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A note on the complexity of Bürgi's algorithm for the computation of sines

Denis Roegel*

7 march 2016

1 Introduction

Folkerts, Launert and Thom have recently announced a very interesting discovery about Jost Bürgi (1552–1632) [4, 5]. Bürgi was a remarkable mechanician, clockmaker and instrument maker, and also the author of a table of progressions which can be viewed as a table of logarithms. In this note, we focus on some specific points concerning Bürgi's newly discovered algorithm.¹

2 Bürgi's algorithm

In his *Fundamentum Astronomiæ* [5], Bürgi provided an interesting and new way to compute a table of sines. Bürgi's algorithm comprises several parts, one of which allows for the simultaneous computation of the values of $\sin\left(\frac{i\pi}{2n}\right)$ for $0 \leq i \leq n$, to any desired accuracy. However, this was only one part of Bürgi's algorithm, as Bürgi made use of other techniques to reconstruct values of sines by accumulating differences.

In this section, we briefly summarize the most original part of Bürgi's algorithm, drawing from our earlier note [8].

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¹For more on Bürgi's work on logarithms, we refer the reader to our critical analysis of Bürgi's work on progressions [7]. A translation of Bürgi's introduction to the tables was recently published by Kathleen Clark [2].

2.1 Dividing the quadrant in nine parts

Folkerts *et al.* [4] give the following table adapted from Bürgi's manuscript dated from the 1580s. Bürgi used instead sexagesimal values, but we can safely convert them to base 10.

| | c_5 | | c_4 | | c_3 | | c_2 | | c_1 |
|----|------------|-----------|---------|--------|--------|-------|-------|----|-------|
| 0 | 0 | | 0 | | 0 | | 0 | | 0 |
| 10 | 2,235,060 | 2,235,060 | 67,912 | 67,912 | 2,064 | 2,064 | 63 | 63 | 2 |
| 20 | 4,402,208 | 2,167,148 | 133,760 | 67,848 | 4,065 | 2,001 | 124 | 61 | 4 |
| 30 | 6,435,596 | 2,033,388 | 195,543 | 61,783 | 5,942 | 1,877 | 181 | 57 | 6 |
| 40 | 8,273,441 | 1,837,845 | 251,384 | 55,841 | 7,638 | 1,696 | 232 | 51 | 7 |
| 50 | 9,859,902 | 1,586,461 | 299,587 | 48,203 | 9,102 | 1,464 | 276 | 44 | 8 |
| 60 | 11,146,776 | 1,286,874 | 338,688 | 39,101 | 10,290 | 1,188 | 312 | 36 | 9 |
| 70 | 12,094,962 | 948,186 | 367,499 | 28,811 | 11,166 | 876 | 339 | 27 | 10 |
| 80 | 12,675,649 | 580,687 | 385,144 | 17,645 | 11,703 | 537 | 356 | 17 | 11 |
| 90 | 12,871,192 | 195,543 | 391,086 | 5,942 | 11,884 | 181 | 362 | 6 | 12 |

The purpose of this table is to compute the values of $\sin 10^\circ$, $\sin 20^\circ$, \dots , $\sin 90^\circ$, to any desired accuracy. Bürgi's algorithm is deceptively simple. There is no bisection, no roots, only two basic operations: many additions, and a few divisions by 2. The computations can be done with integers, or with sexagesimal values, or in any other base.

In order to compute the sines, Bürgi starts with an arbitrary list of values, which can be considered as first approximations of the sines, but need not be. These values are given in column c_1 . In all these columns, c_1 to c_5 , the last value is always the *sinus totus*. That is, the first approximation starts with $\sin 90^\circ = 12$. This gives in modern terms for $\sin 60^\circ$ the value $\frac{9}{12} = 0.75$, an approximation of the actual sine which is $0.866\dots$. It basically matters very little with what values one starts. Only the first value should be 0. It is even possible to take all other initial values equal to 1, for instance, but they can't all be taken equal to 0.

Column c_5 shows the result of that algorithm, here conducted to four steps, but the table could have been extended at will towards the left. Now, in column c_5 , there is a new value for the *sinus totus*, namely 12871192, and therefore we have as a new approximation of $\sin 60^\circ$ the fraction $\frac{11146776}{12871192} = 0.86602515136$, the exact value being $\frac{\sqrt{3}}{2} = 0.866025403\dots$

Bürgi's algorithm is an iterative procedure for computing all the values of column c_{i+1} from those of column c_i . The computations use an intermediate column whose last value is half of the previous *sinus totus*. In the above example, the *sinus totus* in column c_1 is 12, and the last value of the column

between c_2 and c_1 is 6. The last value of the column between c_3 and c_2 is 181, half of 362. If the *sinus totus* is odd, one might take the exact half, but it actually does not matter, as this algorithm leads to increasingly larger numbers, and ignoring a half integer only has marginal consequences on the convergence.

Once the last value of an intermediate column has been obtained, all other values of that column are obtained by adding the values in the previous column, as if the previous column were differences. So, we have $6 + 11 = 17$, $17 + 10 = 27$, and so on.

When the intermediate column has been filled, the new column c_{i+1} is constructed by starting with 0, and adding the values of the intermediate column.

And that's all!

2.2 Bürgi's path to his algorithm

Bürgi did not explain how he obtained his algorithm, but in a recent article, we suggested a very simple way [8]. Recently, other constructions have been provided. In particular, one was announced by Peter Ullrich and will be disclosed in March 2016, and a geometric version of Thom's proof was recently given by Christian Riedweg [6].

3 The accuracy of Bürgi's algorithm

Bürgi seems to have computed a sine table for every $2''$ with 8 places. Whether these 8 places were decimal or sexagesimal, we do not know. We also do not know how accurate this table was, since it is no longer extant. A number of large tables of logarithms have a number of erroneous figures, so that it would be a hastily drawn conclusion to state that Bürgi really had computed a table of sines accurate to 8 places.

Moreover, it seems accepted that Bürgi only used the above algorithm for a number of pivot values, so that eventually smaller gaps could be bridged by adding differences.

It is however interesting to examine the behavior of Bürgi's main algorithm and to somehow measure how it does improve the accuracy of sines. In their article, Folkerts, Launert and Thom stated that for the division of the quadrant in nine parts, the sines were correct to 6 or 7 places, and that an accuracy of 8 (decimal) places can be obtained by adding three more columns [4]. The more general question, which was not answered by Folkerts, Launert and Thom, is how far one has to go to achieve a certain accuracy.

In order to obtain a partial answer to this question, we considered a special case. First, we assume that the quadrant is divided in n parts, so that the purpose is to compute the values of $\sin\left(\frac{i\pi}{2n}\right)$ for $0 \leq i \leq n$. In the above, we have named the columns c_1, c_2 , etc., but now we also name the intermediate (auxiliary) columns a_1, a_2 , etc. Within c_i , the values from top to bottom are $c_{i,0}, c_{i,1}, c_{i,2}, \dots, c_{i,n}$, and within a_i , the values are $a_{i,0}, a_{i,1}, a_{i,2}, \dots, a_{i,n-1}$. We have therefore

$$a_{i,n-1} = \lfloor c_{i,n}/2 \rfloor \quad (1)$$

$$a_{i,j} = a_{i,j+1} + c_{i,j+1} \text{ for } 0 \leq j < n-1 \quad (2)$$

$$c_{i,0} = 0 \quad (3)$$

$$c_{i,j} = c_{i,j-1} + a_{i-1,j-2} \text{ for } 0 < j \leq n \quad (4)$$

For a given i , $c_{i,j}/c_{i,n}$ is an approximation of $\sin\left(\frac{j\pi}{2n}\right)$. But how good is this approximation? In order to answer this question, we choose the following initial values:

$$c_{0,j} = j \text{ for } 0 \leq j \leq n \quad (5)$$

in other words, we start with a linear approximation of the sines.

For a given i , we can now define the error on the j -th value:

$$e_n(i, j) = \left| \sin\left(\frac{j\pi}{2n}\right) - \frac{c_{i,j}}{c_{i,n}} \right| \quad (6)$$

as well as the maximal error in a column:

$$E_n(i) = \max \{e_n(i, j) | 0 \leq j \leq n\} \quad (7)$$

Moreover, we can define the position of the greatest error:

$$G_n(i) = \min \{j | e_n(i, j) = E_n(i)\} \quad (8)$$

We have only taken the minimum value in order to ensure that there is only one integer value.

These two definitions and numerical experimentation lead us to the following conjectures:

Conjecture 1 *The greatest error occurs close to $\arcsin(1/\sqrt{3})$, so that we conjecture that*

$$\lim_{i \rightarrow \infty} \frac{G_n(i)}{n} = \frac{2}{\pi} \arcsin\left(\frac{1}{\sqrt{3}}\right) + \nu(n)$$

where $\lim_{n \rightarrow \infty} \nu(n) = 0$.

Conjecture 2

$$\lim_{i \rightarrow \infty} \frac{E_n(i+1)}{E_n(i)} = \frac{1}{9} + \xi(n)$$

where $\lim_{n \rightarrow \infty} \xi(n) = 0$.

Assuming that these conjectures are true, we can now estimate the number of steps needed to achieve a certain accuracy. Without too much error, we can even assume that the maximal error is divided by 10 for every step, so that each column practically gives a new place of accuracy, or nearly so.

4 The complexity of Bürgi's algorithm

Another question which was not tackled by Folkerts, Launert and Thom is that of the complexity of Bürgi's main algorithm. In other words, how much effort does it take to reach a certain accuracy? From the previous section, we now know that each new step gives a little less than a new place of accuracy. But does this represent a lot of work, or not? In their article, Folkerts, Launert and Thom state that continuing the computations for $n = 9$ up to column 8 is still "a reasonable computational effort." But what if we do divide the right angle in 90 parts? Or 5400 parts? Or more?

Even though Bürgi did not ambition to use his main algorithm to compute his entire sine table, it is theoretically possible. Can we estimate how much time it would have taken? Let us try.

Our aim is to obtain a gross estimate of the complexity of filling the table of section 2.1. The rightmost column c_1 is assumed to be filled, but a_1, c_2, a_2, c_3 , etc., need to be computed. We can estimate the number of one-digit additions as being approximately the number of digits in all these columns. Consider for instance column c_5 . The values are approximately $k \sin\left(\frac{j\pi}{18}\right)$ with $k \approx \mu^4 \times 12$, where $\mu = \frac{\csc^2(\pi/(4n))}{4}$. In general, in our setting, $k \approx \mu^4 \times n$, because $c_{1,n} = n$. $a_{i,j}$ can also be approximated using the fact that the first differences of the sines are close to the cosines and that $c_{i+1,1} = a_{i,0}$. We obtain:

$$c_{i,j} \approx \mu^{i-1} \times n \times \sin\left(\frac{j\pi}{2n}\right) \quad (9)$$

$$a_{i,j} \approx \mu^i \times n \times \cos\left(\frac{j\pi}{2n}\right) \times \sin\left(\frac{\pi}{2n}\right) \quad (10)$$

We stress that these are gross approximations, but they are sufficient for our purpose.

Now, the number of digits of $c_{i,j}$ is approximated by $\log(c_{i,j})$ and since we want to count all these digits, we need to compute

$$d_c(i) = \sum_{j=0}^{j=n} \log(c_{i,j}) \quad (11)$$

which is approximated by an integral:

$$d_c(i) \approx \frac{\int_0^{\pi/2} \log(k \sin x) dx}{\frac{\pi}{2n}} \quad (12)$$

Using the fact that

$$\int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2 \quad (13)$$

we arrive at

$$d_c(i) \approx n \log \left(\frac{\mu^{i-1} \times n}{2} \right) \quad (14)$$

Taking for instance $i = 5$, $n = 9$ and $\mu = 32.9\dots$, we find $d_c(5) \approx 60$, which is close to the actual number of digits in column c_5 .

Using a similar reasoning, we obtain the number of digits d_a of the intermediate columns:

$$d_a(i) \approx n \log \left(\frac{\mu^i \times n \times \sin(\pi/2n)}{2} \right) \quad (15)$$

Adding all these terms and keeping only the largest factors, we obtain an approximation for the total number of digits $D(p, n)$ in columns $a_1, c_2, a_3, \dots, c_p$:

$$D(p, n) = 2p^2 n \log n \quad (16)$$

where $\log n$ is the base 10 logarithm.

For instance, in the above example, with $p = 5$ and $n = 9$, we obtain $D(5, 9) \approx 429$. The actual value is about 340, but, as we said, $D(p, n)$ is only a gross approximation.

Now, if we take for instance $n = 90$ and $p = 8$, we obtain about 22500 digits, and about that many one-digit additions. With $n = 5400$ and $p = 8$, we obtain about 2.6 million digits. And for a $2''$ sine table to 8 places of

accuracy, we have about 100 million digits. Assuming that about 10 one-digit additions can be done per minute, even $n = 5400$ seems out of reach for one person, and $n = 90$ may take perhaps 40 hours to one calculator. This shows, in our opinion, that although Bürgi's algorithm is very simple, it can only be used in conjunction with other methods. Bürgi had certainly found out about the limitations of his main algorithm, and his genius in fact lies in the clever combination of several algorithms. But whether he did in fact compute a table to 8 (decimal) places of accuracy is still not known. . .

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