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Abstract. In this paper we determine the second and the third smallest configuration of hyperplanes in $\mathrm{PG}(N,q)$. We present links with the unique extendability of arcs in $\mathrm{PG}(2,q)$, and with $(k,3)$-arcs having a unique trisecant. These results have links to the study of weights of the d-th order q-ary projective Reed-Muller codes $\mathrm{PRM}(q,d,N)$.

Keywords: configuration of hyperplanes, Reed-Muller codes

1 Introduction

In recent years, the distribution of the weights of the d-th order q-ary projective Reed-Muller codes $\mathrm{PRM}(q,d,N)$ and of the affine Reed-Muller codes $\mathrm{RM}(q,d,N)$ has received the attention of different authors [1, 2, 6–9].

In particular, Serre determined the smallest weight of the projective Reed-Muller code $\mathrm{PRM}(q,d,N)$, for $d \leq q-1$. One of the interesting facts about these results is that they are equivalent to results on the maximal number of points in the projective space $\mathrm{PG}(N,q)$, of dimension N over the finite field $\mathbb{F}_q$ of order q, on the algebraic hypersurfaces of degree d. For instance, the result of Serre geometrically shows that the largest algebraic hypersurfaces of degree $d \leq q-1$ in $\mathrm{PG}(N,q)$ are equal to a union of d hyperplanes passing through a common $(N-2)$-space of $\mathrm{PG}(N,q)$ [11]. The number of points in this configuration $A_1^d(N)$ is

$$dq^{N-1} + \theta_{N-2}, \tag{1}$$

with $\theta_{N-2} = q^{N-2} + q^{N-3} + \cdots + q + 1$. This inspired Sboui to determine the second smallest and the third smallest weight of the code $\mathrm{PRM}(q,d,N)$, for $7 < d < q$. This led to the investigation of the second largest and third largest configuration of hyperplanes in $\mathrm{PG}(N,q)$. In particular he proved that the second weight of the code $\mathrm{PRM}(q,d,N)$ is defined by the algebraic hypersurfaces $\mathcal{A}_2^d(N)$ of degree d which are the union of d hyperplanes, $d-1$ of which meet in a common subspace of dimension $N-2$ and with the d-th hyperplane not passing through this common subspace of dimension $N-2$. The number of points in this configuration is

$$dq^{N-1} + \theta_{N-2} - (d-2)q^{N-2}. \tag{2}$$

Also, the third weight of the code $\mathrm{PRM}(q,d,N)$, with $7<d<q$, is defined by the algebraic hypersurfaces $\mathcal{A}_3^d(N)$ of degree d which are the union of d hyperplanes, $d-2$ of which meet in a common subspace K_1 of dimension $N-2$, and where the last two hyperplanes H_{d-1} and H_d meet in a subspace K_2, different from K_1, such that K_2 is contained in exactly one of the $d-2$ hyperplanes passing through K_1. The number of points in this configuration $\mathcal{A}_3^d(N)$ is

$$dq^{N-1}+\theta_{N-2}-2(d-3)q^{N-2}. \tag{3}$$

All the other configurations different from $\mathcal{A}_1^d(N)$ and $\mathcal{A}_2^d(N)$ are smaller in size than $\mathcal{A}_3^d(N)$ if $d>7$.

Sboui also determined the smallest configuration of d hyperplanes in $\mathrm{PG}(N,q)$, $d<q$, to compare the weights of the codewords of $\mathrm{PRM}(q,d,N)$ arising from algebraic hypersurfaces of degree d consisting of d hyperplanes to the weights of the codewords arising from algebraic hypersurfaces not equal to the union of hyperplanes of $\mathrm{PG}(N,q)$. This configuration is described in the following way: for every $1\leq i,j\leq d$, $i\neq j$, we have $H_i\cap H_j=K_j^i$, where the spaces K_j^i are $\binom{d}{2}$ subspaces of dimension $N-2$, all distinct and meeting in a common subspace Ω of dimension $N-3$. Equivalently, $H_1,\ldots,H_d$ share exactly an $(N-3)$-space Ω and intersect a fixed plane disjoint from Ω in d lines no three of which pass through the same point (that is they form a dual d-arc in the quotient geometry of Ω).

The consequence of this result for the codes $\mathrm{PRM}(q,d,N)$ is the following: the weight w_m^l given by the minimal hyperplane arrangement is the highest weight of a codeword in $\mathrm{PRM}(q,d,N)$ which can be given by any hyperplane arrangement. Moreover, for $q>d(d-1)/2$, any algebraic hypersurface of degree d in $\mathrm{PG}(N,q)$ containing an absolutely irreducible non-linear factor cannot correspond to a codeword with weight less than w_m^l, see [6].

In particular, for d small enough with respect to q, all the configurations of d hyperplanes give weights in $\mathrm{PRM}(q,d,N)$ smaller than the weights corresponding to the hypersurfaces of degree d containing non-linear factors. Therefore, the study of possible configurations of hyperplanes and their number of points is interesting in order to have more information on the codewords of $\mathrm{PRM}(q,d,N)$ having weights close to the minimum distance δ. Unfortunately, due to the complexity of the problem, this investigation is possible only either for the largest configurations of hyperplanes or for the smallest ones.

We continue this study by determining the second and the third smallest configuration of hyperplanes in $\mathrm{PG}(N,q)$. Here, we see that links with the unique extendability of arcs (i.e. set of points no three of which are collinear) in $\mathrm{PG}(2,q)$, and with $(k,3)$-arcs (i.e. set of points no four of which are collinear) having a unique trisecant occur.

We now situate the problems that we will investigate and define the required codes and geometrical structures.

Let $\mathbb{F}_q[X_0,\ldots,X_N]_d^h$ be the set of all homogeneous polynomials of degree d over the finite field $\mathbb{F}_q$ in the variables $X_0,\ldots,X_N$. Consider the projective space

$\mathrm{PG}(N,q)$, and order the points $P_0,\ldots,P_{\theta_N-1}$, where $\theta_N=\frac{q^{N+1}-1}{q-1}$, of $\mathrm{PG}(N,q)$ in a certain way, where we normalize the coordinates of the points P_i by making the leftmost non-zero coordinate equal to one.

The *d-th order q-ary projective Reed-Muller code* $\mathrm{PRM}(q,d,N)$ is the image of the map

$$\Phi : \mathbb{F}_q[X_0,\ldots,X_N]_d^h \cup \{0\} \to \mathbb{F}_q^{\theta_N} : F(X_0,\ldots,X_N) \mapsto (F(P_0),\ldots,F(P_{\theta_N-1})).$$

The parameters of these linear codes are:

- Length of $\mathrm{PRM}(q,d,N)$ is θ_N (see [5]).
- The dimension of $\mathrm{PRM}(q,d,N)$ is $k=\binom{N+d}{N}$, for $d<q$ ([5]), while for $d\le N(q-1)$,

$$k=\sum_{t=d \atop 0<t\le d}^{\mathrm{mod}\ q-1}\left(\sum_{j=0}^{N+1}(-1)^j\binom{N+1}{j}\binom{t-jq+N}{t-jq}\right)$$

  ([12]).
- Minimum distance of $\mathrm{PRM}(q,d,N)$: $\delta=(q-s)q^{N-r-1}$, with $d-1=r(q-1)+s, 0\le s<q-1, d\le N(q-1)$ (see [12, 13]). For $d<q$, this reduces to $\delta=q^N-(d-1)q^{N-1}$.

The non-zero codewords of minimum weight δ of $\mathrm{PRM}(q,d,N)$ correspond to the algebraic hypersurfaces of degree d having the largest number of points in $\mathrm{PG}(N,q)$.

For $d\le q-1$, by the result of Serre [11], the smallest weight codewords correspond to the algebraic hypersurfaces which are the union of d hyperplanes passing through a common $(N-2)$-dimensional subspace of $\mathrm{PG}(N,q)$.

In this article, we investigate the smallest configurations of d hyperplanes, determining in particular the second and the third smallest one; see Section 2. We would like to remark that these results hold only for $d\le\varphi(1)$ and $d\le\varphi(2)$ respectively (see Theorem 1 for the notations) and then the problem of determining the second and the third smallest configurations of hyperplanes in general remains still open. For this reason, we also give a construction which leads to an upper bound on the size of the second smallest configuration of hyperplanes; see Remark 1. Also, we determine the minimum number of points of a configuration of hyperplanes all passing through the same $(N-4)$-space, but not through a common $(N-3)$-space. Finally, in Section 3 we discuss the existence of examples of configurations attaining the bounds of Section 2.

2 Smallest configurations of hyperplanes

In this section we determine the smallest configurations of hyperplanes. Recall that a dual arc in a projective plane is a set of lines intersecting in pairwise

distinct points and a dual $(d,3)$-arc is a set of lines no four of which pass through a fixed point.

Definition 1. *Let $H_1,\dots,H_d$ be d hyperplanes in $\mathrm{PG}(N,q)$ such that $H_1\cap\cdots\cap H_d=S$, with $\dim(S)=N-3$. Let π be a plane not intersecting S and let $\mathcal{L}=\{H_i\cap\pi \mid i\in\{1,\dots,d\}\}$. Consider the following configurations.*

1. *$\mathcal{B}_1^d(N)$: $\mathcal{L}$ is a dual arc in π. Its size is (see also [9, Theorem 3.4])*
$$dq^{N-1}-\frac{d(d-3)}{2}q^{N-2}+\theta_{N-3}. \tag{4}$$
2. *$\mathcal{B}_2^d(N)$: $\mathcal{L}$ is the dual of a $(d,3)$-arc having only one trisecant. Its size is*
$$dq^{N-1}-\frac{d^2-3d-2}{2}q^{N-2}+\theta_{N-3}. \tag{5}$$
3. *$\mathcal{B}_3^d(N)$: $\mathcal{L}$ is the dual of a $(d,3)$-arc having two trisecants. Its size is*
$$dq^{N-1}-\frac{d^2-3d-4}{2}q^{N-2}+\theta_{N-3}. \tag{6}$$

In [9, Theorem 3.4], Sboui proves that the configuration $\mathcal{B}_1^d(N)$ ($\mathcal{A}_4^d$ in [9]) is the smallest possible in terms of number of points.

Two hyperplanes in $\mathrm{PG}(N,q)$ intersect in a common $(N-2)$-space, therefore their union has $2q^{N-1}+\theta_{N-2}$ points. There are only two possible configurations of three hyperplanes: either they share the same $(N-2)$-space and then their union has $3q^{N-1}+\theta_{N-2}$ points or they share the same $(N-3)$-space and then their union has $3q^{N-1}+\theta_{N-3}$ points. The possibilities for four hyperplanes are as follows:

1. $\mathcal{C}_1(N)$: all the hyperplanes pass through a common $(N-2)$-space: the size of this union is $4q^{N-1}+\theta_{N-2}$;
2. $\mathcal{C}_2(N)$: all the hyperplanes pass through a common $(N-3)$-space and three of them pass through a common $(N-2)$-space: the size of this union is $4q^{N-1}-q^{N-2}+\theta_{N-3}$;
3. $\mathcal{C}_3(N)$: all the hyperplanes pass through a common $(N-4)$-space and three of them pass through a common $(N-3)$-space: the size of this union is $4q^{N-1}-2q^{N-2}+2q^{N-3}+\theta_{N-4}$;
4. $\mathcal{C}_4(N)$: all the hyperplanes pass through a common $(N-3)$-space and no three of them pass through a common $(N-2)$-space: the size of this union is $4q^{N-1}-2q^{N-2}+\theta_{N-3}$.

The following theorem presents the result about the second and the third smallest configurations of d hyperplanes in $\mathrm{PG}(N,q)$.

Theorem 1. *1. Let $d>3+\sqrt{2q}$ and*
$$d\le\varphi(1)=\begin{cases}\frac{2}{3}(q+2)+2, & q \text{ odd}\\ \frac{1}{2}q+3, & q \text{ even}\end{cases}.$$
Then, $\mathcal{B}_2^d(N)$ is the second smallest configuration of d hyperplanes in $\mathrm{PG}(N,q)$.

2. Let $d > 3 + 2\sqrt{q}$ *and*

$$d \leq \varphi(2) = \begin{cases} \frac{2}{3}(q+2)+3, & q \text{ odd} \\ \frac{1}{2}q+4, & q \text{ even} \end{cases}.$$

Then, $\mathcal{B}_3^d(N)$ *is the third smallest configuration of* d *hyperplanes in* $\mathrm{PG}(N,q)$.

Let $H_1, \ldots, H_d$, $d > 3$, $d \leq q+2$ if q is odd and $d \leq q+3$ if q is even, be d hyperplanes in $\mathrm{PG}(N,q)$, passing through a common $(N-4)$-space S, but not through a common $(N-3)$-space. It is possible to prove that their size is at least

$$\theta_{N-4} + \left[dq^2 - \frac{d^2-3d}{2}q + \frac{d^2-5d+8}{2} \right] q^{N-3}. \tag{7}$$

Furthermore, if Z is a solid disjoint from S, then equality holds if and only if $d-1$ hyperplanes $H_1, \ldots, H_{d-1}$ pass through a common $(N-3)$-space $R \not\subset H_d$ and the lines $H_1 \cap H_d \cap Z, \ldots, H_{d-1} \cap H_d \cap Z$ form a dual arc of $H_d \cap Z$.

The difference between the size of the configurations $\mathcal{B}_2^d(N)$ and $\mathcal{B}_3^d(N)$, and the size of a configuration of hyperplanes passing through a common $(N-4)$-space but not through a common $(N-3)$-space is greater than

$$\left(\frac{\left(3+2\sqrt{q}\right)^2 - 5\left(3+2\sqrt{q}\right)+6}{2} - q \right) q^{N-3} = q^{N-\frac{5}{2}}$$

and

$$\left(\frac{\left(3+\sqrt{2q}\right)^2 - 5\left(3+\sqrt{2q}\right)+6}{2} - q \right) q^{N-3} = \frac{\sqrt{2}}{2} q^{N-\frac{5}{2}}$$

respectively.

In Section 3 we will show that the bound in Theorem 1 is sharp for q even, whereas for q odd the existence of configurations of hyperplanes attaining the bound is still an open problem.

3 Construction of 3-arcs having a unique trisecant

In this section, we discuss the existence of point sets in the projective plane $\mathrm{PG}(2,q)$ having a fixed number of i-secants. In particular, we will construct some examples of 3-arcs with a unique trisecant which give rise to the second smallest configuration of hyperplanes in $\mathrm{PG}(N,q)$.

Starting from an arc $\mathcal{S}$ of size n which admits a point $P \in \mathrm{PG}(2,q) \setminus \mathcal{S}$ lying on b bisecants, it is possible to construct a 3-arc $\mathcal{S}'$ of size $n-b+2$, having a unique trisecant, simply by adding P and by removing one point of $\mathcal{S}$ from $b-1$ bisecants of $\mathcal{S}$ through P.

For instance, if q is odd, starting from a conic $\mathcal{C}$ and considering for P an external point to $\mathcal{C}$, it is possible to construct a 3-arc of size $\frac{q+7}{2}$ having a unique trisecant.

Moreover, if q is even, starting from a hyperoval $\mathcal{H}$ (i.e. an arc in $PG(2,q)$ of size $q+2$) and considering for P any point not belonging to $\mathcal{H}$, the number of bisecants to $\mathcal{H}$ passing through P is constant and it is equal to $\frac{1}{2}q+1$. Then, it is possible to construct a 3-arc of size $\frac{1}{2}q+3$ having a unique trisecant.

As already mentioned after Theorem 1, it is worth noting that while in the even case this construction reaches the bound of Theorem 1, in the odd case that bound is far from reached.

Also, it is possible to construct 3-arcs with a unique trisecant of size $\frac{q+1}{2}+2$ starting from a particular type of plane cubic curve.

Remark 1. If $d \leq q+1$ (q odd) or $d \leq q+2$ (q even), then it is always possible to construct the smallest configuration of d hyperplanes. Otherwise, from the results of this section and from Corollary 1, if $d \in \left[3+\sqrt{2q}, \frac{q+1}{2}+3\right]$ (q odd) or $d \in \left[3+\sqrt{2q}, \frac{q}{2}+3\right]$ (q even), then it is always possible to construct the second smallest configuration of d hyperplanes. Unfortunately, in the interval $\left[\frac{q+1}{2}+4, q+1\right]$, q odd, no examples of 3-arcs with a unique trisecant are known.

So it is presently unclear if this second smallest configuration of d hyperplanes, with $d \in \left[\frac{q+1}{2}+4, q+1\right]$, q odd, effectively occurs. For this reason, we also give a construction which leads to an upper bound on the size of the second smallest configuration of d hyperplanes.

Note that from a $(d-1,2)$-arc, it is possible to obtain a $(d,3)$-arc having at most $\frac{d-1}{2}$ trisecants. Therefore for $d \leq q+2$ (q odd) or $d \leq q+3$ (q even), and for $\alpha(\mathcal{S}) \leq \frac{d-1}{2}$, it is possible to construct a configuration of d hyperplanes having at most

$$dq^{N-1} - \frac{d^2-3d}{2}q^{N-2} + \frac{d-1}{2}q^{N-2} + \theta_{N-3} = dq^{N-1} - \frac{d^2-4d+1}{2}q^{N-2} + \theta_{N-3}$$

points.

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