Stability Analysis for Impulsive Systems: 2D Vector Lyapunov Function Approach

Hector Ríos, L Hetel, Denis Efimov

Stability Analysis for Impulsive Systems: $2D$ Vector Lyapunov Function Approach*

H. Ríos[1], L. Hetel[2] and D. Efimov[1†]

Abstract—This paper contributes to the stability analysis for impulsive dynamical systems based on a vector Lyapunov function and its divergence operator. The new method relies on a $2D$ time domain representation. The result is illustrated for the exponential stability of linear impulsive systems based on LMIs. The obtained results provide some notions of minimum and maximum dwell-time. Some examples illustrate the feasibility of the proposed approach.

Index Terms—Impulsive systems, Exponential stability, $2D$ Systems.

I. Introduction

Impulsive systems [2] represent an important class of hybrid systems [7] containing discontinuities or jumps in the trajectories of the system governed by discrete dynamics. There exist a large variety of phenomena that are characterized by abrupt changes in the system state at certain instants, *e.g.* power electronics, sample-data systems, models in economics, bursting rhythm models in medicine, *etc*. According to the manner of impulses to be triggered, several types of impulsive systems can be distinguished: systems with *time dependent* impulses, systems with *state dependent* impulses (reset systems), and the combination of both of them (see, *e.g.* [2], [7] and [19]). In the current work impulsive systems with time triggered impulses are studied.

In the context of stability analysis of impulsive systems an important effort has been made for linear dynamics. In [5], a functional-based approach is developed for stability analysis. This method introduces looped functional leading to LMI conditions and allows to establish dwell-time results. Exponential stability, based on Lyapunov functions with discontinuity at the impulses, is proposed by [10] for nonlinear time-varying impulsive systems. Such stability conditions are mainly applied to the linear case. In the sample-data system framework, there exist many constructive works where the impulsive systems are used to describe the behavior of aperiodic sample-data systems (see, *e.g.* [6], [8] and [15]). An equivalent correspondence between continuous and discrete time domains is showed in [14].

In this paper, since hybrid systems are inherently related to $2D$ times due to the continuous and discrete variables [7], based on a $2D$ time domain equivalence (see, *e.g.* [13] and [18]), the exponential stability notion for a class of impulsive systems is studied using a vector Lyapunov function approach [9]. The use of vector Lyapunov functions offers a more flexible framework since each component of the vector Lyapunov function can satisfy less rigid requirements as compared to a single Lyapunov function.

The main result of this work contributes to the development of a *new theoretical framework for the stability analysis* of impulsive dynamical systems. The proposed method is based on a vector Lyapunov function and its divergence operator, that can satisfy less rigid requirements than a single Lyapunov function, in a $2D$ time domain. Next this theory is illustrated for the exponential stability of linear impulsive systems. Some notions of *minimum* and *maximum* dwell-time depending on the structure of the system dynamics, based on LMIs in order to show, on the linear benchmark case, how our approach works.

The outline of this work is as follows. The problem statement and a preliminary definition of exponential stability are given in Section II. The main result as well as the stability conditions are described in Section III. Application of the developed theory to the problem of exponential stability for linear impulsive systems and some simulation example are considered in Section IV. Finally, some concluding remarks are discussed in Section V.

II. Problem Statement

Consider the class of linear impulsive dynamical systems for which the impulse times are external to the system and time dependent, *i.e.*

$$\dot{x}(t) = f_1(x(t)), \ \forall t \in \mathbb{R}_+ \setminus \mathbb{I}, \tag{1}$$

$$x(t) = f_2(x(t^-)), \ \forall t \in \mathbb{I}, \ x(0) = x_0, \tag{2}$$

where $x, \ x_0 \in \mathbb{R}^n$ are the state of the system and the initial condition, respectively. The set of impulse times $\mathbb{I} := \{t_i\}_{i\in\mathbb{N}}$ is a countable subset of $\mathbb{R}_+$ with $t_0 = 0$ and $\lim_{i\to\infty} t_i = +\infty$ in order to avoid any zeno phenomena. The state trajectory is assumed to be right continuous and to have left limits at all times. The notation $x(t^-)$ denotes the left limit of $x(t)$ as t goes to t_i from the left, *i.e.* $x(t^-) = \lim_{t\uparrow t_i} x(t)$. The distance between the impulses, *i.e.* the *dwell-time*, is defined as $T_i := t_{i+1} - t_i$, and it is assumed that any sequence of impulse periods $\{T_i\}_{i\in\mathbb{N}}$ belongs to an interval, *i.e.* $T_i \in [T_{\min}, T_{\max}]$, for all $i \in \mathbb{N}$, where $T_{\min} > 0$ and $T_{\max} > 0$ are the minimum and maximum dwell-time, respectively. The Lipschitz nonlinear functions f_1 and f_2 are such that $f_1(0) = 0$ and $f_2(0) = 0$. It is assumed that for all $x_0 \in \mathbb{R}^n$, f_1 is such that system (1) has a unique solution over any time interval $[t_i, t_{i+1})$ for all $i \in \mathbb{N}$.

[1]Non-A team @ Inria, Parc Scientifique de la Haute Borne, 40 avenue Halley, 59650 Villeneuve d'Ascq, France; and CRIStAL (UMR-CNRS 9189), Ecole Centrale de Lille, BP 48, Cité Scientifique, 59651 Villeneuve-d'Ascq, France. Emails: hector.rios_barajas@inria.fr; denis.efimov@inria.fr

[2]CRIStAL (UMR-CNRS 9189), Ecole Centrale de Lille, BP 48, Cité Scientifique, 59651 Villeneuve-d'Ascq, France. Email: laurentiu.hetel@ec-lille.fr

[†]Department of Control Systems and Informatics, Saint Petersburg State University of Information Technologies Mechanics and Optics (ITMO), Kronverkskiy av. 49, Saint Petersburg, 197101, Russia.

*This work was supported in part by Conseil Region Nord-Pas de Calais (ESTIREZ), the Government of Russian Federation (Grant 074-U01) and the Ministry of Education and Science of Russian Federation (Project 14.Z50.31.0031), and ANR ROCC-SYS (ANR-14-CE27-0008).

The proposed Vector Lyapunov function based approach relies on the embedding of system (1)-(2) into a $2D$ time domain. Indeed, the entire state trajectory $x(t)$ can be viewed as a sequence of the *diagonal dynamics*[2] of the following $2D$ system:

$$\frac{dx_k^t}{dt} = f_1(x_k^t),\ \forall t \in [t_i, t_{i+1})\,,\ t_0 = 0, \tag{3}$$
$$x_{k+1}^{t_{i+1}} = f_2(x_k^{t_{i+1}}),\ \forall i = k \in \mathbb{N},\ x_0^0 = x_0, \tag{4}$$

where $x_k^t = x(t,k) \in \mathbb{R}^n$ is the current state vector, $x_{k+1}^{t_{i+1}} = x(t_{i+1}, k+1) \in \mathbb{R}^n$ represents the reset vector state, $x_k^{t_{i+1}} = x(t_{i+1}, k) \in \mathbb{R}^n$ denotes the value of x just before the switching $k+1$. It is worth saying that the solutions of system (3) are unique for the *diagonal dynamics, i.e.* for all $i = k$ and $\forall t \in \mathbb{R}_+$. Assume that $|q|$ denotes the Euclidean norm of a vector q. The following stability definition is introduced:

Definition 1. *A 2D system described by (3)-(4), is said to be exponentially diagonal stable (EDS) if there exist positive constants $0 < \kappa_1 < 1 - \varepsilon$, $\kappa_2 > 0$, $c > 0$ and $\varepsilon \in (0, 1)$ such that for all $i = k \in \mathbb{N}$*

$$|x_{k+1}^{t_{i+1}}|^2 \le c\kappa_1^{k+1}|x_0|^2, \tag{5}$$
$$|x_k^t|^2 \le \kappa_2 |x_k^{t_i}|^2,\ \forall t \in [t_i, t_{i+1})\,. \tag{6}$$

The goal in this work is to find conditions for the exponential stability of the impulsive systems described by (1)-(2) by means of the $2D$ time representation (3)-(4).

III. Main Result

In order to give the stability conditions a vector Lyapunov approach is used, *i.e.*

$$V(t, x_k^t, x_{k+1}^{t_{i+1}}) = \begin{bmatrix} V_1(t, x_k^t) \\ V_2(x_{k+1}^{t_{i+1}}) \end{bmatrix}, \tag{7}$$

where $V_1(t,\cdot) > 0$, $V_2(\cdot) > 0$, for all $t \ge 0$, and $V_1(t, 0) = 0$, $V_2(0) = 0$, $\forall t \ge 0$. Now, let us introduce the following definition.

Definition 2. *The divergence operator of a function V along the trajectories of system (3)-(4) is defined for all $t \in [t_i, t_{i+1})$ as follows*

$$\mathrm{div}V(t, x_k^t, x_{k+1}^{t_{i+1}}) = \frac{dV_1(t, x_k^t)}{dt} + V_2(x_{k+1}^{t_{i+1}}) - V_2(x_k^{t_{i+1}}). \tag{8}$$

Note that V_1 is differentiable with respect to continuous time t while the difference in V_2 is calculated in discrete time. Based on the previous explanations, the following theorem is established:

Theorem 1. *Assume that there exist positive constants ε, c_1, c_2, c_3, c_4 and c_5 such that the vector Lyapunov function $V(t, x_k^t, x_{k+1}^{t_{i+1}})$ and its divergence along the trajectories of the system* (3)-(4) *satisfy, for all $t \in [t_i, t_{i+1})$, the following inequalities:*

$$c_1|x_k^t|^2 \le V_1(t, x_k^t) \le c_2|x_k^t|^2, \tag{9}$$
$$c_3|x_k^p|^2 \le V_2(x_k^p) \le c_4|x_k^p|^2,\ \forall p = t_i, t_{i+1} \tag{10}$$
$$\mathrm{div}V \le -c_5(|x_k^t|^2 + |x_k^{t_{i+1}}|^2), \tag{11}$$
$$c_2\,(c_4 - c_5) \le c_1 c_5\ \vee\ T_i \le \frac{c_2}{c_5}\alpha = T_{\max}, \tag{12}$$
$$T_{\min} = \frac{c_2}{c_5}\gamma \le T_i, \tag{13}$$

where $\gamma = -\ln\left[\frac{c_3(1-\varepsilon)}{c_5}\right]$ and *$\alpha = -\ln\left[\frac{c_2(c_4-c_5)-c_1c_5}{c_2(c_4-c_5)}\right]$ for all $c_2\,(c_4 - c_5) > c_1c_5$. Then, the $2D$ system* (3)-(4) *is EDS for any sequence $\{T_i\}_{i\in\mathbb{N}}$ such that $T_i \in [T_{\min}, T_{\max}]$, where $T_{\max}$ and $T_{\min}$ are given by* (12) *and* (13)*, respectively.*

Proof: From the divergence definition and inequalities (9), (10) and (11), it follows that

$$\begin{aligned} \frac{dV_1\left(t, x(t,k)\right)}{dt} &\le -c_5(|x_k^t|^2 + |x_k^{t_{i+1}}|^2) - V_2(x_{k+1}^{t_{i+1}}) + V_2(x_k^{t_{i+1}}), \\ &\le -\beta V_1\left(t, x_k^t\right) + \lambda V_2(x_k^{t_{i+1}}) - V_2(x_{k+1}^{t_{i+1}}), \end{aligned} \tag{14}$$

where $\lambda = 1 - \frac{c_5}{c_4}$ and $\beta = \frac{c_5}{c_2}$. By means of the comparison principle, with respect to the time t, from (14), for all $t \in [t_i, t_{i+1})$, it is obtained that

$$\begin{aligned} V_1\left(t, x_k^t\right) &\le e^{-\beta(t-t_i)}V_1\left(t_i, x_k^{t_i}\right) + \int_{t_i}^t e^{-\beta(t-\tau)}\left[\lambda V_2(x_k^{t_{i+1}}) - V_2(x_{k+1}^{t_{i+1}})\right]d\tau, \\ &= e^{-\beta(t-t_i)}V_1\left(t_i, x_k^{t_i}\right) + \rho_i(t)\left[\lambda V_2(x_k^{t_{i+1}}) - V_2(x_{k+1}^{t_{i+1}})\right], \end{aligned} \tag{15}$$

[2]*The diagonal dynamics make reference only to those dynamics given by (3)-(4) corresponding to $i = k$, for all $i, k \in \mathbb{N}$ and for all $t \in \mathbb{R}^+$.*

where $\rho_i(t) = \frac{1-e^{-\beta(t-t_i)}}{\beta} > 0$, for all $t \in [t_i, t_{i+1})$. In order to fulfill the statements given by Definition 1 it is necessary to prove *convergence* and *boundedness*. Thus, let us prove each one separately.

1) Convergence. Evaluating (15) for $t = t_{i+1}$, it gives

$$V_1(t_{i+1}, x_k^{t_{i+1}}) \leq e^{-\beta T_i} V_1(t_i, x_k^{t_i}) + \rho_i(t_{i+1})\lambda V_2(x_k^{t_{i+1}}) - \rho_i(t_{i+1}) V_2(x_{k+1}^{t_{i+1}}). \tag{16}$$

From the inequalities (9) and (10), it follows that $\forall i = k \in \mathbb{N}$

$$\frac{V_2(x_k^p)}{c_4} \leq |x_k^p|^2 \leq \frac{V_1(p, x_k^p)}{c_1}, \ \forall p = t_i, t_{i+1}, \tag{17}$$

$$\frac{V_1(p, x_k^p)}{c_2} \leq |x_k^p|^2 \leq \frac{V_2(x_k^p)}{c_3} \ \forall p = t_i, t_{i+1}. \tag{18}$$

From (17), it is given that $\frac{c_1}{c_4} V_2(x_k^{t_{i+1}}) \leq V_1(t_{i+1}, x_k^{t_{i+1}})$, and therefore from (16), it is obtained that

$$\rho_i(t_{i+1}) V_2(x_{k+1}^{t_{i+1}}) \leq e^{-\beta T_i} V_1(t_i, x_k^{t_i}) + \left(\rho_i(t_{i+1})\lambda - \frac{c_1}{c_4}\right) V_2(x_k^{t_{i+1}}). \tag{19}$$

Let us consider that $c_4 > c_5$, *i.e.* $\lambda \in (0, 1)$. Thus, from (18) and (19) it follows that

$$V_2(x_{k+1}^{t_{i+1}}) \leq \frac{c_2 e^{-\beta T_i}}{c_3 \rho_i(t_{i+1})} V_2(x_k^{t_i}) + \left(\frac{\rho_i(t_{i+1})\lambda - \frac{c_1}{c_4}}{\rho_i(t_{i+1})}\right) V_2(x_k^{t_{i+1}}). \tag{20}$$

Note that if the constraint $\rho_i(t_{i+1})\lambda \leq \frac{c_1}{c_4}$ holds, then the term depended on $V_2(x_k^{t_{i+1}})$ can be disregarded. In this sense, in order to satisfy such a constraint, recalling that $\beta = \frac{c_5}{c_2}$, $\lambda = 1 - \frac{c_5}{c_4}$ and $\rho_i(t_{i+1}) = \frac{1-e^{-\beta T_i}}{\beta}$, the following condition is founded

$$c_2 \left(1 - e^{-\beta T_i}\right)(c_4 - c_5) \leq c_1 c_5,$$
$$\left(1 - e^{-\beta T_i}\right) \leq \frac{c_1 c_5}{c_2 (c_4 - c_5)}.$$

Then, it is clear that if $c_2 (c_4 - c_5) \leq c_1 c_5$ holds then $\rho_i(t_{i+1})\lambda \leq \frac{c_1}{c_4}$ is trivially satisfied. Otherwise

$$e^{-\beta T_i} \geq \frac{c_2 (c_4 - c_5) - c_1 c_5}{c_2 (c_4 - c_5)} \Leftrightarrow T_i \leq \frac{c_2}{c_5}\alpha,$$

where $\alpha = -\ln\left[\frac{c_2(c_4-c_5)-c_1c_5}{c_2(c_4-c_5)}\right] > 0$, for all $c_2 (c_4 - c_5) > c_1 c_5$. Note that these two possibilities, *i.e.* $c_2 (c_4 - c_5) \leq c_1 c_5$ or $T_i \leq \frac{c_2}{c_5}\alpha$, are represented by (12) in Theorem 1. Therefore, if one of them is satisfied, from (20) it is obtained that

$$V_2(x_{k+1}^{t_{i+1}}) \leq \frac{c_2 e^{-\beta T_i}}{c_3 \rho_i(t_{i+1})} V_2(x_k^{t_i}).$$

Then, by induction, it follows that

$$V_2(x_{k+1}^{t_{i+1}}) \leq \left(\frac{c_2 e^{-\beta T_i}}{c_3 \rho_i(t_{i+1})}\right)^{k+1} V_2(x_0). \tag{21}$$

Hence, (21) decreases if the following condition holds

$$\frac{c_2 e^{-\beta T_i}}{c_3 \rho_i(t_{i+1})} \leq 1 - \varepsilon,$$
$$c_5 e^{-\beta T_i} \leq c_3 (1 - \varepsilon)\left(1 - e^{-\beta T_i}\right),$$
$$e^{-\beta T_i} \leq \frac{c_3 (1 - \varepsilon)}{c_5} \Leftrightarrow T_i \geq \frac{c_2}{c_5}\gamma,$$

which is the same that (13), with $\gamma = -\ln\left[\frac{c_3(1-\varepsilon)}{c_5}\right]$. Then, from (17), (18), and (21) , it follows that $\forall i = k \in \mathbb{N}$

$$|x_{k+1}^{t_{i+1}}|^2 \leq c\kappa_1^{k+1} |x_0|^2,$$

with $c = \frac{c_4}{c_3} > 0$ and $0 < \kappa_1 = \frac{c_5}{c_3(1-e^{-\gamma})} < 1 - \varepsilon$, for some small positive ε. Thus, the trajectories of system (3)-(4) are convergent under the constraints $c_4 > c_5$, $c_2 (c_4 - c_5) \leq c_1 c_5$ or $T_i \leq \frac{c_2}{c_5}\alpha$, and $T_i \geq \frac{c_2}{c_5}\gamma$, *i.e.* eq. (5) from Definition 1 is obtained.

Now, let us take into account that $c_5 \geq c_4$, *i.e.* $\lambda \leq 0$. Therefore, from (19), it follows that the term depended on $V_2(x_k^{t_{i+1}})$ can be disregarded, then one gets (21) and just under condition $T_i \geq \frac{c_2}{c_5}\gamma$ convergence is obtained. Thus, it is concluded that the trajectories of system (3)-(4) are convergent under constraints (12)-(13) if $c_4 > c_5$, or only under (13) if $c_5 \geq c_4$ holds. In order to complete the proof, let us prove boundedness between the impulses, *i.e.* $|x_k^t|^2 \leq \kappa_2 |x_k^{t_i}|^2$ for all $t \in [t_i, t_{i+1})$.

2) Boundedness. From (15), it is given that

$$V_1(t, x_k^t) \leq e^{-\beta(t-t_i)} V_1(t_i, x_k^{t_i}) + \rho_i(t_{i+1})\lambda V_2(x_k^{t_{i+1}}). \tag{22}$$

Let us consider the case $c_5 \geq c_4$, *i.e.* $\lambda \leq 0$. Therefore, from (22), it follows that $V_1(t, x_k^t) \leq e^{-\beta(t-t_i)} V_1(t_i, x_k^{t_i})$, $\forall i = k \in \mathbb{N}$, and boundedness is given, *i.e.*

$$|x_k^t|^2 \leq \kappa_2 |x_k^{t_i}|^2, \ \forall t \in [t_i, t_{i+1}) \,,$$

with $\kappa_2 = \frac{c_2}{c_5}$. Finally, for the case $c_4 > c_5$, *i.e.* $\lambda \in (0,1)$, from (22) and evaluating $t = t_{i+1}$, one gets

$$\begin{aligned} \frac{c_1}{c_4} V_2(x_k^{t_{i+1}}) &\leq e^{-\beta T_i} V_1(t_i, x_k^{t_i}) + \rho_i(t_{i+1})\lambda V_2(x_k^{t_{i+1}}), \\ V_2(x_k^{t_{i+1}}) &\leq \frac{e^{-\beta T_i}}{\left(\frac{c_1}{c_4} - \rho_i(t_{i+1})\lambda\right)} V_1(t_i, x_k^{t_i}). \end{aligned} \tag{23}$$

Note that $\rho_i(t_{i+1})\lambda < \frac{c_1}{c_4}$ has to hold in order to satisfy inequality (23). However, as it was previously described, if $c_2\,(c_4 - c_5) \leq c_1 c_5$ holds, $\rho_i(t_{i+1})\lambda < \frac{c_1}{c_4}$ is trivially satisfied, otherwise T_i should be less than or equal to $\frac{c_2}{c_5}\alpha$, *i.e.* $T_i \leq \frac{c_2}{c_5}\alpha$ with $\alpha = -\ln\left[\frac{c_2(c_4-c_5)-c_1c_5}{c_2(c_4-c_5)}\right] > 0$, for all $c_2\,(c_4 - c_5) > c_1 c_5$. Thus, applying (23) in (22), it is given that

$$\begin{aligned} V_1(t, x_k^t) &\leq \left[e^{-\beta(t-t_i)} + \frac{\rho_i(t_{i+1})\lambda e^{-\beta T_i}}{\left(\frac{c_1}{c_4} - \rho_i(t_{i+1})\lambda\right)} \right] V_1(t_i, x_k^{t_i}), \\ &\leq \left[\frac{c_5\,(c_1 - c_4\rho_i(t_{i+1})\lambda) + c_2 c_4}{c_5\,(c_1 - c_4\rho_i(t_{i+1})\lambda)} \right] V_1(t_i, x_k^{t_i}). \end{aligned} \tag{24}$$

Therefore, from (24), boundedness is obtained, *i.e.* $|x_k^t|^2 \leq \kappa_2 |x_k^{t_i}|^2$ for all $t \in \big[t_i, t_{i+1}\big)$ and $\kappa_2 = \frac{c_2(c_1c_5 + c_2c_4)}{c_1^2 c_5 - c_1 c_2 (c_4 - c_5)\left(1 - e^{-\gamma}\right)}$, which clearly is also valid for the case $c_5 \geq c_4$, *i.e.* $\lambda \leq 0$.

Thus, during each interval between impulses, the trajectories of the system are bounded by a constant value as in (6), and due to the convergence property given by (5), according to Definition 1, the $2D$ system described by (3)-(4) is EDS. ■

The statements given by Theorem 1 relies on a vector Lyapunov function approach in contrast to the results given in [7] (similarly in [11]), where *asymptotic stability* is obtained by means of a single Lyapunov function that needs to have a negative semi-definite derivative. Alternatively, our divergence operator, and not each term, needs to satisfy inequality (11).

Remark 1. *Based on the statements given by Theorem 1, the constructive application is illustrated by the Algorithm 1 which provides some notions of* minimum *and* maximum dwell-time *depending on the structure of the system dynamics. In particular, the first and third case for exponential diagonal stability (pseudo-code lines: 5 and 13, Algorithm 1) give conditions for* minimum dwell-time *while the second case (pseudo-code lines: 7, Algorithm 1) provides conditions for* maximum dwell-time.

Algorithm 1 Exponential Stability

```
1:  Define the Lyapunov functions V_1 and V_2
2:  Calculate the constants c_1, c_2, c_3, c_4 and c_5
3:  if c_4 > c_5 then
4:      if c_2 (c_4 − c_5) ≤ c_1 c_5 and T_i > (c_2/c_5) γ then
5:          "System is EDS"
6:      else if (c_2/c_5) α > T_i > (c_2/c_5) γ then
7:          "System is EDS"
8:      else
9:          "No Conclusion"
10:     end
11: else
12:     if T_i > (c_2/c_5) γ then
13:         "System is EDS"
14:     else
15:         "No Conclusion"
16:     end
17: end
18: end Algorithm
```

In the following, the statements given by Theorem 1 are applied to provide numerical tools for the exponential stability analysis of linear impulsive systems.

IV. Case Study: Linear Impulsive Systems

In this section the linear case is presented as an illustration of the main result given by Theorem 1. In the following a direct application of the proposed results is presented, then a simple quadratic Lyapunov function approach and numerical tools based on LMIs are provided to get simple conditions that look similar to existing works.

A. Exponential Stability

Consider a linear impulsive system with a timer variable as follows

$$\frac{dx_k^t}{dt} = Ax_k^t,\ \frac{d\tau_k^t}{dt} = 1,\ \forall t \in [t_i, t_{i+1}), \tag{25}$$

$$x_{k+1}^{t_{i+1}} = Ex_k^{t_{i+1}},\ \tau_{k+1}^{t_{i+1}} = 0,\ \forall i = k \in \mathbb{N}, \tag{26}$$

where $\tau_k^t \in [T_{\min}, T_{\max}] \subset \mathbb{R}_+$ is the timer variable, the constant matrices A and E have corresponding dimensions. Inspired by [7] and [16], the timer variable was introduced in order to extend the state of the system and provide smooth Lyapunov functions with less conservative results. It has been also shown that system (25)-(26) can be used to describe more general impulsive systems (see *e.g.* [12] and the references therein).

In this sense, define an extended vector $z_k^t = [(x_k^t)^T\ \tau_k^t]^T \in \mathbb{R}^{n+1}$; then, system (25)-(26) can be rewritten as follows

$$\frac{dz_k^t}{dt} = A_z z_k^t,\ \forall t \in [t_i, t_{i+1}), \tag{27}$$

$$z_{k+1}^{t_{i+1}} = E_z z_k^{t_{i+1}},\ \forall i = k \in \mathbb{N}, \tag{28}$$

where

$$A_z = \begin{bmatrix} A & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix},\ E_z = \begin{bmatrix} E & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix}.$$

Note that system (25)-(26) has the structure given by (3)-(4), and the main result, *i.e.* Theorem 1, may be directly applied to this extended system. However, since only stability in x_k^t is required, the introduction of a more specific definition of stability is necessary, *i.e. partial stability* or, briefly, x_k^t*-stability* (for more details see [17]).

Definition 3. *A 2D system described by (27)-(28), is said to be exponentially diagonal x_k^t-stable (EDx_k^t-S) if there exist positive constants κ_1, κ_2, κ_3, c and ε such that*

$$|x_{k+1}^{t_{i+1}}|^2 \le c\kappa_1^{k+1}|x_0|^2,\ 0 < \kappa_1 < 1-\varepsilon, \tag{29}$$

$$|x_k^t|^2 \le \kappa_2 |x_k^{t_i}|^2,\ \forall t \in [t_i, t_{i+1}), \tag{30}$$

$$|\tau_k^t| \le \kappa_3,\ \forall t \in [t_i, t_{i+1}), \tag{31}$$

for all $i = k \in \mathbb{N}$, and a small positive ε.

Note that condition (31) holds by definition, *i.e.* $|\tau_k^t| \le \kappa_3,\ \forall t \in [t_i, t_{i+1})$, with $\kappa_3 = T_{\max}$. Hence, based on the extended system (27)-(28) and Definition 3, the following result is provided:

Corollary 1. *Assume that there exist positive constants ε, c_1, c_2, c_3, c_4 and c_5 such that the vector Lyapunov function $V(z_k^t) = [V_1(z_k^t)\ V_2(z_{k+1}^{t_{i+1}})]^T$ and its divergence along the trajectories of the system* (27)-(28) *satisfy, for all $t \in [t_i, t_{i+1})$, the following inequalities:*

$$c_1|x_k^t|^2 \le V_1(z_k^t) \le c_2|x_k^t|^2, \tag{32}$$

$$c_3|x_k^p|^2 \le V_2(z_k^p) \le c_4|x_k^p|^2,\ \forall p = t_i, t_{i+1} \tag{33}$$

$$\mathrm{div}V \le -c_5(|x_k^t|^2 + |x_k^{t_{i+1}}|^2), \tag{34}$$

$$c_2\,(c_4 - c_5) \le c_1 c_5\ \vee\ T_i \le \frac{c_2}{c_5}\alpha = T_{\max}, \tag{35}$$

$$T_{\min} = \frac{c_2}{c_5}\gamma \le T_i, \tag{36}$$

where $\gamma = -\ln\left[\frac{c_3(1-\varepsilon)}{c_5}\right]$ and *$\alpha = -\ln\left[\frac{c_2(c_4-c_5)-c_1c_5}{c_2(c_4-c_5)}\right]$ for all $c_2\,(c_4 - c_5) > c_1c_5$. Then, the system* (27)-(28) *is EDx_k^t-S for any sequence $\{T_i\}_{i\in\mathbb{N}}$ such that $T_i \in [T_{\min}, T_{\max}]$, where $T_{\max}$ and $T_{\min}$ are given by* (35) *and* (36)*, respectively.*

Proof: The procedure to prove the statements given in this theorem follows in spirit the proof given for Theorem 1. In this sense, from (8), (32), (33) and (34), it follows that

$$\frac{dV_1(z_k^t)}{dt} \le -\beta V_1(z_k^t) + \lambda V_2(z_k^{t_{i+1}}) - V_2(z_{k+1}^{t_{i+1}}), \tag{37}$$

where $\lambda = 1 - \frac{c_5}{c_4}$ and $\beta = \frac{c_5}{c_2}$. By means of the comparison principle, with respect to t, from (37), for all $t \in [t_i, t_{i+1})$, it follows

$$V_1(z_k^t) \le e^{-\beta(t-t_i)}V_1(z_k^0) + \rho_i(t_i)(\lambda V_2(z_k^{t_{i+1}}) - V_2(z_{k+1}^{t_{i+1}})), \tag{38}$$

where $\rho_i(t_i) = \frac{1-e^{-\beta(t-t_i)}}{\beta} > 0$, for all $t \in [t_i, t_{i+1})$. Evaluating (38) for $t = t_{i+1}$, it is obtained that

$$V_1(z_k^{t_{i+1}}) \le e^{-\beta T_i}V_1(z_k^{t_i}) + \rho_i(t_{i+1})\lambda V_2(z_k^{t_{i+1}}) - \rho_i(t_{i+1})V_2(z_{k+1}^{t_{i+1}}), \tag{39}$$

with $\rho_i(t_{i+1}) = \frac{1-e^{-\beta T_i}}{\beta} > 0$, for all $T_i \geq 0$. Therefore, from the inequalities (32) and (33), it follows that

$$\frac{V_2(z_k^p)}{c_4} \leq |x_k^p|^2 \leq \frac{V_1(z_k^p)}{c_1},\ \forall p = t_i, t_{i+1} \tag{40}$$

$$\frac{V_1(z_k^p)}{c_2} \leq |x_k^p|^2 \leq \frac{V_2(z_k^p)}{c_3},\ \forall p = t_i, t_{i+1}. \tag{41}$$

From this point forward, based on (39),(40) and (41); the proof repeats the steps given in Theorem 1, from eq. (16) to eq. (24), and then the constraints (35)-(36), as in Theorem 1, ensure the convergence and boundedness of the trajectories x_k^t, *i.e.*

$$|x_{k+1}^{t_{i+1}}|^2 \leq c\kappa_1^{k+1}|x_0|^2,\ 0 < \kappa_1 < 1-\varepsilon,$$
$$|x_k^t|^2 \leq \kappa_2 |x_k^{t_i}|^2,\ \forall t \in [t_i, t_{i+1}),$$

where $\kappa_1 = \frac{c_5}{c_3(1-e^{-\gamma})}$ and $\kappa_2 = \frac{c_2(c_1c_5+c_2c_4)}{c_1^2c_5 - c_1c_2(c_4-c_5)\left(1-e^{-\gamma}\right)}$. Finally, since $|\tau_k^t| \leq \kappa_3,\ \forall t \in [t_i, t_{i+1})$, with $\kappa_3 = T_{\max}$, based on Definition 3, the exponential diagonal x_k^t-stability is proven. ∎

B. Quadratic Lyapunov Functions

Consider that V_1 and V_2 take the following quadratic structure

$$V_l(x_k^t, \tau_k^t) = (x_k^t)^T P_l(\tau_k^t) x_k^t,\ l = 1, 2,$$

where $P_1(\tau_k^t)$ is continuously differentiable with respect to t, symmetric, bounded, and positive definite matrix, while $P_2(\tau_k^p)$, $\forall p = t_i, t_{i+1}$, are symmetric, bounded, and positive definite matrices, *i.e.*

$$0 < c_1 I \leq P_1(\tau_k^t) \leq c_2 I,\ \forall t \in [t_i, t_{i+1}), \tag{42}$$

$$0 < c_3 I \leq P_2(\tau_k^p) \leq c_4 I,\ \forall p = t_i, t_{i+1}. \tag{43}$$

From divergence definition it follows that for all $t \in [t_i, t_{i+1})$

$$\text{div}V = (x_k^t)^T \left(P_1(\tau_k^t)A + A^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt}\right) x_k^\tau + (x_k^{t_{i+1}})^T \left(E^T P_2(0) E - P_2(T_i)\right) x_k^{t_{i+1}}.$$

Then, one gets that $\text{div}V \leq -c_5(|x_k^t|^2 + |x_k^{t_{i+1}}|^2)$, if it is possible to find some matrices $P_1(\tau_k^t)$, $P_2(0)$ and $P_2(T_i)$ such that for all $t \in [t_i, t_{i+1})$ and $i \in \mathbb{N}$

$$\begin{bmatrix} P_1(\tau_k^t)A + A^T P_1(\tau_k^t) + \frac{dP_1(\tau_k^t)}{dt} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & E^T P_2(0) E - P_2(T_i) \end{bmatrix} \leq -c_5 I. \tag{44}$$

Hence, if the constraints (35)-(36) hold; for c_1, c_2, c_3, c_4 and c_5; the system (25)-(26) is EDx_k^t-S; and the following corollary has been proven.

Corollary 2. *Consider the vector Lyapunov function* $V(x_k^t, \tau_k^t)$*, with* $V_l(x_k^t, \tau_k^t) = (x_k^t)^T P_l(\tau_k^t) x_k^t,\ l = 1, 2,$ *for all* $t \in [t_i, t_{i+1})$*. Assume that there exist matrices* $P_1(\tau_k^t) = P_1^T(\tau_k^t) > 0$*, continuously differentiable on t, and* $P_2(\tau_k^p) = P_2^T(\tau_k^p) > 0$, $\forall p = t_i, t_{i+1}$ *satisfying* (42)-(43)*. If the matrix inequality* (44) *holds and constraints* (12)-(13) *are satisfied with* c_1*,* c_2*,* c_3*,* c_4 *and* c_5*, then the system* (27)-(28) *is ED*x_k^t*-S.*

Note that Corollary 2 is able to deal with linear impulsive systems where matrix A is not Hurwitz, and/or E is anti-Schur, respectively. In this sense, Corollary 2 provides a general result due to the introduction of the timer variable.

For the particular case in which matrices $P_1(\tau_k^t)$ and $P_2(\tau_k^t)$ are designed in the same way, *i.e.* $P_1(\tau_k^t) = P_2(\tau_k^t)$, the statements given by Corollary 2 coincide with the previously obtained in [3]. Therefore, Corollary 2 provides a more general way to chose the corresponding matrices $P_1(\tau_k^t)$ and $P_2(\tau_k^t)$ since they can be designed in a different form.

C. Numerical Solution: LMI Approach

It is clear that the problem is now how to design feasible matrices $P_1(\tau_k^t)$ and $P_2(\tau_k^t)$, in terms of t, such that the statements of Corollary 2 are satisfied. Inspired by [1] and [3], the following proposition provides a way to design such matrices in order to ensure the *exponential diagonal* x_k^t*-stability* based on LMIs.

Proposition 1. *Consider that* $P_1(\tau_k^t)$ *and* $P_2(\tau_k^t)$ *have the following structure for all* $t \in [t_i, t_{i+1})$

$$P_1(\tau_k^t) = P_{11} + \frac{(T_i - \tau_k^t)}{T_i} P_{12},\ P_2(\tau_k^t) = P_{21} + \tau_k^t P_{22},$$

where $P_{jl} = P_{jl}^T > 0$*, for* $j, l = 1, 2$*. Then, the system* (25)-(26) *is ED*x_k^t*-S if there exist matrices* P_{jl}*, for* $j, l = 1, 2$ *such that the following LMIs*

$$\Upsilon_0(\Theta) \leq -Q_1, \quad \Upsilon_1(\Theta) \leq -Q_2, \tag{45}$$

for the finite set $\Theta \in \{T_{\min}, T_{\max}\}$*,* $Q_1 = Q_1{}^T > 0$ *and* $Q_2 = Q_2{}^T > 0$*, with*

$$\Upsilon_0(\Theta) = \begin{bmatrix} (P_{11}+P_{12})A + A^T(P_{11}+P_{12}) - \frac{1}{\Theta}P_{12} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & E^T P_{21} E - (P_{21} + \Theta P_{22}) \end{bmatrix},$$
$$\Upsilon_1(\Theta) = \begin{bmatrix} P_{11}A + A^T P_{11} - \frac{1}{\Theta}P_{12} & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & E^T P_{21} E - (P_{21} + \Theta P_{22}) \end{bmatrix},$$

and constraints (12)-(13) *hold; with* $c_1 = \lambda_{\min}(P_{11})$*,* $c_2 = \lambda_{\max}(P_{11}+P_{12})$*,* $c_3 = \lambda_{\min}(P_{21})$*,* $c_4 = \lambda_{\max}(P_{21} + \Theta P_{22})$ *and* $c_5 = \min(\lambda_{\min}(Q_1), \lambda_{\min}(Q_2))$.

Proof: From divergence definition it is obtained that for all $t \in [t_i, t_{i+1})$

$$\mathrm{div}V \leq \begin{bmatrix} x_k^t \\ x_k^{t_{i+1}} \end{bmatrix}^T M(T_i) \begin{bmatrix} x_k^t \\ x_k^{t_{i+1}} \end{bmatrix},$$

where

$$M(T_i) = \begin{bmatrix} P_{11}A + A^T P_{11} - \frac{1}{T_i}P_{12} + \frac{(T_i - \tau_k^t)}{T_1}(P_{12}A + A^T P_{12}) & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & E^T P_{21}E - (P_{21} + T_i P_{22}) \end{bmatrix}.$$

Then, since this LMI is affine in τ_k^t, its negative definiteness is given by the negativeness over the finite set $\tau_k^t \in \{0, T_i\}$; hence, the LMIs (45) are provided. Then, if these LMIs are feasible for the finite set $\Theta \in \{T_{\min}, T_{\max}\}$, the divergence of V satisfies (34) with $c_5 = \min(\lambda_{\min}(Q_1), \lambda_{\min}(Q_2))$.
Finally, it is easy to check the boundedness of $P_1(t)$ and $P_2(t)$, *i.e.*

$$0 < \lambda_{\min}(P_{11})I \leq P_1(\tau_k^t) \leq \lambda_{\max}(P_{11}+P_{12})I,$$
$$0 < \lambda_{\min}(P_{21})I \leq P_2(\tau_k^t) \leq \lambda_{\max}(P_{21}+T_iP_{22})I,$$

for all $t \in [t_i, t_{i+1})$. Therefore, according to the statements given by Corollary 2, system (25)-(26) is EDx_k^t-S, if constraints (12)-(13) are satisfied. ■

Note that Corollary 2 and Proposition 1 provide an easy way to deal with stability by means of simple quadratic Lyapunov functions. Note that more complex tools like *sum-of-squares* [3], *looped-functional approach* [5], or *convex characterizations* [4], may be applied to improve the application of our method for the linear case. In the following two linear examples are provided to depict the LMI approach. The examples illustrate the aperiodic and periodic impulse case, respectively. Both examples are taken from [5] and the obtained results are very similar.

Example 1. Let us consider system (27)-(28) with

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Note that the continuous dynamics is unstable while the discrete one is stable, *i.e.* A is not Hurwitz and E is Schur. Proposition 1 is applied together with a bisection-like approach and the following feasible results are obtained:

$$P_{11} = \begin{bmatrix} 1.3914 & 0.0139 \\ 0.0139 & 1.3597 \end{bmatrix}, \; P_{12} = 10^4 \begin{bmatrix} 1.2638 & -0.6319 \\ -0.6319 & 3.7917 \end{bmatrix},$$
$$P_{21} = \begin{bmatrix} 1.0007 & 0 \\ 0 & 1.0007 \end{bmatrix}, \; P_{22} = \begin{bmatrix} 0.9336 & 0 \\ 0 & 0.9336 \end{bmatrix},$$
$$Q_1 = 0.5I_4, \; Q_2 = I_4,$$

with $c_1 = 1.3545$, $c_2 = 3.9410 \times 10^4$, $c_3 = 1.0007$, $c_4 = 1.3118$ and $c_5 = 0.5000$. Therefore, according to Proposition 1, the impulsive system is EDx_k^t-S for all $0.3333 > T_i > 0$. It is easy to check that $c_4 > c_5$ and $c_2\,(c_4 - c_5) > c_1 c_5$ hold and then the second case for exponential diagonal x_k^t-stability is obtained. The trajectories of the system, for different values of T_i (aperiodic impulse case), are depicted in Fig. 1. When the analysis is restricted to single quadratic Lyapunov functions linear in τ_k^t, it is possible to show stability for $0.2400 > T_i > 0$.

Remark 2. *Note that Algorithm 1 can be easily adapted for the* EDx_k^t-S *case. In this sense, Example 1 (A unstable and E Schur) illustrates the conditions for* maximum dwell-time *given by Theorem 1 and depicted in pseudo-code lines: 7, Algorithm 1 to the* EDx_k^t-S *case.*

Example 2. Let us consider system (27)-(28) with

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

For this example the continuous dynamics is stable while the discrete one is unstable, *i.e.* A is Hurwitz and E is anti-Schur.

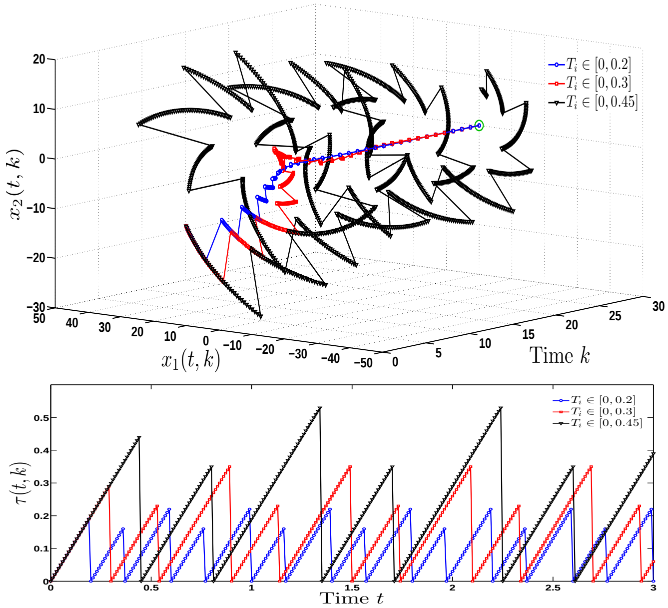

Figure 1. Example 1 (*Unstable-Stable*): Trajectories of the linear impulsive system for different values of T_i (aperiodic impulse case).

Proposition 1 is applied and the following feasible results are obtained:

$$P_{11} = \begin{bmatrix} 3.3096 & 0.4527 \\ 0.4527 & 1.4411 \end{bmatrix}, \; P_{12} = \begin{bmatrix} 0.6648 & 0.0963 \\ 0.0963 & 0.3941 \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} 0.1123 & 0 \\ 0 & 0.1123 \end{bmatrix}, \; P_{22} = \begin{bmatrix} 0.0050 & 0.0006 \\ 0.0006 & 0.0056 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 5.7158 & -0.0941 & 0 & 0 \\ \star & 5.9605 & 0 & 0 \\ \star & \star & 4.5800 & 0 \\ \star & \star & \star & 4.5800 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 5.1473 & -0.0415 & 0 & 0 \\ \star & 5.1724 & 0 & 0 \\ \star & \star & 4.5800 & 0 \\ \star & \star & \star & 4.5800 \end{bmatrix},$$

with $c_1 = 1.3372$, $c_2 = 4.1071$, $c_3 = 0.1123$, $c_4 = 6.05$ and $c_5 = 4.58$. Therefore, according to Proposition 1, the impulsive system is EDx_k^t-S for all $T_i > 3.3254$. It is easy to check that $c_4 > c_5$ and $c_2\left(c_4 - c_5\right) \leq c_1 c_5$ hold and then the first case for exponential diagonal x_k^t-stability is obtained. The trajectories of the system, for different values of T_i (periodic impulse case), are depicted in Fig. 2. If the analysis is restricted to single quadratic Lyapunov functions linear in τ_k^t, it is possible to show stability for $T_i > 5.1000$.

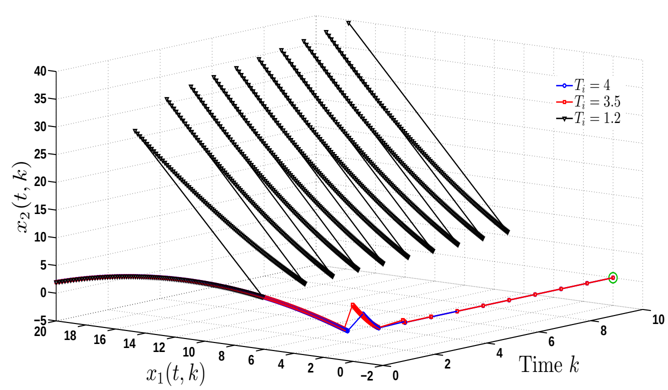

Figure 2. Example 2 (*Stable-Unstable*): Trajectories of the linear impulsive system for different values of T_i (periodic impulse case).

Remark 3. *Note that Example 2 (A Hurwitz and E anti-Schur) illustrates the conditions for* minimum dwell-time *given by Theorem 1 and depicted in pseudo-code lines: 5 and 13, Algorithm 1 to the* $\mathrm{ED}x_k^t$-S *case.*

Remark 4. *The previous examples show numerically that when the analysis is restricted to the same class of Lyapunov functions,* i.e. *linear with respect to τ_t^k, the vector Lyapunov function is less conservative than the single one.*

V. Conclusions

In this paper a method is proposed to the development of a new stability analysis for impulsive dynamical systems based on a vector Lyapunov function and its divergence operator in a $2D$ time domain. The result is illustrated for the exponential stability of linear impulsive systems based on LMIs, and some numerical examples illustrate the feasibility of the proposed approach. The obtained results provide some notions of *minimum* and *maximum* dwell-time. A proper comparison with existing works is in the scope of the future research as well as average dwell-time conditions and the analysis of exponential stability for nonlinear systems.

References

[1] L.I Allerhand and U. Shaked. Robust stability and stabilization of linear switched systems with dwell time. *IEEE Transactions on Automatic Control*, 56(2):381–386, 2011.

[2] D. Bainov and P. Simeonov. *Systems with Impulse Effects: Stability, Theory and Applications*. Academy Press, 1989.

[3] C. Briat. Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints. *Automatica*, 49:3449–3457, 2013.

[4] C. Briat and A. Seuret. Convex dwell-time characterizations for uncertain linear impulsive systems. *IEEE Transactions on Automatic Control*, 57(12):3241–3246, 2012.

[5] C. Briat and A. Seuret. A looped-functional approach for robust stability of linear impulsive systems. *Systems & Control Letters*, 61(10):980–988, 2012.

[6] G.E. Dullerud and S. Lall. Asynchronous hybrid systems with jumps analysis and synthesis. *Systems & Control Letters*, 37(2):61–69, 1999.

[7] R. Goebel, R.G. Sanfelice, and A.R. Teel. *Hybrid Dynamical Systems: Modeling, Stability and Robustness*. Princeton University Press, New Jersey, USA, 2012.

[8] L. Hu, Y. Cao, and H. Shao. Constrained robust sampled-data control of nonlinear uncertain systems. *International Journal of Robust and Nonlinear Control*, 12:447–155, 2002.

[9] V. Lakshmikantham, V. M. Matrosov, and S. Sivasundaram. *Vector Lyapunov Functions and Stability Analysis of Nonlinear Systems*. Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1991.

[10] P. Naghshtabrizi, J.P. Hespanha, and A.R. Teel. Exponential stability of impulsive systems with application to uncertain sampled-data systems. *Systems and Control Letters*, 57(5):378–385, 2008.

[11] D. Nešić, A.R. Teel, and L. Zaccarian. Stability and performance of siso control systems with first-order reset elements. *IEEE Transactions on Automatic Control*, 56(11):2567–2582, 2011.

[12] D. Nešić, L. Zaccarian, and A.R. Teel. Stability properties of reset systems. *Automatica*, 44(8):2019–2026, 2008.

[13] E. Rogers, K. Galkowski, and D. Owens. *Control Systems Theory and Applications for Linear Repetitive Processes*, volume 349 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 2007.

[14] A. Seuret. A novel stability analysis of linear systems under asynchronous samplings. *Automatica*, 48(1):177–182, 2012.

[15] N. Sivashankar and P.P. Khargonekar. Characterization of the $\mathcal{L}_2$-induced norm for linear systems with jumps with applications to sampled-data systems. *SIAM Journal on Control and Optimization*, 32(4):1128–1150, 1994.

[16] W. Sun, K.M. Nagpal, and P.P. Khargonekar. $\mathcal{H}_\infty$ control and filtering for sample-data systems. *IEEE Transactions on Automatic Control*, 38(8):1162–1175, 1993.

[17] V. I. Vorotnikov. *Partial stability and Control*. Birkhauser, Boston, 1998.

[18] N. Yeganefar, N. Yeganefar, M. Ghamgui, and E. Moulay. Lyapunov theory for 2-d nonlinear roesser models: Application to asymptotic and exponential stability. *IEEE Transactions on Automatic Control*, 58(5):1299–1304, 2013.

[19] S.T. Zavalishchin and A.N. Sesekin. *Dynamic Impulse Systems. Theory and Applications*. Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1997.