Continuous branching processes : the discrete hidden in the continuous

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Dedicated to Claude Lobry

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ABSTRACT. Feller diffusion is a continuous branching process. The branching property tells us that for $t > 0$ fixed, when indexed by the initial condition, it is a subordinator (i. e. a positive–valued Lévy process), which is fact is a compound Poisson process. The number of points of this Poisson process can be interpreted as the number of individuals whose progeny survives during a number of generations of the order of $t \times N$, where N denotes the size of the population, in the limit $N \to \infty$. This fact follows from recent results of Bertoin, Fontbona, Martinez [1]. We compare them with older results of de O'Connell [7] and [8]. We believe that this comparison is useful for better understanding these results. There is no new result in this presentation.

RÉSUMÉ. La diffusion de Feller est un processus de branchement continu. La propriété de branchement nous dit que à $t > 0$ fixé, indexé par la condition initiale, ce processus est un subordinateur (processus de Lévy à valeurs positives), qui est en fait un processus de Poisson composé. Le nombre de points de ce processus de Poisson s'interprète comme le nombre d'individus dont la descendance survit au cours d'un nombre de générations de l'ordre de $t \times N$, où N désigne la taille de la population, dans la limite $N \to \infty$. Ce fait découle de résultats récents de Bertoin, Fontbona, Martinez [1]. Nous le rapprochons de résultats plus anciens de O'Connell [7] et [8]. Ce rapprochement nous semble aider à mieux comprendre ces résultats. Cet article ne contient pas de résultat nouveau.

KEYWORDS : Continuous branching, Immortal individuals, Bienaymé–Galton–Watson processes, Lévy processes, Feller diffusion

MOTS-CLÉS : Branchement continu, Individus à progéniture immortelle, Processus de Bienaymé–Galton–Watson, Processus de Lévy, Diffusion de Feller

1. Introduction

Consider the simplest continuous branching process, i. e. a Feller diffusion :

$$X_t(x) = x + \alpha \int_0^t X_s(x)ds + \int_0^t \sqrt{\beta X_s} dB_s,$$

where $\{B_t,\ t \geq 0\}$ is a standard Brownian motion and $x > 0$. It is easy to check (see section 5 below) that this process possesses the branching property, namely

$$X_t(x+y) \overset{(d)}{=} X_t(x) + X'_t(y),$$

where $X'_t(y)$ is a copy of $X_t(y)$ which is independent of $X_t(x)$. One can in fact construct (see Lemma 5.2 below) a version of the two parameter process $\{X_t(x),\ x \geq 0,\ t \geq 0\}$ such for all $x, y > 0$, $X_t(x+y) - X_t(x)$ is independent of $\{X_t(z),\ 0 \leq z \leq x\}$. In other words, for each fixed $t > 0$, $x \to X_t(x)$ is an $\mathbb{R}_+$–valued process with independent and stationary increments. It is in fact a compound Poisson process, i. e. the sum of a finite number of jumps (see section 7). At the same time, for fixed $x > 0$, $t \to X_t(x)$ is a continuous process.

It is well known that, as stated in section 4, $X_t(x)$ is the limit as $N \to \infty$ of $X_t^N := N^{-1}Z^N_{[Nt]}$, where for each $N, k \in \mathbb{N}$, Z_k^N denotes the size of the k–th generation of a population starting with $Z_0^N = [Nx]$, and the offsprings of the various individuals are i. i. d., with mean $1 + \alpha/N$ and variance β. Because of the division by N, $X_t(x)$ does not count individuals. This would be impossible, since in the limit there are intuitively constantly infinitely many individuals which are born, and infinitely many who die.

However, if we consider in the population $Z^N_\cdot$ the number of those individuals in the generation 0 whose progeny is still alive in generation $[Nt]$, that number does not explode (due to the fact that the mean number of offsprings per individual is $1+\alpha/N$, and that we look at a generation of order N), and converges precisely to the number of jumps of the Levy process (called also a subordinator, since it is increasing) $\{y \to X_t(y),\ 0 < y < x\}$, while the contribution of the progeny of each of those in the population X_t^N converges to the size of the corresponding jump. This means that if consider only those individuals whose progeny lives long enough, we should not divided the number of individuals by N, if we want a non trivial limit. The same is true for the number of those individuals whose progeny never goes extinct.

The aim of this article is to explain those points in detail, by presenting both fifteen years old results by Neil O'Connell (see [7] and [8]), and very recent ones by Jean Bertoin, Joaquim Fontbona and Servet Martinez (see [1]). There is no new result in this paper, and in particular nothing is due to the author, except for the way things are presented.

2. Bienaymé–Galton–Watson processes

Consider a Bienaymé–Galton–Watson process, i. e. a process $\{Z_n,\ n \geq 0\}$ with values in $\mathbb{N}$ such that

$$Z_{n+1} = \sum_{k=1}^{Z_n} \xi_{n,k},$$

where $\{\xi_{n,k},\ n \geq 0, k \geq 1\}$ are i. i. d. r. v.'s with as joint law that of ξ whose generating function f satisfies

$$\mu := \mathbb{E}[\xi] = f'(1) = 1 + r, \text{ and } 0 < q := f(0) = \mathbb{P}(\xi = 0) < 1.$$

We call f the probability generating function (p. g. f. in short) of the Bienaymé–Galton–Watson process $\{Z_n,\ n \geq 0\}$. In order to exclude trivial situations, we assume that $\mathbb{P}(\xi = 0) = f(0) > 0$, and that $\mathbb{P}(\xi > 1) > 0$. This last condition implies that $s \to f(s)$, which is increasing on $[0, 1]$, is a strictly convex function.

The process is said to be *subcritical* if $\mu < 1$ ($r < 0$), **critical** if $\mu = 1$ ($r = 0$), and *supercritical* if $\mu > 1$ ($r > 0$). We shall essentially be interested in the supercritical case.

First note that the process $\{Z_n,\ n \geq 0\}$ is a Markov process, which has the so–called branching property, which we now formulate. For $x \in \mathbb{N}$, let $\mathbb{P}_x$ denote the law of the Markov process $\{Z_n,\ n \geq 0\}$ starting from $Z_0 = x$. The law of $\{Z_n,\ n \geq 0\}$ under $\mathbb{P}_{x+y}$ is the same as that of the sum of two independent copies of $\{Z_n,\ n \geq 0\}$, one having the law $\mathbb{P}_x$, the other the law $\mathbb{P}_y$.

We next define

$$T = \inf\{k > 0; Z_k = 0\},$$

which is the time of extinction. We first recall the

Proposition 2.1 *Assume that $Z_0 = 1$. Then the probability of extinction $\mathbb{P}(T < \infty)$ is one in the subcritical and the critical cases, and it is the unique root $\eta < 1$ of the equation $f(s) = s$ in the supercritical case.*

PROOF. Let $f^{\circ n}(s) := f \circ \cdots \circ f(s)$, where f has been composed n times with itself. It is easy to check that $f^{\circ n}$ is the generating function of the r. v. Z_n.

On the other hand, clearly $\{T \leq n\} = \{Z_n = 0\}$. Consequently

$$\begin{aligned} \mathbb{P}(T < \infty) &= \lim_n \mathbb{P}(T \leq n) \\ &= \lim_n \mathbb{P}(Z_n = 0) \\ &= \lim_n f^{\circ n}(0). \end{aligned}$$

Now the function $s \to f(s)$ is continuous, increasing and strictly convex, starts from $q > 0$ at $s = 0$, and ends at 1 at $s = 1$. If $\mu = f'(1) \leq 1$, then $\lim_n f^{\circ n}(0) = 1$. If however $f'(1) = 1 + r > 1$, then there exists a unique $0 < \eta < 1$ such that $f(\eta) = \eta$, and it is easily seen that $\eta = \lim_n f^{\circ n}(0)$. □

Note that the state 0 is absorbing for the Markov chain $\{Z_n,\ n \geq 0\}$, and it is accessible from each state. It is then easy to deduce that all other states are transient, hence either $Z_n \to 0$, or $Z_n \to \infty$, as $n \to \infty$. In other words, the population tends to infinity a. s. on the set $\{T = \infty\}$.

Denote $\sigma^2 = \text{Var}(\xi)$, which is assumed to be finite. We have the

Lemma 2.2

$$\begin{aligned} \mathbb{E}Z_n &= \mu^n \mathbb{E}Z_0 \\ \mathbb{E}[Z_n^2] &= \frac{\mu^{2n} - \mu^n}{\mu^2 - \mu}\sigma^2 \mathbb{E}Z_0 + \mu^{2n}\mathbb{E}(Z_0^2). \end{aligned}$$

PROOF. We have

$$\begin{aligned}\mathbb{E}Z_n &= \mathbb{E}\left[\mathbb{E}\left[\sum_{k=1}^{Z_{n-1}} \xi_{n-1,k}|Z_{n-1}\right]\right]\\ &= \mu\mathbb{E}Z_{n-1}\\ &= \mu^n\mathbb{E}Z_0,\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[Z_n^2] &= \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{k=1}^{Z_{n-1}} \xi_{n-1,k}\right)^2 |Z_{n-1}\right]\right]\\ &= \mu^2\mathbb{E}[Z_{n-1}(Z_{n-1}-1)] + (\sigma^2+\mu^2)\mathbb{E}Z_{n-1}\\ &= \mu^2\mathbb{E}[Z_{n-1}^2] + \sigma^2\mathbb{E}Z_{n-1}\\ &= \mu^2\mathbb{E}[Z_{n-1}^2] + \sigma^2\mu^{n-1}\mathbb{E}Z_0.\end{aligned}$$

Consequently $a_n := \mu^{-2n}\mathbb{E}[Z_n^2]$ satisfies

$$\begin{aligned}a_n &= a_{n-1} + \sigma^2\mu^{-(n+1)}\mathbb{E}Z_0\\ &= a_0 + \sigma^2\mathbb{E}Z_0\sum_{k=1}^{n}\mu^{-(k+1)}.\end{aligned}$$

□

Let now Z_n^* denote the number of individuals in generation n with an infinite line of descent. Under $\mathbb{P}_1$, $\{T=\infty\} = \{Z_0^*=1\}$. ξ denoting a r. v. whose law is that of the number of offsprings of each individual, let $\xi^* \le \xi$ denote the number of those offsprings with an infinite line of descent. Let $\overline{q} := 1-q$. We have the

Proposition 2.3 *Assume that $Z_0 = 1$.*

1) Conditional upon $\{T=\infty\}$, $\{Z_n^*,\ n\ge 0\}$ is again a Bienaymé–Galton–Watson process, whose p. g. f. is given by

$$f^*(s) = [f(q+\overline{q}s)-q]/\overline{q}.$$

2) Conditional upon $\{T<\infty\}$, the law of $\{Z_n,\ n\ge 0\}$ is that of a Bienaymé–Galton–Watson process, whose p. g. f. is given by

$$\tilde{f}(s) = f(qs)/q.$$

3) For all $0\le s,t\le 1$,

$$\begin{aligned}\mathbb{E}\left[s^{\xi-\xi^*}t^{\xi^*}\right] &= f(qs+\overline{q}t)\\ \mathbb{E}\left[s^{Z_n-Z^*_n}t^{Z^*_n}\right] &= f^{\circ n}(qs+\overline{q}t).\end{aligned}$$

4) Conditional upon $\{T=\infty\}$, the law of $\{Z_n,\ n\geq 0\}$ is that of $\{Z_n^*,\ n\geq 0\}$ to which we add individuals with finite line of descent, by attaching to each individual of the tree of the Z_n^*'s N independent copies of a Bienaymé–Galton–Watson tree with p. g. f. $\tilde{f}$, where

$$\mathbb{E}[s^N|Z^*]=\frac{D^n f(qs)}{D^n f(q)},$$

where $D^n f$ denotes the n–th derivative of f, and n is the number of sons of the considered individual in the tree Z^*.

PROOF. Let us first prove the first part of *3*. Consider on the same probability space mutually independent r. v.'s $\{\xi, Y_i,\ i\geq 1\}$, where the law of ξ is given as above, and $\mathbb{P}(Y_i=1)=\overline{q}=1-\mathbb{P}(Y_i=0)$, $\forall i\geq 1$. Note that $\overline{q}$ is the probability that any given individual has an infinite line of descent, so that the joint law of $(\xi-\xi^*,\xi^*)$ is that of

$$\left(\sum_{i=1}^{\xi}(1-Y_i),\sum_{i=1}^{\xi}Y_i\right).$$

$$\begin{aligned}\mathbb{E}\left[s^{\xi-\xi^*}t^{\xi^*}\right]&=\mathbb{E}\left[\mathbb{E}\left[s^{\xi-\xi^*}t^{\xi^*}|\xi\right]\right]\\&=\mathbb{E}\left[s^{\sum_{i=1}^{\xi}(1-Y_i)}t^{\sum_{i=1}^{\xi}Y_i}\right]\\&=\mathbb{E}\left[\mathbb{E}[s^{1-Y_1}t^{Y_1}]^{\xi}\right]\\&=\mathbb{E}[(qs+\overline{q}t)^{\xi}]\\&=f(qs+\overline{q}t).\end{aligned}$$

A similar computation yields the second statement in *3*. Indeed

$$\begin{aligned}\mathbb{E}\left[s^{Z_n-Z_n^*}t^{Z_n^*}\right]&=\mathbb{E}\left[\mathbb{E}\left(s^{Z_n-Z_n^*}t^{Z_n^*}|Z_{n-1}\right)\right]\\&=\mathbb{E}\left[\left(\mathbb{E}\left[s^{\xi-\xi^*}t^{\xi^*}\right]\right)^{Z_{n-1}}\right]\\&=f^{\circ(n-1)}(f(qs+\overline{q}t))\end{aligned}$$

We next prove *1* as follows

$$\begin{aligned}\mathbb{E}\left(t^{\xi^*}|\xi^*>0\right)&=\frac{\mathbb{E}(1^{\xi-\xi^*}t^{\xi^*};\xi^*>0)}{\mathbb{P}(\xi^*>0)}\\&=\frac{\mathbb{E}(1^{\xi-\xi^*}t^{\xi^*})-\mathbb{E}(1^{\xi-\xi^*}t^{\xi^*};\xi^*=0)}{\mathbb{P}(\xi^*>0)}\\&=\frac{f(q+\overline{q}t)-f(q)}{\overline{q}}\\&=\frac{f(q+\overline{q}t)-q}{\overline{q}}.\end{aligned}$$

We now prove *2*. It suffices to compute

$$\begin{aligned}\mathbb{E}\left(s^{\xi}|\xi^*=0\right) &= \mathbb{E}\left(s^{\xi-\xi^*}|\xi^*=0\right)\\ &= \frac{f(sq+0\overline{q})}{q}.\end{aligned}$$

Finally we prove *4*. All we have to show is that

$$\mathbb{E}[s^{\xi-\xi^*}|\xi^*=n] = \frac{D^n f(qs)}{D^n f(q)}.$$

This follows from the two following identities

$$\begin{aligned}n!\mathbb{E}[s^{\xi-\xi^*};\xi^*=n] &= \overline{q}^n D^n f(qs+\overline{q}t)|_{t=0}\\ &= \overline{q}^n D^n f(qs),\\ n!\mathbb{P}(\xi^*=n) &= \overline{q}^n D^n f(qs+\overline{q}t)|_{s=1,t=0}\\ &= \overline{q}^n D^n f(q).\end{aligned}$$

□

3. A continuous time Bienaymé–Galton–Watson process

Consider a continuous time $\mathbb{N}$–valued branching process $Z=\{Z_t^k,\ t\geq 0, k\in\mathbb{N}\}$, where t denotes time, and k is the number of ancestors at time 0. Such a process is a Bienaymé–Galton–Watson process in which to each individual is attached a random vector describing its lifetime and its numbers of offsprings. We assume that those random vectors are i. i. d.. The rate of reproduction is governed by a finite measure on $\mathbb{N}$, satisfying $\mu(1)=0$. More precisely, each individual lives for an exponential time with parameter $\mu(\mathbb{N})$, and is replaced by a random number of children according to the probability $\mu(\mathbb{N})^{-1}\mu$. Hence the dynamics of the continuous time jump Markov process Z is entirely characterized by the measure μ. We have the

Proposition 3.1 *The generating function of the process Z is given by*

$$\mathbb{E}\left(s^{Z_t^k}\right) = \psi_t(s)^k,\ s\in[0,1],\ k\in\mathbb{N},$$

where

$$\frac{\partial\psi_t(s)}{\partial t} = \Phi(\psi_t(s)), \quad \psi_0(s)=s,$$

and the funcion Φ is defined by

$$\Phi(s) = \sum_{n=0}^{\infty}(s^n-s)\mu(n),\ s\in[0,1].$$

PROOF. Note that the process Z is a continuous time $\mathbb{N}$–valued jump Markov process, whose infinitesimal generator is given by

$$Q_{n,m} = \begin{cases} 0, & \text{if } m < n-1, \\ n\mu(m+1-n), & \text{if } m \geq n-1 \text{ and } m \neq n, \\ -n\mu(\mathbb{N}), & \text{if } m = n. \end{cases}$$

Define $f : \mathbb{N} \to [0,1]$ by $f(k) = s^k$, $s \in [0,1]$. Then $\psi_t(s) = P_t f(1)$. It follows from the backward Kolmogorov equation for the process Z (see e. g. Theorem 7.6 in [9]) that

$$\begin{aligned} \frac{dP_t f(1)}{dt} &= (QP_t f)(1) \\ \frac{\partial \psi_t(s)}{\partial t} &= \sum_{k=0}^{\infty} Q_{1,k} \psi_t(s)^k \\ &= \sum_{k=0}^{\infty} \mu(k)\psi_t(s)^k - \psi_t(s) \sum_{k=0}^{\infty} \mu(k) \\ &= \Phi(\psi_t(s)). \end{aligned}$$

□

The branching process Z is called immortal if $\mu(0) = 0$.

4. Convergence to a continuous branching process

To each integer N, we associate a Bienaymé–Galton-Watson process $\{Z_n^N,\ n \geq 0\}$ starting from $Z_0^N = N$. We now define the continuous time process

$$X_t^N := N^{-1} Z_{[Nt]}^N.$$

We shall let the p. g. f. of the Bienaymé–Galton-Watson process depend upon N in such a way that

$$\begin{aligned} \mathbb{E}[\xi_N] &= f_N'(1) = 1 + \frac{\alpha}{N} \\ \text{Var}[\xi_N] &= \beta, \end{aligned}$$

where $\alpha \in \mathbb{R}$, $\beta > 0$, and we assume that the sequence of r. v.'s $\{\xi_N^2,\ N \geq 1\}$ is uniformly integrable. Let $t \in \mathbb{N}/N$ and $\Delta t = N^{-1}$. It is not hard to check that

$$\begin{aligned} \mathbb{E}[X_{t+\Delta t}^N - X_t^N | X_t^N] &= \alpha X_t^N \Delta t, \\ \mathbb{E}[(X_{t+\Delta t}^N - X_t^N)^2 | X_t^N] &= \beta X_t^N \Delta t + \alpha^2 (X_t^N)^2 (\Delta t)^2. \end{aligned}$$

As $N \to \infty$, $X^N \Rightarrow X$, where $\{X_t,\ t \geq 0\}$ solves the SDE

$$dX_t = \alpha X_t dt + \sqrt{\beta X_t} dB_t,\ t \geq 0. \tag{1}$$

A detailed proof of the convergence appear in [10], see also [3].

5. The continuous branching process

Denote by $\{X_t(x),\ x > 0, t > 0\}$ the solution of the SDE (1), starting from x at time $t = 0$, i. e. such that $X_0(x) = x$. For $x > 0$ and $y > 0$ consider $\{X_t(x),\ t > 0\}$ and $\{X'_t(y),\ t > 0\}$, where $\{X'_t(y),\ t > 0\}$ is a copy of $\{X_t(y),\ t > 0\}$ which is independent of $\{X_t(x),\ t > 0\}$. Let $Y_t^{x,y} = X_t(x) + X'_t(y)$. We have

$$\begin{aligned} dY_t^{x,y} &= \alpha(X_t(x) + X'_t(y))dt + \sqrt{\beta X_t(x)}dB_t + \sqrt{\beta X'_t(y)}dB'_t \\ &= \alpha Y_t^{x,y}dt + \sqrt{\beta Y_t^{x,y}}dW_t, \\ Y_0^{x,y} &= x + y \end{aligned}$$

where $\{B_t,\ t \geq 0\}$ and $\{B'_t,\ t \geq 0\}$ are two mutually independent standard Brownian motions, and $\{W_t,\ t \geq 0\}$ is also a standard Brownian motion. Then clearly $\{Y_t^{x,y},\ t \geq 0\}$ and $\{X_t(x+y),\ t \geq 0\}$ have the same law. This shows that $\{X_t(x),\ x > 0, t > 0\}$ possesses the branching property.

This property entails that for all $t,\ \lambda > 0$, there exists $u(t,\lambda)$ such that

$$\mathbb{E}\left[\exp(-\lambda X_t(x))\right] = \exp[-xu(t,\lambda)]. \tag{2}$$

From the Markov property of the process $t \to X_t(x)$, we deduce readily the semigroup identity

$$u(t+s,\lambda) = u(t, u(s,\lambda)).$$

We seek a formula for $u(t,\lambda)$. Let us first get by a formal argument an ODE satisfied by $u(\cdot,\lambda)$. For $t > 0$ small, we have that

$$X_t(x) \simeq x + \alpha x t + \sqrt{\beta x}B_t,$$

hence

$$\mathbb{E}\left(e^{-\lambda X_t(x)}\right) \simeq \exp\left(-\lambda x[1 + \alpha t - \beta\lambda t/2]\right),$$

and

$$\frac{u(t,\lambda) - \lambda}{t} \simeq \alpha\lambda - \frac{\beta}{2}\lambda^2.$$

Assuming that $t \to u(t,\lambda)$ is differentiable, we deduce that

$$\frac{\partial u}{\partial t}(0,\lambda) = \alpha\lambda - \frac{\beta}{2}\lambda^2.$$

This, combined with the semigroup identity, entails that

$$\frac{\partial u}{\partial t}(t,\lambda) = \alpha u(t,\lambda) - \frac{\beta}{2}u^2(t,\lambda), \quad u(0,\lambda) = \lambda. \tag{3}$$

It is easy to solve that ODE explicitly, and we now prove rigorously that u is indeed the solution of (3), without having to go through the trouble of justifying the above argument. Let $\gamma = 2\alpha/\beta$, $\gamma_t = \gamma(1 - e^{-\alpha t})^{-1}$.

Lemma 5.1 *The function* $(t,\lambda) \to u(t,\lambda)$ *which appears in* (2) *is given by the formula*

$$u(t,\lambda) = \frac{\gamma e^{\alpha t}}{e^{\alpha t} - 1 + \gamma/\lambda} = \frac{\lambda\gamma_t}{\lambda + \gamma_t e^{-\alpha t}}, \tag{4}$$

and it is the unique solution of (3).

PROOF. It suffices to show that $\{M_s^x,\ 0 \le s \le t\}$ defined by

$$M_s^x = \exp\left(-\frac{\gamma e^{\alpha(t-s)}}{e^{\alpha(t-s)} - 1 + \gamma/\lambda} X_s(x)\right)$$

is a martingale, which follows from Itô's calculus. □

The function $\Psi : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\Psi(r) = \frac{\beta}{2} r^2 - \alpha r$$

is called the *branching mechanism* of the continuous branching process X.

For each fixed $t > 0$, $x \to X_t(x)$ has independent and homogeneous increments with values in $\mathbb{R}_+$. We shall consider its right–continuous modification, which then is a subordinator. Its Laplace exponent is the function $\lambda \to u(t, \lambda)$, which can be rewritten (like for any subordinator, see the Levy–Kintchin formula in e. g. [4]) as

$$u(t, \lambda) = d(t)\lambda + \int_0^\infty (1 - e^{-\lambda r})\Lambda(t, dr),$$

where $d(t) \ge 0$ and $\int_0^\infty (r \wedge 1)\Lambda(t, dr) < \infty$. Comparing with (4), we deduce that $d(t) = 0$, and

$$\Lambda(t, dr) = p(t)\exp(-q(t)r)dr,$$
$$\text{where } p(t) = \gamma_t^2 e^{-\alpha t},\ q(t) = \gamma_t e^{-\alpha t}. \tag{5}$$

We have defined the two parameter process $\{X_t(x);\ x \ge 0, t \ge 0\}$. $X_t(x)$ is the population at time t made of descendants of the initial population of size x at time 0. We may want to introduce three parameters, if we want to discuss the descendants at time t of a population of a given size at time s. The first point, which is technical but in fact rather standard, is that we can construct the collection of those random variables jointly for all $0 \le s < t$, $x \ge 0$, so that all the properties we may reasonably wish for them are satisfied. More precisely, following [2], we have the

Lemma 5.2 *On some probability space, there exists a three parameter process*

$$\{X_{s,t}(x),\ 0 \le s \le t,\ x \ge 0\},$$

such that

1) For every $0 \le s \le t$, $X_{s,t} = \{X_{s,t}(x),\ x \ge 0\}$ *is a subordinator with Laplace exponent* $u(t-s, \cdot)$.

2) For every $n \ge 2$, $0 \le t_1 < t_2 < \cdots < t_n$, *the subordinators* $X_{t_1,t_2}, \ldots, X_{t_{n-1},t_n}$ *are mutually independent, and*

$$X_{t_1,t_n}(x) = X_{t_{n-1},t_n} \circ \cdots \circ X_{t_1,t_2}(x),\ \forall x \ge 0,\ \text{a. s.}$$

3) The processes $\{X_{0,t}(x),\ t \ge 0, x \ge 0\}$ *and* $\{X_t(x),\ t \ge 0, x \ge 0\}$ *have the same finite dimensional distributions.*

Now consider $\{X_{s,t}(x),\ x \geq 0\}$ for fixed $0 \leq s \leq t$. It is a subordinator with Laplace exponent (the functions p and q are given in (5))

$$u(t-s,\lambda) = p(t-s)\int_0^\infty (1-e^{-\lambda r})e^{-q(t-s)r}dr.$$

We shall give a probabilistic description of the process $\{X_{s,t}(x),\ x \geq 0\}$ in a further section. For now on, we shall write $X_t(x)$ for $X_{0,t}(x)$.

Let us first study the large time behaviour of the process $X_t(x)$. Consider the extinction event

$$E = \{\exists t > 0, \text{ s. t. } X_t(x) = 0\}.$$

We define again $\gamma = 2\alpha/\beta$.

Proposition 5.3 *If $\alpha \leq 0$, $\mathbb{P}_x(E) = 1$ a.s. for all $x > 0$. If $\alpha > 0$, $\mathbb{P}_x(E) = \exp(-x\gamma)$ and on E^c, $X_t(x) \to +\infty$ a. s.*

PROOF. If $\alpha \leq 0$, $\{X_t(x),\ t \geq 0\}$ is a positive supermartingale. Hence it converges a. s. The limit r. v. $X_\infty(x)$ takes values in the set of fixed points of the SDE (1), which is $\{0, +\infty\}$. But from Fatou and the supermartingale property,

$$\mathbb{E}(\lim_{t\to\infty} X_t(x)) \leq \lim_{t\to\infty} \mathbb{E}(X_t(x)) \leq x.$$

Hence $\mathbb{P}(X_\infty(x) = +\infty) = 0$, and $X_t(x) \to 0$ a. s. as $t \to \infty$.

If now $\alpha > 0$, it follows from Itô's formula that

$$e^{-\gamma X_t(x)} = e^{-\gamma x} - \gamma \int_0^t e^{-\gamma X_s(x)}\sqrt{\beta X_s(x)}dB_s,$$

hence $\{M_t = e^{-\gamma X_t(x)},\ t \geq 0\}$ is a martingale with values in $[0,1]$, hence it converges a. s. as $t \to \infty$. Consequently $X_t(x) = -\gamma \log(M_t)$ converges a. s., and as above its limit belongs to the set $\{0, +\infty\}$. Moreover

$$\begin{aligned}\mathbb{P}(E) &= \lim_{t\to\infty} \mathbb{P}(X_t(x) = 0) \\ &= \lim_{t\to\infty} \mathbb{E}[\exp\{-xu(t,\infty)\}] \\ &= \lim_{t\to\infty} \exp\left\{-x\frac{\gamma e^{\alpha t}}{e^{\alpha t}-1}\right\} \\ &= \exp\{-x\gamma\}.\end{aligned}$$

It remains to prove that

$$\mathbb{P}(E^c \cap \{X_t \to 0\}) = 0. \tag{6}$$

Define the stopping times

$$\begin{aligned}\tau_1 &= \inf\{t > 0,\ X_t(x) \leq 1\}, \text{ and for } n \geq 2, \\ \tau_n &= \inf\{t > \tau_{n-1} + 1,\ X_t(x) \leq 1\}.\end{aligned}$$

On the set $\{X_t(x) \to 0, \text{ as } t \to \infty\}$, $\tau_n < \infty$, $\forall n$. Define for $n \geq 1$

$$A_n = \{\tau_{n+1} < \infty, X_{\tau_{n+1}}(x) > 0\}.$$

For all $N > 0$,

$$\begin{aligned}
\mathbb{P}(E^c \cap \{X_t \to 0\}) &\leq \mathbb{P}(\cap_{n=1}^N A_n) \\
&\leq \mathbb{E}\left(\prod_{n=1}^N \mathbb{P}(A_n|\mathcal{F}_{\tau_n})\right) \\
&\leq (\mathbb{P}(X_1(1) > 0))^N \\
&\to 0, \text{ as } N \to \infty,
\end{aligned}$$

where we have used the strong Markov property, and the fact that

$$\mathbb{P}(A_n|X_{\tau_n}) \leq \mathbb{P}(X_1(1) > 0).$$

□

6. Back to Bienaymé–Galton–Watson

6.1. The individuals with an infinite line of descent

Let us go back to the discrete model, indexed by N. For each $t \geq 0$, let Y_t^N denote the number of individuals in the population $Z_{[Nt]}^N$ with an infinite line of descent. Let us describe the law of Y_0^N. Each of the N individuals living at time $t = 0$ has the probability $1 - q_N$ of having an infinte line of descent. It then follows from the branching property that the law of Y_0^N is the binomial law $B(N, 1 - q_N)$. It remains to evaluate q_N, the unique solution in the interval $(0, 1)$ of the equation $f_N(x) = x$. Note that

$$f_N''(1) = \mathbb{E}[\xi_N(\xi_N - 1)] = \beta - \frac{\alpha}{N} + \left(\frac{\alpha}{N}\right)^2.$$

We deduce from a Taylor expansion of f near $x = 1$ that

$$q_N = 1 - \frac{2\alpha}{N\beta} + \circ\left(\frac{1}{N}\right),\ 1 - q_N = \frac{2\alpha}{N\beta} + \circ\left(\frac{1}{N}\right).$$

Consequently, Y_0^N converges in law, as $N \to \infty$, towards a Poisson distribution with parameter $\gamma = 2\alpha/\beta$.

6.2. The individuals whose progeny survives during tN generations

The result of the last section indicates that if we consider only the *prolific individuals*, i. e. those with an infinite line of descent, in the limit $N \to \infty$, we should not divide by N, also $Z_{[Nt]}^N \to +\infty$, as $N \to \infty$, for all $t \geq 0$. If now we consider those individuals whose progeny is still alive at time tN (i. e. those whose progeny contributes to the population at time $t > 0$ in the limit as $N \to \infty$), then again we should not divide by N. Indeed, we have the (we use again the notation $\gamma = 2\alpha/\beta$)

Theorem 6.1 *Under the assumptions from the beginning of section 4, with the notation*

$$\gamma_t = \gamma \left(1 - e^{-\alpha t}\right)^{-1},$$

1) for N large,

$$\mathbb{P}_1(Z^N_{[Nt]} > 0) = \frac{\gamma_t}{N} + \circ\left(\frac{1}{N}\right),$$

and

2) as $N \to \infty$,

$$\mathbb{E}_1\left(\exp[-\lambda Z^N_{[Nt]}/N] | Z^N_{[Nt]} > 0\right) \to \frac{\gamma_t e^{-\alpha t}}{\lambda + \gamma_t e^{-\alpha t}}.$$

PROOF OF 1 : It follows from the branching property that

$$\begin{aligned}\mathbb{P}_1(Z^N_{[Nt]} > 0) &= 1 - \mathbb{P}_1(Z^N_{[Nt]} = 0)\\ &= 1 - \mathbb{P}_N(Z^N_{[Nt]} = 0)^{1/N}\\ &= 1 - \mathbb{P}_1(X^N_t = 0)^{1/N}.\end{aligned}$$

But

$$\begin{aligned}\log\left[\mathbb{P}_1(X^N_t = 0)^{1/N}\right] &= \frac{1}{N}\log \mathbb{P}_1(X^N_t = 0)\\ &= \frac{1}{N}\log \mathbb{P}_1(X_t = 0) + \circ\left(\frac{1}{N}\right).\end{aligned}$$

From (2) and (4), we deduce that

$$\begin{aligned}\mathbb{P}_1(X_t = 0) &= \lim_{\lambda\to\infty} \exp[-u(t,\lambda)]\\ &= \exp(-\gamma_t).\end{aligned}$$

We then conclude that

$$\begin{aligned}\mathbb{P}_1(Z^N_{[Nt]} > 0) &= 1 - \exp\left[-\frac{\gamma_t}{N} + \circ\left(\frac{1}{N}\right)\right]\\ &= \frac{\gamma_t}{N} + \circ\left(\frac{1}{N}\right).\end{aligned}$$

PROOF OF 2 :

$$\begin{aligned}\mathbb{E}_1 \exp[-\lambda Z^N_{[Nt]}/N] &= \left(\mathbb{E}_N \exp[-\lambda Z^N_{[Nt]}/N]\right)^{1/N}\\ &\simeq \left(\mathbb{E}_1 \exp[-\lambda X_t]\right)^{1/N}\\ &= \exp\left(-\frac{\lambda\gamma_t}{N(\lambda + \gamma_t e^{-\alpha t})}\right),\end{aligned}$$

since again from (2) and (4),

$$\mathbb{E}_1 \exp(-\lambda X_t) = \exp\left(-\frac{\lambda\gamma_t}{\lambda + \gamma_t e^{-\alpha t}}\right).$$

But

$$\begin{aligned}
\mathbb{E}_1\left(\exp[-\lambda Z^N_{[Nt]}/N] | Z^N_{[Nt]} > 0\right) &= \frac{\mathbb{E}_1\left(\exp[-\lambda Z^N_{[Nt]}/N]; Z^N_{[Nt]} > 0\right)}{\mathbb{P}_1(Z^N_{[Nt]} > 0)} \\
&= \frac{\mathbb{E}_1\left(\exp[-\lambda Z^N_{[Nt]}/N]\right) - 1 + \mathbb{P}_1(Z^N_{[Nt]} > 0)}{\mathbb{P}_1(Z^N_{[Nt]} > 0)} \\
&= 1 + \frac{\mathbb{E}_1\left(\exp[-\lambda Z^N_{[Nt]}/N]\right) - 1}{\mathbb{P}_1(Z^N_{[Nt]} > 0)} \\
&\simeq 1 - \frac{\lambda}{\lambda + \gamma_t e^{-\alpha t}},
\end{aligned}$$

from which the result follows. □

7. Back to the continuous branching process

Note that the continuous limit $\{X_t\}$ has been obtained after a division by N, so that X_t no longer represents a number of individuals, but a sort of density. The point is that there are constantly infinitely many births and deaths, most individuals having a very short live. If we consider only those individuals at time 0 whose progeny is still alive at some time $t > 0$, that number is finite. We now explain how this follows from the last Theorem, and show how it provides a probabilistic description of the subordinator which appeared at the end of section 4.

The first part of the theorem tells us that for large N, each of the N individuals from the generation 0 has a progeny at the generation $[Nt]$ with probability $\gamma_t/N + \circ(1/N)$, independently of the others. Hence the number of those individuals tends to the Poisson law with parameter γ_t. The second statement says that those individuals contribute to X_t a quantity which follows an exponential random variable with parameter $\gamma_t e^{-\alpha t}$. This means that

$$X_{0,t}(x) = \sum_{i=1}^{Z_x} Y_i,$$

where $Z_x, Y_1, Y_2, \ldots$ are mutually independent, the law of Z_x being Poisson with parameter $x\gamma_t$, and the law of each Y_i exponential with parameter $\gamma_t e^{-\alpha t}$.

Taking into account the branching property, we have more precisely that $\{X_{0,t}(x),\ x \geq 0\}$ is a compound Poisson process, the set of jump locations being a Poisson process with intensity γ_t, the jumps being i. i. d., exponential with parameter $\gamma_t e^{-\alpha t}$. We can recover from this description the formula for the Laplace exponent of $X_t(x)$. Indeed

$$\begin{aligned}\mathbb{E}\exp\left(-\lambda\sum_{i=1}^{Z_x} Y_i\right) &= \sum_{k=0}^{\infty}\left(\mathbb{E}e^{-\lambda Y_1}\right)^k \mathbb{P}(Z_x = k) \\ &= \exp\left(-x\frac{\lambda\gamma_t}{\lambda+\gamma_t e^{-\alpha t}}\right).\end{aligned}$$

We can now describe the genealogy of the population whose total mass follows the SDE (1).

Suppose that Z ancestors from $t = 0$ contribute respectively $Y_1, Y_2, \ldots, Y_Z$ to $X_{0,t}(x)$. Consider now $X_{0,t+s}(x) = X_{t,t+s}(X_{0,t}(x))$. From the Y_1 mass at time t, a finite number Z_1 of individuals, which follows a Poisson law with parameter $Y_1\gamma_s$, has a progeny at time $t+s$, each one contributing an exponential r. v. with parameter $\gamma_s e^{-\alpha s}$ to $X_{0,t+s}(x)$.

Fo any $y, z \geq 0$, $0 \leq s < t$, we say that the individual z in the population at time t is a descendant of the individual y from the population at time s if y is a jump location of the subordinator $x \to X_{s,t}(x)$, and moreover

$$X_{s,t}(y^-) < z < X_{s,t}(y).$$

Note that $\Delta X_{s,t}(y) = X_{s,t}(y) - X_{s,t}(y^-)$ is the contribution to the population at time t of the progeny of the individual y from the population at time s.

8. The prolific individuals

We want to consider again the individuals with an infinite line of descent, but directly in the continuous model. Those could be defined as the individuals such that $\Delta X_{0,t}(y) > 0$, for all $t > 0$. However, it should be clear from Proposition 5.3 that an a. s. equivalent definition is the following

Definition 8.1 *The individual y from the population at time s is said to be prolific if $\Delta X_{s,t}(y) \to \infty$, as $t \to \infty$.*

For any $s \geq 0$, $x > 0$, let

$$\begin{aligned}\mathcal{P}_s(x) &= \{y \in [0, X_s(x)];\ \Delta X_{s,t}(y) \to \infty,\ \text{as } t \to \infty\}, \\ P_s(x) &= \text{card}(\mathcal{P}_s(x)).\end{aligned}$$

Define the conditional probability, given extinction

$$\begin{aligned}\mathbb{P}_e &= \mathbb{P}(\cdot|E) \\ &= e^{x\gamma}\mathbb{P}(\cdot \cap E)\end{aligned}$$

It follows from well–known results on conditioning of Markov processes, see [4] or [10],

Proposition 8.2 *Under $\mathbb{P}_e$, there exists a standard Brownian motion $\{B_t^e,\ t \geq 0\}$ such that $X_\cdot(x)$ solves the SDE*

$$X_t(x) = x - \alpha \int_0^t X_s(x) ds + \int_0^t \sqrt{\beta X_s(x)}\, dB_s^e.$$

The branching mechanism of X under $\mathbb{P}_e$ is given by

$$\Psi_e(r) = \frac{\beta}{2} r^2 + \alpha r = \Psi(\gamma + r).$$

Next we identify the conditional law of $X_t(x)$, given that $P_t(x) = n$, for $n \geq 0$.

Proposition 8.3 *For any Borel measurable $f : \mathbb{R} \to \mathbb{R}_+$,*

$$\mathbb{E}[f(X_t(x))|P_t(x) = n] = \frac{\mathbb{E}_e[f(X_t(x))(X_t(x))^n]}{\mathbb{E}_e[(X_t(x))^n]}.$$

PROOF. Recall that the law of $P_0(x)$ is the Poisson distribution with parameter $x\gamma$. Clearly from the Markov property of $X_\cdot(x)$, the conditional law of $P_t(x)$, given $X_t(x)$, is the Poisson law with parameter $X_t(x)\gamma$. Consequently for $\lambda > 0$, $0 \leq s \leq 1$,

$$\begin{aligned}
\mathbb{E}\left(\exp[-\lambda X_t(x)] s^{P_t(x)}\right) &= \mathbb{E}\left(\exp[-\lambda X_t(x)] \exp[-\gamma(1-s) X_t(x)]\right) \\
&= \mathbb{E}\left(\exp[-(\lambda+\gamma) X_t(x)] \exp[\gamma s X_t(x)]\right) \\
&= \sum_{n=0}^{\infty} \frac{(s\gamma)^n}{n!} \mathbb{E}\left(\exp[-(\lambda+\gamma) X_t(x)](X_t(x))^n\right).
\end{aligned}$$

Now define

$$h(t, \lambda, x, n) = \mathbb{E}\left(\exp[-\lambda X_t(x)] | P_t(x) = n\right).$$

Note that

$$\begin{aligned}
\mathbb{P}(P_t(x) = n) &= \mathbb{E}\left[\mathbb{P}(P_t(x) = n | X(t,x))\right] \\
&= \frac{\gamma^n}{n!} \mathbb{E}\left(e^{-\gamma X_t(x)} (X_t(x))^n\right).
\end{aligned}$$

Consequently, conditioning first upon the value of $P_t(x)$, and then using the last identity, we deduce that

$$\mathbb{E}\left(\exp[-\lambda X_t(x)] s^{P_t(x)}\right) = \sum_{n=0}^{\infty} \frac{(s\gamma)^n}{n!} h(t, \lambda, x, n) \mathbb{E}\left(\exp[-\gamma X_t(x)](X_t(x))^n\right).$$

Comparing the two series, and using the fact that, on $\mathcal{F}_t$, $\mathbb{P}_e$ is absolutely continuous with respect to $\mathbb{P}$, with density $e^{x\gamma} \exp[-\gamma X_t(x)]$, we deduce that for all $n \geq 0$,

$$\begin{aligned}
h(t, \lambda, x, n) &= \frac{\mathbb{E}\left(\exp[-(\lambda+\gamma) X_t(x)](X_t(x))^n\right)}{\mathbb{E}\left(\exp[-\gamma X_t(x)](X_t(x))^n\right)} \\
&= \frac{\mathbb{E}_e\left(\exp[-\lambda X_t(x)](X_t(x))^n\right)}{\mathbb{E}_e\left[(X_t(x))^n\right]}.
\end{aligned}$$

□

To any probability law ν on $\mathbb{R}_+$ with finite mean c, we associate the so–called law of its size–biased picking as the law on $\mathbb{R}_+$ $c^{-1}y\nu(dy)$. We note that the conditional law of $X_t(x)$, given that $P_t(x) = n + 1$ is obtained from the conditional law of $X_t(x)$, given that $P_t(x) = n$ by sized–biased picking.

We now describe the law of $\{P_t(x),\ t \geq 0\}$, for fixed $x > 0$. Clearly this is a continuous time B–G–W process as considered in section 3 above. We have the

Theorem 8.4 *For every $x > 0$, the process $\{P_t(x),\ t \geq 0\}$ is an $\mathbb{N}$–valued immortal Branching process in continuous time, with initial distribution the Poisson law with parameter $x\gamma$, and reproduction measure μ_P given by*

$$\mu_P(n) = \begin{cases} \alpha, & \text{if } n = 2, \\ 0, & \text{if } n \neq 2. \end{cases}$$

In other words, $\{P_t(x),\ t \geq 0\}$ is a Yule tree with the intensity α.

REMARK. — I
f we call Φ_P the Φ–function (with the notations of section 3) associated to the measure μ_P, we have in terms of the branching mechanism Ψ of X

$$\Phi_P(s) = \alpha(s^2 - s) = \frac{1}{\gamma}\Psi(\gamma(1-s)).$$

Note that Ψ_e describes the branching process X, conditioned upon extinction, while Φ_P describes the immortal part of X. Φ_P depends upon the values $\Phi(r)$, $0 \leq r \leq \gamma$, while Ψ_e depends upon the values $\Phi(r)$, $\gamma \leq r \leq 1$. The mapping $\Psi \to (\Psi_e, \Phi_P)$ should be compared with the mapping $f \to (\tilde{f}, f^*)$ from Proposition 2.3. PROOF. The process P inherits its branching property from that of X. The immortal character is obvious. $P_0(x)$ is the number of individuals from the population at time 0, whose progeny survives at time t, for all $t > 0$. Hence it is the limit as $t \to \infty$ of the law of the number of jumps of $\{X_t(y),\ 0 \leq y \leq x\}$, which is the Poisson distribution with parameter $x\gamma$. This coincides with the result in the subsection 6.1, as expected.

Now from the Markov property of X, the conditional law of $P_t(x)$, given $X_t(x)$, is the Poisson law with parameter $X_t(x)\gamma$. Consequently

$$\begin{aligned}\mathbb{E}\left(s^{P_t(x)}\right) &= \mathbb{E}\left(\exp[-(1-s)\gamma X_t(x)]\right) \\ &= \exp[-xu(t,(1-s)\gamma)].\end{aligned}$$

Moreover, if we call $\psi_t(s)$ the generating function of the continuous time B–G–W process $\{P_t(x),\ t \geq 0\}$, we have that

$$\begin{aligned}\mathbb{E}\left(s^{P_t(x)}\right) &= \mathbb{E}\left(\psi_t(s)^{P_0(x)}\right) \\ &= \exp[-x\gamma(1-\psi_t(s))].\end{aligned}$$

Comparing those two formulas, we deduce that

$$1 - \psi_t(s) = \frac{1}{\gamma}u(t,(1-s)\gamma).$$

Taking the derivative with respect to the time variable t, we deduce from the differential equations satisfied by $\psi_t(\cdot)$ and by $u(t,\cdot)$ the identity

$$\Phi_P(\psi_t(s)) = \frac{1}{\gamma}\Psi(u(t,(1-s)\gamma)) = \frac{1}{\gamma}\Psi(\gamma(1-\psi_t(s))).$$

Consequently

$$\Phi_P(r) = \frac{1}{\gamma}\Psi(\gamma(1-r)).$$

The measure μ_P is then recovered easily from Φ_P. □

We next note that the pair $(X_t(x), P_t(x))$, which we now write $(X_t(x), P_t(x))$, enjoys the Branching property, in the following sense. For every $x > 0$, $n \in \mathbb{N}$, denote by $(X_\cdot(x,n), P_\cdot(x,n))$ a version of the process $\{(X_t(x), P_t(x)),\ t \geq 0\}$, conditioned upon $P_0(x) = n$. What we mean here by the branching property is the fact that for all $x, x' > 0$, $n, n' \in \mathbb{N}$,

$$(X_\cdot(x+x', n+n'), P_\cdot(x+x', n+n'))$$

has the same law as

$$(X_\cdot(x,n), P_\cdot(x,n)) + (X'_\cdot(x',n'), P'_\cdot(x',n')),$$

where the two processes $(X_\cdot(x,n), P_\cdot(x,n))$ and $(X'_\cdot(x',n'), P'_\cdot(x',n'))$ are mutually independent.

We now characterize the joint law of $(X_t(x,n), P_t(x,n))$.

Proposition 8.5 *For any $\lambda \geq 0$, $s \in [0,1]$, $t \geq 0$, $x > 0$, $n \in \mathbb{N}$,*

$$\begin{aligned}&\mathbb{E}\left(\exp[-\lambda X_t(x,n)]s^{P_t(x,n)}\right)\\ &= \exp[-x(u(t,\lambda+\gamma)-\gamma)]\left(\frac{u(t,\lambda+\gamma)-u(t,\lambda+\gamma(1-s))}{\gamma}\right)^n.\end{aligned}$$

PROOF. First consider the case $n = 0$. We note that $X_\cdot(x,0)$ is a version of the continuous branching process conditioned upon extinction, i. e. with branching mechanism $\Psi_e(r) = \Psi(\gamma + r)$, while $P_t(x,0) \equiv 0$. Hence

$$\mathbb{E}\left(\exp[-\lambda X_t(x,0)]s^{P_t(x,0)}\right) = \exp[-x(u(t,\lambda+\gamma)-\gamma)]. \tag{7}$$

Going back to the computation in the beginning of the proof of Proposition 8.3, we have

$$\begin{aligned}\mathbb{E}\left(\exp[-\lambda X_t(x)]s^{P_t(x)}\right) &= \mathbb{E}\left(\exp[-(\lambda+\gamma(1-s))X_t(x)]\right)\\ &= \exp[-xu(t,\lambda+\gamma(1-s))].\end{aligned}$$

Since the law of $P_0(x)$ is Poisson with parameter $x\gamma$,

$$\mathbb{E}\left(\exp[-\lambda X_t(x)]s^{P_t(x)}\right) = \sum_{n=0}^{\infty} e^{-x\gamma}\frac{(x\gamma)^n}{n!}\mathbb{E}\left(\exp[-\lambda X_t(x,n)]s^{P_t(x,n)}\right).$$

>From the branching property of (X, P),

$$\begin{aligned}\mathbb{E}\left(\exp[-\lambda X_t(x,n)]s^{P_t(x,n)}\right) = \mathbb{E}\left(\exp[-\lambda X_t(x,0)]s^{P_t(x,0)}\right) \\ \times\left[\mathbb{E}\left(\exp[-\lambda X_t(0,1)]s^{P_t(0,1)}\right)\right]^n.\end{aligned} \tag{8}$$

Combining the four above identities, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(x\gamma)^n}{n!}\left[\mathbb{E}\left(\exp[-\lambda X_t(0,1)]s^{P_t(0,1)}\right)\right]^n \\ &\qquad = \exp\left\{x\left[u(t,\lambda+\gamma)-u(t,\lambda+\gamma(1-s))\right]\right\} \\ &\qquad = \sum_{n=0}^{\infty}\frac{\left\{x\left[u(t,\lambda+\gamma)-u(t,\lambda+\gamma(1-s))\right]\right\}^n}{n!}.\end{aligned}$$

Identifying the coefficients of x in the two series yields

$$\mathbb{E}\left(\exp[-\lambda X_t(0,1)]s^{P_t(0,1)}\right) = \frac{u(t,\lambda+\gamma)-u(t,\lambda+\gamma(1-s))}{\gamma}.$$

The result follows from this, (7) and (8). $\square$

9. Bibliographical comments

We have essentially followed the treatment from [6] in section 2. Section 3 is inspired from [4]. Section 5 owes much to [4], [2] and [5]. The subsection 6.1 is taken from [7], 6.2 from [8]. Section 8 is a translation of the results in [1] to our particular case. Note that [1] considers more general CSBP's, than just the Feller diffusion. This includes the CSBP's which are obtained as limits of BGW processes where the offspring distribution has infinite variance.

10. References

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