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The impact of source terms in the variational representation of traffic flow

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Abstract

This paper revisits the variational theory of traffic flow, now under the presence of continuum lateral inflows and outflows to the freeway say Eulerian source terms. It is found that a VT solution can be easily exhibited only in Eulerian coordinates when source terms are exogenous meaning that they only depend on time and space, but not when they are a function of traffic conditions, as per a merge model. In discrete time, however, these dependencies become exogenous, which allowed us to propose improved numerical solution methods. In Lagrangian and vehicle number-space coordinates, VT solutions may not exist even if source terms are exogenous.

Keywords: traffic flow, source terms, kinematic wave model

1. Introduction

1 The variational theory (VT) applied in traffic flow theory (Daganzo, 2005a,b; Claudel and
2 Bayen, 2010a,b) was an important milestone. Previously, the only analytical solution to the
3 kinematic wave model of Lighthill and Whitham (1955); Richards (1956) was obtained thanks
4 to the method of characteristics, which does not give a global solution in time as one needs
5 to keep track of characteristics crossings (shocks and rarefaction waves) and impose entropy
6 conditions to ensure uniqueness. This means that analytical solutions cannot be formulated
7 except for very simple problems.

8
9 In contrast, VT makes use of the link between conservation laws and the Hamilton-Jacobi
10 partial differential equation (HJ PDE), allowing the kinematic wave model to be solved using
11 the Hopf-Lax formula (Lax, 1957; Olejnik, 1957; Hopf, 1970), better known in transportation as
12 Newell's minimum principle (Newell, 1993) whenever the fundamental diagram or Hamiltonian
13 is piecewise linear. The big advantage is that this representation formula gives an analytical
14 global solution in time that does not require explicit consideration of shocks and/or entropy
15 conditions. Moreover, current approximation methods for the Macroscopic Fundamental Diagram
16 (MFD) of urban networks rely on this approach (Daganzo and Geroliminis, 2008; Geroliminis
17 and Boyacı, 2013; Leclercq and Geroliminis, 2013; Laval and Castrillón, 2015). In addition,
18 when the flow-density fundamental diagram is triangular (or piecewise linear more generally),
19 numerical solutions become exact in the HJ framework, and this is not the case in the con-
20 servation law approach (LeVeque, 1993). It has also been shown that the traffic flow problem

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21 cast in Lagrangian or vehicle number-space coordinates also accepts a VT solution, which can
 22 be used to obtain very efficient numerical solution methods (Leclercq et al., 2007; Laval and
 23 Leclercq, 2013).

24 The traffic flow problem with source term is also of great interest; e.g., it can be used to
 25 approximate (i) long freeways with closely spaced entrances and exits, (ii) the effect of lane-
 26 changing activity on a single lane, or (iii) the effects of turning movements, trip generation
 27 and trip ends in the MFD. But the underlying assumption in the previous paragraphs is that
 28 vehicles are conserved. This begs the twofold question, are representation formulas for the
 29 VT solutions still valid, or even applicable, when there is a Eulerian source term? If not, can
 30 efficient numerical solution methods still be implemented? Recent developments in this area
 31 have not answered these questions as they are primarily concerned with discrete source terms
 32 (e.g., Daganzo, 2014; Costeseque and Lebacque, 2014a,b; Li and Zhang, 2013).

33 To answer these questions this paper is organized as follows. In section 2 we formulate the
 34 general problem and show that in general VT solutions are not applicable; but section 3 shows
 35 that they are when the source term is exogenous. Based on these results, section 4 presents
 36 numerical methods for the endogenous inflow problem that outperform existing methods. Sec-
 37 tion 5 briefly shows that in space-Lagrangian and time-Lagrangian coordinates VT solutions do
 38 not exist even if source terms are exogenous. A discussion of results and outlook is presented
 39 in section 6.

40 2. Problem Formulation

41 Consider a long homogeneous freeway corridor with a large number of entrances and exits
 42 such that the net lateral freeway inflow rate, ϕ , or inflow for short, can be treated as a continuum
 43 variable in time $t \geq 0$ and location $x \in \mathbb{R}$, and has units of veh/time-distance. The inflow is an
 44 endogenous variable consequence of the demand for travel, and could be captured by a function
 45 of the traffic states both in the freeway and the ramps. For simplicity, in this paper the inflow
 46 is assumed to be a function of the density, $k(t, x)$, of the freeway only, i.e. $\phi = \phi(k(t, x))$, but
 47 also the exogenous case $\phi = \phi(t, x)$ will be of interest.

In any case, the traffic flow problem analyzed in this paper is the following conservation law
 with source term:

$$k_t + H(k)_x = \phi, \tag{1a}$$

$$k = g \quad \text{on } \Gamma \tag{1b}$$

where H is the fundamental diagram, g is the data defined on a boundary Γ , and variables in
 subscript represent partial derivatives. Now we define the function $N(t, x)$ such that:

$$N_x = -k \quad \text{say} \quad N(t, x) := \int_x^{+\infty} k(t, y) dy, \tag{2}$$

and integrate (1) with respect to x to obtain its HJ form (Evans, 1998):

$$N_t - H(-N_x) = \Phi, \tag{3a}$$

$$N = G \quad \text{on } \Gamma, \tag{3b}$$

where we have defined:

$$G(t, x) = \oint_{\Gamma} g(t, x) d\Gamma, \quad (t, x) \in \Gamma, \quad \text{and} \quad (4a)$$

$$\Phi(t, x) = - \int_0^x \phi(t, y) dy \quad (4b)$$

where Φ is a potential function; the negative sign in (4b) follows from the traditional counting convention in traffic flow, where further downstream vehicles have lower vehicle numbers.

It is important to note that compared to the traditional case with zero inflow, under a continuum source term the interpretation of N changes: for fixed x it still defines a cumulative count curve due to (2) but its isometrics do no longer give vehicle trajectories. To see this we note that according to (3a) the flow $q = H(k)$ is now:

$$q = N_t - \Phi, \quad (5)$$

The time integration of (5) reveals that cumulative count curves are now given, up to an arbitrary constant, by $\tilde{N}(t, x) + \int_0^t \int_0^x \phi(s, y) ds dy$, where $\tilde{N}(t, x) := \int_0^t q(s, x) ds$ denotes the usual N -curve without source terms and the integral represents the net number of vehicles entering ($\phi > 0$) or exiting ($\phi < 0$) the road segment by time t and upstream of x .

The method to obtain the solution of (3) depends on the dependencies of the potential function. If the inflow function is allowed to depend on the traffic state in the freeway, i.e. $\phi = \phi(k)$ then the potential function Φ depends non-locally on N , since:

$$\Phi(t, x) = \tilde{\Phi}(N, x) = - \int_0^x \phi(-N_x(t, y)) dy, \quad (6)$$

and therefore it is not obvious that (3) accepts a representation formula as a VT solution. To see this, we note that even in the simplest linear case:

$$\phi(k) = a - bk, \quad a, b \geq 0, \quad \text{we get:} \quad (7a)$$

$$\tilde{\Phi}(N, x) = -ax + \left(\int_0^x k(t, y) dy \right) b = -ax - (N(t, x) - c)b, \quad (7b)$$

where $c = \int_0^{+\infty} k(t, y) dy = N(t, 0)$ is an arbitrary constant of integration. This means that (3) becomes the more general HJ equation $N_t - \tilde{H}(x, N, -N_x) = 0$, where \tilde{H} is the Hamiltonian. The Hamiltonian's N -dependency is what complicates matters. Barron et al. (1996); Barron (2015) show that a Hopf-Lax type solution exists in this case only when (among other assumptions) $a = 0$ i.e. $\tilde{H}(x, u, p) = \hat{H}(u, p)$ where \hat{H} does not depend on the space variable and \hat{H} is homogeneous of degree one with respect to N_x , which is of no use in traffic flow since in practice it means that the Hamiltonian has to be monotone linear. As explained in section 5, we argue that this N -dependency prevents formulating an equivalent VT problem (such as (8) below) that can be solved with variational methods.

As shown in the next section, it turns out that when inflows are exogenous a global VT solution can still be identified, albeit not in Hopf-Lax form because optimal paths are no longer straight lines.

66 3. Exogenous inflow

67 3.1. Setting of the VT problem

The results in this section are based on VT (Daganzo, 2005a), where the solution of the HJ equation $N_t - \tilde{H}(t, x, -N_x) = 0$ is given by the solution of the following variational problem:

$$N(P) = \min_{B \in \Gamma^P, \xi \in \mathcal{V}_{BP}} f(B, \xi), \quad \text{with} \quad (8a)$$

$$f(B, \xi) = G(B) + \int_{t_B}^t R(s, \xi(s), \xi'(s)) ds \quad (8b)$$

where P is a generic point with coordinates (t, x) , $B \equiv (t_B, y)$ is a point in the boundary Γ^P , ξ is a member of the set of all valid paths between B and P denoted \mathcal{V}_{BP} , and $\xi(t_B) = y$; see Fig. 1a. The function $R(\cdot)$ gives the maximum passing rates along the observer and corresponds to the (concave) Legendre transform of \tilde{H} , i.e.,

$$R(t, x, v) = \sup_k \left\{ \tilde{H}(t, x, k) - vk \right\}.$$

It is worth mentioning that in the simplest homogeneous case where $\tilde{H} = H(k)$ the VT solution (8) becomes the Hopf-Lax formula:

$$N(P) = \min_{B \in \Gamma^P} \left\{ G(B) + (t - t_B) R \left(\frac{x - y}{t - t_B} \right) \right\}, \quad (9)$$

68 where the minimization over $\xi(t)$ is no longer necessary since characteristics become straight
 69 lines in this case. With a little abuse of notation, consider $R(v) = \sup_k \{H(k) - kv\}$. Un-
 70 fortunately, even the presence of exogenous lateral inflows that vary in time or space make
 71 characteristics not to be straight lines and therefore a Hopf-Lax type solution cannot be de-
 72 vised. A VT solution, however, still exists as shown next.

73

We now formulate VT solution to account for source terms explicitly. In the remaining of the paper, we assume a triangular flow-density diagram. It may be defined by its free-flow speed u , wave speed $-w$ (with $w > 0$) and jam density κ such that

$$H(k) = \min \{ uk, w(\kappa - k) \} \quad \text{for any } k \in [0, \kappa]. \quad (10)$$

74 It follows that the capacity is $Q = \kappa w u / (w + u)$ and the critical density $K = Q/u$.

75

When the potential function is exogenous, i.e. $\Phi = \Phi(t, x)$, the Hamiltonian can be written as the sum of the fundamental diagram and the potential function, i.e.: $\tilde{H}(t, x, k) = H(k) + \Phi(t, x)$. In the case of a triangular fundamental diagram we have $R(t, x, v) = Q - Kv + \Phi(t, x)$ with $v \in [-w, u]$, and the function $f(B, \xi)$ to be minimized reads:

$$f(B, \xi) = G(B) + (t - t_B)Q - (x - y)K + \underbrace{\int_{t_B}^t \Phi(s, \xi(s)) ds}_{=: J}. \quad (11)$$

The J -integral in (11) is what separates this problem from the problems studied so far in traffic flow using VT principles. This integral in terms of ϕ is:

$$J = - \int_{t_B}^t \int_y^{\xi(s)} \phi(s, x) dx ds, \quad (12)$$

76 and represents the net number of vehicles *leaving* the area below the curve $x = \xi(t)$, namely
 77 area $A(\xi)$; see Fig. 1b. Therefore, minimizing J given $y = \xi(t_B)$, can be interpreted as finding
 78 $\xi(t)$ that maximizes the net number of vehicles *entering* $A(\xi)$.

79 3.2. Initial value problems

In the initial value problem (IVP) the boundary Γ is the line $\{t_B = 0\} \times \mathbb{R}$, so that the problem here is (3a) supplemented with:

$$N(0, x) = G(x), \quad x \in \mathbb{R}, \quad (13)$$

where we assume that $G \in C^2(\mathbb{R})$. The candidate set for B, Γ^P is reduced to B 's x -coordinate, y , which is delimited by two points $U = (0, x_U)$ and $D = (0, x_D)$, where:

$$x_U = x - ut, \quad \text{and} \quad x_D = x + wt, \quad (14a)$$

$$x_U < y < x_D \quad (14b)$$

80 see Fig. 1a. The following subsections examine simplified versions of this problem that reveal
 81 considerable insight into the general solution.

82 3.2.1. Constant inflow

Consider the IVP with constant inflow problem:

$$\phi(t, x) = a, \quad (t, x) \in (0, +\infty) \times \mathbb{R}, \quad (15)$$

for some $a \in \mathbb{R} \setminus \{0\}$. It follows from (12) that $J = -aA(\xi)$ and therefore, for a fixed $y = \xi(0)$ the minimum of (11) is obtained by a path that: (i) maximizes $A(\xi)$ when $a > 0$; or (ii) minimizes $A(\xi)$ when $a < 0$. These two optimum paths are the extreme paths in \mathcal{V}_{BP} that define its boundary; see “upper” and “lower” paths in Fig. 2a. This solution is a “bang-bang” solution, typical for this kind of optimal control problems. The reader can verify that the areas under these paths are given by $A(y, x_D)$ and $A(y, x_U)$, respectively, where we have defined:

$$A(y, x_-) = \frac{1}{2} \left((x + x_-)t - \text{sign}(a) \frac{(x_- - y)^2}{u + w} \right) \quad (16)$$

where $\text{sign}(a)$ is the sign of a and x_- is a placeholder for x_D if $a > 0$ or x_U if $a < 0$. Thus the optimization problem (8) has been reduced to a single variable, y

$$N(t, x) = \min f(B, \xi) \Leftrightarrow N(t, x) = \min_{x_U \leq y \leq x_D} f(y), \quad (17a)$$

$$\text{with} \quad f(y) := G(y) + tQ + (y - x)K - aA(y, x_-) \quad (17b)$$

whose first- and second-order conditions for a minimum y^* read:

$$f'(y^*) = \psi(y^* - x_-) + G'(y^*) + K = 0, \quad (18a)$$

$$f''(y^*) = \psi + G''(y^*) > 0, \quad \text{where:} \quad (18b)$$

$$\psi := \frac{|a|}{u + w}, \quad (18c)$$

83 where $|\cdot|$ denotes the absolute value. Notice that $\psi > 0$ as long as $a \neq 0$.

84 Of course, the optimal point y^* needs to satisfy (14b): $x_U \leq y^* \leq x_D$.

85 3.2.2. Constant initial density

Here, in addition to $\phi(t, x) = a$, we assume

$$g(x) = k_0, \Rightarrow G(x) = -k_0x \quad -\infty < x < \infty \quad (19)$$

which implies that the function(17b) to minimize is a parabola:

$$f(y) = -c_0 - c_1y + \frac{\psi}{2}y^2, \quad (20)$$

with constants $c_0 = Kx_U - \frac{\psi}{2} [x_-^2 - \text{sign}(a)(x_- + x)(u + w)t]$ and $c_1 = \psi x_- - (K - k_0)$, and extremum:

$$y^* = x_- - \frac{K - k_0}{\psi}. \quad (21)$$

Notice that (18b) is always satisfied in this case and therefore y^* is always a minimum and should be included so long as $x_U \leq y^* \leq x_D$. Combining this with (21) gives that the final solution can be expressed as:

$$N(t, x) = \begin{cases} f(y^*), & t > (K - k_0)/a > 0 \\ \min\{f(x_U), f(x_D)\}, & \text{otherwise} \end{cases} \quad (22a)$$

$$(22b)$$

86 Notice that $(K - k_0)/a$ represents the time it takes for the system to reach critical density,
87 namely “time-to-capacity”. This means that y^* will be the optimal candidate only once a
88 regime transition occurs. This will happen only if the time to capacity is positive, i.e. when
89 $\text{sign}(a) = \text{sign}(K - k_0)$ or more explicitly, when k_0 is under-critical and ϕ is an inflow (a
90 positive) or when k_0 is over-critical and ϕ is an outflow (a negative).

Somewhat unexpectedly, we note that $-f_x(x_U) = -f_x(x_D) = -f_x(y^*) = k_0 + at$, which means that the density is always given by the traveling wave $k(t, x) = k_0 + at$, which is also the solution of (1) in this case using the method of characteristics. One should also impose feasibility conditions for the density, i.e. $0 \leq k \leq \kappa$, which gives:

$$k(t, x) = \begin{cases} 0, & t > -k_0/a, a < 0 \\ \kappa, & t > (\kappa - k_0)/a, a > 0 \\ k_0 + at, & \text{otherwise} \end{cases} \quad (23a)$$

$$(23b)$$

$$(23c)$$

91 The flow can be obtained using (5), but it is equivalent and simpler to use $q = H(k)$.

92 *3.2.3. Extended Riemann problems*

Riemann problems are the building blocks of Godunov-type numerical solution methods. As illustrated in Fig. 2b, in these problems one is interested in the value of N at $x = x_0 \geq ut$, i.e. at point $P = (t, x_0)$, with initial data typically given by the density at $t = 0$. This is a very special case for with the ‘target’ point P is located on the discontinuity of the initial data at $x = x_0$. However the methodology below can be easily extended to the cases where $P = (t, x)$ with $x < x_0$ or $x > x_0$. Here, we extend the initial data to include the inflow:

$$(g(x), \phi(x)) = \begin{cases} (k_U, a_U), & x \leq x_0 \\ (k_D, a_D), & x > x_0, \end{cases} \quad (24a)$$

$$(24b)$$

which in conjunction with (3) define an extended Riemann problem, or ERP for short. We assume that $(a_U, a_D) \neq (0, 0)$. Notice that now:

$$x_U = x_0 - ut, \text{ and } x_D = x_0 + wt. \quad (25)$$

For simplicity and without loss of generality we set $G(x_0) = 0$, which implies:

$$G(x) = \begin{cases} (x_0 - x)k_U, & x \leq x_0 \\ (x_0 - x)k_D, & x > x_0 \end{cases} \quad (26a)$$

$$(26b)$$

It will be convenient to define:

$$\eta = a_U/a_D, \quad \psi = \frac{a_D}{u+w}, \quad \theta = u/w. \quad (27)$$

93 The J -integral in this case is a weighted average of the portion of $A(\xi)$ upstream and down-
 94 stream of $x = x_0$, weighted by a_U and a_D , respectively. For instance, if $\xi(t) > 0$ for all t , we
 95 have $J = -\int_0^t \int_0^{\xi(t)} \phi(x) dx dt = -\int_0^t (a_U x_0 + a_D(\xi(t) - x_0)) dt = -(a_U x_0 t + a_D \int_0^t \xi(t) - x_0 dt)$,
 96 where $x_0 t$ is the portion of $A(\xi)$ upstream of $x = x_0$ and $\int_0^t \xi(t) - x_0 dt$ is the portion of $A(\xi)$
 97 downstream of $x = x_0$ in this particular case.

It follows that the minimization of the $J(\xi)$ can be achieved analogously to the previous section by considering the upper and lower paths from each candidate $y = \xi(0)$. In addition, however, one has to include 2 middle ‘paths’ that would reach and stay at $x = x_0$ until reaching P ; see Fig. 2b. To formalize, let j_1, j_2 and j_3 be the value of $J(\xi)$ when $y \geq x_0$ along the upper, lower and middle paths, respectively; similarly for j_4, j_5 and j_6 when $y \leq x_0$. Calculating the areas upstream and downstream of $x = x_0$ defined by each path, it can be shown that the j_i ’s can be obtained from:

$$2(j_1(y) - J_0)/\psi = -\theta t^2 w^2 (2\eta(\theta + 1) + 1) + 2twy_0 + y_0^2, \quad (28a)$$

$$2(j_2(y) - J_0)/\psi = -\eta\theta(2\theta + 1)t^2 w^2 + 2\eta\theta twy_0 + y_0^2((\eta - 1)\theta - 1), \quad (28b)$$

$$2(j_3(y) - J_0)/\psi = -(\theta + 1)(2\eta\theta t^2 w^2 + y_0^2), \quad (28c)$$

$$2(j_4(y) - J_0)/\psi = -\theta t^2 w^2 (2\eta(\theta + 1) + 1) + 2twy_0 + y_0^2(\eta\theta + \eta - 1)/\theta, \quad (28d)$$

$$2(j_5(y) - J_0)/\psi = -\eta(\theta(2\theta + 1)t^2 w^2 - 2\theta twy_0 + y_0^2), \quad (28e)$$

$$2(j_6(y) - J_0)/\psi = \eta(\theta + 1)(y_0^2 - 2\theta^2 t^2 w^2) / \theta, \quad (28f)$$

where $y_0 = x_0 - y$ and J_0 is a constant of our problem that represents the net number of vehicles *leaving* the area upstream of x_U , i.e. $-a_U x_U t$ in this case. Notice that $j_1(x_0) = j_4(x_0), j_2(x_0) =$

$j_5(x_0)$ and $j_3(x_0) = j_6(x_0)$, as expected. The function to minimize in this case can be written as:

$$f(y) = G(y) + tQ - (x_0 - y)K + J(y), \quad \text{where:} \quad (29)$$

$$J(y) = \begin{cases} \min\{j_1(y), j_2(y), j_3(y)\}, & y > x_0 \\ \min\{j_4(y), j_5(y), j_6(y)\}, & y \leq x_0 \end{cases} \quad (30a)$$

$$(30b)$$

The solution of (3) under these conditions can be reduced to the evaluation of $f(y)$ at a small number of candidates. In addition to candidates $y = x_U$ and $y = x_D$, we have to consider the discontinuity at $y = x_0$ and the possible minima produced by each of the components in (28). Since $f(y)$ is piecewise quadratic, each one of these components has at most one minimum, namely $y = y_i, i = 1, \dots, 6$, which can be obtained by solving the first-order conditions $f'(y_i) = 0$ associated with each of the j_i 's; i.e.:

$$y_1 = x_D - (K - k_D)/\psi, \quad y_2 = x_0 + \frac{\eta\theta x_D - (K - k_D)/\psi}{\theta(\eta - 1) - 1}, \quad (31a)$$

$$y_3 = x_0 + \frac{K - k_D}{(\theta + 1)\psi}, \quad y_4 = x_0 + \frac{x_U + \theta(K - k_U)/\psi}{1 - \eta(1 + \theta)}, \quad (31b)$$

$$y_5 = x_U + (K - k_U)/(\eta\psi), \quad y_6 = x_0 + \frac{\theta(k_U - K)}{\eta(\theta + 1)\psi}, \quad (31c)$$

For the y_i 's to be valid candidates they must meet the following conditions:

$$x_0 < y_i < x_D, \quad i = 1, 2, 3 \quad (32a)$$

$$x_U < y_i < x_0, \quad i = 4, 5, 6 \quad (32b)$$

which ensure that y_i is on the boundary Γ^P . With all, the sought solution can be expressed as:

$$N(t, x_0) = \min_{y \in \mathcal{Y}} f(y), \quad \text{with: } \mathcal{Y} = \{x_U, x_0, x_D, y_1^*, \dots, y_6^*\} \quad (33)$$

98 where y_i^* is y_i if (32) is met and null otherwise. Notice that this solution method does not
 99 require imposing $f''(y^*) > 0$ because maxima will be automatically discarded in the minimum
 100 operation.

The average flow $\bar{q}(t, x_0)$ during $(0, t)$ can be obtained as follows:

$$\bar{q}(t, x_0) = \frac{1}{t} \int_0^t N_t(s, x_0) - \Phi(s, x_0) ds = \frac{1}{t} (N(t, x_0) - \int_0^t \Phi(s, x_0) ds) \quad (34a)$$

$$= \frac{1}{t} (N(t, x_0) - (J_0 + x_U a_U)t) = N(t, x_0)/t - J_0 - x_U a_U. \quad (34b)$$

101 As shown in Section 4, this formula allows to formulate improved numerical solution methods
 102 for the endogenous problem.

We restrict the space to a segment of road $[\xi, \chi]$ with $\xi < \chi$. In the initial and boundary value problem (IBVP) the boundary Γ is the set defined as

$$(\{t_B = 0\} \times [\xi, \chi]) \cup ((0, +\infty) \times \{x = \xi\}) \cup ((0, +\infty) \times \{x = \chi\}),$$

so that the problem here is (3a) supplemented with:

$$\begin{cases} N(0, x) = G_{ini}(x), & \text{on } [\xi, \chi], \\ N(t, \xi) = G_{up}(t), & \text{on } (0, +\infty), \\ N(t, \chi) = G_{down}(t), & \text{on } (0, +\infty). \end{cases} \quad (35)$$

For obvious compatibility reasons, we request these conditions to satisfy

$$G_{ini}(\xi) = G_{up}(0) \quad \text{and} \quad G_{ini}(\chi) = G_{down}(0).$$

We also consider a $a \neq 0$ and $k_0 \in [0, \kappa]$ such that the inflow rate is given by

$$\varphi(t, x) = a, \quad \text{for any } (t, x) \in [0, +\infty) \times [\xi, \chi]$$

and the initial density

$$g_{ini}(x) = k_0, \quad \text{for any } x \in [\xi, \chi].$$

According to the position of point $P = (t, x)$ with $t > 0$ and $x \in (\xi, \chi)$, we have the following cases to distinguish (see also Jin (2015))

$$(t_U, x_U) = \begin{cases} \left(0, x - \frac{t}{u}\right), & \text{if } x \geq \xi + \frac{t}{u}, \\ \left(t - \frac{x - \xi}{u}, \xi\right), & \text{else,} \end{cases}$$

and

$$(t_D, x_D) = \begin{cases} \left(0, x + \frac{t}{w}\right), & \text{if } x \leq \chi - \frac{t}{w}, \\ \left(t + \frac{x + \chi}{w}, \chi\right), & \text{else,} \end{cases}$$

that define 4 regions (see Figure 6)

Region	Set of points	Upstream point	Downstream point
I :	$\left\{ (t, x) \mid \chi - \frac{t}{w} \geq x \geq \xi + \frac{t}{u} \right\}$	$(t_U, x_U) = \left(0, x - \frac{t}{u}\right)$	$(t_D, x_D) = \left(0, x + \frac{t}{w}\right)$
II :	$\left\{ (t, x) \mid x \leq \min \left\{ \xi + \frac{t}{u}, \chi - \frac{t}{w} \right\} \right\}$	$(t_U, x_U) = \left(t - \frac{x - \xi}{u}, \xi\right)$	$(t_D, x_D) = \left(0, x + \frac{t}{w}\right)$
III :	$\left\{ (t, x) \mid x \geq \max \left\{ \xi + \frac{t}{u}, \chi - \frac{t}{w} \right\} \right\}$	$(t_U, x_U) = \left(0, x - \frac{t}{u}\right)$	$(t_D, x_D) = \left(t + \frac{x + \chi}{w}, \chi\right)$
IV :	$\left\{ (t, x) \mid \xi + \frac{t}{u} \geq x \geq \chi - \frac{t}{w} \right\}$	$(t_U, x_U) = \left(t - \frac{x - \xi}{u}, \xi\right)$	$(t_D, x_D) = \left(t + \frac{x + \chi}{w}, \chi\right)$

Then in Region I, everything is exactly the same than in previous Subsection 3.2 while in the other cases, we will compare initial data G_{ini} coming from (t_D, x_D) with upstream data G_{up} coming from (t_U, x_U) (Region II), initial data G_{ini} from (t_U, x_U) with downstream data G_{down} from (t_D, x_D) (Region III) and finally upstream data G_{up} from (t_U, x_U) and downstream data G_{down} from (t_D, x_D) (Region IV). Everything can be done exactly in the same way and finally the idea is to use the inf-morphism from Aubin et al. (2011)

$$N(t, x) = \min \{N_{ini}(t, x), N_{up}(t, x), N_{down}(t, x)\}$$

105 where N_{ini} , N_{up} and N_{down} denote the partial solutions obtained by considering respectively
 106 initial, upstream and downstream conditions. The interested reader is also referred to (Mazaré
 107 et al., 2011).

108 It is noteworthy that additional constraints could be also addressed in the same framework.
 109 In particular, any internal boundary condition could be added as long as this condition is
 110 exogenous and do not depend on the traffic state. Such internal boundary condition can
 111 represent for instance a moving bottleneck constraint as discussed in Claudel and Bayen (2010b);
 112 Mazaré et al. (2011).

113 4. Numerical solution methods

114 In the following two subsections we formulate two numerical solution methods to find the
 115 global solution of the endogenous inflow problem in conservation law form (1), (7) and in HJ
 116 form (3), (7), respectively. The basic idea is that in discrete time, if the (endogenous) inflows
 117 are computed using the traffic states from the previous time step using an explicit-in-time
 118 numerical scheme, then they become exogenous for the current time step and therefore the VT
 119 solution may be applied.

120 4.1. Godunov's method

This method has been traditionally used to solve the traffic problem in conservation law form without inflows, and constitutes the basis of the well known Cell Transmission (CT) model (Daganzo, 1994) assuming the triangular Hamiltonian (10). In this method, time and space are discretized in increments Δt and $\Delta x = u\Delta t$, respectively, and we let:

$$k_i^j = k(j\Delta t, i\Delta x) \tag{36}$$

be the numerical approximation of the density. The update scheme is the following discrete approximation of the conservation law (1):

$$\frac{k_i^{j+1} - k_i^j}{\Delta t} + \frac{q_{i+1}^j - q_i^j}{\Delta x} = \phi(k_i^j) \tag{37}$$

The key to Godunov's method is the computation of the flow into cell i , q_i^j , which are obtained by solving Riemann problems. Traditionally, inflows have been considered explicitly only in the update scheme (37) but not in the solution of the Riemann problems (Laval and Leclercq, 2010). This implies that the computation of q_i^j corresponds to the original CT rule (also equivalent to the minimum formula between upstream demand and downstream supply (Lebacque, 1996)):

$$q_i^j = \min\{Q, uk_i^j, (\kappa - k_{i+1}^j)w\}, \tag{CT rule} \tag{38}$$

121 Here, we will compare the CT rule with the ERP rule (34b), i.e. the flow based on extended
 122 Riemann problems. Both methods are first-order accurate since both are Godunov-type meth-
 123 ods, and therefore the rate of convergence of both methods should be similar and roughly
 124 proportional to the mesh size. The main difference is in the magnitude of the error, where the
 125 ERP rule should be more accurate because the impacts of inflows are explicitly considered in
 126 the solution of Riemann problems. We illustrate this with the following example.

127

Example. Consider an empty freeway at $t = 0$ subject to an inflow linear in both x and k ;
 i.e.:

$$g(x) = 0, \tag{39a}$$

$$\phi(k) = ax - buk, \quad a, b > 0. \tag{39b}$$

Notice that (Laval and Leclercq, 2010) showed that linear inflow functions arise in the contin-
 uum approximation of the Newell-Daganzo merge model (Newell, 1982; Daganzo, 1994), which
 accounts for the interactions between freeway and on-ramp demands. The particular coeffi-
 cients of the linear function depend upon the state of the freeway and on-ramps. In particular,
 (39b) corresponds to the case where both are in free-flow, ax represent the inflow demand rate
 at x and b the exit probability per unit distance. Using the method of characteristics (Laval
 and Leclercq, 2010) showed that the solution of (1), (39) is :

$$k(t, x) = \frac{a}{b^2u} (bx - 1 + (1 - b(x - tu))e^{-btu}) \tag{40}$$

128 provided $k(t, x) \leq K$. To get an idea of the solution for all densities, Fig. 3 shows the numerical
 129 ERP solution with $\Delta t = 1$ s, with parameters given in its caption. Notice that similarly to
 130 Laval and Leclercq (2010) we chose $\theta = 1$ (say $u = w$) to avoid the numerical errors intrinsic
 131 to Godunov's method and to focus on those caused by the treatment of the inflow.

132 Fig. 4 compares the CT-rule with the ERP-rule numerical solution (with $\Delta t = 40$ s) for
 133 this example *vis-a-vis* the exact solution (40). Parts (a) and (b) show the time evolution of
 134 the density and flow, respectively, at $x = 14$ km obtained with each method. It can be seen
 135 that the main difference, as expected, is in the flow estimates, particularly at $t = 0$ where the
 136 CT rule predicts zero flow since the freeway is empty and it does not consider inflows in its
 137 calculations.

138 To assess the accuracy of each method, part (c) of Fig. 4 shows the density root-mean-
 139 squared error (RMSE) of each method with respect to (40) for varying Δt and only until
 140 $t = 2.35$ s, when the density exceeds the critical density K ; see Fig. 4a. It becomes apparent
 141 that both methods converge to the right solution as $\Delta t \rightarrow 0$, but the accuracy of the proposed
 142 method outperforms the existing method by a factor of two for all values of Δt .

143 Fig. 4d shows the optimal candidate that minimizes $f(y)$ at each time step of the numerical
 144 method, which is an element of the set \mathcal{Y} , for the ERP rule, and of $\{x_U, x_0, x_D\}$ for the CT
 145 rule. It can be seen that both methods coincide except for the time step where the density
 146 approaches the critical density, in which case the proposed method finds the more accurate
 147 optimal candidate y_1^* .

148 4.2. Variational networks

149 Daganzo (2005b) introduced time-space networks to solve the traffic problem without inflows
 150 in variational form using shortest paths. Notice that this is an application of Bellman's dynamic

151 programming principle. Each link i in these “variational networks” is defined by its: (i) slope
 152 v_i : wave speed, (ii) cost c_i : maximum number of vehicles that can pass, (iii) time length τ_i ,
 153 and (iv) distance length $\delta_i = \tau_i v_i$; see inset in Fig. 5.

Since the fundamental diagram is assumed triangular and the freeway homogeneous, there are only three wave speeds to be considered, u , $-w$ and 0 , and the corresponding passing rates are given by

$$\mathcal{L}(v_i) = \begin{cases} w\kappa, & v_i = -w \\ Q, & v_i = 0 \\ 0, & v_i = u \end{cases} \quad \begin{array}{l} (41a) \\ (41b) \\ (41c) \end{array}$$

Let J_i be the contribution of the J -integral in the cost of each link i . It corresponds to the (negative of) integral of $\phi(t, x)$ over the shaded region in Fig. 5, \mathcal{S}_i , and can be approximated by:

$$J_i = -\tau_i \sum_{j \in \mathcal{S}_i} \delta_j a_j, \quad (42a)$$

where a_j is the inflow associated with link j and $j \in \mathcal{S}_i$ means all links that “touch” area \mathcal{S}_i . Finally, the cost to be used in each link becomes:

$$c_i = \mathcal{L}(v_i)\tau_i + J_i. \quad (43)$$

154 The advantage of this method is that it is free of numerical errors (when inflows are exoge-
 155 nous) but it may be cumbersome to implement unless θ is an integer. In that case, as illustrated
 156 in Fig. 5 for $\theta = \frac{u}{w} = 2$, the location of nodes align on a grid pattern. This allows defining
 157 a conventional grid with cell size $\Delta t, \Delta x$ where inflows may be assumed constant. The other
 158 disadvantage is that merge models are typically expressed in terms of flows or densities rather
 159 than N values, and therefore an additional computational layer has to be added.

160 5. Other coordinates

161 As pointed out in Laval and Leclercq (2013) there are two additional coordinate systems
 162 that provide alternative solution methods of the traffic flow problems without inflows. In space-
 163 Lagrangian coordinates the quantity of interest is $X(t, n)$, the position of vehicle n at time t ; in
 164 time-Lagrangian coordinates one is interested in $T(n, x)$, the time vehicle n crosses location x .

165 These representations correspond to the same surface in the three-dimensional space of vehi-
 166 cle number, time and distance, but expressed with respect to a different coordinate system. We
 167 briefly analyze these two alternatives when considering Eulerian source terms and conclude that
 168 even if a HJ equation is still valid in both cases, one cannot expect to get a VT representation
 169 formula even when inflows $\phi(t, x)$ are exogenous.

170 5.1. Space-Lagrangian coordinates: X -models

Let $s(t, n)$ be the spacing of vehicle n at time t . To derive the X -model we multiply the N -
 model (3a) by s to get $sN_t - sH(k) = s\Phi(t, x)$. Noting that $s = -X_n$ and $sN_t = -X_n N_t = X_t$,
 this can be rewritten as:

$$X_t - V(-X_n) = -X_n \Phi(t, X), \quad (44)$$

where $V(s) = sH(1/s)$ is this spacing-speed fundamental diagram. We conclude that (44) is still a HJ PDE but does not admit a VT solution due to the term involving X . The corresponding conservation law can be obtained by taking the partial derivative with respect to n of (44):

$$s_t + V(s)_n = -\phi(t, X)s^2 - \Phi(t, X)s_n, \quad (45)$$

171 Notice that van Wageningen-Kessels et al. (2013) identified (45) but without the term $\Phi(t, X)s_n$
 172 using a different approach, and used it to formulate a numerical solution method in the case of
 173 discrete inflows.

174 5.2. Time-Lagrangian coordinates: T-models

Let $r = T_x$ and $h = 1/q$ be the pace and the headway of vehicle n at location x , and let $F(r)$ be the fundamental diagram in this case, i.e. $h = F(r)$. Here, the T-model is simply $F(r) = 1/q$, where q is given by $q = N_t - \Phi(t, x)$, per (5). Noting that $N_t = 1/T_n$ this can be rewritten as:

$$T_n - \frac{F(T_x)}{1 + \Phi(T, x)F(T_x)} = 0, \quad (46)$$

which, again, is still a HJ PDE but does not admit a VT solution due to the term involving T . The corresponding conservation law can be obtained by taking the partial derivative with respect to x of (46):

$$r_n - \frac{F(r)_x + F(r)^2\phi(T, x)}{(1 + \Phi(T, x)F(r))^2} = 0. \quad (47)$$

175 To summarize, it becomes apparent that in Lagrangian and vehicle number-space coordi-
 176 nates the solution to our problem does not accept VT solutions and therefore becomes more
 177 difficult to solve.

178 6. Discussion

179 We have shown in this paper that VT solutions to the traffic flow problem exist only in
 180 Eulerian coordinates when inflows are exogenous. In all other cases the Hamiltonian is a
 181 non-local function of the independent variable, and the corresponding variational may not be
 182 possible to formulate. Even in the simplest endogenous linear case (7) the reader can appreciate
 183 the mathematical difficulties: using $\Phi = \Phi(s, N(s, \xi(s)))$ in (8a)-(11) turns the problem implicit
 184 in N , and therefore it is no longer a VT problem.

185 Improved numerical solution methods for the endogenous case were derived by taking ad-
 186 vantage of this insight. In other fields, it appears that solving the extended Riemann problems
 187 explicitly considering the inflows has not been possible, and the only alternative has been to use
 188 high-resolution Riemann solvers (Schroll and Winther, 1996; LeVeque, 1998). We have shown
 189 that this is not the case in traffic flow, and that the ERP method presented here is indeed more
 190 accurate.

191 A streamlined version of the ERP method could be envisioned that drastically improves
 192 computation times with minimal impact in the quality of the solution. To see this, recall that
 193 Fig. 4d showed that the candidate y_1^* was optimal only during the time step where the density
 194 approaches the critical density. Based on the discussion following eqn. (41) it is reasonable to
 195 conjecture that candidates $y_1^*, y_2^* \dots y_6^*$ in (33) would be optimal only when a transition takes

196 place. Therefore, the streamlined method would consider only the reduced set $\mathcal{Y} = \{x_U, x_0, x_D\}$
197 in (33), which would induce an error only in the time step where the transition occurs. This
198 error can be made arbitrarily small by decreasing Δt . Notice that this does not mean that the
199 continuum solution (33) can be streamlined in this way; this is only possible in discrete time
200 where N -values are updated at each time step.

201 The implications of our findings in the context of MFD analytical approximation methods
202 considering turns as a continuum inflow are not encouraging. This is because inflows in this
203 case would have to be endogenous for the method to be meaningful, and in such case we have
204 seen that there is no VT solution. This implies that the method of cuts—the only method used
205 so far to provide analytical MFD approximations—is no longer applicable. At the same time,
206 however, a stochastic extension of the method of cuts proved successful in approximating a
207 real-life MFD (Laval and Castrillón, 2015). This would indicate that, at least in the context of
208 the MFD, VT solutions still provide good approximations. Research in this topic is ongoing.

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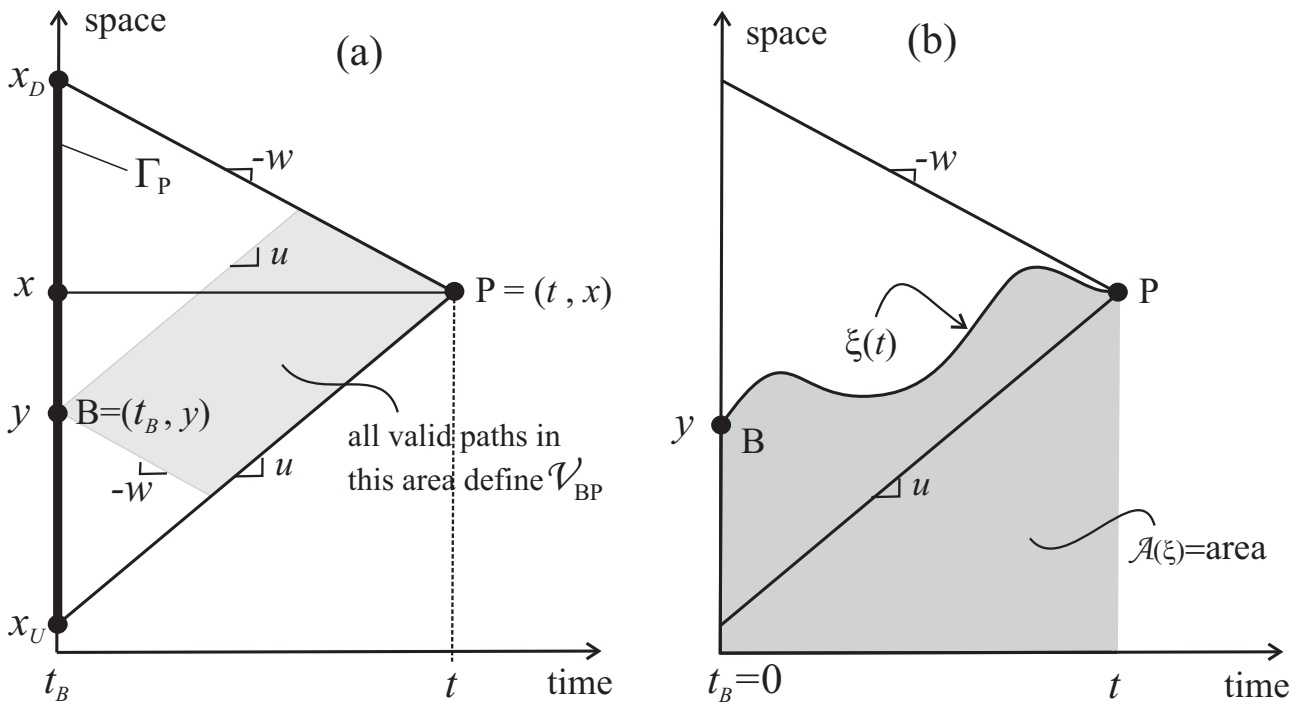


Figure 1: Illustration of key definitions in VT: (a) the initial value problem; (b) the area of the integration to obtain the J -integral.

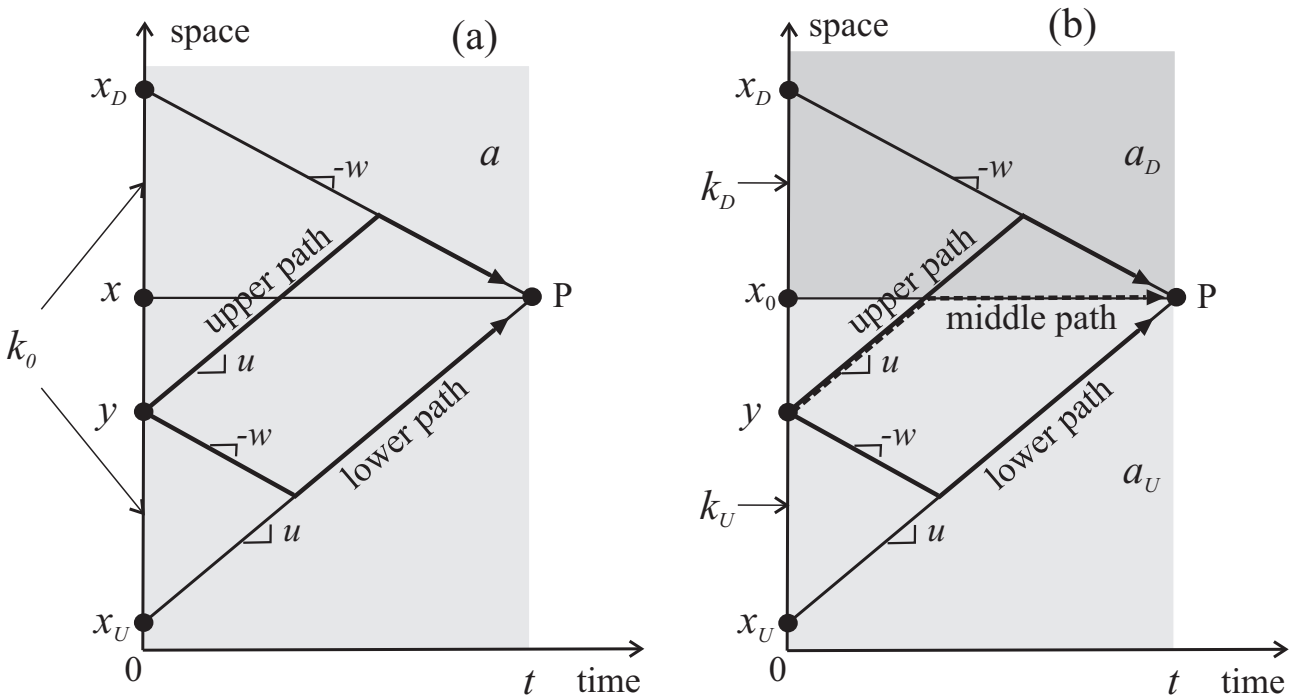


Figure 2: Possible paths to minimize the J -integral: (a) constant initial density; (b) extended Riemann problems.

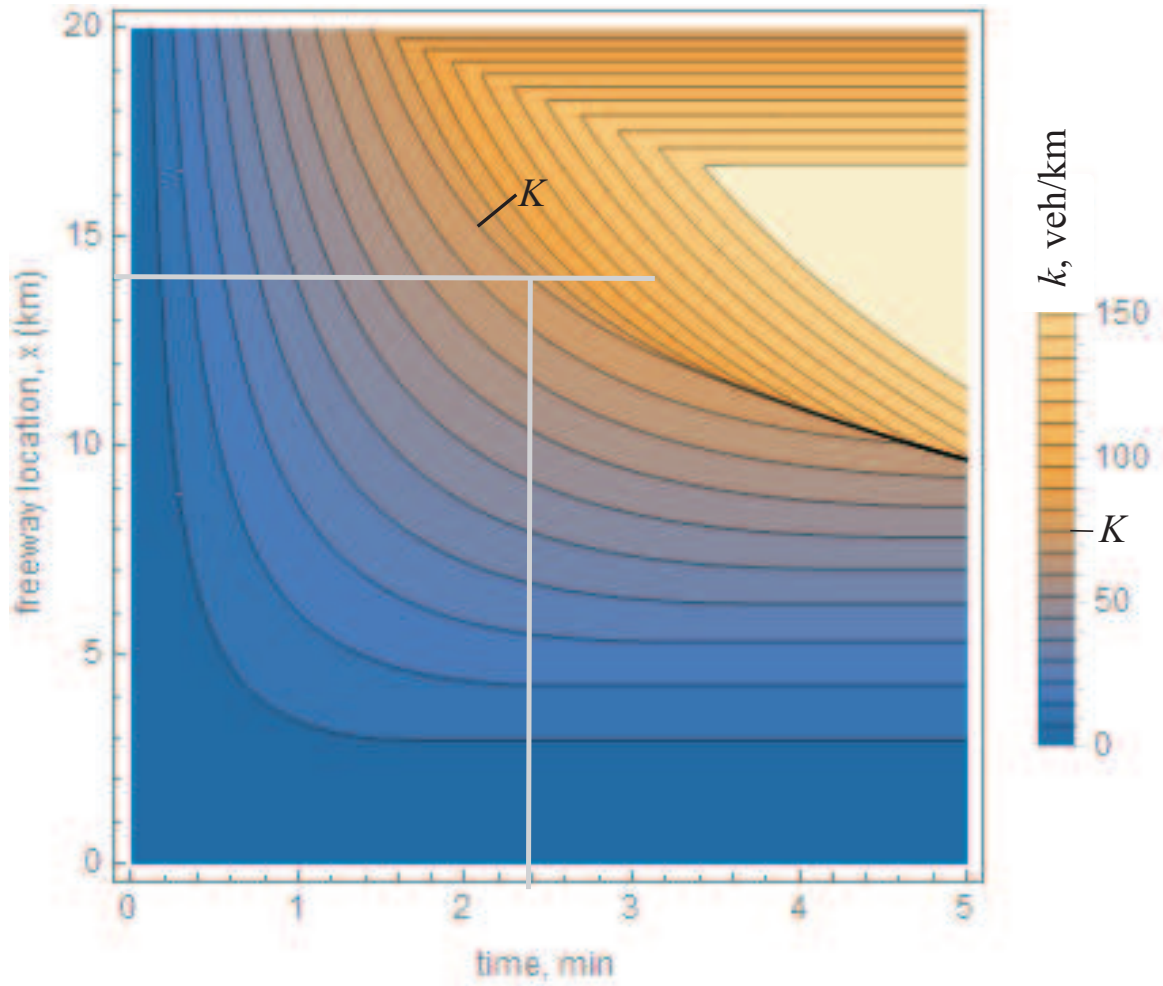


Figure 3: Numerical ERP solution of (39) with $\Delta t = 1$ s and parameters: freeway length $L = 20$ km, $w = 100$ km/hr, $\kappa = 150$ veh/km, $\theta = 1$, $a = 0.5Q/L$, $b = 0.3$.

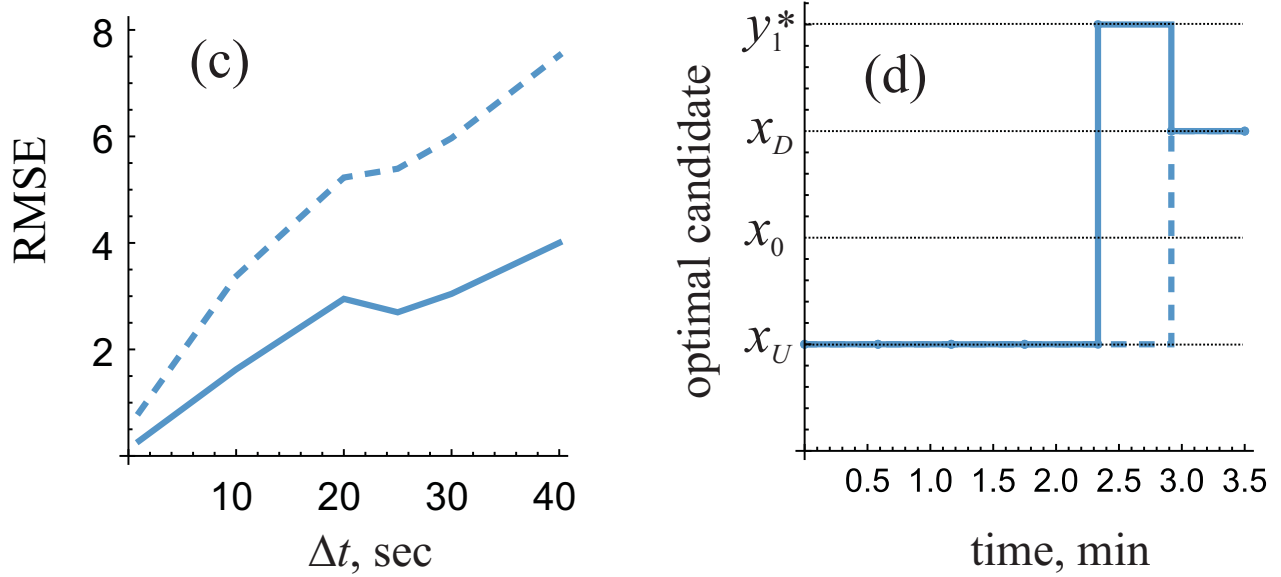
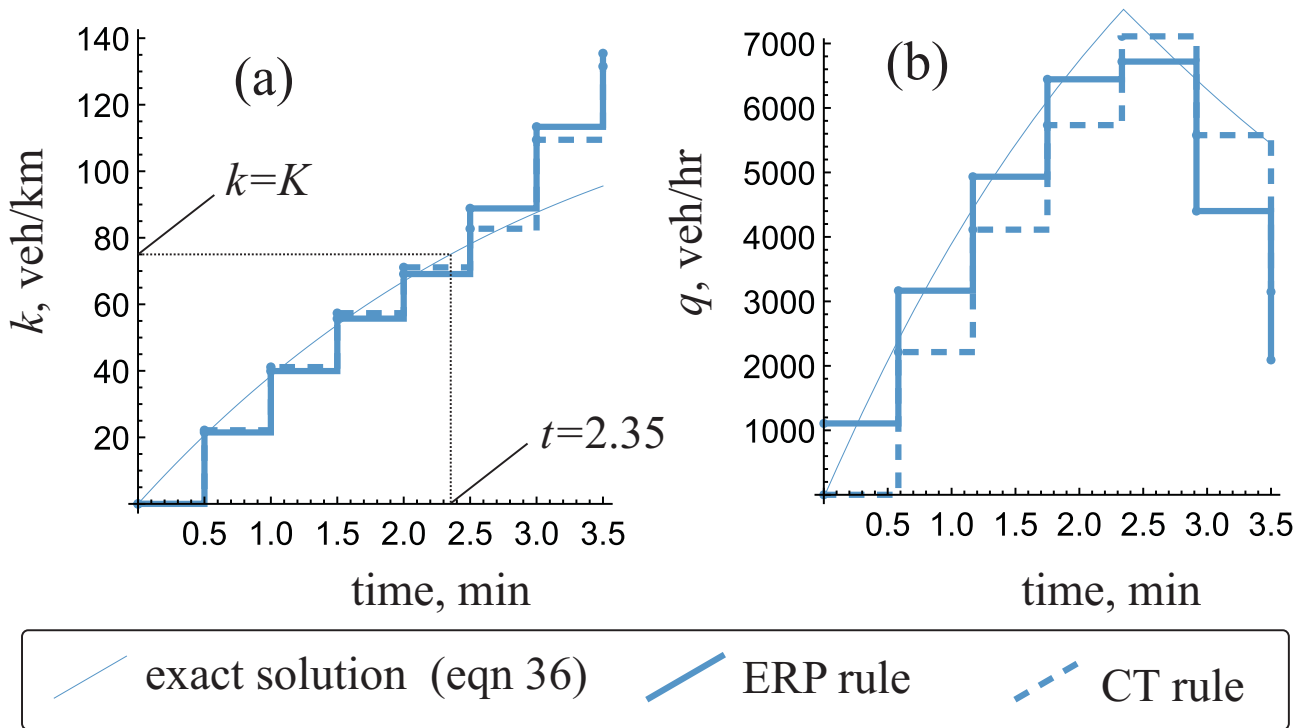


Figure 4: Comparison of the CT rule with the ERP rule solution for the example in §4. (a) and (b): density and flow at $x = 14$ km, (c) candidate number that minimizes $f(y)$, (d) RMSE of each method for varying Δt .

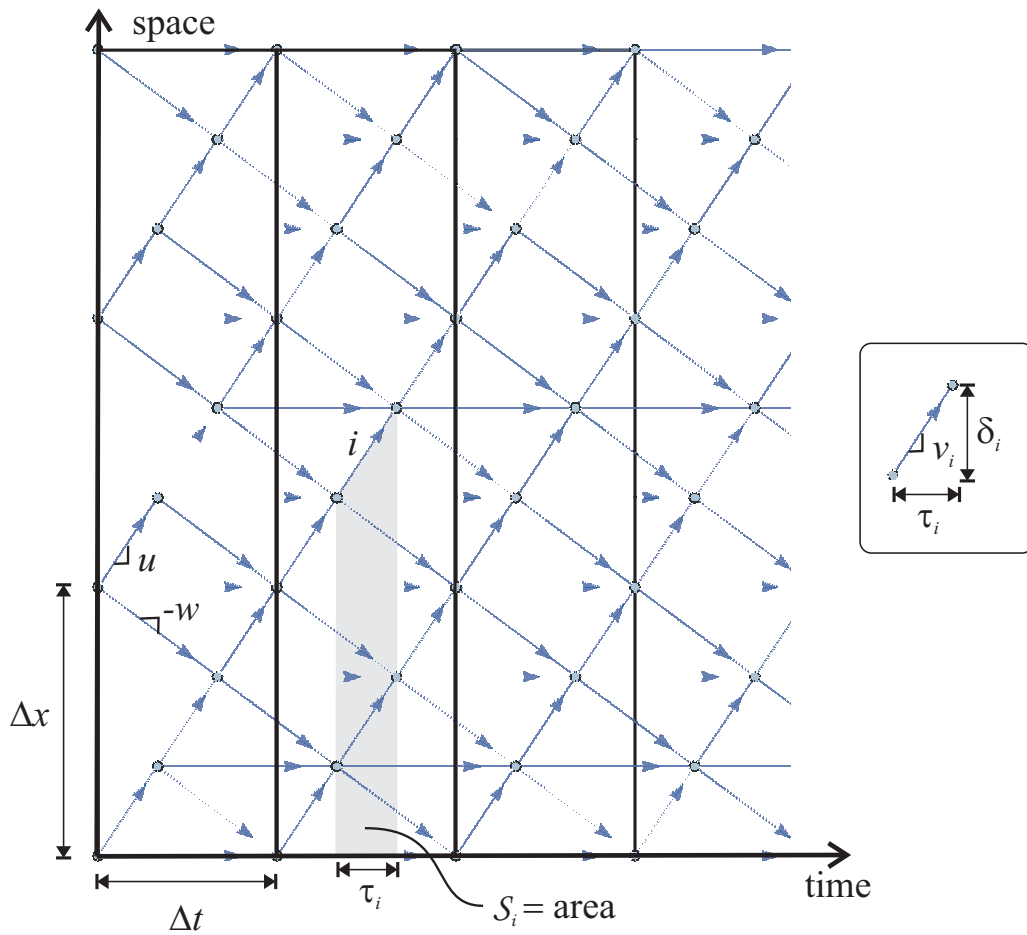


Figure 5: Variational networks: time-space networks to solve the traffic problem with inflows using shortest paths.

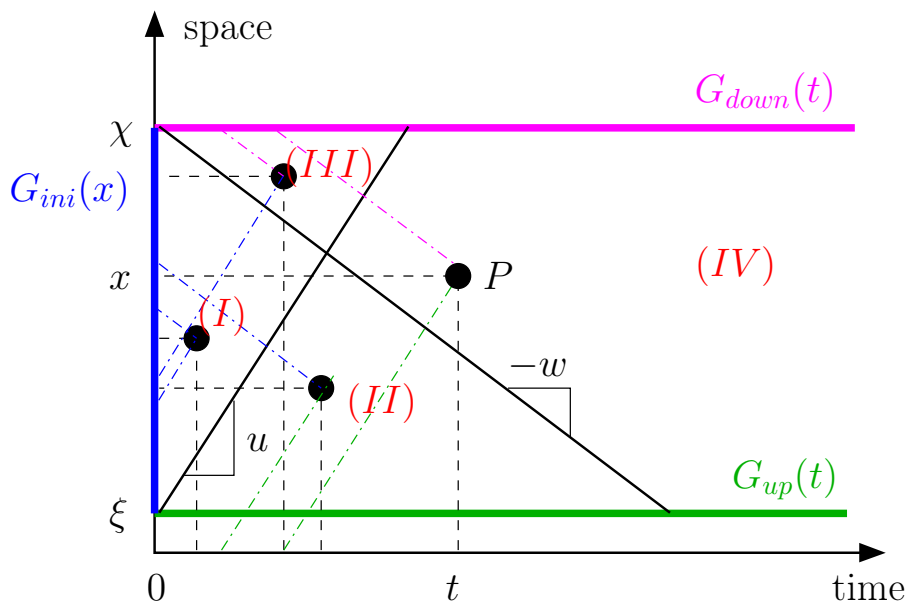


Figure 6: The four different regions to consider for an Initial and Boundary Value Problem with initial data G_{ini} , upstream G_{up} and downstream G_{down} boundary conditions.