P_{0,1}(t) = \pi_1(1 - e^{-\beta t}), \quad P_{1,1}(t) = \pi_1 + \pi_0 e^{-\beta t} \tag{20}$$

where $\pi_0 =: \mu/\beta$ and $\pi_1 := \lambda/\beta$ are the stationary probabilities that the link is in state 1 and in state 0, respectively.

We are interested in calculating $T_N(x)$, the expected path traversal time. We recall that $\bar{x} = 1 - x$.

Proposition 3.4 (Expected path traversal time for on-off links).
Under Assumptions 2 and 4 for any $x = (x_1, \ldots, x_N) \in \{0, 1\}^N$

$$T_1(x) = x_1\gamma_1 + \bar{x}_1\gamma_0 \tag{21}$$

$$T_n(x) = T_{n-1}(x) + \pi_0\gamma_0 + \pi_1\gamma_1 + (\gamma_0 - \gamma_1)(\pi_1\bar{x}_n - \pi_0 x_n)g_{n-1,x}(\beta) \tag{22}$$

for $n = 2, \ldots, N$, which yields

$$T_N(x) = N(\pi_0\gamma_0 + \pi_1\gamma_1) + (\gamma_0 - \gamma_1)\sum_{n=1}^{N}(\pi_1\bar{x}_n - \pi_0 x_n)g_{n-1,x}(\beta) \tag{23}$$

with $g_{0,x}(\beta) := 1$ and where coefficients $\{g_{n-1,x}(\beta), n = 2, \ldots, N\}$ are calculated from the recursions

$$g_{1,x}(k\beta) = x_1F_1(k\beta) + \bar{x}_1F_0(k\beta), \quad k = 1, \ldots, N-1 \tag{24}$$

$$\begin{aligned} g_{n,x}(k\beta) &= \phi(k\beta)g_{n-1,x}(k\beta) + (\bar{x}_n\pi_1 - x_n\pi_0)\psi(k\beta)g_{n-1,x}((k+1)\beta), \\ &\quad k = 1, \ldots, N-n, \; n = 2, \ldots, N-1, \end{aligned} \tag{25}$$

with

$$\phi(s) := \pi_0F_0(s) + \pi_1F_1(s), \quad \psi(s) := F_0(s) - F_1(s). \tag{26}$$

Proof. Since both Assumption 1 (this is obvious) and Assumption 3 (thanks to (19)-(20)) hold when Assumption 4 is satisfied we can use the results in Corollory 3.2. First, note from (19)-(20) that coefficients $c_n(x_n, j, k)$ in (14) are given by

$$c_n(0,0,0) = c_n(1,0,0) = c_n(1,1,1) = -c_n(1,0,1) = \pi_0 \tag{27}$$

$$c_n(0,0,1) = c_n(0,1,0) = c_n(1,1,0) = -c_n(0,1,1) = \pi_1. \tag{28}$$

By writing $c_n(x_n, j, k)$ as

$$c_n(x_n, j, k) = x_nc_n(1, j, k) + \bar{x}_nc_n(0, j, k), \tag{29}$$

and by substituting the values in (27)-(28) into (14), we find after elementary algebra

$$\begin{aligned} T_n(x) &= -g'_{n-1,x}(0) + \pi_0\gamma_0 + \pi_1\gamma_1 + (\gamma_0 - \gamma_1)(\pi_1\bar{x}_n - \pi_0x_n)g_{n-1,x}(\beta) \\ &= T_{n-1}(x) + \pi_0\gamma_0 + \pi_1\gamma_1 + (\gamma_0 - \gamma_1)(\pi_1\bar{x}_n - \pi_0x_n)g_{n-1,x}(\beta) \end{aligned} \tag{30}$$

for $n = 2, \ldots, N-1$, where the latter identity follows from (12). From (13) we get

$$T_1(x) = \gamma_{x_1} = x_1\gamma_1 + \bar{x}_1\gamma_0. \tag{31}$$

Let us focus on the calculation of $g_{n-1,x}(\beta)$ in (30).

From (16), (27)-(28) and (29) we get

$$g_{n,x}(s) = \phi(s)g_{n-1,x}(s) + (\bar{x}_n\pi_1 - x_n\pi_0)\psi(s)g_{n-1,x}(s+\beta) \tag{32}$$

for $n = 2, \ldots, N$, where $\phi(s)$ and $\psi(s)$ are defined in (26), with $g_{1,x}(s) = F_{x_1}(s) = x_1F_1(s) + \bar{x}_1F_0(s)$ from (15). It is easily seen from (32) that coefficients $\{g_{n-1,x}(\beta), n = 2, \ldots, N\}$ are given by (24)-(25)

(Hint: $g_{N,x}(\beta)$ is known if one knows $g_{N-1,x}(\beta)$ and $g_{N-1,x}(2\beta)$ which are both known if one knows $g_{N-2,x}(\beta)$, $g_{N-2,x}(2\beta)$ and $g_{N-2,x}(3\beta)$, etc.).

Finally, (23) follows from (30)-(31) (Hint: $x_1\gamma_1 + \bar{x}_1\gamma_0 - (\gamma_0 - \gamma_1)(\pi_i\bar{x}_1 - \pi_0 x_1) = \pi_0\gamma_0 + \pi_1\gamma_1$), which completes the proof. □

Remark 3.5. Proposition 3.4 extends to more general initial conditions than simply $x_i \in \{0, 1\}$. In particular, given an initial state distribution for all of the links at $t = 0$, $\pi_{n,1}(0)$, $n = 1, \ldots, N$, then Proposition 3.4 continues to hold with the replacement of x_n and $\bar{x}_n$ by $\pi_{n,1}(0)$ and $(1 - \pi_{n,1}(0))$ respectively, for $n = 1, \ldots, N$.

Since $D_n = T_n - T_{n-1}$ we deduce from (22) that

$$\mathbb{E}_x[D_n] = \pi_0\gamma_0 + \pi_1\gamma_1 + (\gamma_0 - \gamma_1)(\pi_1\bar{x}_n - \pi_0 x_n)\mathbb{E}_x[e^{-\beta T_{n-1}}].$$

The sum $\pi_0\gamma_0 + \pi_1\gamma_1$ is nothing but the expectation of D_n when link n is in steady-state at time $t = 0$. Therefore, the second term can be seen as a deviation term when the system is not in steady-state at time 0.

Let us determine the numerical complexity of calculating $\mathbb{E}_x[T_N]$ or, equivalently, from (23) the complexity of calculating $g_{1,x}(\beta), g_{2,x}(\beta), \ldots, g_{N-1,x}(\beta)$. The coefficient $g_{1,x}(\beta)$ can be calculated with (24) in $O(N)$ operations. For each $n = 1, \ldots, N-1$, introduce the $(N-n)$-dimensional vector $\boldsymbol{v}_n := (g_{n,x}(\beta), \ldots, g_{n,x}(\beta(N-n)))^T$ and the $(N-(n-1))$-by-$(N-n)$ matrix $\mathbf{M}_n : [M_n(i, j)]_{i,j}$ where non-zero entries are given by

$$M_n(i, i) := \phi(i\beta) \text{ for } i = 1, \ldots, N - (n-1)$$

and

$$M_n(i, i+1) := (\bar{x}_n\pi_n - x_n\pi_0)\psi(i\beta) \text{ for } i = 1, \ldots, N - (n-1).$$

In this notation, the recursive scheme defined by (24)-(25) can be expressed as

$$\boldsymbol{v}_n = \mathbf{M}_n \boldsymbol{v}_{n-1} \tag{33}$$

in matrix form, for $n = 2, \ldots, N-1$. Since the matrix $\mathbf{M}_n$ has only two non-zero entries per row, the calculation of $\boldsymbol{v}_n$ requires $O(N-(n-1))$ operations and therefore the calculation of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{N-1}$ or, equivalently, that of $g_{1,x}(\beta)$, ...,$g_{N-1,x}(\beta)$ requires $O(N^2)$ operations. Therefore, $O(N^2)$ operations are required to compute $E_x[T_N]$ for any $x \in \{0, 1\}^N$.

We conclude this section by showing that the traversal time across links $1, \ldots, N$ is stochastically minimized (resp. maximized) when all links are initially in state one (resp. state zero). A more general result is actually proved below. The proof relies on the following assumption, which states that the time to traverse a link depends only on the state of the link at the time of arrival and not on the previous history of the link.

Assumption 5: $\mathbb{P}(D_n < u \,|\, \{X_n(t), 0 \le t < T_{n-1}\}, X_n(T_{n-1}) = i) = \mathbb{P}(D_n < u \,|\, Y_n = i)$ *for* $i \in \{0, 1\}$ *and for all* $u > 0$, $n = 1, \ldots, N$.

Let $D_{n,i}$ denote the traversal time for link n when $Y_n = i$, $i \in \{0, 1\}$ and denote by $D_{n,\star}$ the traversal time for link n when the transmission starts at the beginning of an on period (this is the case when the file arrives at a link and finds it in the on state). Clearly $D_{n,0} =_d I + D_{n,\star}$, with I is an exponentially distributed rv with parameter λ, independent of $D_{n,\star}$. Assumptions 4 and 5 ensure that $D_{n,\star} =_d D_{n,1}$, so that

$$D_{n,0} =_d I + D_{n,1} \tag{34}$$

with $D_{n,1}$ independent of I.

Last, Assumptions 4 and 5 can be used to establish the following non-pass property.

LEMMA 3.6.
Under Assumptions 4 and 5 $\mathbb{P}(T_n < u \,|\, T_{n-1} = t) \geq \mathbb{P}(T_n < u \,|\, T_{n-1} = t')$ *for all* $u > 0$, $0 \leq t < t'$, $n = 2, \ldots, N$.

PROOF. This follows from the above relationship between $D_{n,0}$ and $D_{n,1}$ and a standard sample path coupling argument. □

We say that vector $\mathbf{y} = (y_1, \ldots, y_N)$ dominates vector $\mathbf{x} = (x_1, \ldots, x_N)$, and write $\mathbf{x} \leq \mathbf{y}$, if $\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i$ for $n = 1, \ldots, N$.

PROPOSITION 3.7 (STOCHASTIC DOMINANCE).
Under Assumptions 2, 4 and 5 if $\mathbf{x} \leq \mathbf{y}$ *with* $\mathbf{x}, \mathbf{y} \in \{0, 1\}^N$ *then* $\mathbb{P}_{\mathbf{y}}(T_N \leq t) \geq \mathbb{P}_{\mathbf{x}}(T_N \leq t)$.

The proof makes use of the following two lemmas.

LEMMA 3.8.
Under Assumptions 2, 4 and 5 if $\mathbf{x} = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N)$ *and* $\mathbf{y} = (x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_N)$, *then* $\mathbb{P}_{\mathbf{y}}(T_N \leq t) \geq \mathbb{P}_{\mathbf{x}}(T_N \leq t)$, $j = 1, \ldots, N$.

PROOF. Consider n links with initial state configurations $\mathbf{x} = (0, x_2, \ldots, x_n)$ and $\mathbf{y} = (1, x_2, \ldots, x_n)$, $x_j \in \{0, 1\}$ for $j = 2, \ldots, n$ and let $T_n^{\mathbf{x}}$ and $T_n^{\mathbf{y}}$ denote the times at which the file departs link n under $\mathbf{x}$ and $\mathbf{y}$ respectively. We couple the two systems so that $T_n^{\mathbf{x}} \geq T_n^{\mathbf{y}}$. We couple link 1 under $\mathbf{x}$ and $\mathbf{y}$ as follows. Let $D_1^{\mathbf{y}}$ (resp. $D_1^{\mathbf{x}}$) denote the traversal time of link 1 under configuration $\mathbf{y}$ (resp. $\mathbf{x}$). Assumptions 4 and 5 allow us to couple the link under $\mathbf{x}$ and $\mathbf{y}$ as $D_1^{\mathbf{x}} = I + D_1^{\mathbf{y}}$, where I is an exponential rv with parameter λ independent of $D_1^{\mathbf{y}}$. Hence $T_1^{\mathbf{x}} = I + D_1^{\mathbf{y}}$ and $T_1^{\mathbf{y}} = D_1^{\mathbf{y}}$. Since the initial states of the remaining links are the same under $\mathbf{x}$ and $\mathbf{y}$, we can apply Lemma 3.6 to propagate the delay inequality through the links thus arriving at $\mathbb{P}_{\mathbf{y}}(T_n < t) \geq \mathbb{P}_{\mathbf{x}}(T_n < t)$ for all $t > 0$ and $n = 1, \ldots, N$.

Consider now the initial configurations

$$\mathbf{x}^j = (x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_N)$$

and

$$\mathbf{y}^j = (x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_N)$$

for $j = 2, \ldots, N$. We couple links $1, \ldots, j-1, j+1, \ldots, N$ as above under $\mathbf{x}$ and $\mathbf{y}$ and we perform a similar coupling of links j as we did for links 1 above.

The file reaches link j at the same time, say at time W, under both configurations since links $1, \ldots, j-1$ behave the same. By conditioning on the state of link j at time $W = w$ we readily get

$$\begin{aligned}
&\mathbb{P}_{\mathbf{y}^j}(T_N \leq t \,|\, W = w) - \mathbb{P}_{\mathbf{x}^j}(T_N \leq t \,|\, W = w) = \\
&\left[\mathbb{P}_{(1, x'_{j+1}, \ldots, x'_N)}(T_{N-j+1} \leq t - w) - \mathbb{P}_{(0, x'_{j+1}, \ldots, x'_N)}(T_{N-j+1} \leq t - w)\right] \\
&\times (P_{1,1}(w) P_{0,0}(w) - P_{0,1}(w) P_{1,0}(w))
\end{aligned} \tag{35}$$

where $X_i(w) = x'_i$, $i = j+1, \ldots, N$ and we have used the fact that the traversal time across links $j, \ldots, N$ will be the same under configurations $\mathbf{x}$ and $\mathbf{y}$ (thanks to the coupling) when link j is in the same state under $\mathbf{x}$ and $\mathbf{y}$ at time w. We know from the first part of the lemma that

$$\mathbb{P}_{(1, x'_{j+1}, \ldots, x'_N)}(T_{N-j+1} \leq t - w) \geq \mathbb{P}_{0, x'_{j+1}, \ldots, x'_N)}(T_{N-j+1} \leq t - w).$$

On the other hand, we find from (19)-(20) that

$$P_{1,1}(w) P_{0,0}(w) - P_{0,1}(w) P_{1,0}(w) = e^{-\beta t} \tag{36}$$

which proves that the r.h.s. of (35) is nonnegative. Removing the conditioning on w in (35) yields

$$\mathbb{P}_{\mathbf{y}^j}(T_N \leq t) \geq \mathbb{P}_{\mathbf{x}^j}(T_N \leq t).$$

This completes the proof. □

LEMMA 3.9.
Under Assumptions 2, 4 and 5 if $\mathbf{x} = (x_1, \ldots, x_{j-1}, 0, 1, x_{j+2}, \ldots, x_N)$ *and* $\mathbf{y} = (x_1, \ldots, x_{j-1}, 1, 0, x_{j+2}, \ldots, x_N)$*, then* $\mathbb{P}_{\mathbf{y}}(T_N \le t) \ge \mathbb{P}_{\mathbf{x}}(T_N \le t)$*,* $t > 0$*,* $j = 1, \ldots, N-1$.

PROOF. Consider n links with initial state configurations $\mathbf{x} = (0, 1, x_3, \ldots, x_n)$ and $\mathbf{y} = (1, 0, x_3, \ldots, x_n)$, $x_j \in \{0, 1\}$ for $j = 3, \ldots, n$. As in the previous lemma, we will couple the systems under $\mathbf{x}$ and $\mathbf{y}$ so that $T_n^{\mathbf{x}} \ge T_n^{\mathbf{y}}$. We couple the first link under $\mathbf{x}$ and $\mathbf{y}$ as in the proof of Lemma 3.8, namely,

$$D_1^{\mathbf{x}} = I + D_1^{\mathbf{y}} \tag{37}$$

where I is an exponential rv with parameter λ independent of $D_1^{\mathbf{y}}$. We now couple the second link under $\mathbf{x}$ and $\mathbf{y}$. We denote by $D_2^{\mathbf{y}}$ (resp. $D_x^{\mathbf{x}}$) the traversal time of link 2 under configuration $\mathbf{y}$ (resp. $\mathbf{x}$). Clearly $T_2^{\mathbf{x}} = D_1^{\mathbf{x}} + D_2^{\mathbf{x}}$ and $T_2^{\mathbf{y}} = D_1^{\mathbf{y}} + D_2^{\mathbf{y}}$. We distinguish four cases:

(1) under $\mathbf{y}$ (resp. $\mathbf{x}$) link 2 is in state 1 at time $D_1^{\mathbf{y}}$ (resp. $D_1^{\mathbf{x}}$). Then $D_2^{\mathbf{y}} = D_2^{\mathbf{x}}$ yielding $T_2^{\mathbf{y}} = D_1^{\mathbf{y}} + D_2^{\mathbf{y}} \le I + D_1^{\mathbf{y}} + D_2^{\mathbf{x}} = T_2^{\mathbf{x}}$ by using (37);
(2) under $\mathbf{y}$ (resp. $\mathbf{x}$) link 2 is in state 1 (resp. state 0) at time $D_1^{\mathbf{y}}$ (resp. $D_1^{\mathbf{x}}$). Then $D_2^{\mathbf{x}} = I' + D_2^{\mathbf{y}}$ with I' an exponential rv with parameter λ, yielding $T_2^{\mathbf{y}} = D_1^{\mathbf{y}} + D_2^{\mathbf{y}} \le I + I' + D_1^{\mathbf{y}} + D_2^{\mathbf{y}} = T_2^{\mathbf{x}}$ by using (37);
(3) under $\mathbf{y}$ (resp. $\mathbf{x}$) link 2 is in state 0 (resp. state 1) at time $D_1^{\mathbf{y}}$ (resp. $D_1^{\mathbf{x}}$). Then $D_2^{\mathbf{y}} = I + D_2^{\mathbf{x}}$ where I is given in (37), yielding $T_2^{\mathbf{y}} = D_1^{\mathbf{y}} + I + D_2^{\mathbf{x}} = T_2^{\mathbf{x}}$;
(4) under $\mathbf{y}$ (resp. $\mathbf{x}$) link 2 is in state 0 at time $D_1^{\mathbf{y}}$ (resp. D_1^{x}). Then $D_2^{\mathbf{y}} = D_{\mathbf{x}}^2$, yielding $T_2^{\mathbf{y}} = D_1^{\mathbf{y}} + D_2^{\mathbf{y}} \le I + D_1^{\mathbf{y}} + D_2^{\mathbf{y}} = T_2^{\mathbf{x}}$ by using (37).

This shows that $T_2^{\mathbf{y}} \le T_2^{\mathbf{x}}$. Removing the conditioning gives $\mathbb{P}_{\mathbf{y}}(T_2 < t) \ge \mathbb{P}_{\mathbf{x}}(T_2 < t)$ for all $t > 0$. Since the initial states of the remaining links are the same under $\mathbf{x}$ and $\mathbf{y}$, we can apply Lemma 3.6 to propagate the delay inequality through the links thus arriving at $\mathbb{P}_{\mathbf{y}}(T_n < t) \ge \mathbb{P}_{\mathbf{x}}(T_n < t)$ for all $t > 0$ and $n = 1, \ldots, N$.

Consider now the initial configurations

$$\mathbf{x}^j = (x_1, \ldots, x_{j-1}, 0, 1, x_{j+2}, \ldots, x_N)$$

and

$$\mathbf{y}^j = (x_1, \ldots, x_{j-1}, 1, 0, x_{j+2}, \ldots, x_N)$$

for $j = 2, \ldots, N$.
We can use a similar argument to that used in Lemma 3.8 to prove that $\mathbb{P}_{\mathbf{y}}(T_N < t) \ge \mathbb{P}_{\mathbf{x}}(T_N < t)$ for this case. □

We are now in position to prove Proposition 3.7:

PROOF. The proof of Proposition 3.7 proceeds by constructing a sequence of link state vectors $\mathbf{x} = \mathbf{x}^0 \le \mathbf{x}^1 \le \cdots \le \mathbf{x}^m = \mathbf{y}$ where $\mathbf{x}^i$ and $\mathbf{x}^{i+1}$ differ in either only one position, say j with $x_j^i = 0$ and $x_j^{i+1} = 1$, or in two consecutive positions, say j and $j+1$ with $x_j^i = 0$ and $x_{j+1}^i = 1$, and $x_j^{i+1} = 1$ and $x_{j+1}^{i+1} = 0$. When such a construction exists, then Lemmas 3.8 and 3.9 can be applied to yield the desired result. It is a straightforward exercise to show that such a construction exists when $\mathbf{x} \le \mathbf{y}$. This completes the proof. □

COROLLARY 3.10 (STOCHASTIC BOUNDS).
Under Assumptions 2, 4 and 5 for any initial configuration $x = (x_1, \ldots, x_N) \in \{0, 1\}^N$

$$\mathbb{P}_{\mathbf{1}}(T_N \le t) \ge \mathbb{P}_{\mathbf{x}}(T_N \le t) \ge \mathbb{P}_{\mathbf{0}}(T_N \le t)$$

for $t > 0$, where the N-dimensional vectors $\mathbf{1}$ *and* $\mathbf{0}$ *are* $(1, \ldots, 1)$ *and* $(0, \ldots, 0)$ *respectively. In particular,*

$$T_N(x^1) \le T_N(x) \le T_N(x^0). \tag{38}$$

3.5. Calculation of $F_{n,j}(s)$ for exponential transmission times and arbitrary set E_n

In this and the next section, we focus on the calculation of $F_{n,j}(s)$, the LST of the traversal time D_n across link n when the link is in state j at the time the file is ready for transmission (see (2)). This section focuses on general Markov link models and assumes that *normalized* transmission times are iid random variables characterized by an exponential distribution with rate η. The actual transmission time across a link is modulated by rates associated with the link states. Although the length of a file does not change as the file traverses different links, transmission times can vary across links (even when normalized). This can be due to fluctuations in the physical channel or contention with other data transmissions.

Assume that $K_n = |E_n|$. From now on we assume that the link propagation delay is zero. The case when the propagation delay on link n, denoted by Δ_n, is non-zero and constant is obtained from the analysis below by multiplying $F_{n,j}(s)$ by $e^{-s\Delta_n}$ and by adding Δ_n to γ_j for $j \in E_n$.

When link n is in state $j \in E_n$ we assume the transmission rate is constant and given by $r_n(j) \ge 0$. Whenever $r_n(j) > 0$ the file is transmitted at this rate, i.e., the transmission time is exponentially distributed with mean $\mathbb{E}[Z_n] r_j(n)$. If $r_n(j) = 0$ for some $j \in E_n$ transmission halts and resumes as soon as the transmission rate becomes positive. In Section 3.6.1 we will consider the case where transmissions are restarted after a link failure.

To simplify notation we drop the subscript n in K_n, $r_n(j)$, $F_{n,j}(s)$, $\gamma_{n,j}$ as well as in $\mathbf{Q}_n$ (the infinitesimal generator of the Markov chain $\mathcal{X}_n$ governing the behavior of link n – see Section 3.1) for the rest of this section.

It is easily seen that the traversal time D_n of link n is the absorption time of the absorbing continuous-time Markov chain on the set $E_n \times \{a\}$, with infinitesimal generator

$$\mathbf{M} = \begin{pmatrix} \mathbf{Q} - \eta\mathbf{R} & \eta\mathbf{r} \\ \mathbf{0} & 0 \end{pmatrix} \tag{39}$$

where $\mathbf{R} := \mathrm{diag}(r(j), j \in E_n)$, $\mathbf{r} := (r(j), j \in E_n)^T$ and $\mathbf{0}$ is the row vector of dimension K with all entries equal to zero. States in E_n are transient states and $\{a\}$ is an absorbing state corresponding to the file transmission completion time on link n.

The vector of the pdf of the conditional travel times is obtained as (see e.g. [Neuts 1981, Lemma 2.2.2.])

$$(\mathbb{P}(D_n < t \,|\, Y_n = j), j \in E_n)^T = (\mathbf{I} - e^{(\mathbf{Q} - \eta\mathbf{R})t})\mathbf{1}$$

with $\mathbf{1}$ the column vector of dimension K whose entries equal one and $\mathbf{I}$ the $K \times K$ identity matrix. Hence

$$\underline{F}(s) := (F_j(s), j \in E_n) = \left(\int_0^\infty e^{-st} e^{(\mathbf{Q} - \eta\mathbf{R})t} dt \right) (\eta\mathbf{R} - \mathbf{Q})\mathbf{1} = \eta(\mathbf{B}(s) - \mathbf{Q})^{-1}\mathbf{r} \tag{40}$$

where $\mathbf{B}(s) := \mathrm{diag}(s + \eta r(j), j \in E_n)$ (Hint: $(\eta\mathbf{R} - \mathbf{Q})\mathbf{1} = \eta\mathbf{r}$ since $\mathbf{Q}\mathbf{1} = \mathbf{0}^T$).

In particular, the vector of the expected traversal times of link n starting from state $Y_n = j$, $j \in E_n$ (defined in (3)) is given by

$$(\gamma_j, j \in E_n)^T = \eta\, (\mathbf{B}(0) - \mathbf{Q})^{-2}\, \mathbf{r}. \tag{41}$$

The next section considers more general transmission scenarios for the case of on-off links.

3.6. Calculation of $F_{n,j}(s)$ for on-off links, arbitrary transmission times and different transmission scenarios

In this section, we calculate $F_{n,j}(s)$ when link n is described by an on-off Markov process with state space $E_n = \{0, 1\}$ and transition rates $q_n(0,1) = \lambda_n$ and $q_n(1,0) = \mu_n$. We denote by Z_n the file transmission time over link n when it is up. If the link is stable and there is no other data being transmitted, Z_n will be constant. However, transmission times can vary due to varying conditions of the physical channel (fading, shadowing) and contention with other file transmissions. Hence we assume Z_n is a rv and denote by $Z_n(s) := \mathbb{E}[\exp(-sZ_n)]$ its LST. Last, since we consider a single link, we drop the subscript n.

For the reason explained at the beginning of Section 3.5 we assume without loss of generality that link propagation delays are zero.

To simplify notation we drop the subscript n in $F_{n,j}(s)$, $\gamma_{n,j}$, λ_n and μ_n.

For any sequence $(X, X_1, X_2, \ldots)$ of iid rvs X denotes a generic rv with the same distribution as the $X_i's$.

We consider two transmission scenarios. In the first scenario (Section 3.6.1) the file has to be entirely retransmitted if the link switches to the down state during the transmission. We will consider two cases:

(a) Successive retransmission times across a link are all *identical*;
(b) Successive retransmission times across a link iid rvs $\{Z, Z_1, Z_2, \ldots\}$.

Clearly, cases (a) and (b) yield the same result if Z is a constant. In the second scenario the file transmission is resumed after a link failure (Section 3.6.2).

Throughout $\{U, U_1, U_2, \ldots\}$ and $\{V, V_1, V_2, \ldots\}$ are non-negative independent sequences of iid rvs, with $\mathbb{P}(U < x) = 1 - e^{-\mu x}$ and $\mathbb{P}(V < x) = 1 - e^{-\lambda x}$. The rv U_i (resp. V_i) corresponds to the ith on period (resp. off period) of a link since the first attempt to transmit the file.

In scenario 1, case (a) (resp. in scenario 2) we assume that sequences $\{U, U_1, U_2, \ldots\}$ and $\{V, V_1, V_2, \ldots\}$ are independent of Z. In scenario 1, case (b) we assume that sequences $\{U, U_1, U_2, \ldots\}$, $\{V, V_1, V_2, \ldots\}$ and $\{Z, Z_1, Z_2, \ldots, \}$ are mutually independent.

Since Assumption 2 is satisfied here as the link transmission rate is constant, we get from (34) that

$$F_0(s) = \frac{\lambda}{\lambda + s} F_1(s). \tag{42}$$

As a consequence,

$$\gamma_0 = \frac{1}{\lambda} + \gamma_1. \tag{43}$$

Let $\mathbf{1}_A$ be the indicator function of the event A. By convention $\sum_{l=1}^{0} \cdot = 0$.

In the remainder of this section we compute $F_1(s)$ and γ_1 for the different scenarios.

3.6.1. Scenario 1: File entirely retransmitted after link failure.

Case (a):. Conditioned on $X(T_{n-1}) = 1$, we have

$$D_n =_d \sum_{i \geq 0} \mathbf{1}_{\mathcal{B}_i(Z)} \left(Z + \sum_{l=1}^{i} (U_l + V_l) \right)$$

with $\mathcal{B}_i(z) := \{U_1 < z, \ldots, U_i < z, U_{i+1} \geq z\}$. Note that $\mathcal{B}_i(z) \cap \mathcal{B}_j(z) = \emptyset$ for $i \neq j$ and $\sum_{i\geq 0} \mathbf{1}_{\mathcal{B}_i(z)} = 1$ for any z. We have

$$
\begin{aligned}
F_1(s) &= \int_0^\infty \mathbb{E}\left[e^{-s\sum_{i\geq 0}\mathbf{1}_{\mathcal{B}_i(z)}\left(z+\sum_{l=1}^i (U_l+V_l)\right)} \,|\, Z=z\right] d\mathbb{P}(Z<z) \\
&= \int_0^\infty \sum_{i\geq 0} \mathbb{P}(\mathcal{B}_i(z))\mathbb{E}\left[e^{-s\left(z+\sum_{l=1}^i (U_l+V_l)\right)} | \, Z=z, \mathcal{B}_i(z)\right] d\mathbb{P}(Z<z) \\
&= \int_0^\infty e^{-sz} \sum_{i\geq 0} \mathbb{P}(\mathcal{B}_i(z))\mathbb{E}\left[e^{-sV}\right]^i \mathbb{E}\left[e^{-s\sum_{l=1}^i U_l} | \, \mathcal{B}_i(z)\right] d\mathbb{P}(Z<z) \\
&= \int_0^\infty e^{-sz} \sum_{i\geq 0} (1-e^{-\mu z})^i e^{-\mu z}\left(\frac{\lambda}{\lambda+s}\right)^i \left(\prod_{l=1}^i \int_{u_l=0}^z e^{-sx_l}\frac{\mu e^{-u_l\mu} du_l}{1-e^{-\mu z}}\right) d\mathbb{P}(Z<z) \\
&= \int_0^\infty e^{-(\mu+s)z} \sum_{i\geq 0}\left(\frac{\lambda}{\lambda+s}\cdot\frac{\mu}{\mu+s}\left(1-e^{-(\mu+s)z}\right)\right)^i d\mathbb{P}(Z<z) \\
&= (\lambda+s)(\mu+s)\int_0^\infty \frac{e^{-(\mu+s)z}}{(\lambda+s)(\mu+s)-\lambda\mu(1-e^{-(\mu+s)z})} d\mathbb{P}(Z<z). \qquad (44)
\end{aligned}
$$

If Z is exponentially distributed with parameter η, we find

$$F_1(s) = (\lambda+s)(\mu+s)\int_0^\infty \frac{\eta e^{-(\mu+s+\eta)u}}{(\lambda+s)(\mu+s)-\lambda\mu(1-e^{-(\mu+s)u})} du. \qquad (45)$$

The integral in (45) has no explicit form but can be easily computed numerically for given parameters μ, λ, η, s.

If Z is a constant, we find

$$F_1(s) = \frac{(\lambda+s)(\mu+s)e^{-(\mu+s)Z}}{(\lambda+s)(\mu+s)-\lambda\mu(1-e^{-(\mu+s)Z})} \qquad (46)$$

and

$$\gamma_1 = \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\left(e^{\mu Z}-1\right). \qquad (47)$$

Case (b):. Conditioned on $X(T_{n-1}) = 1$, we have

$$D_n =_d \sum_{i\geq 0} \mathbf{1}_{C_i(Z_1,\ldots,Z_{i+1})}\left(Z_{i+1}+\sum_{l=1}^i (U_l+V_l)\right)$$

with

$$C_i(z_1,\ldots,z_{i+1}) := \{U_1 < z_1, \ldots, U_i < z_i, U_{i+1} \geq z_{i+1}\}.$$

Hence,

$$
\begin{aligned}
F_1(s) &= \sum_{i\geq 0} \mathbb{P}(C_i(Z_1,\ldots,Z_{i+1}))\mathbb{E}\left[e^{-s(Z_{i+1}+\sum_{l=1}^i (U_l+V_l))} \,|\, C_i(Z_1,\ldots,Z_{i+1})\right] \\
&= \sum_{i\geq 0} \mathbb{P}(U<Z)^i\mathbb{P}(U>Z)\mathbb{E}\left[e^{-sV}\right]^i \mathbb{E}\left[e^{-s(Z_{i+1}+\sum_{l=1}^i U_l)} \,|\, C_i(Z_1,\ldots,Z_{i+1})\right] \\
&= \sum_{i\geq 0}(1-Z(\mu))^i Z(\mu)\left(\frac{\lambda}{\lambda+s}\right)^i h_i(s) \qquad (48)
\end{aligned}
$$

by using the identities

$$\mathbb{P}(U > Z) = \int_0^\infty \mu e^{-\mu x}\mathbb{P}(Z < x)dx = Z(\mu),$$

where we have set $h_i(s) := \mathbb{E}\left[e^{-s(Z_{i+1}+\sum_{l=1}^i U_l)}\,|\,C_i(Z_1,\ldots,Z_{i+1})\right]$.

Let us evaluate $h_i(s)$. We have

$$\begin{aligned} h_i(s) &= \int_{z_1=0}^\infty \cdots \int_{z_{i+1}=0}^\infty \int_{u_1=0}^{z_1} \cdots \int_{u_i=0}^{z_i} \int_{u_{i+1}=z_{i+1}}^\infty \mu^{i+1} e^{-(\mu+s)\sum_{l=1}^{i+1} u_l} \frac{du_{i+1} d\mathbb{P}(Z_{i+1} < z_{i+1})}{\mathbb{P}(Z_{i+1} < U_{i+1})} \\ &\quad \times \prod_{l=1}^i \frac{du_l d\mathbb{P}(Z < z_l)}{\mathbb{P}(Z_l > U_l)} \\ &= \left(\int_{z=0}^\infty \frac{\mu}{\mu+s}\left(1 - e^{-(\mu+s)z}\right)\frac{d\mathbb{P}(Z<z)}{1-Z(\mu)}\right)^i \int_{z=0}^\infty e^{-(\mu+s)z}\frac{d\mathbb{P}(Z<z)}{Z(\mu)} \\ &= \left(\frac{\mu}{\mu+s}\left(\frac{1-Z(\mu+s)}{1-Z(\mu)}\right)\right)^i \frac{Z(\mu+s)}{Z(\mu)}. \end{aligned}$$

Introducing the above into (48) gives

$$F_1(s) = \frac{(\lambda+s)(\mu+s)Z(\mu+s)}{(\lambda+s)(\mu+s) - \lambda\mu(1-Z(\mu+s))} \tag{49}$$

from which we get

$$\gamma_1 = \left(\frac{1}{\lambda}+\frac{1}{\mu}\right)\left(\frac{1-Z(\mu)}{Z(\mu)}\right). \tag{50}$$

If Z is constant, yielding $Z(\mu+s) = e^{-(\mu+s)Z}$, then (49) (resp. (50)) reduces to (46) (resp. (47)), as expected.

3.6.2. Scenario 2: File retransmission resumes after link failure. Conditioned on $X(T_{n-1}) = 1$, we have

$$D_n = Z + \sum_{i\ge 0} \mathbf{1}_{\mathcal{D}_i(Z)} \sum_{l=1}^i V_l$$

with $\mathcal{D}_i(z) := \{U_1+\ldots+U_i < z \le U_1+\ldots+U_{i+1}\}$.

We have

$$\begin{aligned} F_1(s) &= \int_0^\infty e^{-sz}\mathbb{E}\left[e^{-s\sum_{i\ge1}\mathbf{1}_{\mathcal{D}_i(z)}\sum_{l=1}^i V_l}\,|\,Z=z\right]d\mathbb{P}(Z<z) \\ &= \sum_{i\ge0}\mathbb{E}\left[e^{-sV}\right]^i \int_0^\infty e^{-sz}\mathbb{P}(\mathcal{D}_i(z))\,d\mathbb{P}(Z<z) \\ &= \sum_{i\ge0}\left(\frac{\lambda}{\lambda+s}\right)^i \int_0^\infty e^{-(\mu+s)z}\frac{(\mu z)^i}{i!}d\mathbb{P}(Z<z) \qquad (51) \\ &= \int_0^\infty e^{-(\mu+z-\lambda\mu/(\lambda+s))z}d\mathbb{P}(Z<z) \\ &= Z\left(\frac{(\lambda+s)(\mu+s)-\lambda\mu}{\lambda+s}\right) \qquad (52) \end{aligned}$$

where we have used the identity $\mathbb{P}(\mathcal{D}_i(z)) = \frac{(\mu z)^i}{i!}e^{-\mu z}$ to derive (51) (probability of having i arrivals in an interval of length z for a Poisson process with rate mu). From (52) we find

$$\gamma_1 = \left(1 + \frac{\mu}{\lambda}\right)\mathbb{E}[Z]. \tag{53}$$

To conclude Section 3, let us point out that all results in Sections 3.6-3.5 can be used to calculate the joint LST of the path traversal time obtained in Proposition 3.1 or the expected path traversal time obtained in Proposition 3.4 provided that Assumption 2 is met. This will be the case if the conditional transmission times across links $1, \ldots, N$ are mutually independent.

4. DEPENDENT ON-OFF LINKS IN SERIES: THE CASE OF LOCAL INTERACTIONS

In this section we consider a special class of dependent on-off links where the dependencies are described by local interactions among the links. Consider N on-off links in series, labeled $1, 2, \ldots, N$. Let $X_n(t) \in \{0, 1\}$ be the state of link n at time $t \geq 0$, where link n is up if $X_n(t) = 1$ and down otherwise. Let $X(t) = (X_1(t), \ldots, X_N(t)) \in \{0, 1\}^N$ be the state of all links at time t.

We are interested in constructing a stationary process $\{X(t)\}_t$ where the link states are dependent and where we can induce the Pearson correlation coefficient $\xi_{X_n(t),X_{n+1}(t)} := \xi$ to take any value between -1 and 1, and whose value will be specified shortly. Here, the Pearson correlation coefficient for two real valued rvs X and Y is defined as

$$\xi_{X,Y} = \frac{cov(X, Y)}{\sigma_X \sigma_Y}.$$

Moreover, we would like the link state process $\{X_n(t)\}_t$ to be described by a stationary Markov chain with infinitesimal generator

$$\mathbf{Q} = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix} \tag{54}$$

for all n. In the following, X_n will denote the stationary state of link n. The distribution of X_n is given by $\pi(0) := \mathbb{P}(X_n = 0) = 1/(1 + \rho)$ and $\pi(1) := \mathbb{P}(X_n = 1) = \rho/(1 + \rho)$ where $\rho := \lambda/\mu$.

4.1. Stationary expected path traversal time

We consider two cases, $\xi \geq 0$ (positive correlations) and $\xi < 0$ (negative correlations).

4.1.1. Positive dependencies. We couple the link process at link $n + 1$ to the link process of link n as follows. Here $0 \leq r \leq 1$,

— link n behaves according to a Markov chain with infinitesimal generator $\mathbf{Q}$;
— when link n transitions from zero to one ($X_n(t) : 0 \to 1$) at time t, and link $n + 1$ is in state zero at that time, then link $n + 1$ transitions to one ($X_{n+1}(t) : 0 \to 1$) with probability r and remains unchanged otherwise. Furthermore, link $n + 1$ transitions from state zero to one according to a Poisson process with parameter $\bar{r}\lambda$ independent of the behavior of link n while in state zero;
— if link n transitions from one to zero ($X_n(t) : 1 \to 0$) at time t, and link $n + 1$ is in state one at that time, then link $n+1$ transitions to zero ($X_{n+1}(t) : 1 \to 0$) with probability r and remains unchanged otherwise. Furthermore, link $n+1$ transitions from state one to zero according to a Poisson process with parameter $\bar{r}\mu$ independent of the behavior of link n while in state one;
— otherwise transitions proceed independently for the two links.

Note that the resulting process exhibits a combined spatial-temporal Markov property, i.e.,

$$\mathbb{P}(X_{n_1}(t_1) = x_1 | X_{n_2}(t_2) = x_2, \ldots X_{n_m}(t_m) = x_m) = \mathbb{P}(X_{n_1}(t_1) = x_1 | X_{n_2}(t_2) = x_2), \quad x_j \in \{0, 1\}, 1 \leq j \leq m$$

for all $n_1 \geq n_2 \geq \cdots \geq n_m \geq 1$, $t_1 \geq t_2 \geq \cdots \geq t_m$, and $m = 3, \ldots$. The link states $X_n(t)$ and $X_{n+1}(t)$ evolve according to the following equations:

$$\begin{aligned}
dX_n &= (1 - X_n)dN_{n,0} - X_n dN_{n,1} \\
dX_{n+1} &= (1 - X_n)[(1 - X_{n+1})(B_{n+1}(t)dN_{n,0} + (1 - B_{n+1}(t))dN_{n+1,0}) - X_{n+1}dN_{n+1,1}] \\
&\quad + X_n[(1 - X_{n+1})dN_{n+1,0} - X_{n+1}(B_{n+1}(t)dN_{n,1} + (1 - B_{n+1})(t)dN_{n+1,1})].
\end{aligned}$$

where, $\{N_{i,0}(t)\}_t$ and $\{N_{i,1}(t)\}_t$ are mutually independent Poisson processes with rates λ and μ respectively, and $B_i(t)$ is a Bernoulli rv that takes value one with probability r satisfying $B_i(t_1), \ldots, B_i(t_k)$ is an independent sequence for any $0 < t_1 < \cdots < t_k$, $k = 2, \ldots$, $i = 1, \ldots, N$.

It follows that the process $\{(X_n(t), X_{n+1}(t))\}_t$ is described by a Markov chain with the infinitesimal generator

$$\mathbf{R} = \mu \begin{pmatrix} -(2-r)\rho & \bar{r}\rho & \bar{r}\rho & r\rho \\ 1 & -(1+\rho) & 0 & \rho \\ 1 & 0 & -(1+\rho) & \rho \\ r & \bar{r} & \bar{r} & -(2-r) \end{pmatrix} \tag{55}$$

and has stationary distribution

$$\pi(0,0) = \frac{2 - r + r\rho}{(2-r)(1+\rho)^2}, \quad \pi(0,1) = \pi(1,0) = \frac{2(1-r)\rho}{(2-r)(1+\rho)^2}, \quad \pi(1,1) = \rho\frac{(2-r)\rho + r}{(2-r)(1+\rho)^2}.$$

These can be used to compute $\xi_{X_n,X_{n+1}} := \xi$ yielding

$$\xi = \frac{\pi(1,1) - \pi(1)^2}{\pi(1)\pi(0)} = \frac{r}{2-r}. \tag{56}$$

As expected the correlation coefficient is positive, increases in r, and takes values zero when $r = 0$ and one when $r = 1$. Expressing the stationary distribution in terms of ξ yields

$$\pi(0,0) = \frac{1 + \xi\rho}{(1+\rho)^2}, \quad \pi(0,1) = \pi(1,0) = \frac{\rho(1-\xi)}{(1+\rho)^2}, \quad \pi(1,1) = \frac{\rho(\rho+\xi)}{(1+\rho)^2}. \tag{57}$$

We focus now on the time required to transmit a file through this network. As before, D_n denotes the time required for the file to traverse link n. This time consists of two components. The first is the delay incurred waiting for the link to first come up (it is zero if the link is already up when the file arrives). The second component, S_n, is the time required to service the file once the link comes up for the first time; it depends on the retransmission scenario that is used. We assume that that $S_1, \ldots, S_N$ are mutually independent rvs and define $S_n(s) := \mathbb{E}[\exp(-sS_n)]$.

Our objective is to calculate $G_N(s) := \mathbb{E}[e^{-sT_N}]$, with $T_n = \sum_{n=1}^N D_n$ the path traversal time. Recall that Y_n is the state of link n at the time that the file arrives at that link. Clearly, $\mathbb{P}(Y_1 = i) = \pi(i)$ for $i = 0, 1$ since link 1 is in steady-state at time $t = 0$. The conditional probability distribution for Y_{n+1} given that $Y_n = j$, $j = 0, 1$, is

$$\begin{aligned}
\mathbb{P}(Y_{n+1} = i | Y_n = j) &= \mathbb{P}(X_{n+1} = i | X_n = 1), \quad i, j = 0, 1 \\
&= \pi(1, i)/\pi(1).
\end{aligned}$$

Hence $\{Y_n\}_{n\geq 1}$ is a sequence of mutually independent rvs with distribution given by $\mathbb{P}(Y_1 = i) = \pi(i)$ and for $n \geq 2$,

$$\mathbb{P}(Y_n = i) = \begin{cases} (1-\xi)/(1+\rho) & \text{if } i = 0 \\ (\rho+\xi)/(1+\rho) & \text{if } i = 1. \end{cases}$$

It follows that $\{D_n\}_{n\geq 1}$ is also a sequence of mutually independent rvs, and moreover the sequence $\{D_n\}_{n\geq 2}$ is iid. This yields (Hint: $D_n = S_n + \mathbf{1}_{(Y_n=0)}\tau$ with τ an exponential rv with rate λ, independent of S_n and Y_n)

$$G_N(s) = \frac{(\lambda + (\lambda+s)\rho)(\lambda(1-\xi) + (\lambda+s)\rho)^{N-1}}{(\lambda+s)^N(1+\rho)^N}\prod_{n=1}^{N} S_n(s) \tag{58}$$

where we have used the fact that $P(Y_1 = 0) = 1/(1+\rho)$ and $P(Y_1 = 1) = \rho/(1+\rho)$.

The expected path traversal time is given by

$$\mathbb{E}[T_N] = \sum_{n=1}^{N} E[S_n] + \frac{N}{\lambda(1+\rho)} - \frac{\xi(N-1)}{\lambda(1+\rho)}. \tag{59}$$

Note that the link correlation affects $\mathbb{E}[T_N]$ through the third term and that positive correlation decreases the expected path traversal time in a linear fashion. In the absence of correlation (i.e. when $\xi = 0$) we retrieve the expected result that $\mathbb{E}[T_N] = \sum_{n=1}^{N} \mathbb{E}[S_n] + N/(\lambda(1+\rho))$, since in this case $\mathbb{E}[D_n] = \mathbb{E}[S_n] + \pi(0)/\lambda = \mathbb{E}[S_n] + 1/(\lambda(1+\rho))$. When the correlation is maximum (i.e. $\xi = 1$) we retrieve the expected result that $\mathbb{E}[T_N] = \sum_{n=1}^{N} \mathbb{E}[S_n] + 1/(\lambda(1+\rho))$ where the latter term is the average wait of the file before beginning its transmission on link 1. The expected path traversal time is minimized when $\xi = 1$ since the file will always find an active link except possibly at link 1.

4.1.2. Negative dependencies. As in previous section each link to be governed by a two-state Markov chain with infinitesimal $\mathbf{Q}$ given by (54) but exhibiting negative correlation. Except for the case $\lambda = \mu$, it is not possible to generate a setting where $\xi = -1$ as will be observed below. We propose the following algorithm for constructing the link process at link $n+1$ in terms of the process at link n. We assume that $\lambda \geq \mu$ or equivalently $\rho \geq 1$.

We couple the link process at link $n+1$ to the link process of link n as follows:

— link n behaves according to the Markov chain with infinitesimal generator $\mathbf{Q}$;
— if links n and $n+1$ are in states zero and one respectively, then if link n transitions to state one, then, with probability $r\mu/\lambda$ link $n+1$ transitions to state zero. Furthermore, link $n+1$ transitions from state one to state zero according to a Poisson process with rate $\bar{r}\mu$ independent of the behavior of link n while in state zero;
— if links n and $n+1$ are in states one and zero respectively, then if link n transitions to zero, then with probability r link $n+1$ transitions to state one. Furthermore, link $n+1$ transitions from state zero to state one according to a Poisson process with rate $\lambda - r\mu$ independent of the behavior of link n while in state one;
— otherwise transitions proceed independently for the two links.

This process also exhibits the combined spatial-temporal Markov property (55).

Link states $X_n(t)$ and $X_{n+1}(t)$ evolve according to the following dynamics:

$$\begin{aligned} dX_n &= (1-X_n)dN_{n,0} - X_n dN_{n,1} \\ dX_{n+1} &= (1-X_n)[(1-X_{n+1})dN_{n+1,0} - X_{n+1}(C_{n+1}(t)dN_{n,0} + dN_{n+1,1})] \\ &\quad + X_n[(1-X_{n+1})(B_{n+1}(t)dN_{n,1} + dN_{n+1,2}) - X_{n+1}dN_{n+1,1}]. \end{aligned}$$

Here, $\{N_{i,0}(t)\}_t$, $\{N_{i,1}(t)\}_t$ and $\{N_{i,2}(t)\}_t$ are mutually independent Poisson processes with rates λ, μ and $\lambda - r\mu$ respectively, $B_i(t)$ is a Bernoulli rv taking value one with probability r and $\{B_i(t_j)\}_{j=1}^k$ is an independent sequence of these rv's for any $0 < t_1 < \cdots < t_k$, $k = 2, \ldots$, and $C_i(t)$ is a Bernoulli rv taking value one with probability $r\mu/\lambda$ and $\{C_i(t_j)\}_{j=1}^k$) is an independent sequence of these rv's for any $0 < t_1 < \cdots < t_k$, $k = 2, \ldots, i = 1, \ldots, N$.

It follows that the process $\{(X_n(t), X_{n+1}(t))\}_t$ is described by a Markov chain with the following infinitesimal generator,

$$\mathbf{R} = \mu \begin{pmatrix} -2\rho & \rho & \rho & 0 \\ \bar{r} & -(\bar{r}+\rho) & r & \rho - r \\ \bar{r} & r & -(\bar{r}+\rho) & \rho - r \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

and has stationary distribution

$$\pi(0,0) = \frac{\bar{r}}{(1+\rho)(\bar{r}+\rho)}, \quad \pi(0,1) = \pi(1,0) = \frac{\rho}{(1+\rho)(\bar{r}+\rho)}, \quad \pi(1,1) = \frac{\rho(\rho - r)}{(1+\rho)(\bar{r}+\rho)}. \tag{60}$$

This yields the Pearson correlation coefficient

$$\xi = \frac{\pi(1,1) - \pi(1)^2}{\pi(1)\pi(0)} = \frac{-r}{\bar{r}+\rho}. \tag{61}$$

for $0 \le r \le 1$, which generates a Pearson coefficient within the range $[-1/\rho, 0]$. As expected the correlation is negative and decreases in r. Expressing the stationary distribution (60) in terms of ξ yields

$$\pi(0,0) = \frac{1+\rho\xi}{(1+\rho)^2}, \quad \pi(0,1) = \pi(1,0) = \frac{\rho(1-\xi)}{(1+\rho)^2}, \quad \pi(1,1) = \frac{\rho(\rho+\xi)}{(1+\rho)^2}. \tag{62}$$

Observe that the stationary distributions found for positive (see (57)) and negative (see (62)) correlations are identical as a function of ξ. This implies that in both cases the stationary performance measures will have the same form as function of ξ under the enforced assumption that the rvs $S_1, \ldots, S_N$ are mutually independent rvs, with S_n the transmission time across link n once that link comes up for the first time. In particular (see (59)),

$$\mathbb{E}[T_N] = \sum_{n=1}^{N} \mathbb{E}[S_n] + \frac{N}{\lambda(1+\rho)} - \frac{\xi(N-1)}{\lambda(1+\rho)} \tag{63}$$

for negative correlations. The LST of the path traversal time distribution is given by (58).

The case where $\mu > \lambda$ (corresponding to $\rho < 1$) proceeds along similar lines as the case $\rho \ge 1$ yielding a Pearson correlation coefficient $\xi = -\rho r/(1+\rho\bar{r})$ within the range $[-\rho, 0]$ and same stationary distribution (60) and, in particular the same expected path traversal time (63), as function of ξ. This shows that for every ρ we can generate correlations for $\xi \in [-\min\{\rho, 1/\rho\}, 0]$. As announced earlier this is in contrast with the case of positive correlations where one can generate a Pearson coefficient in the entire range $[0, 1]$ for any value of the model parameter ρ.

We conclude from the analysis in Sections 4.1.1-4.1.2 that the mapping $\xi \to \mathbb{E}[T_N]$ is linear and non-increasing.

5. NUMERICAL RESULTS

We are interested in quantifying the impact of the link initial states on the expected path traversal times. We consider the transmission scenario investigated in Section 3.6.1 where all links are independent and identical on-off links with λ (resp. μ) the transition from state 0 (off) to state 1 (on) (resp. from state 1 to state 0). On each link the transmission rate is r bits/sec. when the link is on, and the propagation delay is taken to be zero (introducing a propagation delay for each link is straightforward as explained at the beginning of Section 3.5). Recall that the stationary distribution to find a link in state 1 (resp. in state 0) is $\pi(0) = \mu/(\lambda+\mu)$ (resp. $\pi(1) = \lambda/(\lambda+\mu)$).

The file transmission times across links $1, \ldots, N$ are taken to be constant and all equal, namely, $Z_1 = \ldots = Z_N$. As defined in Section 3.6, Z_n is the time to transmit the file across link n if this link was always up and stable and if there was no contention with other file transmissions and no propagation delay. To be more concrete, if the file as (constant) size L (in bits) then $Z_n = L/r$ sec. for all n. Without lost of generality we will assume that $Z_n = 1$.

Last, we assume that when the link switches to the off-state during the transmission the file is entirely retransmitted when the link switches back to the on-state, which corresponds to scenario 1 in Section 3.6.1. In this setting and notation, $F_1(s)$ and γ_1 are given in (46) and (47), respectively, and $F_0(s)$ and γ_0 are obtained from (42) and (43), respectively.

We select the expected on period duration, $1/\mu$, so that the file will be successfully transmitted during this period with probability 0.95, i.e. $\mathbb{P}(Z_n < U) = 0.95$ with U an exponential r.v. with parameter μ, so that $\mu = -\log(0.95)$ since $Z_n = 1$ and $1/\mu \approx 19.495$.

Figures 1-4 display the mappings $N \to \log(T_N(x))$ for $N \in \{1, 2, \ldots, 20\}$, $x = x^1$ (all links are initially in state 1), $x = x^0$ (all links are initially in state 0) and $x = x^\star$ (all links are initially in steady-state) and for different values of $(\pi(0), \pi(1))$ (or, equivalently, for different values of λ since μ is fixed). We recall that $T_N(x)$ (resp. $T_N(x^\star)$) is the expected traversal time of N links in series when links are initially in state x (resp. in steady-state). $T_N(x^1)$ and $T_N(x^0)$ are computed by using Proposition 3.4, whereas $T_N(x^\star)$ it is obtained from (5), (43) and (47) as $T_N(x^\star) = e^\mu N/(\lambda + \mu)$.

From these results we can make the following observations : (i) the initial state of the links may have an important impact on the expected path traversal time as shown by the difference between $T_N(x^0)$ and $T_N(x^1)$, with the conclusion that approximating $T_N(x)$ by $T_N(x^\star)$ will in general not give an accurate result and (ii) this impact decreases as N increases, which can be explained by the fact that many links will approach their steady-state by the time they transmit the file.

6. CONCLUDING REMARKS

In this paper, we have investigated the time needed for a file to traverse a series of communication links whose physical conditions may vary over time. We have modeled the dynamics of each link as a finite-state Markov process. For statistically independent links we have obtained a recursive scheme for computing the LST of the transient joint traversal time across the links, from which we have derived a recursive scheme for calculating the transient expected traversal time of a path. When specialized to a series on N on-off links, the latter result yields a scheme of complexity $O(N^2)$. We have also considered particular dependency structures between adjacent links (positive and negative) and have obtained explicit results for the stationary expected traversal time across a series of communication links as a function of the Pearson correlation coefficient between two adjacent links. All of these results require the knowledge of either the LST or the expectation of the link's traversal time given the state of the link when the file becomes available for transmission is known. These quantities have been calculated for various file transmission scenarios. We have also provided numerical examples of the lower and upper bounds that we have established for the transient expected path traversal times across a path composed of independent and identical on-off links and have displayed the expected traversal times of dependent links as a function of the correlation coefficient between adjacent links.

Acknowlegements

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A. APPENDIX: INDEPENDENCE OF LINK TRAVERSAL TIMES WHEN LINKS INITIALLY IN STEADY-STATE

Recall that $\pi_n^\star(i)$ is the stationary probability that link n is in state $i \in E_n$.

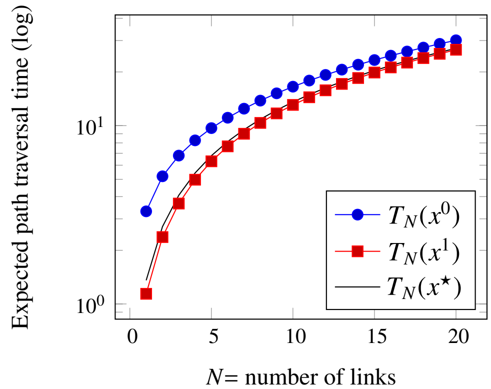

Fig. 1. Expected path traversal times (log) when $(\pi(0), \pi(1)) = (0.1, 0.9)$ (with $\mu = -\log(0.95)$) under scenario 1 with $Z_n = 1$.

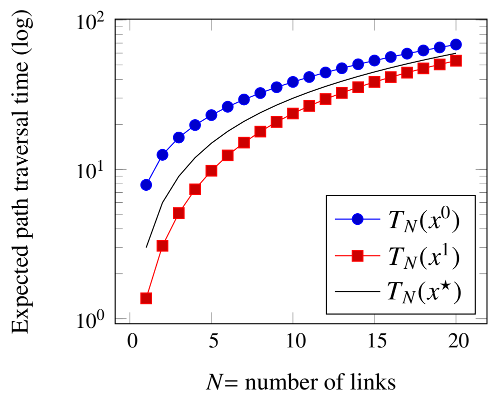

Fig. 2. Expected path traversal times (log) when $(\pi(0), \pi(1)) = (0.25, 0.75)$ (with $\mu = -\log(0.95)$) when transmission scenario 1 is used with $Z_n = 1$.

For any $x_n > 0$, $i_n \in E_n$, $n = 1, \ldots, N$,

$$
\begin{aligned}
\mathbb{P}(D_1 \le x_1, \ldots, D_N \le x_N) &= \sum_{\substack{i_j \in E_j \\ j=1,\ldots,N}} \mathbb{P}(D_1 \le x_1, \ldots, D_N \le x_N, Y_1 = i_1, \ldots, Y_N = i_N) \\
&= \sum_{\substack{i_j \in E_j \\ j=1,\ldots,N}} \mathbb{P}(D_N \le x_N \,|\, D_1 \le x_1, \ldots, D_{N-1} \le x_{N-1}, Y_1 = i_1, \ldots, Y_N = i_N) \\
&\quad \times \mathbb{P}(D_1 \le x_1, \ldots, D_{N-1} \le x_{N-1}, Y_1 = i_1, \ldots, Y_N = i_N) \\
&= \sum_{\substack{i_j \in E_j \\ j=1,\ldots,N}} \mathbb{P}(D_N \le x_N \,|\, Y_N = i_N)\, \mathbb{P}(D_1 \le x_1, \ldots, D_{N-1} \le x_{N-1}, Y_1 = i_1, \ldots, Y_N = i_N)
\end{aligned}
$$

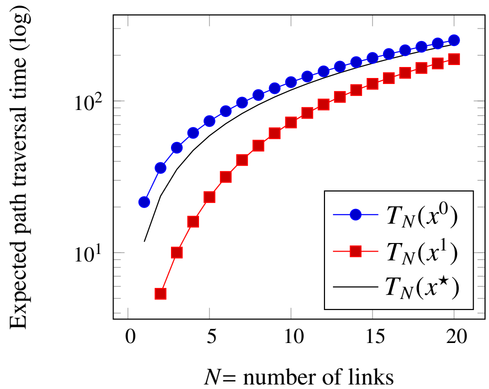

Fig. 3. Expected path traversal times (log0 when $(\pi(0), \pi(1)) = (0.5, 0.5)$ (with $\mu = -\log(0.95)$) under scenario 1 with $Z_n = 1$

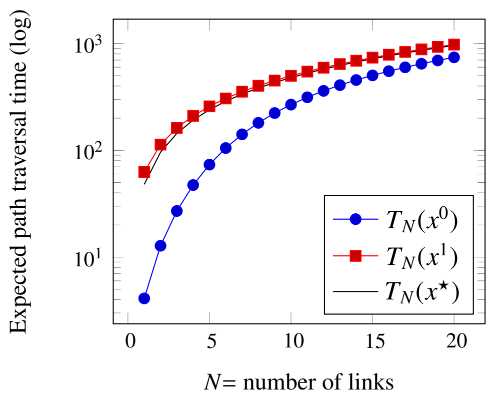

Fig. 4. Expected path traversal times (log) when $(\pi(0), \pi(1)) = (0.75, 0.25)$ (with $\mu = -\log(0.95)$) under scenario 1 with $Z_n = 1$.

by using Assumption 2 with $n = N$. On the other hand,

$$\begin{aligned}
&\mathbb{P}(D_1 \le x_1, \ldots, D_{N-1} \le x_{N-1}, Y_1 = i_1, \ldots, Y_N = i_N) \\
&\quad = P(Y_N = i_N \,|\, D_1 \le x_1, \ldots, D_{N-1} \le x_{N-1}, Y_1 = i_1, \ldots, Y_{N-1} = i_{N-1}) \\
&\qquad \times \mathbb{P}(D_1 \le x_1, D_{N-1} \le x_{N-1}, Y_1 = i_1, \ldots, Y_{N-1} = i_{N-1}) \\
&\quad = \pi^\star_N(i_N)\,\mathbb{P}(D_1 \le x_1, \ldots, D_{N-1} \le x_{N-1}, Y_1 = i_1, \ldots, Y_{N-1} = i_{N-1}).
\end{aligned}$$

To derive the above we have used the property that link N does not depend on links $1, \ldots, N-1$ from Assumption 1, along with the assumption that link N is in steady-state which implies that $\mathbb{P}(Y_N = i_N) = \pi^\star_N(i_N)$. Because links $1, \ldots N-1$ are mutually independent, we can iterate this process to yield

$$\begin{aligned}
\mathbb{P}(D_1 \le x_1, \ldots, D_N \le x_N) &= \sum_{\substack{i_j \in E_j \\ j=1,\ldots,N}} \prod_{k=1}^{N} \mathbb{P}(D_k \le x_k \,|\, Y_k = i_k)\,\pi^\star_k(i_k) \\
&= \mathbb{P}(D_1 \le x_1) \times \cdots \times \mathbb{P}(D_N \le x_N).
\end{aligned}$$

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