A unified view of exact continuous penalties for l2-l0 minimization

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Abstract

Numerous nonconvex continuous penalties have been proposed to approach the ℓ_0 pseudo-norm for optimization purpose. Apart from the theoretical results for convex ℓ_1 relaxation under restrictive hypothesis, only few works have been devoted to analyze the consistency, in terms of minimizers, between the ℓ_0-regularized least square functional and relaxed ones using continuous approximations. In this context, two questions are of fundamental importance: does relaxed functionals preserve global minimizers of the initial one? Does this approximation introduce unwanted new (local) minimizers? In this paper we answer these questions by deriving necessary and sufficient conditions on such ℓ_0 continuous approximations in order that each minimizer of the underlying relaxation is also a minimizer of the ℓ_2-ℓ_0 functional and that all the global minimizers of the initial functional are preserved. Hence, a general class of penalties is provided giving a unified view of exact continuous approximations of the ℓ_0-norm within the ℓ_2-ℓ_0 minimization framework. As the inferior limit of this class of penalties, we get the recently proposed CEL0 penalty. Finally, state of the art penalties, such as MCP, SCAD or Capped-ℓ_1, are analyzed according to the proposed class of exact continuous penalties.

Key words ℓ_0-regularized least squares, exact reformulation, exact ℓ_0 penalties, sparse modeling, underdeterminated linear system, global minimizers, local minimizers, minimizers equivalence

1 Introduction

In this paper, we are concerned with the following ℓ_0-regularized least squares problem,

$$\hat{\mathrm{x}} \in \arg\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0, \tag{1}$$

where $A \in \mathbb{R}^{M\times N}$ (usually $M \ll N$), $d \in \mathbb{R}^M$ and $\|\cdot\|_0$ denotes the so-called "ℓ_0-norm" defined by,

$$\|\mathrm{x}\|_0 = \sharp\left\{\mathrm{x}_i, i = 1,\dots,N : \mathrm{x}_i \neq 0\right\}, \tag{2}$$

with $\sharp$ denoting the cardinality. In other words, the ℓ_0-norm counts the nonzero entries of x. Finally, $\lambda > 0$ is an hyperparameter characterizing a trade-off between data fidelity and sparsity. This problem finds a wide range of applications in signal/image processing, learning and coding areas among many

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others. Constrained versions of (1) have also been extensively used and studied over the last decades. However, due to their nonconvexity, only a partial equivalence between the k-sparsity constrained minimization problem and the ℓ_0-regularized one (1) exists [20]. Moreover, a combinatorial search to tackle such problems is known to be NP-Hard [18] and considerable efforts have been done and continue to be done in order to design efficient methods/algorithms to find (approximate) solutions of (1) (or its constrained forms).

Convex relaxation, where the ℓ_0-norm is replaced by the ℓ_1-norm, became very popular after the theoretical results of Donoho [9] and, Candès, Romberg and Tao [5] showing that under suitable conditions on A (*e.g.* RIP criteria, incoherence assumption...), sufficiently sparse signals can be exactly recovered by ℓ_1 minimization. However, these conditions are quite restrictive for many practical applications. Other common approaches to deal with such problems are greedy methods such as matching pursuit (MP) [17], orthogonal matching pursuit (OMP) [22] or single best replacement (SBR) [24]. Similar recovery guaranties as for the ℓ_1 relaxation have been established for OMP in [26]. Besides, *continuous* nonconvex approximations have received considerable attention and numerous penalties have been proposed to approach the ℓ_0-norm. Among the variety of ℓ_0-like continuous penalties, one can find the *nonnegative garrote* [4], *log-sum penalty* [6], *capped-ℓ_1* [29], *ℓ_p-norms* $(0 < p < 1)$ [11], *smoothly clipped absolute deviation* (SCAD) [10] or *minimax concave penalty* (MCP) [28]. Such penalties have been shown to better promote sparsity than ℓ_1-norm and to avoid (reduce) the bias introduced by the ℓ_1 penalty on large coefficients [30, 10]. Although Fan and Li [10] designed conditions that a "good" penalty function must satisfy (unbiasedness, continuity in data, sparsity) and that Zhang [27] proposed the notion of *sparse convexity* to compare penalties, the choice of one *continuous* nonconvex penalty rather than another remains unclear. Moreover, only few work have been dedicated to analyse the links between the underlying relaxed problems and the original one (involving the ℓ_0-norm).

A class of smooth nonconvex penalties have been proposed to approach the ℓ_0-norm in [7] and asymptotic connections with the ℓ_0 penalized criteria, in terms of global minimizers, have been shown. In [12], the authors showed the equivalence between ℓ_0- and ℓ_p-norm minimization under linear equalities or inequalities. More precisely, after reformulating these two problems as the minimization of a (non-convex) cost function over a bounded feasible region, which is the same for both problems, the main result of that paper states the existence of a vertex of this set which is solution of both ℓ_0- and ℓ_p- minimization problems for some $p \leq 1$. Mixed-Integer Programming (MIP) reformulations of ℓ_0-norm based criteria have recently been proposed in [3] allowing *exact optimization* through branch and bound based algorithms together with cutting plane methods. However, due to computing time issues, such methods are restricted to moderate-size problems involving hundreds of variables. By considering problem (1) with a finite DC (Difference of Convex functions) data term and an additional constraint $\mathrm{x} \in K \subset \mathbb{R}^N$, where K is a polyhedral convex set, the authors in [15] have proposed a family of continuous DC approximations of the ℓ_0-norm and showed links between the approximated and original problems. They proved that any minimizer of the approximated problem is in a ε-neighbourhood of a minimizer of the initial problem. Moreover, when the data term functional is concave and bounded below on K, they showed that optimal solutions of the approximated problem are included in the ones of the initial problem. Then, using an exact continuous reformulation of the problem as a DC program, a stronger result is showed for the *Capped-ℓ_1* penalty (see also [16]), for which global solutions of the approximation exactly coincides with the ones of the initial functional. However these two last results are limited to global solutions and are not available for local minimizers. At the same time, we have proposed the continuous exact ℓ_0 (CEL0) penalty [23] leading to stronger properties (in the case of problem (1)) since we proved that the resulting tight continuous relaxation, G_{CEL0}, preserves global minimizers of G_{ℓ_0} while some local minimizers are removed (the others remain unchanged).

Following the recent works [15] and [23], it seems of interest to give a unified view of ℓ_0-norm continuous approximations leading to nonconvex equivalent (in terms of minimizers) continuous

relaxations of G_{ℓ_0}.

Contributions and outline In this work, we consider continuous approximations of the ℓ_0-norm of the form

$$\Phi(\mathrm{x}) = \sum_{i=1}^{N} \phi_i(\mathrm{x}_i), \tag{3}$$

where ϕ_i are continuous 1D penalties approximating the weighted "0-1" function[1] $\lambda|\cdot|_0$, leading to continuous relaxations of G_{ℓ_0} in (1) defined by

$$\tilde{\mathrm{G}}(\mathrm{x}) = \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \Phi(\mathrm{x}). \tag{4}$$

Given $A \in \mathbb{R}^{M\times N}$, the present paper is devoted to the determination of *necessary and sufficient* conditions on ϕ_i (which may depend on the elements of A but do not require any assumption on A) ensuring the two following properties for all $d \in \mathbb{R}^M$:

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \tilde{\mathrm{G}}(\mathrm{x}) = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}), \tag{P1}$$

$$\hat{\mathrm{x}} \text{ (local) minimizer of } \tilde{\mathrm{G}} \implies \hat{\mathrm{x}} \text{ (local) minimizer of } \mathrm{G}_{\ell_0}. \tag{P2}$$

In other words, we are concerned with the design of a class of continuous relaxations of G_{ℓ_0}, preserving all its global minimizers, and for which any local minimal point is also a local minimizer of the initial functional. From (P2), $\tilde{\mathrm{G}}$ can potentially eliminate local (not global) minimizers of G_{ℓ_0} which is an interesting property for such highly nonconvex functional. Note that the approximating functions ϕ_i are depending on λ, may depend on A but must not depend on d. Indeed, the approximation may depend on the given problem (which is defined by the matrix A) but not on the data so that the same relaxation can be used for all acquired data.

In Section 2, we consider the one dimensional case and prove five *necessary and sufficient* conditions to have (P1) and (P2). More precisely three conditions on the 1D continuous penalties are shown to be necessary and sufficient to have (P1) while two supplementary conditions are required to also have (P2). Then, Section 3 extends this result to the case where the matrix A, in the quadratic data fidelity term, is orthogonal. A further extension to any $A \in \mathbb{R}^{M\times N}$ is proposed in Section 4. This section provides the main contribution of this work where we show that, without any assumption on $A \in \mathbb{R}^{M\times N}$, the designed conditions are necessary and sufficient to have (P1) and (P2). We thereby highlight a class of *continuous* nonsmooth nonconvex penalties approximating the ℓ_0-norm and leading to continuous relaxations of G_{ℓ_0} for which properties (P1) and (P2) hold. It is worth noting that the CEL0 penalty proposed in [23] is the inferior limit of the obtained class of penalties. Finally, Section 5 analyses, within this context of exact continuous relaxation of G_{ℓ_0}, previously proposed "ℓ_0-like" penalties such as MCP, SCAD or Capped-ℓ_1. Using the general study conducted in Section 4, conditions on the parameters of these state of the art penalties are established in order to make (P1) and (P2) valid for the underlying relaxation $\tilde{\mathrm{G}}$.

Notations and definitions We use the same notations as in [19, 23]:

- $\mathbb{I}_N = \{1, \cdots, N\}$,
- $a_i \in \mathbb{R}^M$, the ith column of $A \in \mathbb{R}^{M\times N}$. We assume that $a_i \neq 0_{\mathbb{R}^M}$, $\forall i \in \mathbb{I}_N$,
- $\|\cdot\| = \|\cdot\|_2$ the ℓ_2-norm. Otherwise we will precise the norm with a subscript,

[1] $|\cdot|_0$ is defined by $|u|_0 = 1$ for $u \neq 0$ and $|u|_0 = 0$ for $u = 0$.

- $e_i \in \mathbb{R}^N$, the unitary vector of the standard basis of $\mathbb{R}^N$,
- $\mathrm{x}^{(i)} = (\mathrm{x}_1, \cdots, \mathrm{x}_{i-1}, 0, \mathrm{x}_{i+1}, \cdots, \mathrm{x}_N) \in \mathbb{R}^N$,
- $A_\omega = (a_{\omega[1]}, \cdots, a_{\omega[\sharp\omega]}) \in R^{M \times \sharp\omega}$ for $\omega \subseteq \mathbb{I}_N$, the restriction of $A \in \mathbb{R}^{M\times N}$ to the columns indexed by the elements of $\omega \subseteq \mathbb{I}_N$,
- $\mathrm{x}_\omega = (\mathrm{x}_{\omega[1]}, \cdots, \mathrm{x}_{\omega[\sharp\omega]}) \in \mathbb{R}^{\sharp\omega}$ for $\omega \subseteq \mathbb{I}_N$, the restriction of $\mathrm{x} \in \mathbb{R}^N$ to the entries indexed by the elements of $\omega \subseteq \mathbb{I}_N$,
- $\sigma(\mathrm{x}) = \{i \in \mathbb{I}_N; \mathrm{x}_i \neq 0\} \subseteq \mathbb{I}_N$, the support of $\mathrm{x} \in \mathbb{R}^N$.

We recall the CEL0 penalty [23] which will be used in what follows:

$$\Phi_{\text{CEL0}}(\mathrm{x}) = \sum_{i \in \mathbb{I}_N} \phi_{\text{CEL0}}(\|a_i\|, \lambda; \mathrm{x}_i), \tag{5}$$

where for $a \in \mathbb{R}_+$, $\lambda \in \mathbb{R}_+$ and $u \in \mathbb{R}$,

$$\phi_{\text{CEL0}}(a, \lambda; u) = \lambda - \frac{a^2}{2}\left(|u| - \frac{\sqrt{2\lambda}}{a}\right)^2 \mathbb{1}_{\left\{|u| \leq \frac{\sqrt{2\lambda}}{a}\right\}}. \tag{6}$$

2 One dimensional analysis

Let us consider the one dimensional problem,

$$\hat{u} \in \arg\min_{u \in \mathbb{R}} \mathrm{g}_0(u) := \frac{1}{2}(au - d)^2 + \lambda |u|_0, \tag{7}$$

where $a > 0$ and $d \in \mathbb{R}$. Then, we are interested in a *continuous* approximation of $\lambda|\cdot|_0$, denoted ϕ, leading to the following continuous relaxation of g_0:

$$\tilde{\mathrm{g}}(u) := \frac{1}{2}(au - d)^2 + \phi(u). \tag{8}$$

The goal is then to design *necessary and sufficient* conditions on ϕ such that $\tilde{g}$ satisfies properties (P1) and (P2). In order to avoid penalties ϕ defined up to an additive constant, we introduce the following supplementary condition:

$$\min_{u \in \mathbb{R}} \mathrm{g}_0(u) = \min_{u \in \mathbb{R}} \tilde{\mathrm{g}}(u). \tag{9}$$

Then, we consider functions ϕ defined independently of d (*i.e.* conditions derived on ϕ must not depend on d) to be able to extend the 1D case conditions to the ND case and so that the continuous approximation can be applied whatever the observations for a given problem (*i.e.* a given a). Moreover, we consider ϕ differentiable on $\mathbb{R}$, except for $u \in B$, where B denotes a finite subset of points of $\mathbb{R}$. Finally, we assume that ϕ is at least C^2 (twice differentiable) on $\mathbb{R} \setminus B$.

The following proposition characterizes the global minimizer(s) of g_0.

Proposition 1 (minimizers of g_0)**.** *Let $a > 0$, $\lambda > 0$ and $d \in \mathbb{R}$. Then, a global minimizer, $u^\star \in \mathbb{R}$, of g_0 verifies*

$$u^\star = \begin{cases} 0 & \textit{iff } |d| \leq \sqrt{2\lambda}, \\ \frac{d}{a} & \textit{iff } |d| \geq \sqrt{2\lambda}. \end{cases} \tag{10}$$

Proof. It is clear that g_0 has always two (local) minimizers $u_1 = 0$ and $u_2 = \frac{d}{a}$. Then, noticing that

$$\mathrm{g}_0(0) = \frac{d^2}{2} \text{ and } \mathrm{g}_0\left(\frac{d}{a}\right) = \lambda, \tag{11}$$

completes the proof. $\square$

In order to make $\tilde{\mathrm{g}}$ preserving global minimizers of g_0, we first need to give a characterization of the critical points of $\tilde{\mathrm{g}}$ (*i.e.* points $\hat{u} \in \mathbb{R}$ such that $0 \in \partial\tilde{\mathrm{g}}(\hat{u})$ where $\partial\tilde{\mathrm{g}}$ denotes the generalized gradient of $\tilde{\mathrm{g}}$ [8]) and ensure that global minimizers of g_0 are critical points of $\tilde{\mathrm{g}}$. The following proposition characterizes the critical points of $\tilde{\mathrm{g}}$.

Proposition 2 (critical points of $\tilde{\mathrm{g}}$)**.** *Let $a > 0$, $\lambda > 0$, $d \in \mathbb{R}$ and ϕ be locally Lipschitz [8]. Then $\hat{u} \in \mathbb{R}$ is a critical point of $\tilde{\mathrm{g}}$ (i.e. $0 \in \partial\tilde{\mathrm{g}}(\hat{u})$) if and only if*

$$\begin{cases} ad - a^2\hat{u} \in \left[\underline{\delta}^{\hat{u}}, \bar{\delta}^{\hat{u}}\right] & \textit{iff } \hat{u} \in B, \\ a^2\hat{u} - ad + \phi'(\hat{u}) = 0 & \textit{iff } \hat{u} \in \mathbb{R} \setminus B, \end{cases} \tag{12}$$

where

$$\forall v \in B,\ \underline{\delta}^v = \min\{l_v^-, l_v^+\} \ \textit{ and } \ \bar{\delta}^v = \max\{l_v^-, l_v^+\}, \tag{13}$$

with

$$l_v^- = \lim_{\substack{u \to v \\ u < v}} \phi'(u) \ \textit{ and } \ l_v^+ = \lim_{\substack{u \to v \\ u > v}} \phi'(u). \tag{14}$$

Proof. From [8, Proposition 2.1.2], $\partial\phi(u)$ is a convex set and is reduced to the singleton $\{\phi'(u)\}$ when ϕ is differentiable. Moreover, [8, Theorem 2.5.1] states that $\partial\phi(u)$ is the convex hull of the set of limits of the form $\phi'(u + h_i)$ where $h_i \to 0$ as $i \to +\infty$, *i.e.*

$$\partial\phi(\bar{u}) = \mathrm{co}\left\{\xi \in \mathbb{R} : \xi = \lim_{k \to +\infty} \phi'(u_k), u_k \to \bar{u}, u_k \in D\right\}, \tag{15}$$

where D is the set of points where ϕ is differentiable and "co" denotes the convex hull. Hence, using the definitions (13) and (14), we have,

$$\partial\phi(u) = \begin{cases} \left[\underline{\delta}^u, \bar{\delta}^u\right] & \text{if } u \in B, \\ \{\phi'(u)\} & \text{if } u \in \mathbb{R} \setminus B. \end{cases} \tag{16}$$

Finally, the differentiability of the quadratic term in (8) leads to

$$\partial\tilde{\mathrm{g}}(u) = a(au - d) + \partial\phi(u), \tag{17}$$

which completes the proof. $\square$

From (10) and (12), we are now able to derive conditions on ϕ in order to ensure that the global minimizer(s) $u^\star \in \left\{0, \frac{d}{a}\right\}$ of g_0 are critical point(s) of $\tilde{\mathrm{g}}$.

Lemma 3 (global minimizers of g_0 are critical points of $\tilde{\mathrm{g}}$)**.** *Let $a > 0$ and $\lambda > 0$, then the global minimizer(s) of g_0 are critical point(s) of $\tilde{\mathrm{g}}$ for all $d \in \mathbb{R}$ if and only if ϕ verifies the two following conditions:*

$$\begin{cases} 0 \in B \ \textit{ and } \ \underline{\delta}^0 \le -\sqrt{2\lambda}a \ \textit{ and } \ \bar{\delta}^0 \ge \sqrt{2\lambda}a, & \text{(18a)} \\ B \subset [-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a] \ \textit{ and } \ \forall u \in \mathbb{R} \setminus [-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a],\ \phi'(u) = 0, & \text{(18b)} \end{cases}$$

Proof. Let $u_d^\star$ be a global minimizer of g_0 for $d \in \mathbb{R}$. Then, Proposition 1 states that

$$\forall d \in \mathbb{R}, \left\{ \begin{array}{l} |d| \leq \sqrt{2\lambda} \implies u_d^\star = 0, \\ |d| \geq \sqrt{2\lambda} \implies u_d^\star = \frac{d}{a}. \end{array} \right. \tag{19}$$

Moreover, from Proposition 2, we have the two following equivalences,

$$0 \text{ is a critical point of } \tilde{\mathrm{g}} \iff \left\{ \begin{array}{ll} ad \in \left[\underline{\delta}^0, \bar{\delta}^0\right] & \text{if } 0 \in B, \\ \phi'(0) = ad & \text{if } 0 \in \mathbb{R} \setminus B, \end{array} \right. \tag{20}$$

$$\frac{d}{a} \text{ is a critical point of } \tilde{\mathrm{g}} \iff \left\{ \begin{array}{ll} 0 \in \left[\underline{\delta}^{d/a}, \bar{\delta}^{d/a}\right] & \text{if } \frac{d}{a} \in B, \\ \phi'\left(\frac{d}{a}\right) = 0 & \text{if } \frac{d}{a} \in \mathbb{R} \setminus B. \end{array} \right. \tag{21}$$

Then, if follows from the previous equations that

$$\{\forall d \in \mathbb{R},\ u^\star \text{ global minimizer of } \mathrm{g}_0 \implies u^\star \text{ critical point of } \tilde{\mathrm{g}}\} \tag{22}$$

is equivalent to

$$\forall d \in \mathbb{R}, \begin{cases} |d| \leq \sqrt{2\lambda} \implies \left\{ \begin{array}{ll} ad \in \left[\underline{\delta}^0, \bar{\delta}^0\right] & \text{if } 0 \in B, \\ \phi'(0) = ad & \text{if } 0 \in \mathbb{R} \setminus B. \end{array} \right. & (23) \\ |d| \geq \sqrt{2\lambda} \implies \left\{ \begin{array}{ll} 0 \in \left[\underline{\delta}^{d/a}, \bar{\delta}^{d/a}\right] & \text{if } \frac{d}{a} \in B, \\ \phi'\left(\frac{d}{a}\right) = 0 & \text{if } \frac{d}{a} \in \mathbb{R} \setminus B. \end{array} \right. & (24) \end{cases}$$

Keeping in mind that (23) holds $\forall d \in \mathbb{R}$, one can rewrite it as follows:

$$\left\{ \begin{array}{lll} [-\sqrt{2\lambda}a, \sqrt{2\lambda}a] \subseteq \left[\underline{\delta}^0, \bar{\delta}^0\right] & & \text{if } 0 \in B, \\ \phi'(0) = ad & \forall |d| \leq \sqrt{2\lambda} & \text{if } 0 \in \mathbb{R} \setminus B. \end{array} \right. \tag{25}$$

Clearly, the second line of (25) is impossible for a fixed ϕ. Hence (23) is equivalent to (18a).

Similarly, one can rewrite (24) as

$$\forall |u| \geq \sqrt{2\lambda}/a, \left\{ \begin{array}{ll} 0 \in \left[\underline{\delta}^u, \bar{\delta}^u\right] & \text{if } u \in B, \\ \phi'(u) = 0 & \text{if } u \in \mathbb{R} \setminus B. \end{array} \right. \tag{26}$$

The fact that B contains a finite number of points of $\mathbb{R}$ together with the continuity of ϕ shows that only the second line of (26) can occur. Hence (24) is equivalent to (18b).

Finally, note that if $d = -\sqrt{2\lambda}$ (resp. $d = \sqrt{2\lambda}$) and that $-\sqrt{2\lambda}/a \in B$ (resp. $\sqrt{2\lambda}/a \in B$) we have, from (14) and (18b), $l^-_{-\sqrt{2\lambda}/a} = 0$ (resp. $l^+_{\sqrt{2\lambda}/a} = 0$). Then, it is obvious from (13) that $0 \in [\underline{\delta}^{-\sqrt{2\lambda}/a}, \bar{\delta}^{-\sqrt{2\lambda}/a}]$ (resp. $0 \in [\underline{\delta}^{\sqrt{2\lambda}/a}, \bar{\delta}^{\sqrt{2\lambda}/a}]$) and that $\frac{d}{a}$ is a critical point of $\tilde{\mathrm{g}}$. □

Actually, when $d = \pm\sqrt{2\lambda}$, both 0 and $\frac{d}{a}$ are global minimizers of g_0 and, under (18), are critical points of $\tilde{\mathrm{g}}$. Figure 1 illustrates conditions (18).

Lemma 4. *Let $a > 0$ and $\lambda > 0$, then $\tilde{g}$ do not have any global minimizer(s) within $(-\frac{\sqrt{2\lambda}}{a}, 0) \cup (0, \frac{\sqrt{2\lambda}}{a})$ for all $d \in \mathbb{R}$ if and only if ϕ verifies the following condition:*

$$\forall u \in (-\sqrt{2\lambda}/a, 0) \cup (0, \sqrt{2\lambda}/a),\ \phi(u) > \phi_{\text{CELO}}(a, \lambda; u), \tag{27}$$

where ϕ_{CELO} is given in (6).

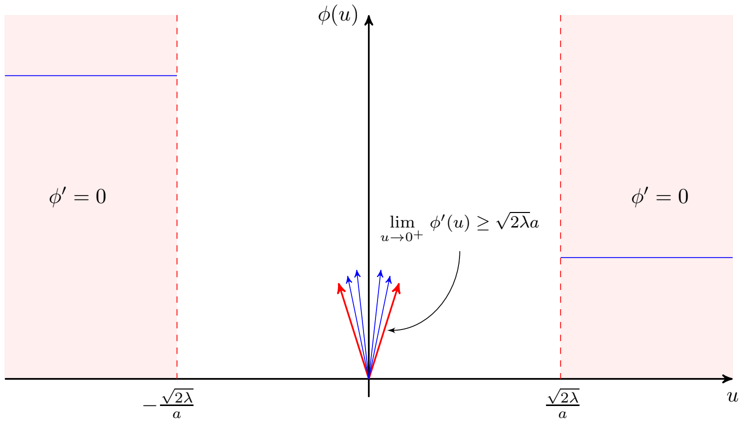

Figure 1: Illustration of the conditions on ϕ given in Lemma 3. The arrows represent condition (18a) (in red) with possible half tangent of ϕ at 0 (in blue) and red areas illustrate (18b) where ϕ is constant.

Proof. We want that

$$\forall u \in (-\sqrt{2\lambda}/a, 0) \cup (0, \sqrt{2\lambda}/a),\ \tilde{g}(u) > \left\{ \begin{array}{ll} \mathrm{g}_0(0) = \frac{d^2}{2} & \text{if } |d| \leq \sqrt{2\lambda}, \\ \mathrm{g}_0\left(\frac{d}{a}\right) = \lambda & \text{if } |d| \geq \sqrt{2\lambda}, \end{array} \right. \ \forall d \in \mathbb{R}, \tag{28}$$

which can be rewritten as,

$$\forall u \in (-\sqrt{2\lambda}/a, 0) \cup (0, \sqrt{2\lambda}/a),\ \phi(u) > \phi_{\min}(u) := \sup_{d \in \mathbb{R}} f_u(d), \tag{29}$$

where

$$f_u(d) = \left\{ \begin{array}{ll} -\frac{a^2u^2}{2} + aud & \text{if } |d| \leq \sqrt{2\lambda}, \\ \lambda - \frac{1}{2}(au - d)^2 & \text{if } |d| \geq \sqrt{2\lambda}. \end{array} \right. \tag{30}$$

Then, simple calculations lead to

$$\sup_{|d| \leq \sqrt{2\lambda}} f_u(d) = -\frac{a^2u^2}{2} + \sqrt{2\lambda}a|u|, \tag{31}$$

and

$$\sup_{|d| \geq \sqrt{2\lambda}} f_u(d) = \left\{ \begin{array}{ll} \lambda & \text{if } |u| \geq \frac{\sqrt{2\lambda}}{a}, \\ -\frac{a^2u^2}{2} + \sqrt{2\lambda}a|u| & \text{otherwise,} \end{array} \right. \tag{32}$$

Finally we obtain

$$\forall u \in (-\sqrt{2\lambda}/a, 0) \cup (0, \sqrt{2\lambda}/a),\ \phi_{\min}(u) = -\frac{a^2u^2}{2} + \sqrt{2\lambda}a|u| = \phi_{\text{CEL0}}(a, \lambda; u), \tag{33}$$

where ϕ_{CEL0} is given in (6). This completes the proof. $\square$

The following theorem gives *necessary and sufficient* conditions on ϕ in order to have (P1) for $\tilde{\mathrm{g}}$.

Theorem 5 (necessary and sufficient conditions for (P1))**.** *Let $a > 0$ and $\lambda > 0$, then $\tilde{g}$ has property (P1) (and (9)) for all $d \in \mathbb{R}$ if and only if ϕ verifies the three following conditions:*

$$\begin{cases} \phi(0) = 0, & \text{(34a)} \\ \forall u \in \mathbb{R} \setminus (-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a),\ \phi(u) = \lambda|u|_0 = \lambda, & \text{(34b)} \\ \forall u \in (-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a) \setminus \{0\},\ \phi(u) > \phi_{\text{CEL0}}(a, \lambda; u), & \text{(34c)} \end{cases}$$

where ϕ_{CEL0} is given in (6).

Proof. We process by showing both implications

(34) $\Rightarrow$ (P1) **(with** (9)**):** Let us first remark that (34) $\Longrightarrow$ (18). Indeed, it is an evidence that (34b) $\Longrightarrow$ (18b) and, by definition of ϕ_{CEL0}, we have $\phi_{\text{CEL0}}(0) = 0$ and

$$\lim_{u \to 0^+} \phi'_{\text{CEL0}}(u) = \sqrt{2\lambda}a \quad \text{and} \quad \lim_{u \to 0^-} \phi'_{\text{CEL0}}(u) = -\sqrt{2\lambda}a. \tag{35}$$

Hence (34a) and (34c) imply that ϕ is not differentiable at 0 (*i.e.* $0 \in B$) and, with (35), that (18a) holds.

Under conditions (34), the critical points characterization of $\tilde{g}$ in (12) becomes,

$$0 \in \partial\tilde{g}(\hat{u}) \iff \begin{cases} d \in \left[\frac{\underline{\delta}^0}{a}, \frac{\bar{\delta}^0}{a}\right] & \text{iff } \hat{u} = 0, \\ ad - a^2\hat{u} \in \left[\underline{\delta}^{\hat{u}}, \bar{\delta}^{\hat{u}}\right] & \text{iff } \hat{u} \in B \setminus \{0\}, \\ a^2\hat{u} - ad + \phi'(\hat{u}) = 0 & \text{iff } \hat{u} \in \left[-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a\right] \setminus B, \\ \hat{u} = \frac{d}{a} & \text{iff } \hat{u} \in (-\infty, -\sqrt{2\lambda}/a) \cup (\sqrt{2\lambda}/a, +\infty). \end{cases} \tag{36}$$

Among these points, there is at least one global minimizer of $\tilde{g}$. Indeed, from the continuity of ϕ and conditions (34a) and (34b), ϕ is bounded. Moreover, the quadratic data fidelity term of $\tilde{g}$ is coercive (since $a > 0$). Thus $\tilde{g}$ is coercive and its continuity ensures the existence of a global minimizer.

Then, from Lemma 4, condition (34c) ensures that $\tilde{g}$ do not have any global minimizer(s) within $(-\sqrt{2\lambda}/a, 0) \cup (0, \sqrt{2\lambda}/a)$ and, from (36), the only "candidates point(s)" to be global minimizer(s) of $\tilde{g}$ are thus given by

$$\hat{u} = \begin{cases} 0 & \text{if } d \in \left[\frac{\underline{\delta}^0}{a}, \frac{\bar{\delta}^0}{a}\right] \underset{(18a)}{\supseteq} [-\sqrt{2\lambda}, \sqrt{2\lambda}] \\ \hat{u} = \frac{d}{a} & \text{if } d \in (-\infty, -\sqrt{2\lambda}] \cup [\sqrt{2\lambda}, +\infty), \end{cases} \tag{37}$$

Moreover, from (34a) and (34b) one gets

$$\tilde{g}(\hat{u}) = g_0(\hat{u}), \tag{38}$$

ensuring, with Proposition 1, that (P1) holds.

(34) $\Leftarrow$ (P1) **(with** (9)**):** Under (P1), all the global minimizer(s) of g_0 are critical points of $\tilde{g}$ which, from Lemma 3, is equivalent to (18). Moreover (9) leads to (34a) and allows to reduce (18b) to (34b). Finally, assume that (34c) does not hold and that there exists $u_0 \in (0, \sqrt{2\lambda}/a)$ such that $\phi(u_0) \leq \phi_{\text{CEL0}}(u_0)$. Then, one can easily get a $d_0 = \sqrt{2\lambda}$ for which g_0 has two global minimizers $\{0, \frac{d_0}{a}\}$ and for which the whole interval $[0, \frac{d_0}{a}] = [0, \frac{\sqrt{2\lambda}}{a}]$ minimizes the CEL0 functional g_{CEL0} (defined by (8) for $\phi = \phi_{\text{CEL0}}$) since g_{CEL0} is the convex hull of g_0 [23, Section 2]. Hence, we have $u_0 \notin \{0, \frac{d_0}{a}\}$ and nevertheless $\tilde{g}(u_0) \leq g_{\text{CEL0}}(u_0) = g_0(0) = g_0(\frac{d_0}{a})$ which contradicts (P1) and completes the proof. $\square$

Remark 6. *One can notice that* (34c) *imposes* ϕ *to be singular at the origin (see the proof of Theorem* 5 *and Figure* 1*). We highlight here a well-known property stating that sparsity is enforced by penalties singular at the origin. Moreover condition* (34b) *leads to penalties which are constant for large* $|u|$*. This was known to be a condition for unbiasedness [10].*

A schema showing conditions (34) is presented on Figure 2. One can see that within the gray zone, the penalty can be totally arbitrary as long as it is continuous, equal to zero at the origin and equal to λ at $\pm\sqrt{2\lambda}/a$. Moreover, for $|u| \geq \sqrt{2\lambda}/a$, the penalty must be constant equal to λ.

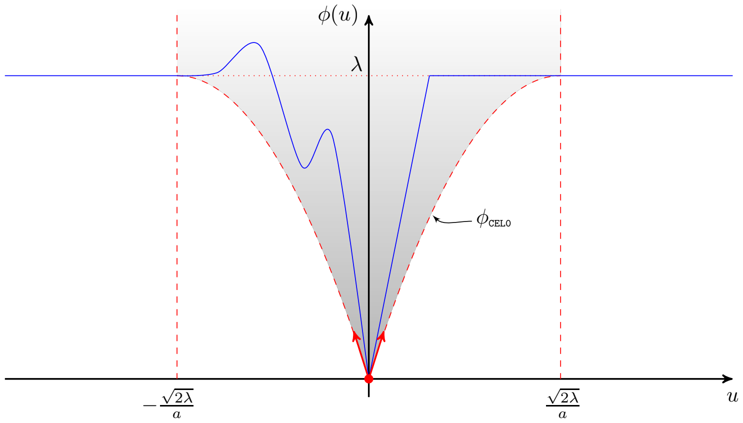

Figure 2: Illustration of the conditions given in Theorem 5. Dashed red curves together with the red point at 0 represent conditions (34). The gray zone is the admissible part of the plan where the penalty must verify condition (34c). An example of such a penalty is showed in blue.

Now, we shall ensure that each (local) minimizer of $\tilde{\mathrm{g}}$ is a (local) minimizer of g_0. We start by proving the following technical lemma.

Lemma 7. *Let* $a > 0$, $\lambda > 0$, $\mathcal{C} = \{u \in \mathbb{R} : 0 \in \partial\tilde{\mathrm{g}}(u)\}$ *be the set of critical points of* $\tilde{\mathrm{g}}$ *and let us suppose that conditions* (34) *hold. Then the two following statements are equivalent:*

$$\textit{1.}\quad \forall d \in \mathbb{R},\ \hat{u} \in \mathcal{C} \setminus \left\{0, \frac{d}{a}\right\} \Longrightarrow \hat{u} \textit{ is } \textbf{\textit{not}} \textit{ a (local) minimizer of } \tilde{\mathrm{g}}, \tag{39}$$

$$\textit{2.}\quad \left\{ \begin{array}{l} \forall u \in [-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a] \setminus B,\ \phi'(u) \neq 0 \implies \phi''(u) < -a^2. \\ \forall u \in B \setminus \{0\},\ l_u^- > l_u^+. \end{array} \right. \tag{40}$$

where l_u^- *and* l_u^+ *are defined in* (14).

Proof. Following the proof of Theorem 5, under conditions (34), the critical point of $\tilde{\mathrm{g}}$ are given by (36). Hence, we have

$$u_0 \in \mathcal{C} \setminus \left\{0, \frac{d}{a}\right\} \underset{(36)}{\iff} \left\{ \begin{array}{ll} \phi'(u_0) = ad - a^2 u_0 & \text{if } u_0 \in [-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a] \setminus \{B \cup \{\frac{d}{a}\}\} \\ ad - a^2 u_0 \in [\underline{\delta}^{u_0}, \bar{\delta}^{u_0}] & \text{if } u_0 \in B \setminus \{0, \frac{d}{a}\} \end{array} \right. \tag{41}$$

$\Longrightarrow$ Assume that (39) holds. Then, from the first line of (41), we have $\forall d \in \mathbb{R}$

$$\phi'(u_0) = ad - a^2 u_0 \text{ for a given } u_0 \in [-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a] \setminus \left\{B \cup \left\{\frac{d}{a}\right\}\right\} \Longrightarrow \phi''(u_0) < -a^2, \tag{42}$$

since $\phi''(u_0) < -a^2 \iff \tilde{g}''(u_0) < 0$ ensuring that $u_0 \in \mathcal{C} \setminus \left\{0, \frac{d}{a}\right\}$ is not a (local) minimizer of $\tilde{g}$. Let us remark that $\phi'(u_0) = ad - a^2 u_0$ is valid only for the given u_0 and does not define the expression of ϕ' on $[-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a] \setminus \left\{B \cup \left\{\frac{d}{a}\right\}\right\}$, then we cannot compute $\phi''(u_0)$ from $ad - a^2 u_0$.

Moreover, $\forall\, u_0 \in [-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a] \setminus B$,

$$\phi'(u_0) \neq 0 \implies \exists\, d_0 := (\phi'(u_0) + a^2 u_0)/a \neq a u_0 \text{ s.t. } \phi'(u_0) = a d_0 - a^2 u_0. \tag{43}$$

Then, (43) with the fact that (42) has to be verified for all $d \in \mathbb{R}$ allows to rewrite (42) as follows:

$$\forall u \in [-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a] \setminus B,\ \phi'(u) \neq 0 \implies \phi''(u) < -a^2. \tag{44}$$

Similarly, always under (39), the second line of (41) leads to: $\forall d \in \mathbb{R}$

$$ad - a^2 u_0 \in \left[\underline{\delta}^{u_0}, \bar{\delta}^{u_0}\right] \text{ for } u_0 \in B \setminus \left\{0, \frac{d}{a}\right\} \tag{45}$$

$$\implies \lim_{\substack{v \to u_0 \\ v < u_0}} \tilde{g}'(v) > \lim_{\substack{v \to u_0 \\ v > u_0}} \tilde{g}'(v), \tag{46}$$

$$\underset{(14)}{\iff} a^2 u_0 - ad + l_{u_0}^- > a^2 u_0 - ad + l_{u_0}^+, \tag{47}$$

$$\iff l_{u_0}^- > l_{u_0}^+, \tag{48}$$

since (46) prevents such a $u_0 \in \mathcal{C} \setminus \left\{0, \frac{d}{a}\right\}$ to be a (local) minimizer of $\tilde{g}$. Indeed, since u_0 is a critical point of $\tilde{g}$ and that $\tilde{g}$ is not differentiable at u_0, the right and left derivative of $\tilde{g}$ at u_0 have different sign. Then, to ensure that u_0 is not a (local) minimizer, inequality (46) is required.

Moreover,

$$\forall u_0 \in B \setminus \{0\},\ \exists\, d_0 \in \mathbb{R} \text{ s.t. } a d_0 - a^2 u_0 \in \left[\underline{\delta}^{u_0}, \bar{\delta}^{u_0}\right] \text{ and } u_0 \neq \frac{d_0}{a}, \tag{49}$$

since by definition $\underline{\delta}^{u_0} \neq \bar{\delta}^{u_0}$. Then, (49) with the fact that (48) has to be verified for all $d \in \mathbb{R}$ allows to rewrite (48) as follows:

$$\forall u \in B \setminus \{0\},\ l_u^- > l_u^+. \tag{50}$$

Hence we have (39) $\implies$ (40).

$\Longleftarrow$ Assume that (40) holds and let $\hat{u} \in \mathcal{C} \setminus \left\{0, \frac{d}{a}\right\}$ for $d \in \mathbb{R}$. Hence, from the critical point characterization (36), one gets that $\hat{u} \in (-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a) \setminus \{0\}$. Moreover, it is obvious from the proof of ($\Longrightarrow$) that (40) $\implies$ {(42),(46)}. Hence, we get

$$\left.\begin{array}{l} \hat{u} \in (-\sqrt{2\lambda}/a, \sqrt{2\lambda}/a) \setminus B \underset{\hat{u} \neq d/a}{\Rightarrow} \tilde{g}''(\hat{u}) < 0, \\ \hat{u} \in B \setminus \{0\} \Rightarrow \lim\limits_{v \to \hat{u}^-} \tilde{g}'(v) > \lim\limits_{v \to \hat{u}^+} \tilde{g}'(v) \end{array}\right\} \Rightarrow \hat{u} \text{ is not a (local) minimizer of } \tilde{g}, \tag{51}$$

which completes the proof. □

Now, the following theorem states *necessary and sufficient* conditions on ϕ to have both (P1) and (P2) for $\tilde{g}$.

Theorem 8 (necessary and sufficient conditions for (P1) and (P2)). *Let $a > 0$ and $\lambda > 0$, then $\tilde{\mathrm{g}}$ has both properties (P1) and (P2) for all $d \in \mathbb{R}$ if and only if, in addition to conditions (34), ϕ verifies the two following conditions:*

$$\begin{cases} \forall u \in B \setminus \{0\}, \ \lim\limits_{\substack{v \to u \\ v < u}} \phi'(v) > \lim\limits_{\substack{v \to u \\ v > u}} \phi'(v), & (52a) \\ \forall u \in (\beta^-, \beta^+) \setminus B, \ \phi''(u) < -a^2. & (52b) \end{cases}$$

for $\beta^- \in \left[-\sqrt{2\lambda}/a, 0\right)$ and $\beta^+ \in \left(0, \sqrt{2\lambda}/a\right]$ defined as the larger (resp. lower) real for which ϕ is constant on the whole interval $(-\infty, \beta^-]$ and (resp. $[\beta^+, +\infty)$). With this definition, $B \subset [\beta^-, \beta^+]$.

Proof. Since g_0 can only have two (local) minimizers ($\hat{u}_1 = 0$ and $\hat{u}_2 = \frac{d}{a}$), one can see that condition (39) is *necessary and sufficient* to have (P2). Then, from Lemma 7, this condition is equivalent to (40) which, combined with conditions (34), can be reduced to (52). □

Remark 9. *For the CEL0 penalty (6), condition (52b) is not verified. Indeed we have,*

$$\forall\, 0 < |u| < \frac{\sqrt{2\lambda}}{a}, \ \phi''_{\text{CEL0}}(a, \lambda; u) = -a^2. \tag{53}$$

However, in this case, when $\phi'_{\text{CEL0}}(a, \lambda; u) = ad - a^2 u \Leftrightarrow |ad| = \sqrt{2\lambda}a$ (by definition of CEL0), the whole interval $[0, -\frac{\sqrt{2\lambda}}{a}]$ (resp. $[\frac{\sqrt{2\lambda}}{a}, 0]$ depending on the sign of the quantity ad) minimizes $\tilde{\mathrm{g}}$ and one can easily get a minimizer of g_0 by simple thresholding [23]. Moreover, condition (34c) is also not verified by the CEL0 penalty. Finally, from conditions (34) and (52), the CEL0 penalty can be seen as the inferior limit of the resulting class of penalties.

Remark 10. *Under the conditions of Theorems 5 and 8, ϕ is strictly concave-decreasing on $[\beta^-, 0]$ (resp. strictly concave-increasing on $[0, \beta^+]$). Hence, from (34c), if $\beta^- = -\frac{\sqrt{2\lambda}}{a}$ (resp. $\beta^+ = \frac{\sqrt{2\lambda}}{a}$) then necessarily $\beta^- \notin B$ (resp. $\beta^+ \notin B$) and $\phi'(\beta^-) = 0$ (resp. $\phi'(\beta^+) = 0$). See Figure 3 for an illustration.*

Finally, Theorems 5 and 8 provide *necessary and sufficient* conditions on ϕ in order to make the continuous relaxation $\tilde{\mathrm{g}}$ verifying properties (P1) and (P2). These conditions are illustrated on Figure 3. In the following of the paper, we extend these results to the N-dimensional case.

3 When A is orthogonal

Let's consider G_{ℓ_0} defined in (1) and $\tilde{\mathrm{G}}$ in (4). Then, following [23, Section 3], the case where the matrix A is orthogonal can easily be deduced from the previous 1D study. Let $\hat{d} = AD^{-2}A^T d$ and $\tilde{z} = D^{-1}A^T d$, where $D \in \mathbb{R}^{N \times N}$ is a diagonal matrix with diagonal elements $d_i = \|a_i\| \, \forall i \in \mathbb{I}_N$, then we have

$$\frac{1}{2}\|A\mathrm{x} - d\|^2 = \frac{1}{2}\|d - \hat{d}\|^2 + \frac{1}{2}\|D\mathrm{x} - \tilde{z}\|^2. \tag{54}$$

Using this relation, G_{ℓ_0} can be rewritten as,

$$\mathrm{G}_{\ell_0}(\mathrm{x}) = \frac{1}{2}\|d - \hat{d}\|^2 + \sum_{i \in \mathbb{I}_N} \frac{1}{2}(\|a_i\|^2 \mathrm{x}_i - \tilde{z}_i)^2 + \lambda|\mathrm{x}_i|_0 \tag{55}$$

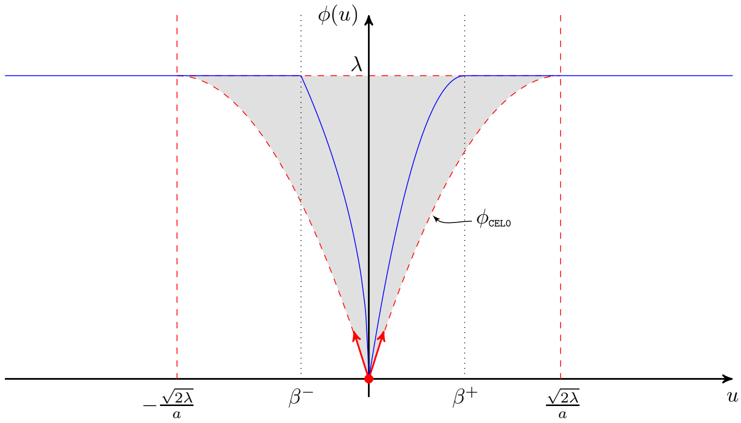

Figure 3: Illustrations of all the conditions given by Theorems 5 and 8 to have (P1) and (P2). Dashed red curves together with the red point at 0 represent the conditions while the blue curve shows an example of exact penalty verifying such conditions. Between β^- and β^+, the penalty must stay in the gray area and verify conditions (52).

and its minimization is reduced to the minimization of N independent 1D functionals. According to Theorems 5 and 8, *necessary and sufficient* conditions on Φ in order to have (P1) and (P2) for all $d \in \mathbb{R}^M$, are given by: $\forall i \in \mathbb{I}_N$,

$$\begin{cases} \phi_i(0) = 0, & \text{(56a)} \\ \exists \beta^{i-} \in \left[-\frac{\sqrt{2\lambda}}{\|a_i\|}, 0\right) \text{ and } \beta^{i+} \in \left(0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right] \text{ s.t. } \forall u \in \mathbb{R} \setminus (\beta^{i-}, \beta^{i+}),\ \phi_i(u) = \lambda, & \text{(56b)} \\ \forall u \in (\beta^{i-}, \beta^{i+}) \setminus \{0\},\ \phi_i(u) > \phi_{\text{CEL0}}(\|a_i\|, \lambda; u), & \text{(56c)} \\ \forall u \in B^i \setminus \{0\},\ \lim\limits_{\substack{v \to u \\ v < u}} \phi_i'(v) > \lim\limits_{\substack{v \to u \\ v > u}} \phi_i'(v), & \text{(56d)} \\ \forall u \in (\beta^{i-}, \beta^{i+}) \setminus B^i,\ \phi_i''(u) < -\|a_i\|^2, & \text{(56e)} \end{cases}$$

where $B^i \ni 0$ is a finite subset of $[\beta^{i-}, \beta^{i+}]$ on which ϕ_i is not differentiable. Let us recall that the N-dimensional penalty Φ is given by,

$$\Phi(\mathrm{x}) = \sum_{i \in \mathbb{I}_N} \phi_i(\mathrm{x}_i). \tag{57}$$

The following proposition gives a relation between conditions (56) which can be useful in practice to define penalties verifying the five conditions (56).

Proposition 11. *Let $i \in \mathbb{I}_N$ and $B^i \subseteq \{\beta^{i-}, 0, \beta^{i+}\}$, then the following implication holds:*

$$\{(56a),(56b),(56e)\} \Longrightarrow (56c).$$

Proof. Let ϕ_i verifying conditions (56a), (56b), (56e) and $f = \phi_i - \phi_{\text{CEL0}}(\|a_i\|, \lambda; \cdot)$. Then we have $f(0) = 0$ and $f(\beta^{i+}) = \lambda - \phi_{\text{CEL0}}(\|a_i\|, \lambda; \beta^{i+}) \geq 0$. Moreover, by assumption on B^i, f is twice differentiable on $(0, \beta^{i+})$ and $f''(u) = \phi_i''(u) + \|a_i\|^2 < 0$ $\forall u \in (0, \beta^{i+})$ which implies that $\forall u \in (0, \beta^{i+})$ $\phi_i(u) > \phi_{\text{CEL0}}(\|a_i\|, \lambda; u)$. Same arguments can be used to show the result on $(\beta^{i-}, 0)$. $\square$

According to Section 2, while the five conditions (56) are necessary and sufficient to have $\{$(P1),(P2)$\}$ for $\tilde{\mathrm{G}}$ and G_{ℓ_0} (Theorem 8), only conditions (56a)-(56c) are required to only have (P1) (Theorem 5). The main question is then to know if conditions (56) remain valid for an arbitrary $A \in \mathbb{R}^{M \times N}$.

4 Extension to an arbitrary matrix $A \in \mathbb{R}^{M\times N}$

This section presents the main result of the paper. We show that conditions (56) are *necessary and sufficient* to have (P1) and (P2) for G_{ℓ_0}, defined in (1), and $\tilde{\mathrm{G}}$, defined in (4), without any assumption on the matrix $A \in \mathbb{R}^{M\times N}$. First of all, since the case where A is orthogonal is a special case of $A \in \mathbb{R}^{M\times N}$, we get from the previous section that if we do not consider any assumption on $A \in \mathbb{R}^{M\times N}$,

- conditions (56) are *necessary* to have {(P1),(P2)},
- only conditions (56a), (56b) and (56c) are *necessary* to have (P1).

Note that one may define weaker conditions for specific $A \in \mathbb{R}^{M\times N}$ and $d \in \mathbb{R}^M$. However, in the present paper, we are concerned with conditions valid for any $d \in \mathbb{R}^M$ and which do not require a special structure of the matrix $A \in \mathbb{R}^{M\times N}$ (but these conditions may be expressed in function of the elements of A). Then, the goal in what follows is to show that these conditions are also *sufficient*.

Theorem 12 (links between global minimizers of $\tilde{\mathrm{G}}$ and G_{ℓ_0})**.** *Let $A \in \mathbb{R}^{M\times N}$, $\lambda > 0$ and $\tilde{\mathrm{G}}$ be defined with Φ verifying conditions (56a), (56b) and (56c). Then, $\forall d \in \mathbb{R}^M$*

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}) = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \tilde{\mathrm{G}}(\mathrm{x}), \tag{58}$$

and,

$$\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}) = \min_{\mathrm{x}\in\mathbb{R}^N} \tilde{\mathrm{G}}(\mathrm{x}). \tag{59}$$

Proof. From, (56a), (56b) and (56c), it is clear that,

$$\forall \mathrm{x} \in \mathbb{R}^N,\ \mathrm{G}_{\text{CEL0}}(\mathrm{x}) \le \tilde{\mathrm{G}}(\mathrm{x}). \tag{60}$$

Let $\hat{\mathrm{x}}$ be a global minimizer of G_{ℓ_0}, then from [19, Proposition 4.1] we have

$$\forall i \in \sigma(\hat{\mathrm{x}}),\ \hat{\mathrm{x}}_i \in \mathbb{R} \setminus \left(-\frac{\sqrt{2\lambda}}{\|a_i\|}, \frac{\sqrt{2\lambda}}{\|a_i\|}\right) \tag{61}$$

which implies, from (56a) and (56b),

$$\mathrm{G}_{\ell_0}(\hat{\mathrm{x}}) = \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}) = \tilde{\mathrm{G}}(\hat{\mathrm{x}}). \tag{62}$$

Then, [23, Theorem 4.5 (i)] ensures that $\hat{\mathrm{x}}$ is also a global minimizer of G_{CEL0} which, with (60) and (62), proves the inclusion $\subseteq$ in (58). Thus, there is at least one point, denoted $\mathrm{x}^\star \in \mathbb{R}^N$, which is a global minimizer of the three functionals (existence of minimizers for G_{ℓ_0} is established in [19, Theorem 4.4 (i)]). Let now $\hat{\mathrm{x}} \in \mathbb{R}^N$ be a global minimizer of $\tilde{\mathrm{G}}$. Clearly, $\tilde{\mathrm{G}}(\hat{\mathrm{x}}) = \tilde{\mathrm{G}}(\mathrm{x}^\star) = \mathrm{G}_{\text{CEL0}}(\mathrm{x}^\star)$ and since $\mathrm{G}_{\text{CEL0}} \le \tilde{\mathrm{G}}$ (eq. 60), $\hat{\mathrm{x}}$ is also a global minimizer of G_{CEL0} and $\tilde{\mathrm{G}}(\hat{\mathrm{x}}) = \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}})$. Combining this last equality with (56c) leads to

$$\forall i \in \sigma(\hat{\mathrm{x}}),\ \hat{\mathrm{x}}_i \notin (\beta^{i-}, \beta^{i+}), \tag{63}$$

and it comes from (56a) and (56b) that $\mathrm{G}_{\ell_0}(\hat{\mathrm{x}}) = \tilde{\mathrm{G}}(\hat{\mathrm{x}}) = \tilde{\mathrm{G}}(\mathrm{x}^\star) = \mathrm{G}_{\ell_0}(\mathrm{x}^\star)$. Thus, $\hat{\mathrm{x}}$ is a global minimizer of G_{ℓ_0} which proves the inclusion $\supseteq$ in (58). Equality (59) is straightforward from the foregoing. □

Hence, Theorem 12 shows that conditions (56a), (56b) and (56c) are sufficient to have (P1). Moreover, we can deduce from this result the existence of minimizers for $\tilde{\mathrm{G}}$ as stated by the following proposition.

Proposition 13 (existence of global minimizers for $\tilde{\mathrm{G}}$)**.** *Let $\tilde{\mathrm{G}}$ be defined as in Theorem 12. Then, the set of global minimizers of $\tilde{\mathrm{G}}$ is nonempty.*

Proof. From [19, Theorem 4.4 (i)], the set of global minimizers of G_{ℓ_0} is nonempty. Then, the result for $\tilde{\mathrm{G}}$ is straightforward from Theorem 12. □

In order to analyse the links between (local) minimizers of $\tilde{\mathrm{G}}$ and G_{ℓ_0}, we start by showing two preliminary results.

Proposition 14. *Let $A \in \mathbb{R}^{M\times N}$, $\lambda > 0$ and $\tilde{\mathrm{G}}$ be defined with Φ verifying conditions (56b), (56d) and (56e). Then, $\forall i \in \mathbb{I}_N$, $\tilde{\mathrm{G}}^i$, the restriction of $\tilde{\mathrm{G}}$ to the ith variable, is strictly concave on $(\beta^{i-}, 0)$ and on $(0, \beta^{i+})$ and strictly convex beyond.*

Proof. Let $i \in \mathbb{I}_N$ and consider the restriction of $\tilde{\mathrm{G}}$ to the ith variable,

$$f(t) = \tilde{\mathrm{G}}(\mathrm{x}^{(i)} + e_i t) = \frac{\|a_i\|^2}{2}t^2 + t\langle a_i, A\mathrm{x}^{(i)} - d\rangle + \phi_i(t) + C, \tag{64}$$

where $C = \frac{1}{2}\|A\mathrm{x}^{(i)} - d\|^2 + \sum_{j\in\mathbb{I}_N\setminus\{i\}} \phi_j(\mathrm{x}_j)$ is a constant independent of t. Then, from (56e), one can easily get that $\forall t \in (\beta^{i-}, \beta^{i+}) \setminus B^i$ $f''(t) = \|a_i\|^2 + \phi_i''(t) < 0$. Moreover (56d) leads to

$$\forall t \in B^i \setminus \{0\},\ \lim_{\substack{u\to t\\ u<t}} f'(u) > \lim_{\substack{u\to t\\ u>t}} f'(u).$$

These two results prove the strict concavity of f on $(\beta^{i-}, 0)$ and $(0, \beta^{i+})$. Then, it comes from (56b) that $\forall t \in (-\infty, \beta^{i-}) \cup (\beta^{i+}, +\infty)$ $f''(t) = \|a_i\|^2 > 0$ which completes the proof. □

A consequence of Proposition 14 is given by the following result.

Proposition 15. *Let $\tilde{\mathrm{G}}$ be defined as in Proposition 14. If $\tilde{\mathrm{G}}$ reaches a (local) minimum at $\hat{\mathrm{x}} \in \mathbb{R}^N$, then*

$$\forall i \in \sigma(\hat{\mathrm{x}}), \quad \hat{\mathrm{x}}_i \in (-\infty, \beta^{i-}] \cup [\beta^{i+}, +\infty). \tag{65}$$

Moreover, if $\beta^{i-} \in B$ (resp. $\beta^{i+} \in B$), then the interval $(-\infty, \beta^{i-}]$ (resp. $[\beta^{i+}, +\infty)$) in (65) can be reduced to $(-\infty, \beta^{i-})$ (resp. $(\beta^{i+}, +\infty)$).

Proof. The proof of (65) is straightforward from Proposition 14 which states that the restriction of $\tilde{\mathrm{G}}$ to the ith variable, is strictly concave on $(\beta^{i-}, 0)$ and on $(0, \beta^{i+})$. Then, the fact that we can consider open intervals when $\beta^{i-} \in B$ (resp. $\beta^{i+} \in B$) comes from the same arguments as the ones used in the proof of Proposition 14 for points belonging to B^i . □

Remark 16. *It is shown in [23] that local (not global) minimizers of G_{ℓ_0} for which $\exists i \in \sigma(\hat{\mathrm{x}})$ such that $|\hat{\mathrm{x}}_i| < \frac{\sqrt{2\lambda}}{\|a_i\|}$ are eliminated with the CEL0 functional. Proposition 15 extends this result to $\tilde{\mathrm{G}}$ (if conditions (56b), (56d) and (56e) hold) and shows that such a $\tilde{\mathrm{G}}$ eliminates minimizers $\hat{\mathrm{x}}$ of G_{ℓ_0} for which*

$$\exists i \in \sigma(\hat{\mathrm{x}}) \ \ s.t. \ \ \hat{\mathrm{x}}_i \in (\beta^{i-}, \beta^{i+}). \tag{66}$$

Hence, from (56b), $\tilde{\mathrm{G}}$ potentially eliminates fewer local minimizers of G_{ℓ_0} than G_{CEL0}.

We are now able to derive the following result between (local) minimizers of $\tilde{\mathrm{G}}$ and G_{ℓ_0}.

Theorem 17 (links between (local) minimizers of $\tilde{\mathrm{G}}$ and G_{ℓ_0})**.** *Let $A \in \mathbb{R}^{M\times N}$, $\lambda > 0$ and $\tilde{\mathrm{G}}$ be defined with Φ verifying conditions (56). Then, $\forall d \in \mathbb{R}^M$*

$$\hat{\mathrm{x}} \text{ (local) minimizer of } \tilde{\mathrm{G}} \implies \hat{\mathrm{x}} \text{ (local) minimizer of } \mathrm{G}_0, \tag{67}$$

and

$$\tilde{\mathrm{G}}(\hat{\mathrm{x}}) = \mathrm{G}_{\ell_0}(\hat{\mathrm{x}}). \tag{68}$$

Proof. Let $\hat{\mathrm{x}} \in \mathbb{R}^N$ be a (local) minimizer of $\tilde{\mathrm{G}}$ and $\hat{\sigma} = \sigma(\hat{\mathrm{x}})$. Thus we have $0_{\mathbb{R}^N} \in \partial\tilde{\mathrm{G}}(\hat{\mathrm{x}})$ which is a necessary condition for a point to be a local optimum of $\tilde{\mathrm{G}}$. Since the quadratic term in $\tilde{\mathrm{G}}$ is differentiable we have,

$$\forall \mathrm{x} \in \mathbb{R}^N,\ \partial\tilde{\mathrm{G}}(\mathrm{x}) = A^T(A\mathrm{x} - d) + \partial\Phi(\mathrm{x}) = \prod_{i \in \mathbb{I}_N} \left[A^T(A\mathrm{x} - d)\right]_i + \partial\phi_i(\mathrm{x}_i). \tag{69}$$

From Proposition 15, $\hat{\mathrm{x}}$ verifies (65) which, combined with conditions (56b) and (56c) leads to

$$0_{\mathbb{R}^N} \in \partial\tilde{\mathrm{G}}(\hat{\mathrm{x}}) \iff \forall i \in \mathbb{I}_N, \begin{cases} \langle a_i, d - A\hat{\mathrm{x}}\rangle \in \left[\underline{\delta}^0, \bar{\delta}^0\right] & \text{if } i \notin \hat{\sigma}, \\ \langle a_i, d - A\hat{\mathrm{x}}\rangle = 0 & \text{if } i \in \hat{\sigma}. \end{cases} \tag{70}$$

Then, the second line of (70) can be rewritten as follows:

$$(A_{\hat{\sigma}})^T A_{\hat{\sigma}} \hat{\mathrm{x}}_{\hat{\sigma}} = (A_{\hat{\sigma}})^T d, \tag{71}$$

which ensures that $\hat{\mathrm{x}}$ is a (local) minimizer of G_{ℓ_0} [19, Corollary 2.5]. Finally, equality (68) comes from Proposition 15 and conditions (56a), (56b). ☐

Theorem 17 is thus the counterpart of Theorem 12 for property (P2) showing that conditions (56) are *sufficient* to have (P2) between $\tilde{\mathrm{G}}$ and G_{ℓ_0}.

Finally, we have shown in this section that the three first conditions in (56) are *necessary and sufficient* to have (P1) while all conditions (56) are required to have {(P1) ,(P2)}. Hence, conditions (56) define a class of exact continuous penalties approximating the ℓ_0-norm for problem (1). As outlined in Remark 9, the CEL0 penalty is the inferior limit of this class of functions. Then, for all penalties belonging to this class, the resulting functional $\tilde{\mathrm{G}}$ eliminates some local (not global) minimizers of G_{ℓ_0}. However, as stated by Remark 16, the amount of removed local (not global) minimizers is potentially lower than the one corresponding to G_{CEL0}. Moreover, G_{CEL0} is the only one leading to the convex hull of G_{ℓ_0} in the orthogonal case [23] and being convex with respect to each component for any matrix $A \in \mathbb{R}^{M \times N}$. The limiting penalty Φ_{CEL0} seems thus to be the most suited to continuously approximate the ℓ_0-norm in problem (1) (see [23] for detail about the special case CEL0).

5 State of the art penalties analysis

This part is devoted to the analysis of state of the art penalties in the proposed framework of exact continuous relaxation of the ℓ_0-norm. Thanks to conditions (56), we highlight bounds on the parameters defining such penalties ensuring that the resulting relaxation $\tilde{\mathrm{G}}$ verifies (P1) or {(P1),(P2)}.

5.1 Capped-ℓ_1

The Capped-ℓ_1 (or truncated-ℓ_1) penalty [29] is defined by

$$\Phi_{\text{Cap}}(\mathrm{x}) := \sum_{i \in \mathbb{I}_N} \lambda \min\left\{\theta_i|\mathrm{x}_i|, 1\right\}, \tag{72}$$

where $\theta_i \in \mathbb{R}_+^\star$ for all $i \in \mathbb{I}_N$.

As stated in the following proposition, by choosing properly the parameters θ_i ($i \in \mathbb{I}_N$), one can ensures that the relaxed functional G_{Cap} verifies property (P1).

Proposition 18. *Property* (P1) *holds for* G_{Cap} *if and only if*

$$\forall i \in \mathbb{I}_N, \ \lambda\theta_i \geq \sqrt{2\lambda}\|a_i\|. \tag{73}$$

Moreover, G_{Cap} *cannot verify* (P2).

Proof. By definition of Φ_{Cap}, condition (56a) is verified for all $\theta_i > 0$. Then, one can easily see that for the Capped-ℓ_1 penalty, we have

$$\forall i \in \mathbb{I}_N, \ \beta^{i-} = -\frac{1}{\theta_i} \ \text{ and } \ \beta^{i+} = \frac{1}{\theta_i} \tag{74}$$

Then,

$$(73) \implies \forall i \in \mathbb{I}_N, \ \frac{1}{\theta_i} \leq \frac{\sqrt{\lambda}}{\sqrt{2}\|a_i\|} \leq \frac{\sqrt{2\lambda}}{\|a_i\|} \underset{(74)}{\implies} (56b). \tag{75}$$

Moreover,

$$\forall i \in \mathbb{I}_N, \ \lim_{u \to 0^+} \phi'_{\text{Cap}}(\theta_i, \lambda; u) = \lambda\theta_i \ \text{ and } \lim_{u \to 0^-} \phi'_{\text{Cap}}(\theta_i, \lambda; u) = -\lambda\theta_i, \tag{76}$$

where $\phi_{\text{Cap}}(\theta, \lambda; u) = \lambda \min\{\theta|u|, 1\}$ for $u \in \mathbb{R}$, $\theta \in \mathbb{R}_+^\star$ and $\lambda \in \mathbb{R}_+^\star$.

Then, the fact that $\phi'_{\text{CEL0}}(\|a_i\|, \lambda; u) \to \sqrt{2\lambda}\|a_i\|$ as $u \to 0^+$ (resp. $-\sqrt{2\lambda}\|a_i\|$ as $u \to 0^-$) together with (76) and the fact that, on $[\beta^-, 0]$ and $[0, \beta^+]$, ϕ_{Cap} is linear and ϕ_{CEL0} strictly concave, shows that (56c) $\iff$ (73). Finally, one can easily see that (56e) cannot be satisfied since ϕ_{Cap} is linear on $[\beta^-, 0]$ and $[0, \beta^+]$. ☐

Figure 4 shows Capped-ℓ_1 penalties for which θ has been tuned according to Proposition 18. One can see on the right graph that the global minimizer is preserved and this will be true for any value of $d \in \mathbb{R}$ since (P1) holds for such values of θ. However, on these examples, a local minimizer of G_{Cap} which is not a minimizer for G_{ℓ_0} exists when $\lambda\theta = \sqrt{2\lambda}a$. According to the calculations conducted in the proof of Lemma 7, this local minimizer, denoted $\hat{u}$, verifies

$$\lambda\theta = ad - a^2\hat{u} \iff \hat{u} = \frac{ad - \lambda\theta}{a^2}. \tag{77}$$

In fact, for any $u_0 \in (0, 1/\theta)$, there exist $d_0 = (\lambda\theta + a^2 u_0)/a \in \mathbb{R}$ for which u_0 is a local minimizer of the associated relaxation G_{Cap} which is not a minimizer for G_{ℓ_0}. This illustrates the fact that (P2) cannot be verified with the Capped-ℓ_1 penalty.

Similar results for the Capped-ℓ_1 penalty have been shown in [15, 16].

5.2 Smoothly clipped absolute deviation (SCAD)

The SCAD penalty, which can be seen as a smoothed version of the Capped-ℓ_1, has been proposed by Fan and Li [10] and reads as follows:

$$\Phi_{\text{SCAD}}(\mathrm{x}) := \sum_{i \in \mathbb{I}_N} \phi_{\text{SCAD}}(\gamma_i, \tilde{\lambda}_i; \mathrm{x}_i), \tag{78}$$

where $\gamma_i \in (2, +\infty)$, $\tilde{\lambda}_i \in \mathbb{R}_+^\star$ for all $i \in \mathbb{I}_N$ and ϕ_{SCAD} is given by: $\forall u \in \mathbb{R}$,

$$\phi_{\text{SCAD}}(\gamma, \tilde{\lambda}; u) := \tilde{\lambda}\left(|u|\mathbb{1}_{\{|u| \leq \tilde{\lambda}\}} - \frac{\tilde{\lambda}^2 - 2\gamma\tilde{\lambda}|u| + u^2}{2(\gamma - 1)\tilde{\lambda}}\mathbb{1}_{\{\tilde{\lambda} < |u| \leq \gamma\tilde{\lambda}\}} + \frac{(\gamma + 1)\tilde{\lambda}}{2}\mathbb{1}_{\{|u| > \gamma\tilde{\lambda}\}}\right). \tag{79}$$

Then one can get a similar result as the one given in Proposition 18 for the Capped-ℓ_1 penalty.

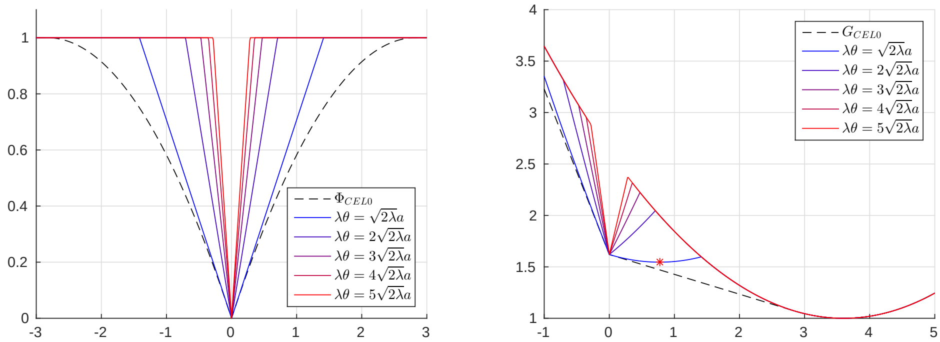

Figure 4: Examples of Capped-ℓ_1 penalties (left) and their associated continuous relaxations G_{Cap} (right) for which (P1) holds for $a = 0.5$, $\lambda = 1$ and $d = 1.8$. The red star represents a local minimizer of G_{Cap} which is not a minimizer for G_{ℓ_0}.

Proposition 19. *Let $\|a_i\| < 1/\sqrt{3}$ for all $i \in \mathbb{I}_N$. Then, Property (P1) holds for G_{SCAD} if and only if*

$$\forall i \in \mathbb{I}_N, \; \frac{(\gamma_i + 1)\tilde{\lambda}_i^2}{2} = \lambda \;\; \text{and} \;\; 2 < \gamma_i \leq \frac{1}{\|a_i\|} - 1. \tag{80}$$

Moreover, G_{SCAD} cannot verify (P2).

Proof. The proof is given in Appendix A. □

Assumption $\|a_i\| < 1/\sqrt{3}$ ($i \in \mathbb{I}_N$) in Proposition 19 can always be verified by normalizing the columns of A and then multiplying the matrix by a real $\zeta < 1/\sqrt{3}$ since this does not change the problem (it corresponds to a change of variable in G_{ℓ_0}). Figure 5 presents SCAD penalties, with the associated relaxations G_{SCAD}, verifying the conditions of Proposition 19. The same conclusions as for the Capped-ℓ_1 (Fig. 4) can be done.

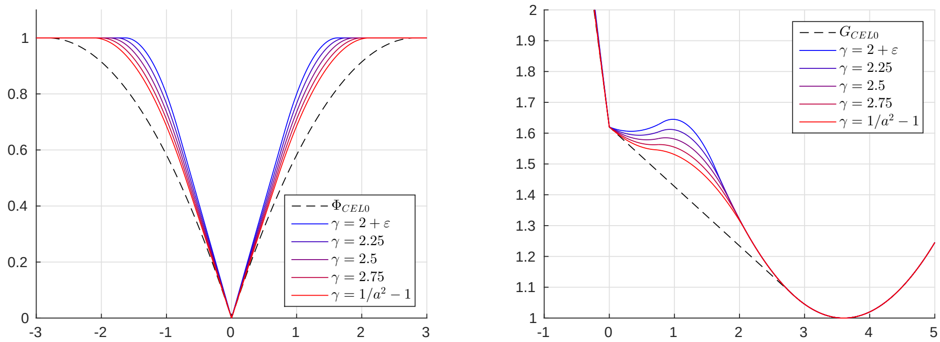

Figure 5: Examples of SCAD penalties (left) and their associated continuous relaxations G_{SCAD} (right) for which (P1) holds for $a = 0.5 < 1/\sqrt{3}$, $\lambda = 1$ and $d = 1.8$.

5.3 Minimax concave penalty (MCP)

The MCP, proposed by Zhang in [28], is defined by

$$\Phi_{\text{MCP}}(\mathrm{x}) := \sum_{i \in \mathbb{I}_N} \phi_{\text{MCP}}(\gamma_i, \tilde{\lambda}_i; \mathrm{x}_i), \tag{81}$$

where $\gamma_i \in \mathbb{R}_+^\star$, $\tilde{\lambda}_i \in \mathbb{R}_+^\star$ for all $i \in \mathbb{I}_N$ and ϕ_{MCP} writes as follows: $\forall u \in \mathbb{R}$,

$$\phi_{\text{MCP}}(\gamma, \tilde{\lambda}; u) := \tilde{\lambda} \int_0^{|u|} \left(1 - x/(\gamma\tilde{\lambda})\right)_+ dx = \tilde{\lambda} \left(\frac{\gamma\tilde{\lambda}}{2} \mathbb{1}_{\{|u| > \gamma\tilde{\lambda}\}} + \left(|u| - \frac{u^2}{2\gamma\tilde{\lambda}}\right) \mathbb{1}_{\{|u| \leq \gamma\tilde{\lambda}\}} \right). \tag{82}$$

From conditions (56) established in the previous section, we can prove that under suitable assumptions on the parameters γ_i and $\tilde{\lambda}_i$ ($i \in \mathbb{I}_N$), properties (P1) and (P2) hold for the relaxed functional G_{MCP}. The following proposition states this result.

Proposition 20. *Properties* (P1) *and* (P2) *hold for* G_{MCP} *if and only if*

$$\forall i \in \mathbb{I}_N, \ \frac{\gamma_i \tilde{\lambda}_i^2}{2} = \lambda \ \text{ and } \ \gamma_i < \frac{1}{\|a_i\|^2}. \tag{83}$$

Proof. By definition of Φ_{MCP}, conditions (56a) and (56d) are verified for all $\gamma_i > 0$ and $\tilde{\lambda}_i > 0$ ($i \in \mathbb{I}_N$). Then, from the results of Section 4, the proof consists in showing that (83) is equivalent to the three conditions (56b), (56c) and (56e). From the definition of Φ_{MCP}, one can easily get that (by symmetry we restrict the proof to $\mathbb{R}_+$),

$$(56\text{b}) \iff \forall i \in \mathbb{I}_N, \ \frac{\gamma_i \tilde{\lambda}_i^2}{2} = \lambda \ \text{ and } \ \gamma_i \tilde{\lambda}_i = \beta^{i+} \leq \frac{\sqrt{2\lambda}}{\|a_i\|} \tag{84}$$

$$\iff \forall i \in \mathbb{I}_N, \ \frac{\gamma_i \tilde{\lambda}_i^2}{2} = \lambda \ \text{ and } \ \gamma_i \leq \frac{1}{\|a_i\|^2} \tag{85}$$

$$(56\text{e}) \iff \forall i \in \mathbb{I}_N, \forall u \in (-\gamma_i\tilde{\lambda}_i, \gamma_i\tilde{\lambda}_i) \setminus \{0\}, \ \phi''_{\text{MCP}}(u) = -\frac{1}{\gamma_i} < -\|a_i\|^2 \tag{86}$$

$$\iff \forall i \in \mathbb{I}_N, \ \gamma_i < \frac{1}{\|a_i\|^2} \tag{87}$$

Clearly, we have that (83) is equivalent to the set of conditions $\{$(56b),(56e)$\}$. Hence, it only remains to show that (83) $\Longrightarrow$ (56c) which is direct from Proposition 11 and completes the proof. ☐

Let Φ_{MCP} be defined according to the conditions of Proposition 20, then it can be rewritten using only the parameter γ_i (*i.e.* removing the dependance on $\tilde{\lambda}_i$),

$$\Phi_{\text{MCP}}(\mathrm{x}) = \sum_{i \in \mathbb{I}_N} \lambda - \frac{1}{2\gamma_i} \left(|\mathrm{x}_i| - \sqrt{2\lambda\gamma_i}\right)^2 \mathbb{1}_{\{|\mathrm{x}_i| \leq \sqrt{2\lambda\gamma_i}\}}. \tag{88}$$

Then, $\forall i \in \mathbb{I}_N$, $\forall \gamma_i < \frac{1}{\|a_i\|^2}$, the associated functional, G_{MCP} verifies (P1) and (P2). This defines a sub-family of MCP which are exact continuous approximations of the ℓ_0-norm. Finally, it is worth noting that the inferior limit of the sub-family of MCP is given by

$$\forall \mathrm{x} \in \mathbb{R}^N, \quad \lim_{\substack{\gamma_i \to 1/\|a_i\|^2 \\ \forall i \in \mathbb{I}_N}} \Phi_{\text{MCP}}(\mathrm{x}) = \Phi_{\text{CEL0}}(\mathrm{x}). \tag{89}$$

Examples of MCP defined following Proposition 20 are shown on Figure 6. This illustrates the fact that, under conditions (83), the MCP is an exact ℓ_0 penalty ensuring to have both (P1) and (P2).

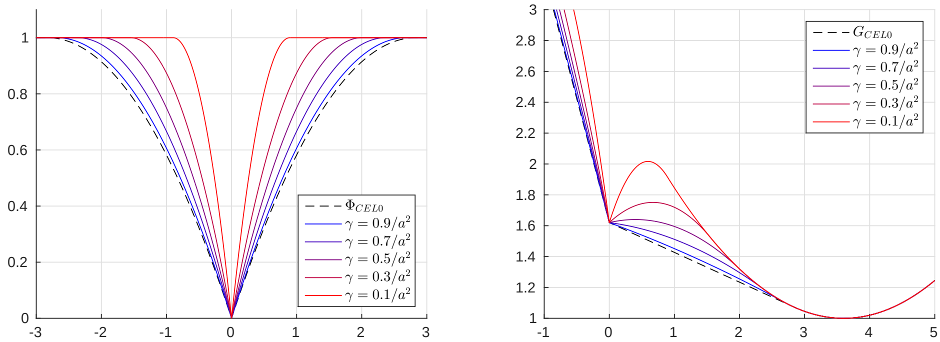

Figure 6: Examples of MCP (left) and their associated continuous relaxations G_{MCP} (right) for which (P1) and (P2) hold for $a = 0.5$, $\lambda = 1$ and $d = 1.8$.

5.4 Truncated-ℓ_p $(0 < p < 1)$

It is an evidence that approximations of the ℓ_0-norm such as the log-sum penalty or the ℓ_p-norms $(0 < p < 1)$ cannot lead to (P1) neither (P2) since they do not verify (56b). However, one can define truncated versions of such penalties in the same fashion as the Capped-ℓ_1 studied in Section 5.1. In the following, we analyse the Truncated-ℓ_p penalties defined by:

$$\Phi_{\text{TLp}}(\mathrm{x}) := \sum_{i \in \mathbb{I}_N} \lambda \min\left\{\theta_i |\mathrm{x}_i|^{p_i}, 1\right\}, \tag{90}$$

where $\theta_i > 0$ and $p_i \in (0, 1)$ for all $i \in \mathbb{I}_N$.

As for the MCP, there exist parameters θ_i and p_i $(i \in \mathbb{I}_N)$ for which the continuous relaxation G_{TLp} verifies properties (P1) and (P2).

Proposition 21. *Properties* (P1) *and* (P2) *hold for* G_{TLp} *if and only if*

$$\forall i \in \mathbb{I}_N, \; \theta_i \geq \theta_0^i := \left(\frac{\|a_i\|^2}{p_i(1-p_i)\lambda}\right)^{p_i/2} \tag{91}$$

Proof. Clearly, conditions (56a) and (56d) are verified, by definition of Φ_{TLp}, for all $\theta_i > 0$ and $0 < p_i < 1$ $(i \in \mathbb{I}_N)$. Then, (56e) is equivalent to (by symmetry we restrict the proof to $\mathbb{R}_+$),

$$\forall i \in \mathbb{I}_N, \; \forall u \in \left(0, \theta_i^{-1/p_i}\right), \; \phi''_{\text{TLp}}(\theta_i, p_i, \lambda; u) = p_i(p_i - 1)\lambda\theta_i u^{p_i - 2} \leq -\|a_i\|^2, \tag{92}$$

where $\phi_{\text{TLp}}(\theta, p, \lambda; u) := \lambda \min\{\theta|u|^p, 1\}$ for $u \in \mathbb{R}$, $\theta \in \mathbb{R}_+^\star$, $p \in (0,1)$ and $\lambda \in \mathbb{R}_+^\star$. Since $\phi''_{\text{TLp}}(\theta_i, p_i, \lambda; \cdot)$ is strictly increasing, (92) reduces to

$$\begin{aligned} \forall i \in \mathbb{I}_N, \; \phi''_{\text{TLp}}(\theta_i, p_i, \lambda; \theta_i^{-1/p_i}) = p_i(p_i-1)\lambda\theta_i\theta_i^{(2-p_i)/p_i} &\leq -\|a_i\|^2, \\ \iff \theta_i \geq \left(\frac{\|a_i\|^2}{p_i(1-p_i)\lambda}\right)^{p_i/2} &. \end{aligned}$$

Hence (91) $\iff$ (56e). Then, one can see that

$$(91) \implies \forall i \in \mathbb{I}_N, \; \beta^{i+} = \theta_i^{-1/p_i} \leq \frac{\sqrt{p_i(1-p_i)\lambda}}{\|a_i\|} \leq \frac{\sqrt{2\lambda}}{\|a_i\|}, \tag{93}$$

implying that (56b) holds. Finally the foregoing together with Proposition 11 shows that (91) $\implies$ (56c) and completes the proof. ∎

Note that one can derive a lower bound on θ_i than the one given by Proposition 21 in order to have only (P1). A similar analysis could be done for a truncated version of the log-sum penalty or any penalty which is not constant for large values of $|u|$. Finally, Figure 7 shows Truncated-ℓ_p penalties for which the associated continuous relaxation $\mathrm{G}_{\mathrm{TLp}}$ verifies (P1) and (P2).

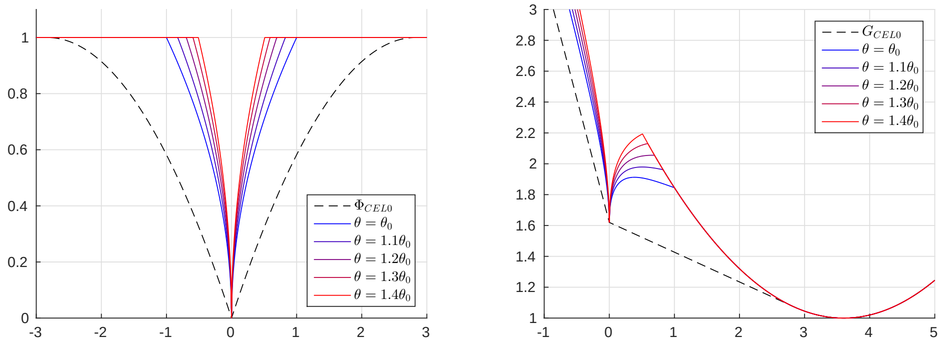

Figure 7: Examples of Truncated-ℓ_p penalties (left) and their associated continuous relaxations $\mathrm{G}_{\mathrm{TLp}}$ (right) for which (P1) and (P2) hold for $a = 0.5$, $\lambda = 1$ and $d = 1.8$.

6 Conclusion

The variety of "ℓ_0-like" continuous penalties proposed in the literature motivates the analysis of the consistency between the minimizers of the initial and relaxed functionals. Following this idea, we proposed a unified view of *exact continuous relaxations* for the ℓ_0-regularized least-squares minimization problem. More precisely, we established five *necessary and sufficient* conditions on the continuous penalty approximating the ℓ_0-norm, such that the resulting continuous objective functional $\tilde{\mathrm{G}}$ preserves all the global minimizers of the initial one G_{ℓ_0} (Theorem 12), and that (local) minimizers of $\tilde{\mathrm{G}}$ are also minimizers for G_{ℓ_0} (Theorem 17). Although the resulting minimization problem is still nonconvex, an interesting point is that some local minimizers of the initial functional can be removed by the relaxation (Remark 16). Moreover, one can take benefit from the current advances in nonsmooth nonconvex optimization [1, 13, 14, 21] to deal with such an equivalent *continuous* reformulation. It would be also of interest to consider regularization path strategies as the ones developed in [28, 2] for the MCP in order to obtain a "ℓ_0-regularization path" as done by the recently proposed greedy algorithms in [25].

Finally, the present paper offers a new way to compare continuous penalties approximating the ℓ_0-norm for G_{ℓ_0}. We showed that, for different penalties proposed over the past, a proper choice of the parameters leads to an exact continuous relaxation of G_{ℓ_0} verifying properties (P1) and (P2) (for MCP and the proposed Truncated-ℓ_p) while one can only ensure (P1) for other penalties as Capped-ℓ_1 or SCAD. However, in the light of Remark 16, the CEL0 penalty, which is the inferior limit of the derived class of exact penalties, is convex with respect to each variable of the $\mathbb{R}^N$ basis and leads to the convex hull in the orthogonal case, seems to be the best choice that one can do to approach the ℓ_0-norm.

A Proof of Proposition 19

By definition of Φ_{SCAD}, condition (56a) holds for all $\gamma_i > 2$ and $\tilde{\lambda}_i > 0$ ($i \in \mathbb{I}_N$). Moreover, one can easily get

$$(56b) \iff \forall i \in \mathbb{I}_N,\ \frac{(\gamma_i+1)\tilde{\lambda}_i^2}{2} = \lambda \ \text{ and } \ \gamma_i \tilde{\lambda}_i = \beta^{i+} \leq \frac{\sqrt{2\lambda}}{\|a_i\|}, \tag{94}$$

$$(56c) \implies \forall i \in \mathbb{I}_N,\ \tilde{\lambda}_i \geq \sqrt{2\lambda}\|a_i\|, \tag{95}$$

where the last implication comes from the same argument as used in the proof of Proposition 18 (note that by symmetry $\beta^{i-} = -\beta^{i+}$). Let us now verify if the above conditions on γ_i and $\tilde{\lambda}_i$ can hold simultaneously. We have, $\forall i \in \mathbb{I}_N$

$$\tilde{\lambda}_i = \sqrt{\frac{2\lambda}{\gamma_i+1}} \ \text{ and } \ \frac{\gamma_i}{\sqrt{\gamma_i+1}} \leq \frac{1}{\|a_i\|} \ \text{ and } \ \frac{1}{\sqrt{\gamma_i+1}} \geq \|a_i\|, \tag{96}$$

$$\underset{\gamma_i>2}{\iff} \tilde{\lambda}_i = \sqrt{\frac{2\lambda}{\gamma_i+1}} \ \text{ and } \ \|a_i\| \leq \frac{1}{\sqrt{\gamma_i+1}} < \frac{\sqrt{\gamma_i+1}}{\gamma_i} \ \text{ and } \ \|a_i\| < \frac{1}{\sqrt{3}}. \tag{97}$$

$$\iff \tilde{\lambda}_i = \sqrt{\frac{2\lambda}{\gamma_i+1}} \ \text{ and } \ \gamma_i \leq \frac{1}{\|a_i\|^2} - 1 \ \text{ and } \ \|a_i\| < \frac{1}{\sqrt{3}}. \tag{98}$$

To conclude the proof, we need to show that (80) $\implies$ (56c). By symmetry, we restrict the proof to $\mathbb{R}_+$. Clearly, using the same arguments as in the proof of Proposition 18, we have under (80)

$$\forall i \in \mathbb{I}_N, \forall u \in (0, \tilde{\lambda}_i],\ \phi_{\text{CEL0}}(\|a_i\|, \lambda; u) < \phi_{\text{SCAD}}(\gamma_i, \tilde{\lambda}_i; u). \tag{99}$$

since (80) $\implies$ (95). Then, $\forall i \in \mathbb{I}_N$, $\forall u \in [\tilde{\lambda}_i, \gamma_i\tilde{\lambda}_i]$, we have

$$\phi_{\text{CEL0}}(\|a_i\|, \lambda; u) = P_1(u) = -\frac{\|a_i\|^2}{2}u^2 + \sqrt{2\lambda}\|a_i\|u,$$

$$\phi_{\text{SCAD}}(\gamma_i, \tilde{\lambda}_i; u) = P_2(u) = -\frac{\tilde{\lambda}_i^2 - 2\gamma_i\tilde{\lambda}_i|u| + u^2}{2(\gamma_i - 1)}$$

where $(P_1, P_2) \in (\mathbb{R}^2[X])^2$ are two order 2 polynomials. Let us consider the order two polynomial $Q = P_2 - P_1$. Then it follows from (94) and (99) that $Q(\tilde{\lambda}_i) > 0$ and $Q(\gamma_i\tilde{\lambda}_i) \geq 0$ (since (80) $\implies$ (94)). Moreover, one can see that

$$\forall u \in [\tilde{\lambda}_i, \gamma_i\tilde{\lambda}_i],\ Q''(u) = \|a_i\|^2 - \frac{1}{\gamma_i - 1} \underset{(80)}{<} 0 \tag{100}$$

Hence Q is strictly concave on $[\tilde{\lambda}_i, \gamma_i\tilde{\lambda}_i]$ which, combined with $Q(\tilde{\lambda}_i) > 0$ and $Q(\gamma_i\tilde{\lambda}_i) \geq 0$, implies that $Q(u) > 0$ $\forall u \in [\tilde{\lambda}_i, \gamma_i\tilde{\lambda}_i)$. This shows that (80) $\implies$ (56c). Finally, the fact that G_{SCAD} cannot verify (P2) follows from similar argument as the ones used in the proof of Proposition 18. □

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