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Ould Ahmed Izid Bih Isselkou

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Université de Nouakchott, B.P. 798 Mauritanie,
isselkou@univ-nkc.mr

ABSTRACT. It is quite usual to transform elliptic PDE problems of second order into fixed point integral problems, via the Green's function. But it is not easy, in general, to handle integrals involved in such a formulation. When it comes to the Laplacian operator on balls of $\Re^n$, we give here a radially symmetrical Green's function which, under some nonlinearity assumptions, makes the Green's Integral representation formula easier to use; we give three examples of application.

RÉSUMÉ. Il est courant de transformer un problème, donné sous forme d'EDP elliptique de second ordre, en un problème intégral de point fixe, et ce en utilisant la fonction de Green. En général, les intégrales intervenant dans une telle formulation, sont de maniement difficile. Lorsqu'il s'agit de l'opérateur du Laplacien sur des boules de $\Re^n$, nous montrons l'existence d'une fonction de Green à symétrie radiale; elle permet, moyennant des hypothèses adéquates sur la non linéarité, de faciliter l'usage de la Formule de représentation de Green; nous donnons trois exemples d'application.

KEYWORDS : Green's function, radially symmetrical, representation formula.

MOTS-CLÉS : Fonction de Green, symétrie radiale, Formule de réprésentation.

1. Introduction

Let us put $B_r = \{x \in \Re^n, \|x\| < r\}$ and

$$\Gamma(r) = \begin{cases} \frac{1}{nw_n(2-n)} r^{2-n},\ if\ n \geq 3, \\ \frac{1}{2\pi}\log(r),\ if\ n = 2 \\ \frac{1}{2}r,\ if\ n = 1, \end{cases}$$

where w_n is the Lebesgue measure of the unit ball B_1.
The classical Green's function(cf. [5]) for the Laplacian, on B_1, is
$G(x,y) = \Gamma(\|x-y\|) - \Gamma(\|\|y\|x - \frac{y}{\|y\|}\|),\ if\ y \neq O$
$G(x,O) = \Gamma(\|x\|) - \Gamma(1)$.
Let us put

$$I_1(x,y) = G(x,O)\chi_{\{\|y\|<\|x\|\}}(y) + G(O,y)\chi_{\{\|y\|>\|x\|\}}(y),$$

where the function χ is the characteristic one. We get

$$\Delta_y I_1(x,.) = \delta_x,\ on\ \mathcal{D}_{rad}(B_1),$$

where $\mathcal{D}_{rad}(B_1)$ are radially symmetrical test functions and δ_x is the Dirac's distribution at the point x.
As examples of application, we deal with the Helmholtz's problem, Talenti's formula and the Lane-Emden function.

2. The radially symmetrical Green's function

Definition 2.1 *Let $\mathcal{D}(B_1)$ be the set of functions which are $\mathcal{C}^\infty$, with compact supports in B_1 and $\mathcal{D}_{rad}(B_1)$, the set of functions of $\mathcal{D}(B_1)$, which are radially symmetrical, with respect to the origin.*

Remark 2.2 $\forall x \in B_1,\ I_1(x,.) \in L^1(B_1) \subset \mathcal{D}'(B_1)$, *where $\mathcal{D}'(B_1)$ is the Schwartz distributions set over B_1.*

For every $\varphi \in \mathcal{D}(B_1)$, one defines

$$M(\varphi)(x) = \frac{1}{|\partial B_{\|x\|}|}\int_{\partial B_{\|x\|}} \varphi(x)dS(x),\ M(\varphi)(O) = \varphi(O),$$

$M(\varphi)$ is the mean of φ, over the sphere $\partial B_{\|x\|} = \{y \in B_1,\ \|y\| = \|x\|\}$.

Proposition 2.3 $\forall x \in B_1,\ \forall \varphi \in\ \mathcal{D}(B_1), \forall \phi \in\ \mathcal{D}_{rad}(B_1),$

$$< \Delta_y I_1(x,.), \varphi >= M(\varphi)(x),\ < \Delta_y I_1(x,.), \phi >=< \delta_x, \phi >,$$

where δ_x is the Dirac distribution at the point x.

Proof.
If $n \geq 3, \forall x \in B_1$, we have

$$< \Delta_y I(x,.), \varphi >=$$

$$c_n \left\{ (\|x\|^{2-n} - 1) \int_{B_{\|x\|}} \Delta\varphi(y) dy + \int_{B_1 - \overline{B}_{\|x\|}} (\|y\|^{2-n} - 1) \Delta\varphi(y) dy \right\}.$$

If $x = O$, we have only the second integral, $c_n \int_{B_1} (\|y\|^{2-n} - 1)\Delta\varphi(y)dy$. As $\Delta_y \left(c_n \left(\|y\|^{2-n} - 1\right)\right) = \delta_O$, we get

$$c_n \int_{B_1} (\|y\|^{2-n} - 1)\Delta\varphi(y)dy =< c_n \Delta(\|y\|^{2-n} - 1), \varphi >= c_n < \Delta \left(\|y\|^{2-n} - 1\right), \varphi >$$

$$=< \delta_O, \varphi >= \varphi(O).$$

Let us put $\psi(y) = \|y\|^{2-n} - 1$; $\Delta\psi = 0$, in $B_1 - \overline{B}_{\|x\|}$. Using the Divergence Theorem and the Green's Identity, we get

$$< \Delta_y I_1(x,.), \varphi >=$$

$$c_n \left\{ (\|x\|^{2-n} - 1) \int_{\partial B_{\|x\|}} \frac{\partial \varphi(y)}{\partial \nu} ds + \int_{\partial\left(B_1 - \overline{B}_{\|x\|}\right)} \left(\psi(y) \frac{\partial \varphi}{\partial \nu}(y) - \varphi(y) \frac{\partial \psi}{\partial \nu}(y) \right) ds \right\},$$

where ν is the outer normal. Using the fact that $\psi(y) = \varphi(y) = 0,\ if\ y \in \partial B_1$, we get

$$< \Delta_y I_1(x,.), \varphi >= c_n \left\{ (\|x\|^{2-n} - 1) \int_{\partial B_{\|x\|}} \frac{\partial \varphi(y)}{\partial \nu} ds \right.$$

$$\left. - \int_{\partial B_{\|x\|}} \left(\psi(y) \frac{\partial \varphi}{\partial \nu}(y) - \varphi(y) \frac{\partial \psi}{\partial \nu}(y) \right\} ds$$

As $\forall y \in \partial B_{\|x\|}$, $\psi(y) = \|x\|^{2-n} - 1$ and $\frac{\partial \psi}{\partial \nu}(y) = (2-n)\|x\|^{1-n}$, we obtain

$$< \Delta_y I_1(x,.), \varphi >= c_n \int_{\partial B_{\|x\|}} \varphi(y)(2-n)\|x\|^{1-n} ds = c_n (2-n)\|x\|^{1-n} \int_{\partial B_{\|x\|}} \varphi(y) ds$$

$$= M(\varphi)(x).$$

If ϕ is radially symmetrical, we infer that

$$< \Delta_y I(x,.), \phi >= \phi(\|x\|) c_n \|x\|^{1-n} (2-n) |\partial B_{\|x\|}| = \phi(\|x\|).$$

If $n = 2$, then, as in the case $n \geq 3$, we use the Divergence Theorem, the Green's Identity and set $\psi(y) = \log\|y\|$. We get

$$< \Delta_y I_1(x,.), \varphi >= \frac{1}{2\pi} \left(log\|x\| \int_{B_{\|x\|}} \Delta\varphi(y) dy + \int_{\overline{B_1 - B_{\|x\|}}} \psi(y) \Delta\varphi(y) dy \right)$$

$$= \frac{1}{2\pi} \left(log\|x\| \int_{\partial B_{\|x\|}} \frac{\partial \varphi}{\partial \nu}(y) ds - \int_{\partial B_{\|x\|}} \psi(y) \frac{\partial \varphi}{\partial \nu}(y) ds + \int_{\partial B_{\|x\|}} \varphi(y) \frac{\partial \psi}{\partial \nu}(y) ds \right)$$

$$= \frac{1}{2\pi}\frac{1}{\|x\|}\int_{\partial B_{\|x\|}} \varphi(y)ds = M(\varphi)(x).$$

If $\phi \in D_{rad}(B_1)$, we obtain

$$< \Delta_y I_1(x,.), \phi >= M(\phi)(x) = \frac{1}{2\pi}\frac{1}{\|x\|}|\partial B_{\|x\|}|\phi(\|x\|) = \phi(\|x\|).$$

If $n = 1$, let φ be an even (radially symmetrical) function, which belongs to $\mathcal{D}\left(]-1,1[\right)$, then

$$\begin{array}{rcl} < \Delta_y I_1(t,.), \varphi > & = & < I_1(t,.), \varphi" > \\ & = & \frac{1}{2}\left((|t|-1)\int_{-|t|}^{|t|}\varphi"(s)ds + \int_{|s|>|t|}(|s|-1)\varphi"(s)ds\right) \\ & = & \frac{1}{2}\left((|t|-1)\int_{-|t|}^{|t|}\varphi"(s)ds - \int_{-1}^{-|t|}(s+1)\varphi"(s)ds\right. \\ & + & \left.\int_{|t|}^{1}(s-1)\varphi"(s)ds\right) \\ & = & \frac{1}{2}\left((|t|-1)\varphi'(s)]_{-|t|}^{|t|} - (s+1)\varphi'(s)]_{-1}^{-|t|}\right. \\ & + & \left.(s-1)\varphi'(s)]_{|t|}^{1} + \int_{-1}^{-|t|}\varphi'(s)ds - \int_{|t|}^{1}\varphi'(s)ds\right) \end{array}$$

As $\varphi(-1) = \varphi(1) = 0$, we get

$$\begin{aligned} < \Delta_y I_1(t,.), \varphi >= \frac{1}{2}\left((|t|-1)\left(\varphi'(|t|) - \varphi'(-|t|)\right) - (-|t|+1)\varphi'(-|t|)\right) \\ +\frac{1}{2}\left(-(|t|-1)\varphi'(|t|) + \varphi(-|t|) + \varphi(|t|)\right)) \\ = \frac{1}{2}\left(\varphi(-|t|) + \varphi(|t|)\right) = M(\varphi)(t). \end{aligned}$$

We then get

$$< \Delta_y I_1(t,.), \phi >= M(\phi)(t) = \phi(|t|).$$

If one considers a ball B_r of radius $r > 0$, instead of B_1, and defines

$$I_r(x,y) = c_n\left((\|x\|^{2-n} - r^{2-n})\chi_{B_{\|x\|}}(y) + (\|y\|^{2-n} - r^{2-n})\chi_{B_r - \overline{B}_{\|x\|}}(y)\right),\ for\, n \geq 3,$$

$$I_r(x,y) = \frac{1}{2\pi}\left((\log\|x\| - \log r)\chi_{B_{\|x\|}}(y) + (\log\|y\| - \log r)\chi_{B_r - \overline{B}_{\|x\|}}(y)\right),\ for\, n = 2,$$

$$I_r(x,y) = \frac{1}{2}\left((|x| - r)\chi_{\{|y|<|x|\}}(y) + (|y| - r)\chi_{\{|x|<|y|\}}(y)\right),\ for\, n = 1,$$

we obtain,

Proposition 2.4 $\forall\varphi \in D(B_r)$ *and* $\forall\phi \in D_{rad}(B_r)$,

$$< \Delta_y I_r(x,.), \varphi >= M(\varphi)(x)\ and\ < \Delta_y I_r(x,.), \phi >=< \delta_x, \phi > .$$

Proof.
The same proof, as in the unit ball case(cf. Proposition 1).

Remark 2.5 *Since* $I_r(x,y) = I_r(\|x\|, \|y\|)$, *we have called it a Radially Symmetrical Green's Function.*

3. Applications

Let us give three examples of applications.

3.1. The Helmholtz's problem

Let us suppose $n \geq 3$ and look for $(u,\ \lambda > 0)$, solution of

$$(*)\begin{cases} -\Delta u(x) = \lambda u(x),\ in\ B_1 \subset \Re^n, \\ u = 0\ on\ \partial B_1, \end{cases}$$

We put

$$v(r) = a_0 \Sigma_{k=0}^{\infty} (-1)^k \frac{1}{2^k k! \Pi_{j=0}^{k-1}(n+2j)} r^{2k}.$$

Proposition 3.1 *The problem $(*)$ admits the analytical solutions $u_\lambda(r) = v(\sqrt{\lambda} r)$, where λ is any positive "zero" of v.*

$Proof$.
One can use results from [4], to see that every regular solution of (*) is radially symmetrical, with respect to the origin. Using the polar coordinates, one can rewrite the problem (*) as following

$$\begin{cases} -\left(u"(r) + \frac{2}{r}u'(r)\right) = \lambda u(r), r\ in\ [0,1[,\ \lambda > 0 \\ u(1) = 0. \end{cases}$$

Using Frobenius's theorem, one infers the previous problem admits a analytical solution v near the origin. Let us suppose the convergence radius of v greater than 1, using I_1 instead of the classical Green's function, we get that any solution of (*) can be written as follows

$$u(r) = -\lambda \int_{B_1} I_1(x,y) u(y) dy$$

$$= -\lambda c_n \left(\int_{\|x\| > \|y\|} (\|x\|^{2-n} - 1) u(\|y\|) dy + \int_{\|x\| < \|y\|} (\|y\|^{2-n} - 1) u(\|y\|) dy \right).$$

Using polar coordinates, we get

$$u(r) = -\frac{\lambda}{2-n} \left((r^{2-n} - 1) \int_0^{\|x\|} s^{n-1} u(s) ds + \int_{\|x\|}^1 (s^{2-n} - 1) s^{n-1} u(s) ds \right).$$

Replacing $u(s)$ by the unknown series, $\Sigma_{k=0}^{\infty} a_k r^k$, interchanging the signs $\int$ and Σ and performing standard computations, one obtains

$$a_{2k+1} = 0\ and\ a_{2k} = a_0 (-1)^k \lambda^k \frac{1}{2^k k! \Pi_{j=0}^{k-1}(n+2j)},\ \forall k > 1.$$

So we get $u(r) = v(\sqrt{\lambda} r)$ and this ends the proof.

Remark 3.2 *If $n = 3$, as $2^k k! \Pi_{j=0}^{k-1}(n+2j) = (2k+1)!$, we get the wellknown result:*

$$u_\lambda(r) = a_0 \frac{sin\left(\sqrt{\lambda}r\right)}{\sqrt{\lambda}r} \; and \; \lambda = j^2\pi^2, j > 1.$$

If $n = 4$, as $2^k k! \Pi_{j=0}^{k-1}(n+2j) = 2^{2k} k!(k+1)!$, another wellknown result,

$$u_\lambda(r) = \frac{a_0}{\sqrt{\lambda}r} J_1\left(\sqrt{\lambda}r\right),$$

where λ is any positive zero of Bessel's first kind function J_1. If we use the "fzero" subroutine in Matlab, the first zero of J_1 is about 3.8317.

3.2. Talenti's formula

The Talenti's formula gives an isoperimetric inequality for solutions of rearranged linear elliptic problems (cf. [13]).
Let us consider the semilinear elliptic problem

$$(1) \begin{cases} \Delta u(x) + g(\|x\|, u(x)) = 0, \; in \; B_1 \subset \Re^3, \\ u = 0 \; on \; \partial B_1, \end{cases}$$

where g, say, is continuous. Let u be a radially symmetrical nonnegative solution(cf.[4]) of (1), we have

Proposition 3.3

$$u(x) = \frac{1}{n^2 w_n^{\frac{2}{n}}} \int_{w_n\|x\|^n}^{w_n} t^{\frac{2}{n}-2} \int_0^t g\left(\left(\frac{r}{w_n}\right)^{\frac{1}{n}}, u\left(\left(\frac{r}{w_n}\right)^{\frac{1}{n}}\right)\right) drdt.$$

$Proof.$
Replacing the classical Green's function G (for Δ, on B_1), by the new one I_1, and using again the Green's Representation Formula, we get

$$u(x) = -\int_{B_1} I_1(x,y) g(y, u(y)) dy$$

$$= c_n \left\{ (1 - \|x\|^{2-n}) \int_{B_{\|x\|}} g(y, u(y)dy + \int_{B_1 - \overline{B_{\|x\|}}} (1 - \|y\|^{2-n}) g(y, u(y)) dt \right\}$$

$$= \frac{1}{2-n} \left\{ (1 - \|x\|^{2-n}) \int_0^{\|x\|} r^{n-1} g(r, u(r)dr + \int_{\|x\|}^1 r^{n-1}(1 - r^{2-n}) g(r, u(r)) dr \right\}$$

$$= \int_{|x\|}^1 s^{1-n} ds \int_0^{\|x\|} r^{n-1} g(r, u(r)dr + \int_{\|x\|}^1 r^{n-1} \int_r^1 s^{1-n} ds \; g(r, u(r)) dr.$$

Interchanging the order of integration(by Fubini's Theorem), we get

$$u(x) = \int_{\|x\|}^1 s^{1-n} \int_0^s r^{n-1} g(r, u(r)) drds.$$

If we put $w_n r^n = t$ (and use r again), we get

$$u(x) = \frac{1}{nw_n} \int_{\|x\|}^{1} s^{1-n} \int_0^{w_n s^n} g\left((\frac{r}{w_n})^{\frac{1}{n}}, u\left((\frac{r}{w_n})^{\frac{1}{n}} \right) \right) drds.$$

Putting $t = w_n s^n$ (and using s again), we get

$$u(x) = \frac{1}{n^2 w_n^{\frac{2}{n}}} \int_{w_n \|x\|^n}^{w_n} s^{\frac{2}{n}-2} \int_0^{s} g\left((\frac{r}{w_n})^{\frac{1}{n}}, u\left((\frac{r}{w_n})^{\frac{1}{n}} \right) \right) drds.$$

Let us now suppose $g(r,u)$ is a nonnegative function, decreasing with respect to the first variable and increasing with respect to the second one or $g(r,u) = f(r)$, where f is a nonnegative decreasing function. We also suppose $g(r,0) \neq 0$, then we get Talenti's formula

Corollary 3.4 *For every nonnegative solution u of (1), we have*

$$u(x) = \frac{1}{n^2 w_n^{\frac{2}{n}}} \int_{|u>u(x)|}^{|B_1|} t^{\frac{2}{n}-2} F(t) dt,$$

where $F(t) = \int_0^t h_*(s)ds$, h_* is the decreasing rearrangement of the function $h(r) = g(r, u(r))$ and $w_n = |B_1|$.

$Proof.$
Using the Maximum Principle and [4], we see that any nonnegative solution u of (1) is a radially decreasing function and $h(r) = g(r, u(r))$ is also a decreasing function. The standard properties of the rearrangement(cf [13]) give us

$$\int_0^r h\left(\left(\frac{s}{w_n} \right)^{\frac{1}{n}} \right) dr = \int_0^r h_*(s) ds.$$

As u is radially decreasing, we get $w_n \|x\|^n = |u > u(x)|$. So one can use the previous Proposition to end the proof.

3.3. The Lane-Emden function

Let us consider the semilinear problem,

$$(P_\lambda) \begin{cases} \Delta u + \lambda(1+u)^2 = 0 \; in \; B_1 \subset \Re^3, \\ u \geq 0, \; in \; B_1 \\ u = 0 \; on \; \partial B_1. \end{cases}$$

This Problem is a particular case of a class of mathematical models (cf.[3]). It is known (cf. [11],[3], [1],[8], [10]) that there exists a critical eigenvalue $\lambda^*(2)$, such that (P_λ) admits two solutions if $0 < \lambda < \lambda^*(2)$, one solution if $\lambda = \lambda^*(2)$ and no solution if $\lambda > \lambda^*(2)$. The minimal solution is analytical(cf. [12],[10], [9]). Let u be a solution of (P_λ), then u is a radially symmetrical decreasing function (cf. [4]).
Let φ be the Lane-Emden function (cf.[2],[6],[7] and [14]), solution of

$$(E) \begin{cases} v"(r) + 2v'(r) + rv^2(r) = 0 \; , \\ v(0) = 1, v'(0) = 0. \end{cases}$$

Proposition 3.5 *Let us suppose $0 < \lambda \leq \lambda^*(2)$, if u is the minimal solution of (P_λ), then*

$$u(r) = \frac{c}{\lambda}\varphi(r\sqrt{c}) - 1,$$

where c is such that $\lambda = c\varphi(\sqrt{c})$.

$Proof.$
Let $u(r) = \Sigma_{i=0}^{\infty} a_i r^i$, be the minimal solution (near 0) of (P_λ), then using I_1 instead of G, we get

$$\Sigma_{i=0}^{\infty} a_i r^i = \lambda\left((r^{-1} - 1)\int_0^r t^2 \Sigma_{j=0}^{\infty} c_j t^j dt + \int_r^1 t^2(t^{-1} - 1)\Sigma_{j=0}^{\infty} c_j t^j dt\right),$$

where $\Sigma_{j=0}^{\infty} c_j r^j = (1 + u(r))^2$. We get

$$\begin{aligned}
\Sigma_{i=0}^{\infty} a_i r^i &= \lambda\left((r^{-1} - 1)\int_0^r t^2 \Sigma_{j=0}^{\infty} c_j t^j dt + \int_r^1 (t - t^2)\Sigma_{j=0}^{\infty} c_j t^j dt\right) \\
&= \lambda\left(\Sigma_{j=0}^{\infty} c_j \frac{r^{2+j}}{3+j} - \Sigma_{j=0}^{\infty} c_j \frac{r^{3+j}}{3+j} + \Sigma_{j=0}^{\infty} \frac{c_j}{j+2} - \Sigma_{j=0}^{\infty} \frac{c_j}{3+j}\right. \\
&\qquad \left. - \Sigma_{j=0}^{\infty} c_j \frac{r^{2+j}}{2+j} + \Sigma_{j=0}^{\infty} c_j \frac{r^{3+j}}{3+j}\right) \\
&= \lambda\left(\Sigma_{j=2}^{\infty} c_{j-2} \frac{r^j}{1+j} + \Sigma_{j=0}^{\infty} \frac{c_j}{j+2} - \Sigma_{j=0}^{\infty} \frac{c_j}{3+j} - \Sigma_{j=2}^{\infty} c_{j-2} \frac{r^j}{j}\right) \\
&= \lambda\left(\Sigma_{j=2}^{\infty} c_{j-2} r^j \left(\frac{1}{1+j} - \frac{1}{j}\right) + \Sigma_{j=0}^{\infty} c_j \left(\frac{1}{2+j} - \frac{1}{j+3}\right)\right), \\
c_j &= \Sigma_{k=0}^{j} a'_k a'_{j-k},\ with\ a'_i = a_i, \forall i \neq 0\ and\ a'_0 = 1 + a_0.
\end{aligned}$$

We infer that,

$$a_0 = \Sigma_{j=0}^{\infty} c_j \left(\frac{1}{2+j} - \frac{1}{j+3}\right),\ a_1 = 0\ and\ a_j = \lambda\left(\frac{1}{1+j} - \frac{1}{j}\right) c_{j-2},\ if\ j > 2.$$

Let us put,

$$d_0 = 1,\ d_{2i} = \left(\frac{1}{2i+1} - \frac{1}{2i}\right)\Sigma_{k=0}^{i-1} d_{2k} d_{2i-2-2k},\ and\ d_{2i+1} = 0,\ \forall i \geq 1. \tag{1}$$

By induction, we see that

$$a_{2i} = d_{2i}\lambda^i (1 + a_0)^{i+1}\ and\ a_{2i+1} = 0,\ \forall\, i \geq 1.$$

Let us put $\varphi(r) = \Sigma_{i=0}^{\infty} a_{2i} r^{2i}$, one can use (1) to infer that

$$\varphi^2(r) = -\frac{1}{r}(r\varphi(r))''.$$

As $n = 3$, we have

$$\frac{1}{r}(r\varphi(r))'' = \Delta\varphi(r).$$

So we infer that φ is the Lane-Emden function, for $\varphi(0) = d_0 = 1,\ \varphi'(0) = d_1 = 0$. We also get

$$u(r) = \Sigma_{i=0}^{\infty} a_i r^i = a_0 + \Sigma_{i=1}^{\infty} d_{2i} \lambda^i (1 + a_0)^{i+1} r^{2i}$$
$$= a_0 + 1 + \Sigma_{i=1}^{\infty} d_{2i} \lambda^i (1 + a_0)^{i+1} r^{2i} - 1$$
$$= \frac{1}{\lambda} \lambda (1 + a_0) \Sigma_{i=0}^{\infty} d_{2i} \lambda^i (1 + a_0)^i r^{2i} - 1 = \frac{1}{\lambda} c\varphi(r\sqrt{c}) - 1,$$

where $c = \lambda(1 + a_0)$. We have $u(1) = \frac{c}{\lambda}\varphi(\sqrt{c}) - 1 = 0 \ or \ \lambda = c\varphi(\sqrt{c})$.

Proposition 3.6 *The power series*

$$\Sigma_{i=0}^{\infty} d_{2i} r^{2i},$$

is alternating and has a radius of convergence $r_0 \geq 1$.

$Proof.$
$d_0 = 1 \ and \ d_2 = -\frac{1}{6}$, we will infer that $d_{2i} = (-1)^i |d_{2i}|$. Let us suppose the previous relation is true for every $i \in [0\ n]$, then

$$d_{2(n+1)} = -\frac{1}{(2(n+1)+1)(2n)} \Sigma_{i=0}^{n} d_{2i} d_{2(n+1-i)}.$$

Using the recurrence hypothesis, we get

$$d_{2(n+1)} = -(-1)^n \frac{1}{(2(n+1)+1)(2n)} \Sigma_{i=0}^{n} |d_{2i}||d_{2(n-i)}|,$$

which gives the conclusion.
Let us now show that

$$\lim_{n\to\infty} d_{2n} = 0 \ and \ |d_{2(n+1)}| < |d_{2n}|.$$

It is immediate to see(by induction) that $|d_{2n}| \leq 1$. So we get

$$|d_{2(n+1)}| = \frac{1}{(2(n+1)+1)(2n+2)} \Sigma_{i=0}^{n} |d_{2i}||d_{2(n+1-i)}|$$
$$\leq \frac{1}{(2(n+1)+1)(2n+2)} (n+1)$$
$$\leq \frac{1}{2(2(n+1)+1)} \to 0, \ as \ n \to \infty.$$

We have, $|d_2| = \frac{1}{6} < d_0 = 1$, let us suppose that

$$|d_{2(i+1)}| < |d_{2i}|, \ \forall\, 0 \leq i \leq n.$$

$$|d_{2n}| = \frac{1}{(2n+1)(2n)} \Sigma_{i=0}^{n-1} |d_{2i}||d_{2(n-1-i)}|.$$

Using the recurrence hypothesis ($|d_{2(n-1-i)}| \geq |d_{2(n-i)}|$), we get

$$|d_{2n}| \geq \frac{1}{(2n+1)(2n)} \Sigma_{i=0}^{n-1} |d_{2i}||d_{2(n-i)}|$$

$$= \frac{1}{(2n+1)(2n)} \left((2n+3)(2(n+1))|d_{2(n+1)}| - |d_{2n}|\right).$$

So we infer that

$$|d_{2(n+1)}| \leq \frac{4n^2+2n+1}{4n^2+10n+6}|d_{2n}| < |d_{2n}|.$$

We deduce from the previous inequality that $r_0 \geq 1$. We have

$$\frac{1}{r_0} = \limsup_{n\to\infty} |d_{2n}|^{\frac{1}{2n}} = \lim_{n\to\infty} |d_{2n}|^{\frac{1}{2n}}.$$

Let us consider the problem,

$$(P) \begin{cases} \Delta v + v^2 = 0 \; in \; B_1, \\ v > 0 \; in \; B_1 \\ v = 0 \; on \; \partial B_1, \end{cases}$$

Proposition 3.7 *Let u be the unique solution of (P), then $\sqrt{u(0)}$ is the first zero of φ.*

$Proof$.

Let u be the unique solution of (P) and let us define $v(x) = \frac{1}{u(0)} u\left(\frac{x}{\sqrt{u(0)}}\right)$. As the function u is radially symmetrical and analytical on $[0,1]$, so is v on $[0, \sqrt{u(0)}]$. The Maximum Principle implies that,
$v(r) > 0 forall\ 0 \leq r < \sqrt{u(0)}$, $v\left(\sqrt{u(0)}\right) = \frac{1}{u(0)}u(1) = 0$. As $u'(0) = 0$, we get $v'(0) = 0$. As $v(0) = 1$ and v verifies

$$\Delta v(x) + v^2(x) = 0,$$

we get

$$v(r) = \varphi(r), \; \forall\, 0 \leq r \leq \sqrt{u(0)}.$$

4. Numerical Computations

Let us call ξ_1, the first zero of φ. From a modelling point of view, the constants ξ_1 and r_0 are important, numerical estimations of these constants exist (cf. ([2], p. 95) and ([7], p. 360)).
As

$$r_0 = \lim_{n\to\infty} |d_{2n}|^{\frac{-1}{2n}},$$

we have computed (using Maple) some values of $|d_{2n}|^{\frac{-1}{2n}}$ and obtained

n	$\|d_{2n}\|^{\frac{-1}{2n}}$
200	3.908246029
300	3.924291150
400	3.932914538
500	3.938350910
600	3.942113057
700	3.944881693
800	3.947010199
900	3.948701028
1000	3.950078638
1100	3.951224204

In order to get an estimation of $\lambda^*(2)$, one can use Proposition 5, plot the curve $r \to r^2\varphi(r)$, on the interval $[0, \xi_1]$ and get its maximum.

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