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The Main Conjecture for Near-MDS Codes

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1 Introduction

Near-MDS have been introduced in 1995 in [11]. They are defined by weakening some restrictions in the definition of the MDS codes. The most popular definition is via generalized Hamming weights. A linear $[n, k]_q$ -code C is called a near-MDS code if

$$d_i(C) = n - k + 1 \text{ for } i = 2, \dots, k, \quad d_1(C) = n - k.$$

Of course, it is enough to require $d_1(C) = n - k$ and $d_2(C) = n - k + 2$. From the properties of the generalized Hamming weights one can easily deduce that the dual of a near-MDS code is again a near-MDS code. The following propositions characterize near-MDS codes and can serve as alternative definitions. The proofs can be found in [11].

Proposition 1. *A linear $[n, k]_q$ -code C is a near-MDS code if and only if any parity-check matrix H_C of C satisfies the conditions:*

- (1) *any $n - k - 1$ columns of H_C are linearly independent;*
- (2) *there exist $n - k$ linearly dependent columns;*
- (3) *any $n - k + 1$ columns of H_C are of rank $n - k$.*

Proposition 2. *A linear $[n, k]_q$ -code C is near-MDS if and only if any generator matrix G_C of C satisfies the conditions:*

- (1) *any $k - 1$ columns of G_C are linearly independent;*
- (2) *there exist k linearly dependent columns;*
- (3) *any $k + 1$ columns of G_C are of rank k .*

Proposition 3. *A linear $[n, k]_q$ code is a near-MDS code if and only if $d(C) + d(C^\perp) = n$.*

Closely related to near-MDS codes are the so-called almost-MDS codes introduced by de Boer [8,9]. Almost-MDS are defined as $[n, k]_q$ -codes with minimum distance $d = n - k$, or, in other words, as codes with Singleton defect 1. Not every almost-MDS code is near-MDS, as pointed out in [11], but for large n both notions coincide.

Proposition 4. *If $n > k + q$ every $[n, k, n - k]_q$ -code is a near-MDS code.*

The weight distribution of a near-MDS code can be determined up to a single parameter. In the theorem below it is taken to be the number of words of minimal weight.

Theorem 1. *Let C be an $[n, k]_q$ near-MDS code. Let (A_i) and (A'_i) be the spectra of C and C^\perp , respectively. Then*

$$A_{n-k+s} = \binom{n}{k-s} \sum_{j=0}^{s-1} (-1)^j \binom{n-k+s}{j} (q^{s-j} - 1) + (-1)^s \binom{k}{s} A_{n-k},$$

where $s = 1, \dots, k$, and

$$A'_{k+s} = \binom{n}{k+s} \sum_{j=0}^{s-1} (-1)^j \binom{k+s}{j} (q^{s-j} - 1) + (-1)^s \binom{n-k}{s} A'_k,$$

where $s = 1, \dots, n - k$.

For almost-MDS codes the situation is more complicated. The numbers of the parameters depends on the Singleton defect of the orthogonal code (cf. [16]). Theorem 1 gives a simple upper bound on the number of words of minimal weight.

Corollary 1. *For an $[n, k]_q$ near-MDS code*

$$A_{n-k} \leq \binom{n}{k-1} \frac{q-1}{k},$$

with equality if and only if $A_{n-k+1} = 0$. By duality,

$$A'_k \leq \binom{n}{k+1} \frac{q-1}{n-k},$$

with equality if and only if $A'_{k+1} = 0$.

2 The Geometric View at Near-MDS Codes

It is known that with every $[n, k, d]_q$ -code of full length one can associate a multiset of points in $\text{PG}(k-1, q)$ (possibly in a non-unique way) so that isomorphic codes are associated with projectively equivalent multisets (cf. [14]). This implies that the existence of an $[n, k]_q$ near-MDS code is equivalent to that of a set \mathcal{K} of points in $\text{PG}(k-1, q)$ with the following properties:

- (1) every $k-1$ points from \mathcal{S} are in general position (generate a hyperplane)
- (2) there exist k points from \mathcal{S} that lie in a hyperplane
- (3) every $k+1$ points from \mathcal{S} generate $\text{PG}(k-1, q)$.

In particular, if $k=3$ a near-MDS code is equivalent to an $(n, 3)$ -arc in $\text{PG}(2, q)$. The nonexistence of maximal $(n, 3)$ -arcs, i.e. arcs with $n=2q+3$ was ruled out originally by Thas [26]. This result is a part of a more general theorem about the nonexistence of maximal arcs in $\text{PG}(2, q)$ for odd q proved by Ball, Blokhuis and Mazzocca [4,5]. Since every $(2n+2, 3)$ -arc is extendable one gets that the size of an $(n, 3)$ -arc is bounded by $n \leq 2q+1$. This provides the best upper bound on the length of a near-MDS code (cf. Theorem 2(vi)).

Almost-MDS codes are equivalent to so-called n -tracks. An n -track is a set of points in $\text{PG}(r, q)$ such that every r of them are in general position. Tables containing exact values and bounds on the maximal size of an n -track are contained in [3,8,9,20].

3 Near-MDS Codes over Small Fields

With no loss of generality, we consider only codes with $k \leq 2q$ and $n \geq 2k$. Near-MDS codes of dimension greater than $\frac{n}{2}$ are obtained as orthogonal to near-MDS codes with $k \leq \frac{n}{2}$.

In the binary case we can list all near-MDS codes. These are the extended Hamming $[8, 4, 4]$ -code, the simplex $[7, 3, 4]$ -code, the $[6, 3, 3]$ -codes obtained by shortening the Hamming code of length 7, as well as, several trivial codes of dimensions one and two.

In the ternary case, we have one $[9, 3, 6]_3$ -code associated with the affine plane $\text{AG}(2, 3)$, one $[10, 4, 6]_3$ -code, one $[11, 5, 6]_3$ -code (the orthogonal to the Golay code) and one $[12, 6, 6]_3$ -code (the extended ternary Golay code).

For codes over \mathbb{F}_4 , there exist three non-isomorphic $[9, 3, 6]_4$ -codes, associated with the three non-equivalent $(9, 3)$ -arcs in $\text{PG}(2, 4)$, two $[10, 4, 6]_4$ -codes, exactly one $[11, 5, 6]_4$ -code and exactly one $[12, 6, 6]_4$ -code [12,13]. It should be noted that

the $[12, 6, 6]_4$ was constructed by Dumer-Zinoviev in [15] as the first member of an infinite family of uniformly packed codes. Remarkably, this code yields a cascade representation of the extended binary Golay code.

There exist two non-isomorphic $[11, 3, 8]_5$ codes associated with the two $(11, 3)$ -arcs in $\text{PG}(2, 5)$. One of them extends to a $[12, 4, 8]_5$ code which cannot be further extended. A $[12, 6, 6]_5$ -code does exist. It was constructed in [10] using a computer. Later on, Abatangelo and Larato [2] constructed six non-isomorphic codes with these parameters. They extended by two points the elliptic curve Γ_6 of degree 6 in $\text{PG}(5, q)$ arising from a non-singular cubic curve of $\text{PG}(2, q)$ via the canonical Veronese embedding

$$\nu : (X : Y : Z) \rightarrow (X^2 : XY : Y^2 : XZ : YZ : Z^2).$$

4 Near-MDS Codes of Maximal Length

Let us denote by $m'(k, q)$ the maximum possible length for which there exists a $[n, k]_q$ near-MDS code. The following theorem summarizes some straightforward observations about $m'(k, q)$.

Theorem 2. *Let k be a positive integer and let q be a prime power. Then*

- (i) $m'(2, q) = 2q + 2$;
- (ii) $m'(k, q) \leq m'(k - \alpha, q) + \alpha$, for every α with $0 \leq \alpha \leq k$;
- (iii) $m'(k, q) = k + 1$ for $k > 2q$;
- (iv) $m'(2q, q) = 2q + 2$;
- (v) $m'(2q - 1, q) = 2q + 1$.
- (vi) $m'(k, q) \leq 2q + k - 2$;

Near-MDS codes with parameters $[n, k]_q$ can be constructed from elliptic curves over \mathbb{F}_q having exactly n rational points [27] (cf. also [1,2,17,18]). Such codes are referred to as elliptic codes. For every prime power $q = p^r$, p a prime, near-MDS codes exist for lengths up to $N_q(1)$, where $N_q(1)$ denotes the maximum number of \mathbb{F}_q -rational points an elliptic curve defined over \mathbb{F}_q can have. By a result of Waterhouse [28], we know that for every $q = p^e$

$$N_q(1) = \begin{cases} q + \lfloor 2\sqrt{q} \rfloor & \text{for } p \nmid \lfloor 2\sqrt{q} \rfloor \text{ and odd } e, \\ q + \lfloor 2\sqrt{q} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Due to extensive computational work by Bartoli, Marcugini, Milani and Pambianco [7,13,22,25], the exact values of $m'(k, q)$ were determined for all fields

of order $q \leq 9$, as well as lower and upper bounds on the size of the longest near-MDS code for some larger fields. Even more results were obtained for the case of dimension three, which corresponds to the problem of the maximal size of an $(n, 3)$ -arc in $\text{PG}(2, q)$ (cf [21,23,24]). These results are summarized in the table below.

q/k	2	3	4	5	7	8	9	11	13	16
2	6	8	10	12	16	18	20	24	28	34
3	7	9	9	11	15	15	17	21	23	28
4	8	10	10	12	14	16	16	20-21	21-24	
5		11	11	11	13	15	16	18-22	21-25	
6		12	12	12	13	14	16	18-23	21-36	
7			9	11	14	15	17	18-24	21-27	
8			10	12	13	16	18	18-25	21-28	
9				11	13	14	19	19-26	21-29	
10				12	14	15	20	20-27	21-30	
11					14	15	16	18-28	21-31	
12					15	16	16	18-29	21-32	
13					15	15	16	18-30	21-33	
14					16	16	17	18-31	21-34	
15						17	17	18-32	21-35	
16						18	18	18-33	21-36	

5 An Upper Bound on the Maximal Length of a Near-MDS Code

According to Theorem 2(vi), we have $m'(k, q) \leq 2q + k - 2$. It can be seen from the table above that equality is achieved for several pairs (k, q) . The following theorem gives an improvement over Theorem 2(vi) for sufficiently large dimensions.

Theorem 3. *There exist no $[2q + k - 2, k]_q$ near-MDS codes for $k \geq q + 4$ and $q \geq 9$.*

Proof. We are going to use the geometric interpretation of near-MDS codes. Fix an integer $q + 4 < k < q + \sqrt{q} + 3$. Let \mathcal{K} be an arc with $2q + k - 2$ points in $\text{PG}(k - 1, q)$, associated with a $[2q + k - 2, k]_q$ near-MDS code. Furthermore, let P_1, \dots, P_{k-2} be points from \mathcal{K} . By the properties of the arcs associated with near-MDS codes, these $k - 2$ points are in general position. Set

$$S = \langle P_1, P_2, \dots, P_{k-2} \rangle,$$

$$S_i = \langle P_1, \dots, P_{i-1}, P_{i+1}, \dots, P_{k-2} \rangle, \quad i = 1, \dots, k - 2.$$

Obviously, $\dim S = k - 3$, and $\dim S_i = k - 4$. Let H be a hyperplane in $\text{PG}(k - 1, q)$ which does not contain any of the points P_1, \dots, P_{k-2} .

At first, we are going to prove that there exists a point $Q \in S \cap H$ which is not contained in any of the subspaces S_i . Assume for a contradiction that such a point Q does not exist and the subspaces S_i cover all points of $S \cap H$. This means that the subspaces $S_i \cap H$ also cover all points of $S \cap H$. By duality, we get that there exist $k - 2$ points in $\widetilde{S \cap H}$, the dual space to $S \cap H$, that block all hyperplanes. By the restriction on k , we have that $k - 2 < q + \sqrt{q} + 1$, and a well-known result of Heim [19] implies that this blocking set should contain a line. In other words, the subspaces S_i meet in a subspace of codimension 2 in S . This is a contradiction, because $\bigcap_{i=1}^{k-2} S_i = \emptyset$. This follows from the fact that the points P_1, \dots, P_{k-2} are in general position.

Now we can construct a plane π in H which does not meet any of the subspaces S_i . Fix a line L in H which is disjoint from $S \cap H$. The existence of such a line follows from the dimension formula. The plane $\pi = \langle L, Q \rangle$ meets S only in the point Q and is therefore disjoint from any subspaces S_i . Note that from the properties of the near-MDS codes and the arcs associated with them $|\mathcal{K} \cap \pi| \leq 3$.

Now consider a projections $\varphi_i, i = 1, \dots, k - 2$, from each S_i to π given by

$$\varphi_i: \begin{cases} \mathcal{P} \setminus S_i & \rightarrow \pi \\ P & \rightarrow \langle S_i, P \rangle \cap \pi \end{cases},$$

where \mathcal{P} is the set of points of $\text{PG}(k - 1, q)$. Obviously, all induced arcs \mathcal{K}^{φ_i} are plane arcs with parameters $(2q + 1, 3)$.

If $P \in \mathcal{K} \cap \pi$, then P is contained in all induced arcs. Obviously $Q = S \cap \pi$ is also contained in all arcs \mathcal{K}^{φ_i} . Note that there are at most three such points. If R is a point from π which is contained neither in \mathcal{K} nor in S , then it is contained in at most two of \mathcal{K}^{φ_i} (since a hyperplane contains at most k points from \mathcal{K}). Counting the pairs $(P, \mathcal{K}^{\varphi_i})$ with $P \in \mathcal{K}^{\varphi_i}$ in two possible ways, we get

$$(k - 2)(2q + 1) \leq 2(q^2 + q - 3) + 4(k - 2),$$

whence $k \leq q + 4$, a contradiction. Now it remains to use Theorem 2(ii) to obtain the nonexistence for all dimensions $k > q + 4$.

Let us note that the proof of the nonexistence of $(2q + 1, 3)$ -arcs would immediately imply the nonexistence of $[2q + k - 2, k]_q$ near-MDS codes. All numerical evidence suggests that this is true for all $q \geq 8$, but no proof of the nonexistence of $(2q + 1, 3)$ -arcs in $\text{PG}(2, q)$ for large q seems to be known for the time being. There is a problem for $(n, 3)$ -arcs in $\text{PG}(2, q)$ suggested by A. Blokhuis asking to determine a constant c such that $n/q < c < 2$ for q large enough, or a construction where $n/q > c > 1$ [6].

We finish with two conjectures for near-MDS codes that are similar to the famous Main Conjecture for MDS codes.

Weak Main Conjecture for NMDS codes. For all positive integers k and all prime powers q it holds that $m'(k, q) \leq 2(q + 1)$.

Strong Main Conjecture for NMDS codes. There exists a universal constant c (not depending on q) such that $m'(k, q) \leq N_1(q) + c$.

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