# On one class of permutation polynomials over finite fields of characteristic two *

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**Abstract.** Polynomials of type $x^{q^3+q^2+q+2}+bx$ over the field $\mathbb{F}_{q^4}$, where $q = 2^m$, $m \geq 2$, are considered. All cases when these polynomials are permutation polynomials are classified.


## 1 Introduction

Let $q = p^m$ be a prime power and let $\mathbb{F}_q$ denote the finite field of order $q$. A polynomial $f(x) \in \mathbb{F}_q[x]$ is called a permutation polynomial (PP) of $\mathbb{F}_q$ if $f(x)$ induces a one-to-one mapping of $\mathbb{F}_q$ onto itself. It is well known that permutation polynomials have applications in a variety of areas, including such areas as cryptography for the secure transmission of information, and in combinatorics for the construction of various kinds of combinatorial designs (see [4, 6] and references there). Recently, permutation polynomials have been studied extensively in the literature (see [3, 7, 9, 12], for example).

A polynomial $f(x) \in \mathbb{F}_q[x]$ is called a complete permutation polynomial (CPP), if both $f(x)$ and $f(x) + x$ are PP over $\mathbb{F}_q$. Although there are some results on CPP over $\mathbb{F}_q$ [3, 4, 6, 8, 10, 12], still very few classes of them are known, even for monomial functions. For a positive integer $d$ and $a \in \mathbb{F}_q^*$, a monomial function $ax^d$ is a CPP over $\mathbb{F}_q$ if and only if $\gcd(d, q-1) = 1$ and $ax^d + x$ is a PP over $\mathbb{F}_q$.

In [1, 2] we gave a complete description of PP of type

$$f(x) = x^{1+\frac{q^2-1}{q-1}} + bx$$

over $\mathbb{F}_{q^2}$ (or, monomial CPP of type $f(x) = ax^d$, $d = (q^2-1)/(q-1)+1$) and

$$f(x) = x^{1+\frac{q^3-1}{q-1}} + bx$$


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over $\mathbb{F}_{q^3}$ (or, monomial CPP of type $f(x) = ax^d$, $d = (q^3-1)/(q-1)+1$) for the fields of any characterictic. Here we extend such description to polynomials of type

$$f(x) = x^{1+\frac{q^4-1}{q-1}} + bx, \tag{1}$$

over $\mathbb{F}_{q^4}$ (or, monomial CPP of type $f(x) = ax^d$, $d = (q^4-1)/(q-1)+1$) but only for the fields of even characteristic ($q = 2^m$).

For odd $m$, such polynomials were considered [12]. In particular, it was proved (on base of Dickson polynomials) the following result.

**Theorem 1** [12] *Let $q = 2^m$ where $m \geq 3$ is odd. A polynomial of the type* (1) *is a PP over $\mathbb{F}_{q^4}$ if the element $b$ looks as follows:*

$$\left.\begin{array}{ll} 1)\ b = u\,(1+\beta+\beta^2) & + \, v\,\beta^3; \\ 2)\ b = u\,(1+\beta+\beta^3) & + \, v\,(\beta+\beta^2) \\ \text{or } b = u\,(1+\beta^3) & + \, v\,(1+\beta+\beta^2); \\ 3)\ b = u\,(\beta+\beta^3) & + \, v\,(\beta^2+\beta^3), \end{array}\right\} \tag{2}$$

*where $u, v$ run through $\mathbb{F}_q$, $(u,v) \neq (0,0)$, and where $\beta$ is a root of $x^4+x+1$.*

However, it was not proved that there do not exist other elements $b$ for which polynomials $x^{1+\frac{q^4-1}{q-1}} + bx$ are also permutation polynomials. Here we fill this gap and prove, firstly, these sufficient conditions from [12] are also necessary; secondly, these sufficient and necessary conditions are fulfilled only for $b$ from [12]; and, thirdly, permutation polynomials of the type (1) do not exist for the even $m \geq 4$ (for $m = 2$ such polynomials exist).

Our approach as well as in [1, 2] is based on the following lemma from [8], which in our special case of $f(x) \in \mathbb{F}_{q^4}[x]$ can be reformulated as follows:

**Lemma 1.** *The polynomial*

$$f(x) = x^{1+\frac{q^4-1}{q-1}} + bx$$

*over $\mathbb{F}_{q^4}$ is a permutation polynomial if and only if $b \in \mathbb{F}_q^4 \setminus \mathbb{F}_q$ and the following inequality:*

$$x(b+x)^{q^3+q^2+q+1} \neq y(b+y)^{q^3+q^2+q+1} \tag{3}$$

*holds for all $x, y \in \mathbb{F}_q$, such that $x \neq 0, y \neq 0, x \neq y$.*

## 2 Polynomials $x^{1+\frac{q^4-1}{q-1}} + bx$, $q = 2^m$, $m \geq 2$

Following our approach in [1, 2], first, we reduce the inequality (3) to the equation over $x$ and $y$. Using that

$$\left.\begin{array}{l} x(b+x)^{q^3+q^2+q+1} + y(b+y)^{q^3+q^2+q+1} = \\ = z\left(x^4 + B_2x^2 + (z^3+B_2z)x + z^4 + B_1z^3 + B_2z^2 + B_3z + B_4\right), \end{array}\right\} \tag{4}$$

where

$$\left.\begin{array}{l} B_1 = b + b^q + b^{q^2} + b^{q^3}, \\ B_2 = b^{1+q} + b^{1+q^2} + b^{1+q^3} + b^{q+q^2} + b^{q+q^3} + b^{q^2+q^3}, \\ B_3 = b^{1+q+q^2} + b^{1+q+q^3} + b^{1+q^2+q^3} + b^{q+q^2+q^3}, \\ B_4 = b^{1+q+q^2+q^3}. \end{array}\right\} \tag{5}$$

and $z = x + y$, we can rewrite the lemma 1 as follows:

**Lemma 2.** *Let $q = 2^m$. The polynomial*

$$f(x) = x^{1+\frac{q^4-1}{q-1}} + bx,$$

*over $\mathbb{F}_{q^4}$ is a permutation polynomial if and only if $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ and the equation*

$$x^4 + B_2x^2 + (z^3 + B_2z)x + z^4 + B_1z^3 + B_2z^2 + B_3z + B_4 = 0 \tag{6}$$

*has no solutions $x, z \in \mathbb{F}_q$ such that $z \neq 0$ (for $x = 0$ and $x = z$, i.e. $y = 0$, this equation has no solutions).*

Using a new variable $w = x + B_1$, we obtain the equation

$$w^4 + B_2w^2 + (z^3 + zB_2)w + z^4 + B_2z^2 + Dz + E = 0, \tag{7}$$

where we denote

$$D = B_1B_2 + B_3 \text{ and } E = B_1^4 + B_1^2B_2 + B_4. \tag{8}$$

Introducing again a new variable $\gamma = w/z$ we arrive to the equivalent equation (recall that $z \neq 0$):

$$\gamma^4 + \gamma + \frac{B_2}{z^2} \cdot (\gamma^2 + \gamma + 1) + 1 + \frac{D}{z^3} + \frac{E}{z^4} = 0. \tag{9}$$

Thus, we reduced the problem of solution of equation (6) to the problem of solution of equation (9) with $z \neq 0$. For convenience we denote, for an integer $s \geq 1$ and $q = 2^m$, the trace function

$$\mathrm{Tr}_{q^s}(a) = a + a^2 + a^4 + \cdots + a^{2^{sm-1}}, \quad a \in \mathbb{F}_{q^s}$$

and the relative trace function for integers $s, r \geq 1$

$$\mathrm{Tr}_{q^{sr} \to q^s}(a) = a + a^{q^s} + a^{q^{2s}} + \cdots + a^{q^{s(r-1)}}, \quad a \in \mathbb{F}_{q^{sr}}$$

The cases $B_2 \neq 0$ and $B_2 = 0$ we consider separately.

### 2.1 The case $B_2 \neq 0$.

Because

$$\gamma^4+\gamma+\frac{B_2}{z^2}\cdot(\gamma^2+\gamma+1)+1+\frac{D}{z^3}+\frac{E}{z^4}$$
$$=(\gamma^2+\gamma+1)^2+(\gamma^2+\gamma+1)(1+\frac{B_2}{z^2})+1+\frac{D}{z^3}+\frac{E}{z^4}, \tag{10}$$

in this case the problem of existence of a solution of the equation (9) is reduced to the existence of solutions of the two following equations:

$$\xi^2+\xi\ (1+\frac{B_2}{z^2})+1+\frac{D}{z^3}+\frac{E}{z^4}=0. \tag{11}$$

and

$$\gamma^2+\gamma+1=\xi. \tag{12}$$

When $z \neq \sqrt{B_2}$ the equation (11) has a solution if and only if

$$\mathrm{Tr}_q\left(\frac{1+\frac{D}{z^3}+\frac{E}{z^4}}{(1+\frac{B_2}{z^2})^2}\right)=\mathrm{Tr}_q(1+Cv+Dv^3)=0.$$

where

$$C^4=D\sqrt{B_2}+E+B_2^2 \ \text{ and } \ v=\frac{1}{z+\sqrt{B_2}}.$$

**The subcase** $(C,D)=(0,0)$**.**
In this subcase the equation (11) looks as follows:

$$\xi^2+\xi\ \left(1+\frac{B_2}{z^2}\right)+\left(1+\frac{B_2}{z^2}\right)^2=0. \tag{13}$$

Here we have

**Proposition 2** *Let $q=2^m$ and $m\geq 3$ be odd. Let $b\in\mathbb{F}_{q^4}\setminus\mathbb{F}_q$, $B_2\neq 0$, and the following two conditions be satisfied:*

$$B_3=B_1B_2, \tag{14}$$

$$B_4=B_1^4+B_1^2B_2+B_2^2. \tag{15}$$

*Then the polynomial $x^{1+\frac{q^4-1}{q-1}}+bx$ is a permutation polynomial over $\mathbb{F}_{q^4}$.*

If $D=0$ and $E=B_2^2$, then for even $m$ the equation (11) has solutions for all $z$, $z\neq 0$, including $z=\sqrt{B_2}$ and, obviously, in this case there exist many solutions of the equation (12). Therefore, we have the following

**Proposition 3** *Let $q=2^m$ and $m\geq 2$ be even. Let $b\in\mathbb{F}_{q^4}\setminus\mathbb{F}_q$, $B_2\neq 0$, and the conditions* (14) *and* (15) *be satisfied. Then the polynomial $x^{1+\frac{q^4-1}{q-1}}+bx$ is not a permutation polynomial over $\mathbb{F}_{q^4}$.*

**The subcase** $(C, D) \neq (0, 0)$**.**

For this subcase we are going to prove that for all $m \geq 6$ solutions for the both equations (11) and (12) always exist. For this purpose it is enough to prove the existence of $v \neq 0$ and $v \neq \frac{1}{\sqrt{B_2}}$, such that

$$\mathrm{Tr}_q(1 + Cv + Dv^3) = 0 \tag{16}$$

and

$$\mathrm{Tr}_q\left(1 + \frac{B_2}{z^2}\right) = 1. \tag{17}$$

Hence we obtain

**Proposition 4** *Let $q = 2^m$, $B_2 \neq 0$ and $(C, D) \neq (0, 0)$. Then the polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ is not a permutation polynomial over $\mathbb{F}_{q^4}$ for all $m \geq 6$.*

### 2.2 The case $B_2 = 0$

**The subcase** $D \neq 0$**.**

Set $u = \frac{1}{z}$. The number $N$ of $\mathbb{F}_q$-rational points of the plane absolutely irreducible non-singular curve $P$ over $\mathbb{F}_q$,

$$P = \{(\gamma, u) :\ \gamma^4 + \gamma + 1 + Du^3 + Eu^4 = 0\},$$

satisfies the following known (see [11]) inequality: $N \geq q - 6\sqrt{q}$. Because the number of points with $u = 0$ does not exceed 4, when $q - 6\sqrt{q} > 4$ there exists a solution $\gamma, z \in \mathbb{F}_q$, $z \neq 0$, of the equation (9). For this case, the following proposition holds.

**Proposition 5** *Let $q = 2^m$, $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ and*

$$B_2 = 0, \quad B_3 \neq 0.$$

*Then the polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ is not a permutation polynomial over $\mathbb{F}_{q^4}$ for $m \geq 6$.*

**The subcase** $D = 0$**.**

Here we are obliged to consider separately odd and even $m$. For odd $m$ the analysis is simple: if $E \neq 0$, the equation (9) has a solution for any $\gamma$ because $z^4$ is an automorphism of the field $F_{2^m}$ and $\gamma^4 + \gamma + 1 \neq 0$ for odd $m$ and $\gamma \in \mathbb{F}_{2^m}$. If $E = 0$, then a solution of (9) does not exist. But the conditions $B_2 = 0, D = 0, E = 0$ represent a special case of the conditions (14) and (15) and so the following proposition holds.

**Proposition 6** *Let $q = 2^m$ and $m \geq 3$ be odd. Let $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$ and*

$$B_2 = 0, \quad B_3 = 0.$$

*Then the polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ is a permutation polynomial over $\mathbb{F}_{q^4}$ if and only if $B_4 = B_1^4$ .*

For even $m$ the situation is more complicated since the equation (9), for the case $E = 0$, has no solutions in $\mathbb{F}_{2^m}$ when $m \equiv 2 \pmod 4$. Hence for $b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q$, such that $B_2 = 0$, $B_3 = 0$, and $B_4 = B_1^4$, the polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ is a permutation polynomial over $\mathbb{F}_{q^4}$ for $m \equiv 2 \pmod 4$. But it turned out to be that such $b$ does not exist for even $m$ (for odd $m$ such $b$ exists).

**Lemma 3.** *Let $b \in \mathbb{F}_{q^4}$ where $q = 2^m$ and let $B_2$ and $B_3$ be the expressions* (5) *obtained from $b$. Then, for even $m$,*

$$B_2 = B_3 = 0$$

*if and only if $b$ is an element of $\mathbb{F}_q$.*

## 3 The main results

Now we give the results for the cases $m = 2, 3, 4, 5$.

The direct calculations show that for $m = 4$ there are no permutation polynomials of the type $x^{1+\frac{q^4-1}{q-1}} + bx$ over $\mathbb{F}_{q^4}$, $q = 2^4$. But for $m = 2$ there are 48 such polynomials (for $b = \alpha^i, i = 3, 11, 37, 61, 63, 91$, and their cyclotomic classes $L_i = \{i, 2i, 2^2 i, \ldots, 2^8 i\}$ modulo 255 where $\alpha$ is a primitive element of $\mathbb{F}_{4^4}$).

From here, Proposions 3, 4, 5, and Lemma 3 the following theorem is valid.

**Theorem 7** *Let $q = 2^m$ and $m \geq 4$ be even. The polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ over $\mathbb{F}_{q^4}$ is not a permutation polynomial for any $b \in \mathbb{F}_{q^4}^*$.*

For $m = 3$ and $m = 5$ the direct calculations show that the polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ is a permutation polynomial over $\mathbb{F}_{q^4}$ if and only if the conditions of Proposition 3 are satisfied. From here and Proposions 2, 4, 5, and 6 the following theorem is valid.

**Theorem 8** *Let $q = 2^m$ and $m \geq 3$ be odd. The polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ over $\mathbb{F}_{q^4}$ is a permutation polynomial if and only if the following conditions are satisfied:*

$$b \in \mathbb{F}_{q^4} \setminus \mathbb{F}_q, \ \ B_3 + B_1 B_2 = 0, \quad \textit{and} \ \ B_4 + B_1^4 + B_1^2 B_2 + B_2^2 = 0.$$

As we mentioned already, in [12] it was proved in a different way that conditions from Theorem 8 are sufficient conditions.

Now, for odd $m$, we show that we find all solutions for $b$ when the polynomial $x^{1+\frac{q^4-1}{q-1}} + bx$ is a permutation polynomial. Present any element $x$ of $\mathbb{F}_{q^4}$ as a polynomial of degree 3 over $\mathbb{F}_q$:

$$x = x_0 + x_1\beta + x_2\beta^2 + x_3\beta^3, \ \ x_i \in \mathbb{F}_q,$$

where $\beta$ is a primitive element of $\mathbb{F}_{2^4}$, i.e. it is a root of the equation $1+\beta+\beta^4=0$. Now express $B_1, B_2, B_3, B_4$ in terms of $x_i$ (since they are elements of $\mathbb{F}_q$):

$$\begin{aligned}
B_1 &= x_3,\\
B_2 &= x_0x_3 + x_1x_2 + x_3^2,\\
B_3 &= x_0^2x_3 + x_1^3 + x_1x_2x_3 + x_1x_3^2 + x_2^3 + x_2^2x_3 + x_3^3,\\
B_4 &= x_0^4 + x_0^3x_3 + x_0^2x_1x_2 + x_0^2x_3^2 + x_0x_1^3 + x_0x_1x_2x_3 + x_0x_1x_3^2 + x_0x_2^3 +\\
&+ x_0x_2^2x_3 + x_0x_3^3 + x_1^4 + x_1^2x_2x_3 + x_1x_2^3 + x_1x_3^3 + x_2^4 + x_2x_3^3 + x_3^4.
\end{aligned}$$

Using these expressions we obtain:
the condition $B_1B_2 = B_3$ is equivalent to the condition:

$$x_0x_3(x_0+x_3) + x_1(x_1+x_3)^2 + x_2^2(x_2+x_3) = 0 \tag{18}$$

and the condition $B_4 = B_1^4 + B_1^2B_2 + B_2^2$ is equivalent to the condition

$$\left.\begin{array}{r} x_0^4 + x_0^3x_3 + x_0^2x_1x_2 + x_0x_1^3 + x_0x_2^3 + x_0x_2^2x_3 + x_0x_1x_3^2 + \\ + x_0x_1x_2x_3 + x_1^4 + x_1^2x_2^2 + x_1^2x_2x_3 + x_1x_2x_3^2 + x_1x_3^3 + x_2^4 + x_2x_3^3 = 0 \end{array}\right\} \tag{19}$$

**Theorem 9** *All solutions of the system of two equations* (18) *and* (19) *for odd $m$ are:*

$$\begin{array}{llll}
(x_0 = x_2, & x_1 = x_2, & x_2, & x_3),\\
(x_0 = x_2 + x_3, & x_1 = x_2, & x_2, & x_3),\\
(x_0 = 0, & x_1 = x_2 + x_3, & x_2, & x_3),\\
(x_0 = x_3, & x_1 = x_2 + x_3, & x_2, & x_3),
\end{array}$$

*where $x_2, x_3$ run over $\mathbb{F}_q$ and $(x_2, x_3) \neq (0,0)$.*

Just these solutions were given in [12], but it was not proved there that the other solutions do not exist. Thus, here we filled this gap. In the same paper [12] the authors counted also the number of their different solution: $2(2q+1)(q-1)$, that naturally coincides with the sum of the number of solutions for $x_3 \neq 0$ equal $4q(q-1)$, and for $x_3 = 0$ equal $2(q-1)$.

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