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Subidvisions de cycles orientés dans les graphes dirigés de fort nombre chromatique

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Résumé : Un *cycle orienté* est l'orientation d'un cycle. Nous prouvons que pour tout cycle orienté C il existe des graphes dirigés sans subdivisions de C (en tant que sous graphe) et de nombre chromatique arbitrairement grand. Par ailleurs, nous prouvons que pour tout cycle a deux bloques, tout graphe dirigé fortement connexe de nombre chromatique suffisamment grand contient une subdivision de C. Nous prouvons aussi un resultat semblable sur le cycle antidirigé de taille quatre (avec deux sommets de degré sortant 2 et deux sommets de degré entrant 2).

Mots-clés : Subdivisions, graphes dirigés, nombre chromatique

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Abstract: An *oriented cycle* is an orientation of a undirected cycle. We first show that for any oriented cycle C, there are digraphs containing no subdivision of C (as a subdigraph) and arbitrarily large chromatic number. In contrast, we show that for any C is a cycle with two blocks, every strongly connected digraph with sufficiently large chromatic number contains a subdivision of C. We prove a similar result for the antidirected cycle on four vertices (in which two vertices have out-degree 2 and two vertices have in-degree 2).

Key-words: Subdivisions, digraphs, chromatic number

1 Introduction

What can we say about the subgraphs of a graph G with large chromatic number ? Of course, one way for a graph to have large chromatic number is to contain a large complete subgraph. However, if we consider graphs with large chromatic number and small clique number, then we can ask what other subgraphs must occur. We can avoid any graph H that contains a cycle because, as proved by Erdős [8], there are graphs with arbitrarily high girth and chromatic number. Reciprocally, one can easily show that every n-chromatic graph contains every tree of order n as a subgraph.

The following more general question attracted lots of attention.

Problem 1. Which are the graph classes $\mathcal{G}$ such that every graph with sufficiently large chromatic number contains an element of $\mathcal{G}$?

If such a class is finite, then it must contain a tree, by the above-mentioned result of Erdős. If it is infinite however, it does not necessary contains a tree. For example, every graph with chromatic number at least 3 contains an odd cycle. This was strengthened by Erdős and Hajnal [9] who proved that every graph with chromatic number at least k contains an odd cycle of length at least k. A counterpart of this theorem for even length was settled by Mihók and Schiermeyer [17] : every graph with chromatic number at least k contains an even cycle of length at least k. Further results on graphs with prescribed lengths of cycles have been obtained [12, 17, 21, 16, 15].

In this paper, we consider the analogous problem for directed graphs, which is in fact a generalization of the undirected one. The *chromatic number* $\chi(D)$ of a digraph D is the chromatic number of its underlying graph. The *chromatic number* of a class of digraphs $\mathcal{D}$, denoted by $\chi(\mathcal{D})$, is the smallest k such that $\chi(D) \leqslant k$ for all $D \in \mathcal{D}$, or $+\infty$ if no such k exists. By convention, if $\mathcal{D} = \emptyset$, then $\chi(\mathcal{D}) = 0$. If $\chi(\mathcal{D}) \neq +\infty$, we say that $\mathcal{D}$ *has bounded chromatic number*.

We are interested in the following question : which are the digraph classes $\mathcal{D}$ such that every digraph with sufficiently large chromatic number contains an element of $\mathcal{D}$? Let us denote by $\mathrm{Forb}(H)$ (resp. $\mathrm{Forb}(\mathcal{H})$) the class of digraphs that do not contain H (resp. any element of $\mathcal{H}$) as a subdigraph. The above question can be restated as follows :

Problem 2. Which are the classes of digraphs $\mathcal{D}$ such that $\chi(\mathrm{Forb}(\mathcal{D})) < +\infty$?

This is a generalization of Problem 1. Indeed, let us denote by $\mathrm{Dig}(\mathcal{G})$ the set of digraphs whose underlying digraph is in $\mathcal{G}$; Clearly, $\chi(\mathcal{G}) = \chi(\mathrm{Dig}(\mathcal{G}))$.

An *oriented graph* is an orientation of a (simple) graph ; equivalently it is a digraph with no directed cycles of length 2. Similarly, an *oriented path* (resp. *oriented cycle*, *oriented tree*) is an orientation of a path (resp. cycle, tree). An oriented path (resp., an oriented cycle) is said *directed* if all nodes have in-degree and out-degree at most 1.

Observe that if D is an orientation of a graph G and $\mathrm{Forb}(D)$ has bounded chromatic number, then $\mathrm{Forb}(G)$ has also bounded chromatic number, so G must be a tree. Burr proved that every $(k-1)^2$-chromatic digraph contains every oriented tree of order k. This was slightly improved by Addario-Berry et al. [2] who proved the following.

Theorem 3 (Addario-Berry et al. [2])**.** *Every $(k^2/2-k/2+1)$-chromatic oriented graph contains every oriented tree of order k. In other words, for every oriented tree T of order k, $\chi(\mathrm{Forb}(T)) \leqslant k^2/2 - k/2$.*

Conjecture 4 (Burr [6])**.** Every $(2k-2)$-chromatic digraph D contains a copy of any oriented tree T of order k.

For special oriented trees T, better bounds on the chromatic number of Forb(T) are known. The most famous one, known as Gallai-Roy Theorem, deals with directed paths (a *directed path* is an oriented path in which all arcs are in the same direction) and can be restated as follows, denoting by $P^+(k)$ the directed path of length k.

Theorem 5 (Gallai [11], Hasse [13], Roy [18], Vitaver [20]). $\chi(\text{Forb}(P^+(k))) = k$.

The chromatic number of the class of digraphs not containing a prescribed oriented path with two blocks (*blocks* are maximal directed subpaths) has been determined by Addario-Berry et al. [1].

Theorem 6 (Addario-Berry et al. [1]). *Let P be an oriented path with two blocks on n vertices.*

— *If $n = 3$, then $\chi(\text{Forb}(P)) = 3$.*

— *If $n \geqslant 4$, then $\chi(\text{Forb}(P)) = n - 1$.*

In this paper, we are interested in the chromatic number of Forb($\mathcal{H}$) when $\mathcal{H}$ is an infinite family of oriented cycles. Let us denote by S-Forb(D) (resp. S-Forb($\mathcal{D}$)) the class of digraphs that contain no subdivision of D (resp. any element of $\mathcal{D}$) as a subdigraph. We are particularly interested in the chromatic number of S-Forb($\mathcal{C}$), where $\mathcal{C}$ is a family of oriented cycles.

Let us denote by $\vec{C}_k$ the directed cycle of length k. For all k, $\chi(\text{S-Forb}(\vec{C}_k)) = +\infty$ because transitive tournaments have no directed cycle. Let us denote by $C(k, \ell)$ the oriented cycle with two blocks, one of length k and the other of length ℓ. Observe that the oriented cycles with two blocks are the subdivisions of $C(1, 1)$. As pointed Gyárfás and Thomassen (see [1]), there are acyclic oriented graphs with arbitrarily large chromatic number and no oriented cycles with two blocks. Therefore $\chi(\text{S-Forb}(C(k, \ell))) = +\infty$. We first generalize these two results to every oriented cycle.

Theorem 7. *For any oriented cycle C,*

$$\chi(\text{S-Forb}(C)) = +\infty.$$

In fact, we show a stronger theorem (Theorem 20) : for any positive integer b, there are digraphs of arbitrarily high chromatic number that contains no oriented cycles with less than b blocks. It directly implies the following generalization of the previous theorem.

Theorem 8. *For any finite family $\mathcal{C}$ of oriented cycles,*

$$\chi(\text{S-Forb}(\mathcal{C})) = +\infty.$$

In contrast, if $\mathcal{C}$ is an infinite family of oriented cycles, S-Forb($\mathcal{C}$) may have bounded chromatic number. By the above argument, such a family must contain a cycle with at least b blocks for every positive integer b. A cycle C is *antidirected* if any vertex of C has either in-degree 2 or out-degree 2 in C. In other words, it is an oriented cycle in which all blocks have length 1. Let us denote by $\mathcal{A}_{\geqslant 2k}$ the family of antidirected cycles of length at least $2k$. In Theorem 13, we prove that $\chi(\text{Forb}(\mathcal{A}_{\geqslant 2k})) \leqslant 8k - 8$. Hence we are left with the following problem.

Problem 9. What are the infinite families of oriented cycles $\mathcal{C}$ such that Forb($\mathcal{C}$) $< +\infty$? What are the infinite families of oriented cycles $\mathcal{C}$ such that S-Forb($\mathcal{C}$) $< +\infty$?

On the other hand, considering strongly connected (strong for short) digraphs may lead to dramatically different result. An example is provided by the following celebrated result due to Bondy [4] : *every strong digraph of chromatic number at least k contains a directed cycle of length at least k*. Denoting the class of strong digraphs by $\mathcal{S}$, this result can be rephrased as follows.

Theorem 10 (Bondy [4])**.** $\chi(\text{S-Forb}(\vec{C}_k) \cap \mathcal{S}) = k - 1$.

Inspired by this theorem, Addario-Berry et al. [1] posed the following problem.

Problem 11. Let k and ℓ be two positive integers. Does S-Forb$(C(k, \ell) \cap \mathcal{S})$ have bounded chromatic number ?

In Subsection 5.2, we answer to this problem in the affirmative. In Theorem 23 we prove

$$\chi(\text{S-Forb}(C(k, \ell) \cap \mathcal{S}) \leqslant (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1), \text{ for all } k \geqslant \ell \geqslant 2, k \geqslant 3.$$

Note that since $\chi(\text{S-Forb}(C(k', \ell') \cap \mathcal{S}) \leqslant \chi(\text{S-Forb}(C(k, \ell) \cap \mathcal{S})$ if $k' \leqslant k$ and $\ell' \leqslant \ell$, this gives also an upper bound when k or ℓ are small. However, in those cases, we prove better upper bounds. In Corollary 32, we prove

$$\chi(\text{S-Forb}(C(k, 1)) \cap \mathcal{S}) \leqslant \max\{k + 1, 2k - 4\} \text{ for all } k.$$

We also give in Subsection 5.2 the exact value of S-Forb$(C(k, \ell) \cap \mathcal{S})$ for $(k, \ell) \in \{(1, 2), (1, 3), (2, 3)\}$.

More generally, one may wonder what happens for other oriented cycles.

Problem 12. Let C be an oriented cycle with at least four blocks. Is $\chi(\text{S-Forb}(C) \cap \mathcal{S})$ bounded ?

In Section 7, we show that $\chi(\text{S-Forb}(\hat{C}_4) \cap \mathcal{S}) \leqslant 24$ where $\hat{C}_4$ is the antidirected cycle of order 4.

2 Definitions

We follow [5] for basic notions and notations. Let D be a digraph. $V(D)$ denotes its vertex-set and $A(D)$ its arc-set.

If $uv \in A(D)$ is an arc, we sometimes write $u \to v$ or $v \leftarrow u$.

For any $v \in V(D)$, $d^+(v)$ (resp. $d^-(v)$) denotes the out-degree (resp. in-degree) of v. $\delta^+(D)$ (resp. $\delta^-(D)$) denotes the minimum out-degree (resp. in-degree) of D.

An *oriented path* is any orientation of a *path*. The *length* of a path is the number of its arcs. Let $P = (v_1, \dots, v_n)$ be an oriented path. If $v_i v_{i+1} \in A(D)$, then $v_i v_{i+1}$ is a *forward arc* ; otherwise, $v_{i+1} v_i$ is a *backward arc*. P is a *directed path* if all of its arcs are either forward or backward ones. For convenience, a directed path with forward arcs only is called a *dipath*. A *block* of P is a maximal directed subpath of P. A path is entirely determined by the sequence $(b_1, \dots, b_p)$ of the lengths of its blocks and the sign $+$ or $-$ indicating if the first arc is forward or backward respectively. Therefore we denote by $P^+(b_1, \dots, b_p)$ (resp. $P^-(b_1, \dots, b_p)$) an oriented path whose first arc is forward (resp. backward) with p blocks, such that the ith block along it has length b_i.

Let $P = (x_1, x_2, \dots, x_n)$ be an oriented path. We say that P is an (x_1, x_n)-*path*. For every $1 \leqslant i \leqslant j \leqslant n$, we note $P[x_i, x_j]$ (resp. $P]x_i, x_j[$, $P[x_i, x_j[$, $P]x_i, x_j]$) the oriented subpath $(x_i, \dots, x_j)$ (resp. $(x_{i+1}, \dots, x_{j-1})$, $(x_i, \dots, x_{j-1})$, $(x_{i+1}, \dots, x_j)$).

The vertex x_1 is the *initial vertex* of P and x_n its *terminal vertex*. Let P_1 be an (x_1, x_2)-dipath and P_2 an (x_2, x_3)-dipath which are disjoint except in x_2. Then $P_1 \odot P_2$ denotes the (x_1, x_3)-dipath obtained from the concatenation of these dipaths.

The above definitions and notations can also be used for oriented cycles. Since a cycle has no initial and terminal vertex, we have to choose one as well as a direction to run through the cycle.Therefore if $C = (x_1, x_2, \dots, x_n, x_1)$ is an oriented cycle, we always assume that $x_1 x_2$ is an arc, and if C is not directed that $x_1 x_n$ is also an arc.

A path or a cycle (not necessarily directed) is *Hamiltonian* in a digraph if it goes through all vertices of D.

The digraph D is *connected* (resp. *k-connected*) if its underlying graph is connected (resp. k-connected). It is *strongly connected*, or *strong*, if for any two vertices u, v, there is a (u, v)-dipath in D. It is *k-strongly connected* or *k-strong*, if for any set S of $k-1$ vertices $D - S$ is strong. A *strong component* of a digraph is an inclusionwise maximal strong subdigraph. Similarly, a *k-connected component* of a digraph is an inclusionwise maximal k-connected subdigraph.

3 Antidirected cycles

The aim of this section is to prove the following theorem, that establish that $\chi(\mathrm{Forb}(\mathcal{A}_{\geqslant 2k})) \leqslant 8k-8$.

Theorem 13. *Let D be an oriented graph and k an integer greater than 1. If $\chi(D) \geqslant 8k-7$, then D contains an antidirected cycle of length at least $2k$.*

A graph G is *k-critical* if $\chi(G) = k$ and $\chi(H) < k$ for any proper subgraph H of G. Every graph with chromatic number k contains a k-critical graph. We denote by $\delta(G)$ the minimum degree of the graph G. The following easy result is well-known.

Proposition 14. *If G is a k-critical graph, then $\delta(G) \geqslant k-1$.*

Let (A, B) be a bipartition of the vertex set of a digraph D. We denote by $E(A, B)$ the set of arcs with tail in A and head in B and by $e(A, B)$ its cardinality.

Lemma 15 (Burr [7])**.** Every digraph D contains a partition (A, B) such that $e(A, B) \geqslant |E(D)|/4$.

Lemma 16 (Burr [7])**.** *Let G be a bipartite graph and p be an integer. If $|E(G)| \geqslant p|V(G)|$, then G has a subgraph with minimum degree at least $p+1$.*

Lemma 17. *Let $k \geqslant 1$ be an integer. Every bipartite graph with minimum degree k contains a cycle of order at least $2k$.*

Démonstration. Let G be a bipartite graph with bipartition (A, B). Consider a longest path P in G. Without loss of generality, we may assume that one of its ends a is in A. All neighbours of a are in P (otherwise P can be lengthened). Let b be the furthest neighbour of a in B along P. Then $C = P[a, b] \cup ab$ is a cycle containing at least k vertices in B, namely the neighbours of a. Hence C has length at least $2k$, since G is bipartite. □

Proof of Theorem 13. It suffices to prove that every $(8k-7)$-critical oriented graph contains an antidirected cycle of length at least $2k$.

Let D be a $(8k-7)$-critical oriented graph. By Proposition 14, it has minimum degree at least $8k-8$, so $|E(D)| \geqslant (4k-4)|V(D)|$. By Lemma 15, D contains a partition such that $e(A, B) \geqslant |E(D)|/4 \geqslant (k-1)|V(D)|$. Consequently, by Lemma 16, there are two sets $A' \subseteq A$ and $B' \subseteq B$ such that every vertex in A' (resp. B') has at least k out-neighbours in B' (resp. k in-neighbours in A'). Therefore, by Lemma 17, the bipartite oriented graph induced by $E(A', B')$ contains a cycle of length at least $2k$, which is necessarily antidirected. □

Problem 18. Let ℓ be an even integer. What the minimum integer $a(\ell)$ such that every oriented graph with chromatic number at least $a(\ell)$ contains an antidirected cycle of length at least ℓ ?

4 Acyclic digraphs without cycles with few blocks

The aim of this section is to establish Theorems 7 and 8. To do so we will use a result on hypergraph colouring.

A *cycle* of *length* $k \geqslant 2$ in a hypergraph $\mathcal{H}$ is an alternating cyclic sequence $e_0, v_0, e_1, v_1, \dots e_{k-1}, v_{k-1}, e_0$ of distinct hyperedges and vertices in $\mathcal{H}$ such that $v_i \in e_i \cap e_{i+1}$ for all i modulo k. The *girth* of a hypergraph is the length of a shortest cycle.

A hypergraph $\mathcal{H}$ on a ground set X is said to be *weakly c-colourable* if there exists a colouring of the elements of X with c colours such that no hyperedge of $\mathcal{H}$ is monochromatic. The *weak chromatic number* of $\mathcal{H}$ is the least c such that $\mathcal{H}$ is weakly c-colourable. Erdős and Lovász [10] (and more recently Alon *et al.*[3]) proved the following result :

Theorem 19. *[10, Theorem 1'], [3] For $k, g, c \in \mathbb{N}$, there exists a k-uniform hypergraph with girth larger than g and weak chromatic number larger than c.*

Our construction relies on the hypergraphs whose existence is established by Theorem 19.

Theorem 20. *For any positive integers b, c, there exists an acyclic digraph D with $\chi(D) \geqslant c$ in which all oriented cycles have more than b blocks.*

Démonstration. We shall prove the result by induction on c, the result holding trivially for $c = 2$ with D the directed path on two vertices. We thus assume our claim to hold for a graph D_c with $\chi(D_c) = c$, and show how extend it to $c + 1$.

Let p be the number of proper c-colourings of D_c, and let those colourings be denoted by $col_c^1, \dots, col_c^p$. By Theorem 19 there exists a $c \times p$-uniform hypergraph $\mathcal{H}$ with weak chromatic number $> p$ and girth $> b/2$. Let $X = \{x_1, \dots, x_n\}$ be the ground set of $\mathcal{H}$.

We construct D_{c+1} from n disjoint copies $D_c^1, \dots, D_c^n$ of D_c as follows. For each hyperedge $S \in \mathcal{H}$, we do the following (see Figure 1) :

— We partition S into p sets $S_1, \dots, S_p$ of cardinality c.

— For each set $S_i = \{x_{k_1}, \dots, x_{k_c}\}$, we choose vertices $v_{k_1} \in D_c^{k_1}, \dots, v_{k_c} \in D_c^{k_c}$ such that $col_c^i(v_{k_1}) = 1, \dots, col_c^i(v_{k_c}) = c$, and add a new vertex $w_{S,i}$ with $v_{k_1}, \dots, v_{k_c}$ as in-neighbours.

Let us denote by W the set of vertices of D_{c+1} that do not belong to any of the copies of D_c (i.e. the $w_{S,i}$). We now prove that the resulting digraph D_{c+1} is our desired digraph.

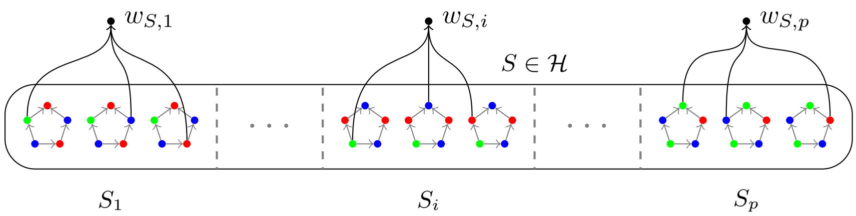

FIGURE 1 – Construction of D_{c+1}

Firstly it is acyclic, as we only add sinks (the $w_{S,i}$) to disjoint copies of D_c, which are acyclic by the induction hypothesis.

Secondly, every oriented cycle C in D_{c+1} has more than b blocks. If C is in a copy of D_c, then we have the result by the induction hypothesis. Henceforth we may assume that S contains some vertices in W, say $w_1, \dots, w_{b'}$ in cyclic order around C. As the vertices of W are all sinks, the number of blocks of C is at least $2b'$. Let us denote by S_{w_i} the hyperedge of $\mathcal{H}$ which triggered the creation of w_i. Then two consecutive $S_{w_i}, S_{w_{i+1}}$ (indices are modulo b') have a vertex x_i of X in common (indeed, the vertices between w_i and w_{i+1} in C belong to some copy D_c^i). Therefore the sequence $x_{b'}, S_{w_1}, x_1, S_{w_2}, x_2, \dots, S_{w_{b'}}, x_{b'}$ contains a cycle in $\mathcal{H}$. Hence by our choice of $\mathcal{H}$, $b' > b/2$, so C has more than b blocks.

Finally, let us prove that $\chi(D_{c+1}) = c + 1$. We added a stable set to the disjoint union of copies of D_c, so $\chi(D_{c+1}) \leqslant \chi(D_c) + 1 = c + 1$.
Now suppose for a contradiction that D_{c+1} admits a proper c-colouring ϕ. It induces on $\mathcal{H}$ the p-colouring ψ where $\psi(x_k)$ is the index of the colouring of D_c on D_c^k, i.e. the restriction of ϕ on D_c^k is the colouring $col_c^{\psi(x_k)}$. Now since $\mathcal{H}$ is $(p+1)$-chromatic, there exists an hyperedge S of $\mathcal{H}$ which is monochromatic. Let i be the integer such that $\psi(x) = i$ for all $x \in S$. Consider $S_i = \{x_{k_1}, \dots, x_{k_c}\}$ and let $v_{k_1} \in D_c^{k_1}, \dots, v_{k_c} \in D_c^{k_c}$ be the in-neighbours of $w_{S,i}$. By construction, $col_c^i(v_{k_1}) = 1, \dots, col_c^i(v_{k_c}) = c$, so $\phi(v_{k_1}) = 1, \dots, \phi(v_{k_c}) = c$. Consequently $w_{S,i}$ has the same colour (by ϕ) as one of its in-neighbours. This contradicts the fact that ϕ is proper. Hence $\chi(D_{c+1}) \geqslant c + 1$. □

Theorems 7 and 8 directly follow from Theorem 20, since a cycle and its subdivision have the same number of blocks.

5 Cycles with two blocks in strong digraphs

In this section we first prove that S-Forb$(C(k,\ell)) \cap \mathcal{S}$ has bounded chromatic number for every k, ℓ. We need some preliminaries.

5.1 Definitions and tools

5.1.1 Levelling

In a digraph D, the *distance* from a vertex x to another y, denoted by $\text{dist}_D(x,y)$ or simply $\text{dist}(x,y)$ when D is clear from the context, is the minimum length of an (x,y)-dipath or $+\infty$ if no such dipath exists. For a set $X \subseteq V(D)$ and vertex $y \in V(D)$, we define $\text{dist}(X,y) = \min\{\text{dist}(x,y) \mid x \in X\}$ and $\text{dist}(y,X) = \min\{\text{dist}(y,x) \mid x \in X\}$, and for two sets $X, Y \subseteq V(D)$, $\text{dist}(X,Y) = \min\{\text{dist}(x,y) \mid x \in X, y \in Y\}$.

An *out-generator* in a digraph D is a vertex u such that for any $x \in V(D)$, there is an (u,x)-dipath. Observe that in a strong digraph every vertex is an out-generator.

Let u be an out-generator of D. For every nonnegative integer i, the *ith level from u* in D is $L_i^u = \{v \mid \text{dist}_D(u,v) = i\}$. Because u is an out-generator, $\bigcup_i L_i^u = V(D)$. Let v be a vertex of D, we set $\text{lvl}^u(v) = \text{dist}_D(u,v)$, hence $v \in L^u_{\text{lvl}(v)}$. In the following, the vertex u is always clear from the context. Therefore, for sake of clarity, we drop the superscript u.

The definition immediately implies the following.

Proposition 21. *Let D be a digraph having an out-generator u. If x and y are two vertices of D with* $\text{lvl}(y) > \text{lvl}(x)$*, then every (x,y)-dipath has length at least* $\text{lvl}(y) - \text{lvl}(x)$.

Let D be a digraph and u be an out-generator of D. A *Breadth-First-Search Tree* or *BFS-tree* T with *root* u, is a sub-digraph of D spanning $V(D)$ such that T is an oriented tree and, for any $v \in V(D)$, $dist_T(u,v) = \text{dist}_D(u,v)$. It is well-known that if u is an out-generator of D, then there exist BFS-trees with root u.

Let T be a BFS-tree with root u. For any vertex x of D, there is an unique (u,x)-dipath in T. The *ancestors* of x are the vertices on this dipath. For an ancestor y of x, we note $y \geqslant_T x$. If y is an ancestor of x, we denote by $T[y,x]$ the unique (y,x)-dipath in T. For any two vertices v_1

and v_2, the *least common ancestor* of v_1 and v_2 is the common ancestor x of v_1 and v_2 for which $\mathrm{lvl}(x)$ is maximal. (This is well-defined since u is an ancestor of all vertices.)

5.1.2 Decomposing a digraph

The *union* of two digraphs D_1 and D_2 is the digraph $D_1 \cup D_2$ with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$. Note that $V(D_1)$ and $V(D_2)$ are not necessarily disjoint.

The following lemma is well-known.

Lemma 22. *Let D_1 and D_2 be two digraphs. $\chi(D_1 \cup D_2) \leqslant \chi(D_1) \times \chi(D_2)$.*

Démonstration. Let $D = D_1 \cup D_2$. For $i \in \{1, 2\}$, let c_i be a proper colouring of D_i with $\{1, \dots, \chi(D_i)\}$. Extend c_i to $(V(D), A(D_i))$ by assigning the colour 1 to all vertices in V_{3-i}. Now the function c defined by $c(v) = (c_1(v), c_2(v))$ for all $v \in V(D)$ is a proper colouring of D with colour set $\{1, \dots, \chi(D_1)\} \times \{1, \dots, \chi(D_2)\}$. □

5.2 General upper bound

Theorem 23. *Let k and ℓ be two positive integers such that $k \geqslant \max\{\ell, 3\}$, and let D be a digraph in* S-Forb$(C(k, \ell)) \cap \mathcal{S}$*. Then, $\chi(D) \leqslant (k + \ell - 2)(k + \ell - 3)(2\ell + 2)(k + \ell + 1)$.*

Démonstration. Since D is strongly connected, it has an out-generator u. Let T be a BFS-tree with root u. We define the following sets of arcs.

$$
\begin{array}{rcl}
A_0 & = & \{xy \in A(D) \mid \mathrm{lvl}(x) = \mathrm{lvl}(y)\}; \\
A_1 & = & \{xy \in A(D) \mid 0 < |\mathrm{lvl}(x) - \mathrm{lvl}(y)| < k + \ell - 3; \\
A' & = & \{xy \in A(D) \mid \mathrm{lvl}(x) - \mathrm{lvl}(y) \geqslant k + \ell - 3\}.
\end{array}
$$

Since $k + \ell - 3 > 0$ and there is no arc xy with $\mathrm{lvl}(y) > \mathrm{lvl}(x) + 1$, (A_0, A_1, A') is a partition of $A(D)$. Observe moreover that $A(T) \subseteq A_1$. We further partition A' into two sets A_2 and A_3, where $A_2 = \{xy \in A' \mid y \text{ is an ancestor of } x \text{ in } T\}$ and $A_3 = A' \setminus A_2$. Then (A_0, A_1, A_2, A_3) is a partition of $A(D)$. Let $D_j = (V(D), A_j)$ for all $j \in \{0, 1, 2, 3\}$.

Claim 23.1. $\chi(D_0) \leqslant k + \ell - 2$.

Subproof. Observe that D_0 is the disjoint union of the $D[L_i]$ where $L_i = \{v \mid \mathrm{dist}_D(u, v) = i\}$. Therefore it suffices to prove that $\chi(D[L_i]) \leqslant k + \ell - 2$ for all non-negative integer i.

$L_0 = \{u\}$ so the result holds trivially for $i = 0$.

Assume now $i \geqslant 1$. Suppose for a contradiction $\chi(D[L_i]) \geqslant k + \ell - 1$. Since $k \geqslant 3$, by Theorem 6, $D[L_i]$ contains a copy Q of $P^+(k-1, \ell-1)$. Let v_1 and v_2 be the initial and terminal vertices of Q, and let x be the least common ancestor of v_1 and v_2. By definition, for $j \in \{1, 2\}$, there exists a (x, v_j)-dipath P_j in T. By definition of least common ancestor, $V(P_1) \cap V(P_2) = \{x\}$, $V(P_j) \cap L_i = \{v_j\}$, $j = 1, 2$, and both P_1 and P_2 have length at least 1. Consequently, $P_1 \cup P_2 \cup Q$ is a subdivision of $C(k, \ell)$, a contradiction. ◊

Claim 23.2. $\chi(D_1) \leqslant k + \ell - 3$.

Subproof. Let ϕ_1 be the colouring of D_1 defined by $\phi_1(x) = \mathrm{lvl}(x) \pmod{k + \ell - 3}$. By definition of D_1, this is clearly a proper colouring of D_1. ◊

Claim 23.3. $\chi(D_2) \leqslant 2\ell + 2$.

Subproof. Suppose for a contradiction that $\chi(D_2) \geqslant 2\ell + 3$. By Theorem 6, D_2 contains a copy Q of $P^-(\ell+1, \ell+1)$, which is the union of two disjoint dipaths which are disjoint except in there initial vertex y, say $Q_1 = (y_0, y_1, y_2, \dots, y_{\ell+1})$ and $Q_2 = (z_0, z_1, z_2, \dots, z_{\ell+1})$ with $y_0 = z_0 = y$. Since Q is in D_2, all vertices of Q belong to $T[u, y]$. Without loss of generality, we can assume $z_1 \geqslant_T y_1$.

If $z_{\ell+1} \geqslant_T y_{\ell+1}$, then let j be the smallest integer such that $z_j \geqslant_T y_{\ell+1}$. Then the union of $T[y_1, y] \odot Q_2[y, z_j] \odot T[z_j, y_{\ell+1}]$ and $Q_1[y_1, y_{\ell+1}]$ is a subdivision of $C(k, \ell)$, because $T[y_1, y]$ has length at least $k-2$ as $\mathrm{lvl}(y) \geqslant \mathrm{lvl}(y_1) + k + \ell - 3$. This is a contradiction.

Henceforth $y_{\ell+1} \geqslant_T z_{\ell+1}$. Observe that all the z_j, $1 \leqslant j \leqslant \ell+1$ are in $T[y_{\ell+1}, y_1]$. This, by the Pigeonhole principle, there exists $i, j \geqslant 1$ such that $y_{i+1} \geqslant_T z_{j+1} \geqslant_T z_j \geqslant_T y_i \geqslant_T z_{j-1}$.

If $\mathrm{lvl}(z_{j-1}) \geqslant \mathrm{lvl}(y_i) + \ell - 1$, then $T[y_i, z_{j-1}] \odot (z_{j-1}, z_j)$ has length at least ℓ. Hence its union with $(y_i, y_{i+1}) \odot T[y_{i+1}, z_j]$, which has length greater than k, is a subdivision of $C(k, \ell)$, a contradiction.

Thus $\mathrm{lvl}(z_{j-1}) < \mathrm{lvl}(y_i) + \ell - 1$ (in particular, in this case, $j > 1$ and $i > 2$). Therefore, by definition of A', $\mathrm{lvl}(y_i) \geqslant \mathrm{lvl}(z_j) + k - 1$ and $\mathrm{lvl}(y_{i-1}) \geqslant \mathrm{lvl}(z_{j-1}) + k - 1$. Hence both $T[z_{j-1}, y_{i-1}]$ and $T[z_j, y_i]$ have length at least $k-1$. So the union of $T[z_{j-1}, y_{i-1}] \odot (y_{i-1}, y_i)$ and $(z_{j-1}, z_j) \odot T[z_j, y_i]$ is a subdivision of $C(k, k)$ (and thus of $C(k, \ell)$), a contradiction. $\lozenge$

Claim 23.4. $\chi(D_3) \leqslant k + \ell + 1$.

Subproof. In this claim, it is important to note that $k + \ell - 3 \geqslant k - 1$ because $\ell \geqslant 2$. We use the fact that $\mathrm{lvl}(x) - \mathrm{lvl}(y) \geqslant k - 1$ if xy is an edge in A_3.

Suppose for a contradiction that $\chi(D_3) \geqslant k + \ell + 1$. By Theorem 6, D_3 contains a copy Q of $P^-(k, \ell)$ which is the union of two disjoint dipaths which are disjoint except in there initial vertex y, say $Q_1 = (y_0, y_1, y_2, \dots, y_k)$ and $Q_2 = (z_0, z_1, z_2, \dots, z_\ell)$ with $y_0 = z_0 = y$.

Assume that a vertex of $Q_1 - y$ is an ancestor of y. Let i be the smallest index such that y_i is an ancestor of y. If it exists, by definition of A_3, $i \geqslant 2$. Let x be the common ancestor of y_i and y_{i-1} in T. By definition of A_3, y_i is not an ancestor of y_{i-1}, so x is different from y_i and y_{i-1}. Moreover by definition of A', $\mathrm{lvl}(y) - k \geqslant \mathrm{lvl}(y_{i-1}) - k \geqslant \mathrm{lvl}(y_i) - 1 \geqslant \mathrm{lvl}(x)$. Hence $T[x, y_{i-1}]$ and $T[x, y]$ have length at least k. Moreover these two dipaths are disjoint except in x. Therefore, the union of $T[x, y_{i-1}]$ and $T[x, y] \odot Q_1[y, y_{i-1}]$ is a subdivision of $C(k, k)$ (and thus of $C(k, \ell)$), a contradiction.

Similarly, we get a contradiction if a vertex of $Q_2 - y$ is an ancestor of y. Henceforth, no vertex of $V(Q_1) \cup V(Q_2) \setminus \{y\}$ is an ancestor of y.

Let x_1 be the least common ancestor of y and y_1. Note that $|T[x_1, y]| \geqslant k$ so $|T[x_1, y_1]| < k$, for otherwise G would contain a subdivision of $C(k, k)$. Therefore $\mathrm{lvl}(y_1) - \mathrm{lvl}(x_1) < k$. We define inductively $x_2, \dots, x_k$ as follows : x_{i+1} is the least common ancestor of x_i and y_i. As above $|T[x_i, y_{i-1}]| \geqslant k$ so $\mathrm{lvl}(y_i) - \mathrm{lvl}(x_i) < k$. Symmetrically, let t_1 be the least common ancestor of y and z_1 and for $1 \leqslant i \leqslant \ell - 1$, let t_{i+1} be the least common ancestor of t_i and z_i. For $1 \leqslant i \leqslant \ell$, we have $\mathrm{lvl}(z_i) - \mathrm{lvl}(t_i) < k$. Moreover, by definition all x_i and t_j are ancestors of y, so they all are on $T[u, y]$.

Let P_y (resp. P_z) be a shortest dipath in D from y_k (resp. z_ℓ) to $T[u, y] \cup Q_1[y_1, y_{k-1}] \cup Q_2[z_1, z_{\ell-1}]$. Note that P_y and P_z exist since D is strongly connected. Let y' (resp. z') be the terminal vertex of P_y (resp. P_z). Let w_y be the last vertex of $T[x_k, y_k]$ in P_y (possibly, $w_y = y_k$.) Similarly, let w_z be the last vertex of $T[t_\ell, z_\ell]$ in P_z (possibly, $w_z = z_\ell$.) Note that $P_y[w_y, y']$ is a shortest dipath from w_y to y' and $P_z[w_z, z']$ is a shortest dipath from w_z to z'.

If $y' = y_j$ for $0 \leqslant j \leqslant k-1$, consider $R = T[x_k, w_y] \odot P_y[w_y, y_j]$ is an (x_k, y_j)-dipath. By Proposition 21, R has length at least k because $\text{lvl}(y_j) - \text{lvl}(x_k) \geqslant \text{lvl}(y_j) - \text{lvl}(y_k) + 1 \geqslant k$. Therefore the union of R and $T[x_k, y] \cup Q_1[y, y_j]$ is a subdivision of $C(k,k)$, a contradiction.

Similarly, we get a contradiction if z' is in $\{z_1, \dots, z_{\ell-1}\}$. Consequently, P_y is disjoint from $Q_1[y, y_{k-1}]$ and P_z is disjoint from $Q_2[y, z_{\ell-1}]$.

If P_y and P_z intersect in a vertex s. By the above statement, $s \notin V(Q) \setminus \{y_k, z_\ell\}$. Therefore the union of $Q_1 \odot P_y[y_k, s]$ and $Q_2 \odot P_z[z_\ell, s]$ is a subdivision of $C(k,\ell)$, a contradiction. Henceforth P_y and P_z are disjoint.

Assume both y' and z' are in $T[u, y]$. If $y' \geqslant_T z'$, then the union of $Q_1 \odot P_y \odot T[y', z']$ and $Q_2 \odot P_z$ form a subdivision of $C(k,\ell)$; and if $z' \geqslant_T y'$, then the union of $Q_1 \odot P_y$ and $Q_2 \odot P_z \odot T[z', y']$ form a subdivision of $C(k,\ell)$. This is a contradiction.

Henceforth a vertex among y' and z' is not in $T[u, y]$. Let us assume that y' is not in $T[u, y]$ (the case $z' \notin T[u, y]$ is similar), and so $y' = z_i$ for some $1 \leqslant i \leqslant \ell - 1$. If $\text{lvl}(y') \geqslant \text{lvl}(x_k) + k$, then both $T[x_k, w_y] \odot P_y[w_y, y']$ and $T[x_k, y] \odot Q_2[y, z_i]$ have length at least k by Proposition 21, so their union is a subdivision of $C(k,k)$, a contradiction. Hence $\text{lvl}(x_k) \geqslant \text{lvl}(z_i) - k + 1 \geqslant \text{lvl}(z_\ell) \geqslant \text{lvl}(t_\ell)$.

If $z' = y_j$ for some j, then necessarily $\text{lvl}(z') \geqslant \text{lvl}(x_k) + k \geqslant \text{lvl}(t_\ell) + k$ and both $T[t_\ell, w_z] \odot P_z[w_z, z']$ and $T[t_\ell, y] \odot Q_1[y, y_j]$ have length at least k, so their union is a subdivision of $C(k,k)$, a contradiction.

Therefore $z' \in T[u, y]$. The union of $T[t_\ell, z']$ and $T[t_\ell, w_z] \odot P_z[w_z, z']$ is not a subdivision of $C(k,k)$ so by Proposition 21, $\text{lvl}(z') \leqslant \text{lvl}(t_\ell) + k - 1 \leqslant \text{lvl}(z_\ell) + k - 1 \leqslant \text{lvl}(z_{\ell-1})$.

If $\text{lvl}(z') \leqslant \text{lvl}(x_k)$, then the union of Q_1 and $Q_2 \odot P_z \odot T[z', y_k]$ is a subdivision of $C(k,\ell)$, a contradiction. Hence $\text{lvl}(z') > \text{lvl}(x_k)$. Therefore $\text{lvl}(y') = \text{lvl}(z_i) \leqslant \text{lvl}(x_k) + k - 1 \leqslant \text{lvl}(z') + k - 2 \leqslant \text{lvl}(z_\ell) + 2k - 3$, which implies that $i = \ell - 1$ that is $y' = z_i = z_{\ell-1}$. Now the union of $[T[x_1, y_1]] \odot Q_1[y_1, y_k] \odot P_y$ and $T[x_1, y] \odot Q_2[y, z_{\ell-1}]$ is a subdivision of $C(k,\ell)$, a contradiction. $\Diamond$

Claims 23.1, 23.2, 23.3, and 23.4, together with Lemma 22 yield the result. $\square$

5.3 Better bound for Hamiltonian digraphs

We now improve on the bound of Theorem 23 in case of digraphs having a Hamiltonian directed cycle. Therefore we define

$$\phi(k,\ell) = \max\{\chi(D) \mid D \in \text{S-Forb}(C(k,\ell)) \text{ and } D \text{ has a Hamiltonian directed cycle}\}.$$

This section aims at proving that $\phi(k,k) \leqslant 6k - 6$.

Let D be a digraph and let $C = (v_1, \dots, v_n, v_1)$ be a Hamiltonian cycle in D (C may be directed or not).

For any $i, j \leqslant n$, let $d_C(v_i, v_j)$ be the distance between v_i and v_j in the undirected cycle C. That is, $d_C(v_i, v_j) = \min\{j - i, n - j + i\}$ if $j > i$ and $d_C(v_i, v_j) = \min\{i - j, n - i + j\}$ otherwise.

A *chord* is an arc of $A(D) \setminus A(C)$. The *span* $\text{span}_C(a)$ of a chord $a = v_i v_j \in F$ is $d_C(i, j)$. We denote by $\text{span}_C(D)$ be the maximum span of a chord in D.

Lemma 24. *If D is a digraph with a Hamiltonian cycle C and at least one chord, then $\chi(D) < 2 \cdot \text{span}_C(D)$.*

Démonstration. Set $C = (v_1, \dots, v_n, v_1)$ and set $\ell = \text{span}_C(D)$. If $n < 2\ell$, then the result trivially holds. Let us assume that $n = k\ell + r$ with $k \geqslant 2$ and $r < \ell$. Consider the following colouring. For any $1 \leqslant i \leqslant k\ell$, let us colour v_i with colour $i - \lfloor i/\ell \rfloor \ell$. For any $1 < t \leqslant r$, let us colour $v_{k\ell+t}$ with $\ell + t - 1$. This colouring uses the $\ell + r$ colours of $\{0, \dots, \ell + r - 1\}$.

Moreover, for any $1 \leqslant i \leqslant n$, all neighbours (in-neighbours and out-neighbours) of v_i belong to $\{v_{i-\ell}, \dots, v_{i-1}\} \cup \{v_{i+1}, \dots, v_{i+\ell}\}$ (all indices must be taken moduo n), for otherwise there would be a chord with span strictly larger than ℓ. Hence, the colouring is proper. □

Let $A \subseteq V(D)$, let $N(A) \subseteq V(D) \setminus A$ be the set of vertices not in A that are adjacent to some vertex in A.

Lemma 25. *Let D be a digraph and let (A, B) be a partition of $V(D)$. Then*

$$\chi(D) = \max\{\chi(D[A]) + |N(A)|, \chi(D[B])\}.$$

Démonstration. Let us consider a proper colouring of $D[B]$ with colour set $\{1, \dots, \chi(D[B])\}$. W.l.o.g., vertices in $N(A)$ have received colours in $\{1, \dots, |N(A)|\}$. Let us colour $D[A]$ using colours in $\{|N(A)|+1, \dots, |N(A)|+\chi(D[A])\}$. We obtain a proper colouring of D using $\max\{\chi(D[A])+ |N(A)|, \chi(D[B])\}$ colours. □

Lemma 26. *Let D be a digraph containing no subdivision of $C(k, k)$ and having a Hamiltonian directed cycle $C = (v_1, \dots, v_n, v_1)$. Assume that D contains a chord v_iv_j with span at least $2k-2$ and let $A = \{v_{i+1}, \dots, v_{j-1}\}$ and $B = \{v_{j+1}, \dots, v_{i-1}\}$ (indices are taken modulo n). Then $|N(A)| \leqslant 2k + 1$ and $|N(B)| \leqslant 2k + 1$.*

Démonstration. W.l.o.g., assume that D has a chord v_1v_j with $2k - 1 \leqslant j \leqslant n - 2k + 3$.

Assume first that v_av_b is an arc from A to B.

(1) we cannot have $a \leqslant j - k$ and $b \leqslant n - k + 1$, for otherwise the two dipaths $C[v_a, v_j]$ and $(v_a, v_b) \odot C[v_b, v_1] \odot (v_1, v_j)$ have length at least k and so their union is a subdivision of $C(k, k)$, a contradiction.

(2) we cannot have $a \geqslant k$ and $b \geqslant j + k - 1$, for otherwise the two dipaths $C[v_1, v_a] \odot (v_a, v_b)$ and $(v_1, v_j) \odot C[v_j, v_b]$ have length at least k and so their union is a subdivision of $C(k, k)$, a contradiction.

Since $j \geqslant 2k-1$, either $a \leqslant j-k$ or $a \geqslant k$, so $v_b \in \{v_{j+1}, \dots, v_{j+k-2}\} \cup \{v_{n-k+2}, \dots, v_n\}$. Similarly, since $j \leqslant n-2k+3$, either $b \leqslant n-k+1$ or $b \geqslant j+k-1$, so $v_a \in \{v_2, \dots, v_{k-1}\} \cup \{v_{j-k+1}, \dots, v_{j-1}\}$.

Analogously, if v_bv_a is an arc from B to A, we obtain that $v_a \in \{v_2, \dots, v_k\} \cup \{v_{j-k+2}, \dots, v_{j-1}\}$ and $v_b \in \{v_{j+1}, \dots, v_{j+k-2}\} \cup \{v_{n-k+3}, \dots, v_n\}$.

Therefore $N(A) \subseteq \{v_1, \dots, v_k\} \cup \{v_{j-k+1}, \dots, v_j\}$, and $N(B) \subseteq \{v_j, \dots, v_{j+k-2}\} \cup \{v_{n-k+2}, \dots, v_n, v_1\}$. Hence $|N(A)| \leqslant 2k + 1$ and $|N(B)| \leqslant 2k + 1$. □

Theorem 27. *Let D be a digraph and let $k \geqslant 1$ be an integer. If D has a Hamiltonian directed cycle and $\chi(D) > 6k - 6$, then D contains a subdivision of a $C(k, k)$. In other words, $\phi(k, k) \leqslant 6k - 6$.*

Démonstration. If $k = 2$, then we have the result by Theorem 37. Henceforth, we assume $k \geqslant 3$.

For sake of contradiction, let us consider a counterexample (i.e a digraph D with a Hamiltonian directed cycle, $\chi(D) > 6k - 6$ and no subdivision of $C(k, k)$) with the minimum number of vertices.

Let $C = (v_1, \dots, v_n, v_1)$ be a Hamiltonian directed cycle of D. By Lemma 24 and because $\chi(D) \geqslant 4k - 4$, D contains a chord of span at least $2k - 2$. Let s be the minimum span of a chord of span at least $2k - 2$ and consider a chord of span s. W.l.o.g., this chord is v_1v_{s+1}. Let $D_1 = D[v_1, \dots, v_{s+1}]$ and let $D_2 = D[v_{s+1}, \dots, v_n, v_1]$. By minimality of the span of v_1v_{s+1}, either D_1 or D_2 contains no chord of span at least $2k - 2$. There are two cases to be considered.

— Assume first that D_1 contains no chord of span at least $2k-2$. By Lemma 24, $\chi(D_1) \leqslant 4k-7$. Let $A = \{v_2, \ldots, v_s\}$. We have $\chi(D[A]) \leqslant \chi(D_1) \leqslant 4k-7$. Moreover, by Lemma 26, $|N(A)| \leqslant 2k+1$.

Now D_2 has a Hamiltonian directed cycle and contains no subdivision of $C(k,k)$. Therefore, $\chi(D_2) \leqslant 6k-6$ since D has been chosen minimum. Finally, by Lemma 25, since $\chi(D[A]) + |N(A)| \leqslant 6k-6$ and $\chi(D_2) \leqslant 6k-6$, we get that $\chi(D) \leqslant 6k-6$, a contradiction.

— Assume now that D_2 contains no chord of span at least $2k-2$. Set $B = \{v_{s+1}, \ldots, v_n\}$. Similarly as in the previous case, we have $\chi(D[B]) \leqslant \chi(D_2) \leqslant 4k-7$ and $|N(B)| \leqslant 2k+1$.

Let D'_1 be the digraph obtained from D_1 by reversing the arc v_1v_s. Clearly D'_1 is Hamiltonian. Moreover, D'_1 contains no subdivision of a $C(k,k)$; indeed if it had such a subdivision S, replacing the arc v_sv_1 by $C[v_s, v_1]$ if it is in S, we obtain a subdivision of $C(k,k)$ in D, a contradiction. Therefore $\chi(D_1) = \chi(D'_1) \leqslant 6k-6$, by minimality of D.

Hence by Lemma 25, since $\chi(D[B]) + |N(B)| \leqslant 6k-6$ and $\chi(D_1) \leqslant 6k-6$, we get that $\chi(D) \leqslant 6k-6$, a contradiction.

□

5.4 Better bound when $\ell = 1$

We now improve on the bound of Theorem 23 when $\ell = 1$. To do so, reduce the problem to digraphs having a Hamiltonian directed cycle. Recall that

$$\phi(k,\ell) = \max\{\chi(D) \mid D \in \text{S-Forb}(C(k,\ell)) \text{ and } D \text{ has a Hamiltonian directed cycle}\}.$$

Theorem 28. *Let k be an integer greater than 1. $\chi(\text{S-Forb}(C(k,1)) \cap \mathcal{S}) \leqslant \max\{2k-4, \phi(k,1)\}$.*

To prove this theorem, we shall use the following lemma.

Lemma 29. *Let D be a digraph containing a directed cycle C of length at least $2k-3$. If there is a vertex y in $V(D-C)$ and two distinct vertices $x_1, x_2 \in V(C)$ such that for $i = 1, 2$, there is a (x_i, y)-dipath P_i in D with no internal vertices in C, then D contains a subdivision of $C(k,1)$.*

Démonstration. Since C has length at least $2k-3$, then one of $C[x_1, x_2]$ and $C[x_2, x_1]$ has length at least $k-1$. Without loss of generality, assume that $C[x_1, x_2]$ has length at least $k-1$. Let z be the first vertex along P_2 which is also in P_1. Then the union of $C[x_1, x_2] \odot P_2[x_2, z]$ and $P_1[x_1, z]$ is a subdivision of $C(k,1)$. □

Proof of Theorem 28. Suppose for a contradiction that there is a strong digraph D with chromatic number greater than $\max\{2k-4, \phi(k,1)\}$ that contains no subdivision of $C(k,1)$. Let us consider the smallest such counterexample.

All 2-connected components of D are strong, and one of them has chromatic number $\chi(D)$. Hence, by minimality, D is 2-connected. Let C be a longest directed cycle in D. By Bondy's theorem (Theorem 10), C has length at least $2k-3$, and by definition of $\phi(k,1)$, C is not Hamiltonian.

Because D is strong, there is a vertex $v \in C$ with an out-neighbour $w \notin C$. Since D is 2-connected, $D-v$ is connected, so there is a (not necessarily directed) oriented path in $D-v$ between $C-v$ and w. Let $Q = (a_1, \ldots, a_q)$ be such a path so that all its vertices except the initial one are in $V(D) \setminus V(C)$. By definition $a_q = w$ and $a_1 \in V(C) \setminus \{v\}$.

— Let us first assume that $a_1a_2 \in A(D)$. Let t be the largest integer such that there is a dipath from $C-v$ to a_t in $D-v$. Note that $t > 1$ by the hypothesis. If $t = q$, then by Lemma 29, C contains a subdivision of $C(k,1)$, a contradiction. Henceforth we may assume that $t < q$. By definition of t, $a_{t+1}a_t$ is an arc. Let P be a shortest (v, a_{t+1})-dipath in D. Such a dipath exists because D is strong. By maximality of t, P has no internal vertex in $(C - v) \cup Q[a_1, a_t]$. Hence, $a_t \in D - C$ and there are an (a_1, a_t)-dipath and a (v, a_t)-dipath with no internal vertices in C. Hence, by Lemma 29, D contains a subdivision of $C(k, 1)$, a contradiction.

— Now, we may assume that any oriented path $Q = (a_1, \dots, a_q)$ from $C - v$ to w starts with a backward arc, i.e., $a_2a_1 \in A(D)$. Let W be the set of vertices x such that there exists a (not necessarily directed) oriented path from w to x in $D - C$. In particular, $w \in W$.

  By the assumption, all arcs between $C - v$ and W are from W to $C - v$. Since D is strong, this implies that, for any $x \in W$, there exists a directed (w, x)-dipath in W. In other words, w is an out-generator of W. Let T_w be a BFS-tree of W rooted in w (see definitions in Section 5.1.1).

  Because D is strong and 2-connected, there must be a vertex $y \in C - v$ such that there is an arc ay from a vertex $a \in W$ to y.

  For purpose of contradiction, let us assume that there exists $z \in C - y$ such that there is an arc bz from a vertex $b \in W$ to z. Let r be the least common ancestor of a and b in T_w. If $|C[y, z]| \geqslant k$, then $T_w[r, a] \odot (a, y) \odot C[y, z]$ and $T_w[r, b] \odot (b, z)$ is a subdivision of $C(k, 1)$. If $|C[z, y]| \geqslant k$, then $T_w[r, a] \odot (a, y)$ and $T_w[r, b] \odot (b, z) \odot C[z, y]$ is a subdivision of $C(k, 1)$. In both cases, we get a contradiction.

  From previous paragraph and the definition of W, we get that all arcs from W to $D \setminus W$ are from W to $y \neq v$, and there is a single arc from $D \setminus W$ to W (this is the arc vw). Note that, since D is strong, this implies that $D - W$ is strong.

  Let D_1 be the digraph obtained from $D - W$ by adding the arc vy (if it does not already exist). D_1 contains no subdivision of $C(k, 1)$, for otherwise D would contain one (replacing the arc vy by the dipath $(v, w) \odot T_w[w, a] \odot (a, y)$). Since D_1 is strong (because $D - W$ is strong), by minimality of D, $\chi(D_1) \leqslant \max\{2k - 4, \phi(k, 1)\}$.

  Let D_2 be the digraph obtained from $D[W \cup \{v, y\}]$ by adding the arc yv. D_2 contains no subdivision of $C(k, 1)$, for otherwise D would contain one (replacing the arc yv by the dipath $C[y, v]$). Moreover, D_2 is strong, so by minimality of D, $\chi(D_2) \leqslant \max\{2k - 4, \phi(k, 1)\}$.

  Consider now D^* the digraph $D_1 \cup D_2$. It is obtained from D by adding the two arcs vy and yv (if they did not already exist). Since $\{v, y\}$ is a clique-cutset in D^*, we get $\chi(D^*) \leqslant \max\{\chi(D_1), \chi(D_2)\} \leqslant \max\{2k - 4, \phi(k, 1)\}$. But $\chi(D) \leqslant \chi(D^*)$, a contradiction.

□

From Theorem 28, one easily derives an upper bound on $\chi(\text{S-Forb}(C(k, 1)) \cap \mathcal{S})$.

Corollary 30. *$\chi(\text{S-Forb}(C(k, 1)) \cap \mathcal{S}) \leqslant 2k - 1$.*

Démonstration. By Theorem 28, it suffices to prove $\phi(k, 1) \leqslant 2k - 1$.

Let $D \in \text{S-Forb}(C(k, 1))$ with a Hamiltonian directed cycle $C = (v_1, \dots, v_n, v_1)$. Observe that if v_iv_j is an arc, then $j \in C[v_{i+1}, v_{i+k-1}]$ for otherwise the union of $C[v_i, v_j]$ and (v_i, v_j) would be a subdivision of $C(k, 1)$. In particular, every vertex had both its in-degree and out-degree at most $k - 1$, and so degree at most $2k - 2$. As $\chi(D) \leqslant \Delta(D) + 1$, the result follows. □

The bound $2k - 1$ is tight for $k = 2$, because of the directed odd cycles. However, for larger values of k, we can get a better bound on $\phi(k, 1)$, from which one derives a slightly better one for $\chi(\text{S-Forb}(C(k, 1)) \cap \mathcal{S})$.

Theorem 31. *$\phi(k, 1) \leqslant \max\{k + 1, \frac{3k-3}{2}\}$.*

Démonstration. For $k = 2$, the result holds because $\phi(2,1) \leqslant \phi(2,2) \leqslant 3$ by Corollary 38.

Let us now assume $k \geqslant 3$. We prove by induction on n, that every digraph $D \in$ S-Forb($C(k,1)$) with a Hamiltonian directed cycle $C = (v_1, \dots, v_n, v_1)$ has chromatic number at most $\max\{k+1, \frac{3k-3}{2}\}$, the result holding trivially when $n \leqslant \max\{k+1, \frac{3k-3}{2}\}$.

Assume now that $n \geqslant \max\{k+1, \frac{3k-3}{2}\} + 1$ All the indices are modulo n. Observe that if v_iv_j is an arc, then $j \in C[v_{i+1}, v_{i+k-1}]$ for otherwise the union of $C[v_i, v_j]$ and (v_i, v_j) would be a subdivision of $C(k,1)$. In particular, every vertex had both its in-degree and out-degree at most $k-1$.

Assume that D contains a vertex v_i with in-degree 1 or out-degree 1. Then $d(v_i) \leqslant k$. Consider D_i the digraph obtained from $D - v_i$ by adding the arc $v_{i-1}v_{i+1}$. Clearly, D_i has a Hamiltonian directed cycle. Moreover is has no subdivision of $C(k,1)$ for otherwise, replacing the arc $v_{i-1}v_{i+1}$ by (v_{i-1}, v_i, v_{i+1}) if necessary, yields a subdivision of $C(k,1)$ in D. By the induction hypothesis, D_i a $\max\{k+1, \frac{3k-3}{2}\}$-colouring which can be extended to v_i because $d(v_i) \leqslant k$.

Henceforth, we may assume that $\delta^-(D), \delta^+(D) \geqslant 2$.

Claim 31.1. *$d^+(v_i) + d^-(v_{i+1}) \leqslant 3k - n - 3$ for all i.*

Subproof. Let v_{i^+} be the first out-neighbour of v_i along $C[v_{i+2}, v_{i-1}]$ and let v_{i^-} be the last in-neighbour of v_{i+1} along $C[v_{i+3}, v_i]$. There are $d^+(v_i) - 1$ out-neigbours of v_i in $C[v_{i^+}, v_{i-1}]$ which all must be in $C[v_{i^+}, v_{i+k-1}]$ by the above observation. Therefore $i^+ \leqslant i + k - d^+(v_i)$. Similarly, $i^- \geqslant i - k + d^-(v_{i+1})$.

- if $v_i \in C[v_{i^-}, v_{i^+}]$, $C[v_{i^-}, v_{i^+}]$ has length $i^+ - i^- \leqslant 2k - d^+(v_i) - d^-(v_{i+1})$. Hence $C[v_{i^+}, v_{i^-}]$ has length at least $n - 2k + d^+(v_i) + d^-(v_{i+1})$. But the union of $(v_i, v_{i^+}) \odot C[v_{i^+}, v_{i^-}] \odot (v_{i^-}, v_{i+1})$ and (v_i, v_{i+1}) is not a subdivision of $C(k,1)$, so $C[v_{i^+}, v_{i^-}]$ has length at most $k-3$. Hence, $k - 3 \geqslant n - 2k + d^+(v_i) + d^-(v_{i+1})$, so $d^+(v_i) + d^-(v_{i+1}) \leqslant 3k - n - 3$.
- otherwise, $v_{i^+} \in C[v_{i^-}, v_{i+1}]$ and $v_{i^-} \in C[v_i, v_{i^+}]$. Both $C[v_{i^-}, v_{i+1}]$ and $C[v_i, v_{i^+}]$ have length less than k as $v_{i^-}v_{i+1}$ and $v_{i^-}v_{i+1}$ are arcs. Moreover, the union of these two dipaths is C and their intersection contains the three distinct vertices v_i, v_{i+1}, v_{i^-}. Consequently, $n = |C| \leqslant |C[v_{i^-}, v_{i+1}]| + |C[v_i, v_{i^+}]| - 3 \leqslant 2k - 3$. Let v_{i_0} be the last out-neighbour of v_i along $C[v_{i+2}, v_{i-1}]$. All the out-neighbours of v_i and all the in-neighbours of v_{i+1} are in $C[v_i, v_{i_0}]$ which has length less than k because $v_iv_{i_0}$ is an arc. Hence $d^+(v_i) + d^-(v_{i+1}) \leqslant k$, so $d^+(v_i) + d^-(v_{i+1}) \leqslant 3k - n - 3$ because $n \geqslant 2k - 3$. $\Diamond$

But $n \geqslant \frac{3k-1}{2}$, so by the above claim, $d^+(v_i) + d^-(v_{i+1}) \leqslant \frac{3k-5}{2}$ for all i.

Summing these inequalities over all i, we get $\sum_{i=1}^n (d^+(v_i) + d^-(v_{i+1}) \leqslant \frac{3k-5}{2} \cdot n$. Thus $\sum_{i=1}^n d(v_i) = \sum_{i=1}^n (d^+(v_i) + d^-(v_i)) \leqslant \frac{3k-5}{2} \cdot n$. Therefore there exists an index i such that v_i has degree at most $\frac{3k-5}{2}$. Consider the digraph D_i defined above. It is Hamiltonian and contains no subdivision of $C(k,1)$. By the induction hypothesis, D_i has a $\max\{k+1, \frac{3k-3}{2}\}$-colouring which can be extended to v because $d(v_i) \leqslant \frac{3k-5}{2}$. $\square$

Corollary 32. *Let k be an integer greater than 1. Then $\chi(\text{S-Forb}(C(k,1)) \cap \mathcal{S}) \leqslant \max\{k+1, 2k-4\}$.*

Démonstration. By Theorems 28 and 31, $\chi(\text{S-Forb}(C(k,1)) \cap \mathcal{S}) \leqslant \max\{2k-4, k+1, \frac{3k-3}{2}\} = \max\{k+1, 2k-4\}$. $\square$

6 Small cycles with two blocks in strong digraphs

6.1 Handle decomposition

Let D be a strongly connected digraph. A *handle* h of D is a directed path $(s, v_1, \ldots, v_\ell, t)$ from s to t (where s and t may be identical) such that :

— $d^-(v_i) = d^+(v_i) = 1$, for every i, and

— removing the internal vertices and arcs of h leaves D strongly connected.

The vertices s and t are the *endvertices* of h while the vertices v_i are its *internal vertices*. The vertex s is the *initial vertex* of h and t its *terminal vertex*. The *length* of a handle is the number of its arcs, here $\ell + 1$. A handle of length 1 is said to be *trivial*.

Given a strongly connected digraph D, a *handle decomposition* of D starting at $v \in V(D)$ is a triple $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$, where $(D_i)_{0 \leqslant i \leqslant p}$ is a sequence of strongly connected digraphs and $(h_i)_{1 \leqslant i \leqslant p}$ is a sequence of handles such that :

— $V(D_0) = \{v\}$,

— for $1 \leqslant i \leqslant p$, h_i is a handle of D_i and D_i is the (arc-disjoint) union of D_{i-1} and h_i, and

— $D = D_p$.

A handle decomposition is uniquely determined by v and either $(h_i)_{1 \leqslant i \leqslant p}$, or $(D_i)_{0 \leqslant i \leqslant p}$. The number of handles p in any handle decomposition of D is exactly $|A(D)| - |V(D)| + 1$. The value p is also called the *cyclomatic number* of D. Observe that $p = 0$ when D is a singleton and $p = 1$ when D is a directed cycle.

A handle decomposition $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$ is *nice* if all handles except the first one h_1 have distinct endvertices (i.e., for any $1 < i \leqslant p$, the initial and terminal vertices of h_i are distinct).

A digraph is *robust* if it is 2-connected and strongly connected. The following proposition is well-known (see [5] Theorem 5.13).

Proposition 33. *Every robust digraph admits a nice handle decomposition.*

Lemma 34. *Every strong digraph D with $\chi(D) \geqslant 3$ has a robust subdigraph D' with $\chi(D') = \chi(D)$ and which is an oriented graph.*

Démonstration. Let D be a strong digraph D with $\chi(D) \geqslant 3$. Let D' be a 2-connected components of D with the largest chromatic number. Each 2-connected component of a strong digraph is strong, so D' is strong. Moreover, $\chi(D') = \chi(D)$ because the chromatic number of a graph is the maximum of the chromatic numbers of its 2-connected components. Now by Bondy's Theorem (Theorem 10), D' contains a cycle C of length at least $\chi(D') \geqslant 3$. This can be extended into a handle decomposition $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$ of D such that $D_1 = C$. Let D'' be the digraph obtained from D' by removing the arcs (u, v) which are trivial handles h_i and such that (v, u) is in $A(D_{i-1})$, we obtain an oriented graph D'' which is robust and with $\chi(D'') = \chi(D') = \chi(D)$. □

6.2 $C(1, 2)$

Proposition 35. *A robust digraph containing no subdivision of $C(1, 2)$ is a directed cycle.*

Démonstration. Let D be a robust digraph containing no subdivision of $C(1, 2)$. Assume for a contradiction that a robust digraph of D is not a directed cycle. By Proposition 33, it contains a directed cycle C and a nice handle h_2 from u to v. Now the union of h_2 and $C[u, v]$ is a subdivision of $C(1, 2)$. □

Corollary 36. $\chi(\text{S-Forb}(C(1,2)) \cap \mathcal{S}) = 3$.

Démonstration. Lemma 34, Proposition 35, and the fact that every directed cycles is 3-colourable imply $\chi(\text{S-Forb}(C(1,2)) \cap \mathcal{S}) \leqslant 3$.

The directed cycles of odd length have chromatic number 3 and contain no subdivision of $C(1,2)$. Therefore, $\chi(\text{S-Forb}(C(1,2)) \cap \mathcal{S}) = 3$. □

6.3 $C(2,2)$

Theorem 37. *Let D be a strong digraph. If $\chi(D) \geqslant 4$, then D contains a subdivision of $C(2,2)$.*

Démonstration. By Lemma 34, we may assume that D is robust.

By Proposition 33, D has a nice handle decomposition. Consider a nice decomposition $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$ that maximizes the sequence $(\ell_1, \dots, \ell_p)$ of the length of the handles with respect to the lexicographic order.

Let q be the largest index such that h_q is not trivial.

Assume first that $q \neq 1$. Let s and t be the initial and terminal vertex of h_q respectively. There is an (s,t)-path P in D_{q-1}. If $P = (s,t)$, let r be the index of the handle containing the arc (s,t). Obviously, $r < q$. Now replacing h_r by the handle h'_r obtained from it by replacing the arc (s,t) by h_q and replacing h_q by (s,t), we obtain a nice handle decomposition contradicting the minimality of $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$. Therefore P has length at least 2. So $P \cup h_q$ is a subdivision of $C(2,2)$.

Assume that $q = 1$, that is D has a hamiltonian directed cycle C. Let us call *chords* the arcs of $A(D) \setminus A(C)$. Suppose that two chords (u_1, v_1) and (u_2, v_2) *cross*, that is $u_2 \in C]u_1, v_1[$ and $v_2 \in C]v_1, u_1[$. Then the union of $C[u_1, u_2] \odot (u_2, v_2)$ and $(u_1, v_1) \odot C[v_1, v_2]$ forms a subdivision of $C(2,2)$.

If no two chords cross, then one can draw C in the plane and all chords inside it without any crossing. Therefore the graph underlying D is outerplanar and has chromatic number at most 3. □

Since the directed odd cycles are in S-Forb$(C(2,2))$ and have chromatic number 3, Theorem 37 directly implies the following.

Corollary 38. $\chi(\text{S-Forb}(C(2,2)) \cap \mathcal{S}) = 3$.

6.4 $C(1,3)$

Theorem 39. *Let D be a strong digraph. If $\chi(D) \geqslant 4$, then D contains a subdivision of $C(1,3)$.*

Démonstration. By Lemma 34, we may assume that D is robust. Thus, by Proposition 33, D has a nice handle decomposition. Consider a nice decomposition $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$ that maximizes the sequence $(\ell_1, \dots, \ell_p)$ of the length of the handles with respect to the lexicographic order.

Let q be the largest index such that h_q is not trivial.

<u>Case 1</u> : Assume first that $q \neq 1$. Let s and t be the initial and terminal vertex of h_q respectively. Since D_{q-1} is strong, there is an (s,t)-dipath P in D_{q-1}. If $P = (s,t)$, let r be the index of the handle containing the arc (s,t). Obviously, $r < q$. Now replacing h_r by the handle h'_r obtained from it by replacing the arc (s,t) by h_q and replacing h_q by (s,t), we obtain a nice handle decomposition contradicting the minimality of $(v, (h_i)_{1 \leqslant i \leqslant p}, (D_i)_{0 \leqslant i \leqslant p})$. Therefore P has length at least 2. If either P or h_q has length at least 3, then $P \cup h$ is a subdivision of $C(1,3)$. Henceforth,

we may assume that both P and h_q have length 2. Set $P = (s, u, t)$ and $h = (s, x, t)$. Observe that $V(D) = V(D_{q-1}) \cup \{x\}$.

Assume that x has a neighbour t' distinct from s and t. By directional duality (i.e., up to reversing all arcs), we may assume that $x \to t'$. Considering the handle decomposition in which h_q is replaced by (s, x, t') and (x, t') by (x, t), we obtain that there is a dipath (s, u', t') in D_{q-1}. Now, if $u' = t$, then the union of (s, x, t') and (s, u, t, t') is a subdivision of $C(1, 3)$. Henceforth, we may assume that $t \notin \{s, u, u', t'\}$. Since D_{q-1} is strong, there is a dipath Q from t to $\{s, u, u', t'\}$, which has length at least one by the preceding assumption. Note that $x \notin Q$ since Q is a dipath in D_{q-1}. Whatever vertex of $\{s, u, u', t'\}$ is the terminal vertex z of Q, we find a subdivision of $C(1, 3)$:

— If $z = s$, then the union of (x, t') and $(x, t) \odot Q \odot (s, u', t')$ is a subdivision of $C(1, 3)$;

— If $z = u$, then the union of (s, u) and $h_q \odot Q$ is a subdivision of $C(1, 3)$;

— If $z = u'$, then the union of (s, u') and $h_q \odot Q$ is a subdivision of $C(1, 3)$;

— If $z = t'$, then the union of (s, x, t') and $(s, u, t) \odot Q$ is a subdivision of $C(1, 3)$.

Case 2 : Assume that $q = 1$, that is D has a hamiltonian directed cycle C. Assume that two chords (u_1, v_1) and (u_2, v_2) cross. Without loss of generality, we may assume that the vertices u_1, u_2, v_1 and v_2 appear in this order along C. Then the union of $C[u_2, v_1]$ and $(u_2, v_2) \odot C[v_2, u_1] \odot (u_1, v_1)$ forms a subdivision of $C(1, 3)$.

If no two chords cross, then one can draw C in the plane and all chords inside it without any crossing. Therefore the graph underlying D is outerplanar and has chromatic number at most 3. □

Since the directed odd cycles are in S-Forb($C(1, 3)$) and have chromatic number 3, Theorem 39 directly implies the following.

Corollary 40. $\chi(\text{S-Forb}(C(1, 3)) \cap \mathcal{S}) = 3$.

6.5 $C(2, 3)$

Theorem 41. *Let D be a strong directed graph. If $\chi(D) \geqslant 5$, then D contains a subdivision of $C(2, 3)$.*

Démonstration. By Lemma 34, we may assume that D is a robust oriented graph. Thus, by Proposition 33, D has a nice handle decomposition. Let HD $= ((h_i)_{1 \leqslant i \leqslant p}, (D_i)_{1 \leqslant i \leqslant p})$ be a nice decomposition that maximizes the sequence $(\ell_1, \dots, \ell_p)$ of the length of the handles with respect to the lexicographic order. Recall that D_i is strongly connected for any $1 \leqslant i \leqslant p$. In particular, h_1 is a longest directed cycle in D. Let q be the largest index such that h_q is not trivial. Observe that for all $i > q$, h_i is a trivial handle by definition of q and, for $i \leqslant q$, all handles h_i have length at least 2.

Claim 41.1. *For any $1 < i \leqslant q$, h_i has length exactly 2.*

Subproof. For sake of contradiction, let us assume that there exists $2 \leqslant r \leqslant q$ such that $h_r = (x_1, \dots, x_t)$ with $t \geqslant 4$. Since D_{r-1} is strong, there is a (x_1, x_t)-dipath P in D_{r-1}. Note that P does not meet $\{x_2, \dots, x_{t-1}\}$. If P has length at least 2, then $P \cup h_r$ is a subdivision of $C(2, 3)$. If $P = (x_1, x_t)$, let r' be the handle containing the arc $h_{r'}$. Now the handle decomposition obtained from HD by replacing $h_{r'}$by the handle derived from it by replacing the arc (x_1, x_t) by h_r, and replacing h_r by (x_1, x_t), contradicts the maximality of HD. ◊

For $1 < i \leqslant q$, set $h_i = (a_i, b_i, c_i)$. Since h_1 is a longest directed cycle in D and $\chi(D) \geqslant 5$, by Bondy's Theorem, h_1 has length at least 5. Set $h_1 = (u_1, \dots, u_m, u_1)$.

A *clone* of u_i is a vertex whose unique out-neighbour in D_q is u_{i+1} and whose unique in-neighbour in D_q is u_{i-1} (indices are taken modulo m).

Claim 41.2. *Let $v \in V(D) \setminus V(D_1)$. Let $1 < i \leqslant q$ such that $v = b_i$, the internal vertex of h_i. There is an index j such that b_i is a clone of u_j, that is $a_i = u_{j-1}$ and $c_i = u_{j+1}$.*

Subproof. We prove the result by induction on i.

By the induction hypothesis (or trivially if $i = 2$), there exists i^- and i^+ such that a_i is u_{i^-} or a clone of u_{i^-} and c_i is u_{i^+} or a clone of u_{i^+}. If $i^+ \notin \{i^- + 1, i^- + 2\}$, then the union of h_i and $(a_i, u_{i^-+1}, \dots, u_{i^+-1}, c_i)$ is a subdivision of $C(2,3)$, a contradiction If $i^+ = i^- - 1$, then $(a_i, b_i, c_i, h_1[u_{i^++1}, \dots, u_{i^--1}], a_i)$ is a cycle longer than h_1, a contradiction. Henceforth $i^+ = i^- + 2$. If c_i is not u_{i^+}, then it is a clone of u_{i^+}. Thus the union of $(a_i, b_i, c_i, u_{i^++1})$ and $(a_i, u_{i^-+1}, u_{i^+}, u_{i^++1})$ is a subdivision of $C(2,3)$, a contradiction Similarly, we obtain a contradiction if $a_i \neq u_{i^-}$. Therefore, $a_i = u_{i^--1}$ and $c_i = u_{i^-+1}$, that is b_i is a clone of u_{i^-+1}. Moreover all $b_{i'}$ for $i' < i$ are not adjacent to b_i and thus are still clones of some u_j. $\lozenge$

For $1 \leqslant i \leqslant m$, let S_i be the set of clones of u_i.

Claim 41.3.

(i) If $S_i \neq \emptyset$, then $S_{i-1} = S_{i+1} = \emptyset$.

(ii) If $x \in S_i$, then $N_D^+(x) = \{u_{i+1}\}$ and $N_D^-(x) = \{u_{i-1}\}$.

Subproof. (i) Assume for a contradiction, that both S_i and S_{i+1} are non-empty, say $x_i \in S_i$ and $x_{i+1} \in S_{i+1}$. Then the union of $(u_{i-1}, u_i, x_{i+1}, u_{i+2})$ and $(u_{i-1}, x_i, u_{i+1}, u_{i+2})$ is a subdivision of $C(2,3)$, a contradiction.

(ii) Let $x \in S_i$. Assume for a contradiction that x has an out-neighbour y distinct from u_{i+1}. By (i), $y \notin S_{i-1}$, and $y \neq u_{i-1}$ because D is an oriented graph. If $y \in S_i \cup \{u_i\}$, then $(x, y, h_1[u_{i+1}, u_{i-1}], x)$ is a directed cycle longer than h. If $y \in S_j \cup \{u_j\}$ for $j \notin \{i-2\}$, then the union of (u_{i-1}, x, y, u_{j+1}) and $h_1[u_{i-1}, u_{j+1}]$ is a subdivision of $C(2,3)$, a contradiction. If $y \in S_{i-2}$, then the union of (x, y, u_{i-1}) and $(x, h_1[u_{i+1}, u_{i-1}]$ is a subdivision of $C(2,3)$, a contradiction. If $y = u_j$ for $j \notin \{i-1, i, i+1\}$, then the union of (u_{i-1}, x, y) and $h_1[u_{i-1}, y]$ is a subdivision of $C(2,3)$, a contradiction. $\lozenge$

This implies that $q = 1$. Indeed, if $q \geqslant 2$, then there is $i \leqslant m$ such that $b_2 \in S_i$. But $D - b_q = D_{q-1}$ is strong, and $\chi(D - b_q) \geqslant 5$, because $\chi(D) \geqslant 5$ and b_q has only two neighbours in D by Claim 41.3-(ii). But then by minimality of D, $D - b_q$ contains a subdivision of $C(2,3)$, which is also in D, a contradiction.

Hence $m = |V(D)|$. Because $\chi(D) \geqslant 5$, D is not outerplanar, so there must be $i < j < k < \ell < i + m$ such that $(u_i, u_k) \in A(D)$ and $(u_j, u_\ell) \in A(D)$. We must have $j = i + 1$ and $\ell = k + 1$ since otherwise $(u_i, \dots, u_j, u_\ell)$ and $(u_i, u_k, \dots, u_\ell)$ form a subdivision of $C(2,3)$. In addition, $k = j + 1$ since otherwise, $(u_j, u_\ell, \dots, u_i, u_k)$ and $(u_j, \dots, u_k)$ form a subdivision of $C(2,3)$. Therefore, any two "crossing" arcs must have their ends being consecutive in D_1. This implies that $N^+(u_j) = \{u_{j+1}, u_{j+2}\}$, $N^-(u_j) = \{u_{j-1}\}$, $N^+(u_k) = \{u_{k+1}\}$ and $N^-(u_k) = \{u_{k-1}, u_{k-2}\}$.

Now let D' be the digraph obtained from $D - \{u_j, u_k\}$ by adding the arc (u_i, u_ℓ). Because u_j and u_k have only three neighbours in D, $\chi(D') \geqslant 5$. By minimality of D, D' contains a subdivision of $C(2,3)$, which can be transformed into a subdivision of $C(2,3)$ in D by replacing the arc (u_i, u_ℓ) by the directed path $(u_i, u_j, u_{k,l})$. $\square$

Since every semi-complete digraph of order 4 does not contain $C(2,3)$ (which has order 5), we have the following.

Corollary 42. $\chi(\text{S-Forb}(C(2,3)) \cap \mathcal{S}) = 4$.

7 Cycles with four blocks in strong digraphs

Theorem 43. *Let D be a digraph in* S-Forb$(\hat{C}_4)$. *If D admits an out-generator, then* $\chi(D) \leqslant 24$.

Démonstration. The general idea is the same as in the proof of Theorem 23.

Suppose that D admits an out-generator u and let T be an BFS-tree with root u (See Subsubsection 5.1.1.). We partition $A(D)$ into three sets according to the levels of u.

$$\begin{aligned} A_0 &= \{(x,y) \in A(D) \mid \text{lvl}(x) = \text{lvl}(y)\}; \\ A_1 &= \{(x,y) \in A(D) \mid |\text{lvl}(x) - \text{lvl}(y)| = 1\}; \\ A_2 &= \{(x,y) \in A(D) \mid \text{lvl}(y) \leqslant \text{lvl}(x) - 2\}. \end{aligned}$$

For $i = 0, 1, 2$, let $D_i = (V(D), A_i)$.

Claim 43.1. $\chi(D_0) \leqslant 3$.

Subproof. Suppose for a contradiction that $\chi(D) \geqslant 4$. By Theorem 6, it contains a $P^-(1,1)$ (y_1, y, y_2), that is y, y_1 and y, y_2 are in $A(D_0)$. Let x be the least common ancestor of y_1 and y_2 in T. The union of $T[x, y_1]$, (y, y_1), (y, y_2), and $T[x, y_2]$ is a subdivision of $\hat{C}_4$, a contradiction. ♢

Claim 43.2. $\chi(D_1) \leqslant 2$.

Subproof. Since the arc are between consecutive levels, then the colouring ϕ_1 defined by $\phi_1(x) = lvl(x) \mod 2$ is a proper 2-colouring of D_1. ♢

Let $y \in V_i$ we denote by $N'(y)$ the out-degree of y in $\bigcup_{0 \leqslant j \leqslant i-1} V_j$. Let $D' = (V, A')$ with $A' = \cup_{x \in V}\{(x,y), y \in N'(x)\}$ and $D_x = (V, A_x)$ where A_x is the set of arc inside the level and from V_i to V_{i+1} for all i. Note that $A = A' \cup A_x$ and

Claim 43.3. $\chi(D_2) \leqslant 4$.

Subproof. Let x be a vertex of $V(D)$. If y and z are distinct out-neighbours of x in D_2, then their least common ancestor w is either y or z, for otherwise the union of $T[w, y]$, (x, y), (x, z), and $T[w, z]$ is a subdivision of $\hat{C}_4$. Consequently, there is an ordering $y_1, \dots, y_p$ of $N^+_{D_2}(y)$ such that the y_i appear in this order on $T[u, x]$.

Let us prove that $N^+(y_i) = \emptyset$ for $2 \leqslant i \leqslant p-1$. Suppose for a contradiction that y_i has an out-neighbour z in D_2. Let t be the least common ancestor of y_1 and z. If $t = z$, then the union of $(y_i, z) \odot T[z, y_1]$, (x, y_1), (x, y_p), and $T[y_i, y_p]$ is a subdivision of $\hat{C}_4$; if $t = y \neq z$, then the union of (y_i, z), $(x, y_1) \odot T[y_1, z]$, (x, y_p), and $T[y_i, y_p]$ is a subdivision of $\hat{C}_4$. Otherwise, if $t \notin \{y, z\}$, $T[t, y_1]$, $T[t, z]$, $(x, y_i) \odot (y_i, z)$ and (x, y_1) is a subdivision of $\hat{C}_4$.

Henceforth, in D_2, every vertex has at most two out-neighbours that are not sinks. Let V_0 be the set of sinks in D_2. It is a stable set in D_2. Furthermore $\Delta^+(D_2 - V_0) \leqslant 2$, so $D_2 - C$ is 3-colourable, because D_2 (and so $D_2 - V_0$) is acyclic. Therefore $\chi(D_2) \leqslant 4$. ♢

Claims 43.1, 43.2, 43.3, and Lemma 22 implies $\chi(D) \leqslant 24$. □

8 Further research

The upper bound of Theorem 23 can be lowered when considering 2-strong digraphs.

Theorem 44. *Let k and ℓ be two integers such that, $k \geqslant \ell$, $k+\ell \geqslant 4$ and $(k,\ell) \neq (2,2)$. Let D be a 2-strong digraph. If $\chi(D) \geqslant (k+\ell-2)(k-1)+2$, then D contains a subdivision of $C(k,\ell)$.*

Démonstration. Let D be a 2-strong digraph with chromatic number at least $(k+\ell-2)(k-1)+2$. Let u be a vertex of D. For every positive integer i, let $L_i = \{v \mid \mathrm{dist}_D(u,v) = i\}$.

Assume first that $L_k \neq \emptyset$. Take $v \in L_k$. In D, there are two internally disjoint (u,v)-dipaths P_1 and P_2. Those two dipaths have length at least k (and ℓ as well) since $\mathrm{dist}_D(u,v) \geqslant k$. Hence $P_1 \cup P_2$ is a subdivision of $C(k,\ell)$.

Therefore we may assume that L_k is empty, and so $V(D) = \{u\} \cup L_1 \cup \cdots \cup L_{k-1}$. Consequently, there is i such that $\chi(D[L_i]) \geqslant k+\ell-1$. Since $k+l-1 \geqslant 3$ and $(k-1,\ell-1) \neq (1,1)$, by Theorem 6, $D[L_i]$ contains a copy Q of $P^+(k-1,\ell-1)$. Let v_1 and v_2 be the initial and terminal vertices of Q. By definition, for $j \in \{1,2\}$, there is a (u,v_j)-dipath P_j in D such that $V(P_j) \cap L_i = \{v_j\}$. Let w be the last vertex along P_1 that is in $V(P_1) \cap V(P_2)$. Clearly, $P_1[w,v_1] \cup P_2[w,v_2] \cup Q$ is a subdivision of $C(k,\ell)$. □

To go further, it is natural to ask what happens if we consider digraphs which are not only strongly connected but k-strongly connected (k-strong for short).

Proposition 45. *Let C be an oriented cycle of order n. Every $(n-1)$-strong digraph contains a subdivision of C.*

Démonstration. Set $C = (v_1, v_2, \dots, v_n, v_1)$. Without loss of generality, we may assume that $(v_1, v_n) \in A(C)$. Let D be an $(n-1)$-strong digraph. Choose a vertex x_1 in $V(D)$. Then for $i = 2$ to n, choose a vertex x_i in $V(D) \setminus \{x_1, \dots, x_{i-1}\}$ such that $x_{i-1}x_i$ is an arc in D if $v_{i-1}v_i$ is an arc in C and $x_i x_{i-1}$ is an arc in D if $v_i v_{i-1}$ is an arc in C. This is possible since every vertex has in- and out-degree at least $n-1$. Now, since D is $(n-1)$-strong, $D - \{x_2, \dots, x_{n-1}\}$ is strong, so there exists a (x_1, x_n)-dipath P in $D - \{x_2, \dots, x_{n-1}\}$. The union of P and $(x_1, x_2, \dots, x_n)$ is a subdivision of C. □

Let $\mathcal{S}_p$ be the class of p-strong digraphs. Proposition 45 implies directly that S-Forb$(C) \cap \mathcal{S}_p = \emptyset$ and so $\chi(\text{S-Forb}(C) \cap \mathcal{S}_p) = 0$ for any oriented cycle C of length $p+1$. This yields the following problems.

Problem 46. Let C be an oriented cycle and p a positive integer. What is $\chi(\text{S-Forb}(C) \cap \mathcal{S}_p)$?

Note that $\chi(\text{S-Forb}(C) \cap \mathcal{S}_{p+1}) \leqslant \chi(\text{S-Forb}(C) \cap \mathcal{S}_p)$ for all p, because $\mathcal{S}_{p+1} \subseteq \mathcal{S}_p$.

Problem 47. Let C be an oriented cycle.

1) What is the minimum integer p_C such that $\chi(\text{S-Forb}(C) \cap \mathcal{S}_{p_C}) < +\infty$?
2) What is the minimum integer p_C^0 such that $\chi(\text{S-Forb}(C) \cap \mathcal{S}_{p_C^0}) = 0$?

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