Silent and equivalent magnetic distributions on thin plates

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Abstract

In geosciences and paleomagnetism, estimating the remanent magnetization in old rocks is an important issue to study past evolution of the Earth and other planets or bodies. However, the magnetization cannot be directly measured and only the magnetic field that it produces can be recorded.

In this paper we consider the case of thin samples, to be modeled as a planar set $S \subset \mathbb{R}^2 \times \{0\}$, carrying a magnetization $\boldsymbol{m}$ (a 3-dimensional vector field supported on S). This setup is typical of scanning microscopy that was developed recently to measure a single component of a weak magnetic field, close to the sample. Specifically, one is given a record of $b_3[\boldsymbol{m}]$ (tiny: a few nano Teslas), the vertical component of the magnetic field produced by $\boldsymbol{m}$, on a planar region $Q \subset \mathbb{R}^2 \times \{h\}$ located at some fixed height $h > 0$ above the sample plane. We assume that both S and Q are Lipschitz-smooth bounded connected open sets in their respective planes, and that the magnetization $\boldsymbol{m}$ belongs to $[L^2(S)]^3$, whence $b_3[\boldsymbol{m}] \in L^2(Q)$. Such magnetizations possess net moments $\langle \boldsymbol{m} \rangle \in \mathbb{R}^3$ defined as their integral on S.

Recovering the magnetization $\boldsymbol{m}$ or its net moment $\langle \boldsymbol{m} \rangle$ from available measurements of $b_3[\boldsymbol{m}]$ are inverse problems for the Poisson-Laplace equation in the upper half-space $\mathbb{R}^3_+$ with right hand side in divergence form. Indeed, Maxwell's equations in the quasi-static approximation identify the divergence of $\boldsymbol{m}$ with the Laplacian of a scalar magnetic potential in $\mathbb{R}^3_+$ whose normal derivative on Q coincides with $b_3[\boldsymbol{m}]$. Hence Neumann data $b_3[\boldsymbol{m}]$ are available on $Q \subset \mathbb{R}^3_+$, and we aim at recovering $\boldsymbol{m}$ or $\langle \boldsymbol{m} \rangle$ on S. We thus face recovery issues on the boundary of the harmonicity domain from (partial) data available inside.

Such inverse problems are typically ill-posed and call for regularization. Indeed, magnetization recovery is not even unique, due to the existence of silent sources $\boldsymbol{m} \neq 0$ such that $b_3[\boldsymbol{m}] = 0$. And though such sources have vanishing moment so that net moment recovery is unique, estimation of the latter turns out to be unstable with respect to measurements errors.

The present work investigates silent sources, equivalent magnetization of minimal $L^2(S)$-norm to some given $\boldsymbol{m} \in [L^2(S)]^3$ (two magnetizations are called equivalent if their difference is silent), as well as density / instability results.

1 Introduction

The motivation for this article is rooted in a problem arising in geosciences, when studying magnetic properties of ancient rocks. More precisely, they get magnetized as they are formed in the presence of an external magnetic field: typically, an igneous rock is formed when lava cools down after a volcanic eruption and, during this process, the ambient magnetic field magnetizes the rock. Once the rock has solidified, this so-called remanent magnetization remains stable for a very long period, unless subsequent events result in heating the rock again. Thus, by dating old rocks and studying their remanent magnetization, geophysicists can estimate what the Earth magnetic field was at the time of their formation and hence figure out the history of this magnetic field. Now, understanding when the Earth magnetic field started and putting up a record of its reversals are key questions in geosciences, and analogous issues arise for the moon, Mars and other bodies from the solar system. This is why recovering information on a magnetization from measurements of the magnetic field it generates is a fundamental issue to the area. This is the kind of topic we address below.

The remanent magnetization is a characteristic of the rock that can be modeled, from a mathematical point of view, as a vector field (*i.e.* magnetic moment per unit volume) defined at every point of the rock sample. Unfortunately, magnetization is not directly accessible to measurement. However, it produces an external magnetic field which can be measured. Classical magnetometers allow one to get the three components of the external magnetic field away from the sample. Comparing them to the field that a magnetic dipole located inside the sample would produce, this provides an estimate of the total net moment of the magnetization (*i.e.*, the integral of the magnetization over the sample). This estimate is valid if the sample is not too big, and is far enough from the magnetometer to be considered as being essentially localized [8]. But since the method requires the measuring device to be placed at some distance from the sample, it is not applicable to weakly magnetized samples.

In this connection, modern scanning magnetic microscopes break new ground. They typically measure only one component of the external field, but they can be placed fairly close to the sample and have high sensitivity, therefore they can measure tiny magnetic fields (with amplitude of a few nano Teslas) corresponding to quite weak magnetizations. The framework in which they are used is the following [12]: the microscope is moved above a horizontal piece of rock, thus giving a map of one component of the magnetic field on some planar region at fixed distance from the sample. In the simplest case, in order to ensure that the sample is indeed planar, the rock is stuck on some support (like a plate of Plexiglas) and sanded down to a very thin slab. The thickness of the latter can be considered to be much smaller than other characteristic dimensions involved in the process, in particular it is small with respect to the height at which the microscope operates. Thus, the sample can be regarded as being 2-dimensional from the mathematical point of view. This framework is recapped in Figure 1.

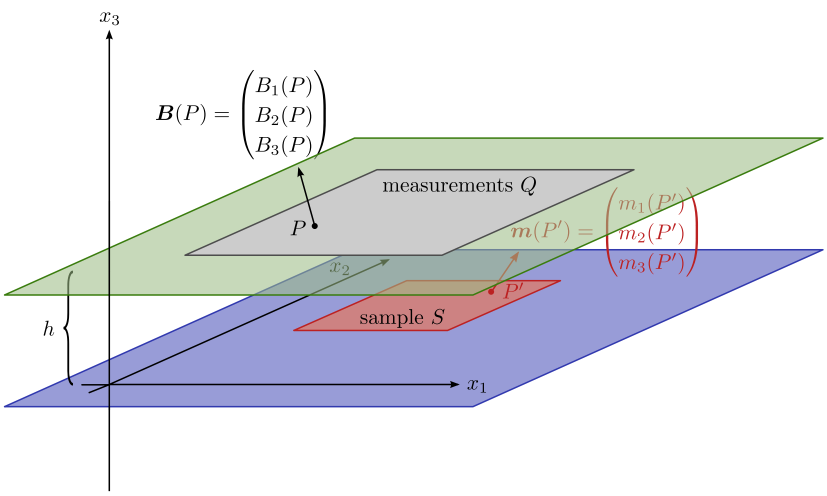

Figure 1: Measurements framework and notations.

The present study addresses some theoretical issues related to the recovery of certain key features of the magnetization, given measurements of the kind we just described. It may be viewed as a sequel to previous investigation of similar inverse magnetization problems in [5]. We model the sample (resp. the area where measurements are performed) by a Lipschitz-smooth bounded connected open set $S \subset \mathbb{R}^2$ (resp. $Q \subset \mathbb{R}^2$). The measurements are performed at height $h > 0$ above the sample plane.

We typically face the following inverse problems: being given measurements $b_3[\boldsymbol{m}]$ on $Q \times \{h\} \subset \mathbb{R}^2 \times \{h\} \subset \mathbb{R}^3_+$ of the vertical component $B_3 = B_3[\boldsymbol{m}]$ of the magnetic field produced by a planar rock sample with unknown magnetization $\boldsymbol{m}$ supported on $\overline{S} \times \{0\} \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$, we want to recover $\boldsymbol{m}$ (as we shall see this is ill-posed for $\boldsymbol{m}$ is not uniquely defined by $b_3[\boldsymbol{m}]$) or some equivalent version thereof (we shall investigate magnetizations of minimum L^2-norm producing the same field as $\boldsymbol{m}$), or simply its net moment $\langle \boldsymbol{m} \rangle \in \mathbb{R}^3$ (defined as the integral of $\boldsymbol{m}$ on S), which is already a valuable piece of information. Observe in the present setting that the vertical component is normal to the measurement plane.

The functions $\boldsymbol{m}$ and $b_3[\boldsymbol{m}]$ are supported on different planes lying at height 0 and h respectively. More precisely, $\boldsymbol{m}$ is supported in $\overline{S} \times \{0\}$ while $b_3[\boldsymbol{m}]$ is supported in $\overline{Q} \times \{h\}$. Thus, they can be viewed as functions of two variables. Of course, $b_3[\boldsymbol{m}]$ is nothing but the restriction to $Q \times \{h\}$ of a function defined more generally on $\mathbb{R}^3 \setminus (\overline{S} \times \{0\})$, namely the normal component of the magnetic field generated by $\boldsymbol{m}$. In particular, we shall find it convenient to regard $b_3[\boldsymbol{m}]$ as the restriction to $Q \times \{h\}$ of $B_3[\boldsymbol{m}]$, the normal component of the magnetic field on the plane $\mathbb{R}^2 \times \{h\}$. We will assume that the $\mathbb{R}^3$-valued function $\boldsymbol{m}$ lies in $[L^2(S)]^3$. As we shall see, this entails that $b_3[\boldsymbol{m}] \in L^2(Q)$ (with values in $\mathbb{R}$).

The present paper points at non-uniqueness and instability properties of the inverse magnetization problem. These call for regularization and suggest that solving certain extremal problems for harmonic gradients is useful in this context. A detailed study of them, however, is left here for further research. We rely on standard tools from harmonic analysis as the expression of $b_3[\boldsymbol{m}]$ involves Poisson and Riesz kernels. This follows from the Maxwell equations in Magnetostatics [11, Ch. 5, Sec. 5.9], which are to the effect that:

- the vertical component $B_3[\boldsymbol{m}]$ of the magnetic field is (proportional to) the vertical derivative of a scalar magnetic potential $\Lambda[\boldsymbol{m}]$ in $\mathbb{R}^3_+$
- this potential $\Lambda[\boldsymbol{m}]$ satisfies the Poisson-Laplace equation with right-hand side term in divergence form: $\Delta_3 \Lambda[\boldsymbol{m}] = \operatorname{div} \boldsymbol{m}$ in $\mathbb{R}^3_+$ (with Δ_3 the Laplace operator on $\mathbb{R}^3$). In particular, $\Lambda[\boldsymbol{m}]$ is a harmonic function in the upper half-space $\mathbb{R}^3_+$, since $\boldsymbol{m}$ is supported on $\overline{S} \subset \mathbb{R}^2 \times \{0\}$.

The set of magnetizations $\boldsymbol{m}$ that produce the same field component $b_3[\boldsymbol{m}]$, and therefore the same measurements, depends on the kernel of the operator b_3 (acting on $[L^2(S)]^3$). Those magnetization within $\operatorname{Ker} b_3$ will be called silent sources (or S-silent sources), as they correspond to silent or invisible source terms supported on $\overline{S}$. They account for non-uniqueness in the inverse magnetization problem. Two magnetizations in $[L^2(S)]^3$ will be called equivalent (or S-equivalent) if their difference is silent. We shall characterize silent sources and give a characterization of the (unique) equivalent magnetization of minimal $L^2(S)$-norm to some given $\boldsymbol{m} \in [L^2(S)]^3$.

In contrast, since S-silent sources have vanishing net moment $\langle \boldsymbol{m} \rangle$, the problem of recovering $\langle \boldsymbol{m} \rangle$ from $b_3[\boldsymbol{m}]$ given on Q has a unique solution. In fact, a preliminary estimation of the net moment would help to recover the magnetization. However, such an estimation is unstable and needs regularization again.

The above considerations reflect the typical ill-posedness of inverse potential problems from incomplete Neumann data.

The overview is the following. In Section 2, we fix notation and recall standard properties of Poisson and Riesz transforms, along with the Hodge decomposition of vector fields on $\mathbb{R}^2$. These are needed to study the magnetic operators Λ and b_3 in Section 3. In Section 4, we characterize silent sources and magnetizations of minimum norm which are equivalent to a given one. We also consider moment recovery issues. This will require the not-so-classical Hardy-Hodge decomposition of $\mathbb{R}^3$-valued vector fields on $\mathbb{R}^2$. Some perspectives are discussed in Section 5.

2 Notation, definitions, preliminary properties

2.1 Notation, definitions

The upper half space, with boundary $\mathbb{R}^2 \times \{0\}$, will be denoted by

$$\mathbb{R}^3_+ = \{(\boldsymbol{x}, x_3) \in \mathbb{R}^3\,,\ \boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2\,,\ x_3 > 0\}.$$

We put ∂_{x_j} for the partial derivative with respect to the coordinate x_j, $j = 1, 2, 3$. The 2- and 3-dimensional gradient operators are respectively denoted with ∇_2 and ∇_3, *i.e.*, $\nabla_2\, g = (\partial_{x_1} g,\, \partial_{x_2} g)$ and $\nabla_3\, g = (\partial_{x_1} g,\, \partial_{x_2} g,\, \partial_{x_3} g)$. Similarly, the 2- and 3-dimensional divergence operators are denoted with $\nabla_2 \cdot$ and $\nabla_3 \cdot$, *e.g.*, $\nabla_2 \cdot \boldsymbol{g} = \partial_{x_1} g_1 + \partial_{x_2} g_2$. The 2-dimensional rotational operator is written $\nabla_2 \times$, *i.e.*, $\nabla_2 \times \boldsymbol{g} = \partial_{x_1} g_2 - \partial_{x_2} g_1$.

For $\Omega \subseteq \mathbb{R}^2$ an open set, $L^2(\Omega)$ and $W^{1,2}(\Omega)$ are the usual Lebesgue space and Sobolev space (of functions belonging to $L^2(\Omega)$ together with their first distributional derivatives), equipped respectively with their usual inner products [6, 9], [17, Ch. 2].

If Ω is a bounded Lipschitz-smooth open set (such that its boundary $\partial\Omega$ is locally the graph of a Lipschitz continuous function), let $W^{1,2}_0(\Omega)$ be the $W^{1,2}(\Omega)$ closure of the space of C^∞ smooth functions with compact support in Ω. We refer to [1], [6, Ch. 9], [7, Ch. 2], [15, Ch. V, VI], [17, Ch. 2-4] for properties of Sobolev spaces .

For $f \in L^2(\Omega)$, we let $\widetilde{f} = f \vee 0 \in L^2(\mathbb{R}^2)$ indicate the function equal to f on Ω and to 0 outside Ω. Similarly, for $\boldsymbol{f} \in \left[L^2(\Omega)\right]^3$ with components f_j, $j = 1, 2, 3$, we let $\widetilde{\boldsymbol{f}} = \boldsymbol{f} \vee 0 \in \left[L^2(\mathbb{R}^2)\right]^3$ indicate the function with components $\widetilde{f_j} = f_j \vee 0$. We put χ_Ω for the characteristic function on Ω: $\chi_\Omega = 1 \vee 0$.

We write inner products with $\langle \cdot, \cdot \rangle$ for all Hilbert spaces when the context is obvious, *e.g.*, $\langle \boldsymbol{x}, \boldsymbol{x}' \rangle = x_1 x_1' + x_2 x_2'$ for points in $\mathbb{R}^2$, or $\langle f, g \rangle = \iint \langle f(\boldsymbol{x}), g(\boldsymbol{x}) \rangle \, \mathrm{d}\boldsymbol{x}$ for functions in $[L^2(\mathbb{R}^2)]^3$. When the context might be unclear, we use a subscript to specify the space, *e.g.*, $\langle f, g \rangle_{W^{1,2}(\mathbb{R}^2)} = \langle f, g \rangle_{L^2(\mathbb{R}^2)} + \langle f', g' \rangle_{L^2(\mathbb{R}^2)}$.

2.2 Poisson kernel and Riesz transforms

We will make intensive use of Poisson and Riesz transforms. We recall in this section important properties of these transforms as operators $L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. For a general exposition, the reader might consult, *e.g.* [9, 16].

Hereafter $\boldsymbol{x}$ always denotes a point $(x_1, x_2) \in \mathbb{R}^2$ and $x_3 > 0$; in other words $(\boldsymbol{x}, x_3)$ belongs to the upper half-space $\mathbb{R}^3_+$. We also denote by $h > 0$ a positive real number. We repeatedly define functions on $\mathbb{R}^2$ by restricting functions defined on the upper half-space to the plane at height h.

2.2.1 The Poisson kernel of $\mathbb{R}^3_+$

The Poisson kernel is the function from $\mathbb{R}^3_+ \to \mathbb{R}$ defined by $(\boldsymbol{x}, x_3) \mapsto P_{x_3}(\boldsymbol{x}) = \frac{x_3}{2\,\pi\, d_{x_3}(\boldsymbol{x})^3}$ with

$$d_{x_3}(\boldsymbol{x}) = \left(|\boldsymbol{x}|^2 + {x_3}^2\right)^{1/2} = \left({x_1}^2 + {x_2}^2 + {x_3}^2\right)^{1/2} .$$

It is obviously C^∞ on $\mathbb{R}^3_+$. By restriction to the plane at height h, P_h is a function of $\boldsymbol{x} \in \mathbb{R}^2$ that is summable (see [16, Ch. I, Lem. 1.17]) and

$$\|P_h\|_{L^1(\mathbb{R}^2)} = 1. \tag{1}$$

Likewise, for $j = 1, 2$, the derivative $\partial_{x_j} P_h$ is a summable function of $\boldsymbol{x} \in \mathbb{R}^2$. Indeed, a direct computation shows that $\partial_{x_1} P_h(\boldsymbol{x}) = \frac{-3hx_1}{2\pi d_h(\boldsymbol{x})^5}$ and therefore, for any $R > 0$,

$$\int_{-R}^{R}\int_{-R}^{R} |\partial_{x_1} P_h(\boldsymbol{x})| \,\mathrm{d}x_1 \,\mathrm{d}x_2 = -2\int_{-R}^{R}\int_{0}^{R} \partial_{x_1} P_h(\boldsymbol{x}) \,\mathrm{d}x_1 \,\mathrm{d}x_2 = -2\int_{-R}^{R} \Big[P_h(\boldsymbol{x})\Big]_{x_1=0}^{R} \mathrm{d}x_2.$$

Now, observing that $\frac{1}{d_h(\boldsymbol{x})^3} = \partial_{x_2} \frac{x_2}{(x_1^2+h^2)\, d_h(\boldsymbol{x})}$ and letting R tend to $+\infty$, we see that the norm equals $\frac{2}{\pi h}$. Interchanging the roles of x_1 and x_2 we get the norm of $\partial_{x_2} P_h$ the same way. To sum up, we have, for $j = 1, 2$:

$$\|\partial_{x_j} P_h\|_{L^1(\mathbb{R}^2)} = \frac{2}{\pi h}. \tag{2}$$

Finally, the restriction to the plane at height h of $\partial_{x_3}\Big((\boldsymbol{x}, x_3) \mapsto P_{x_3}(\boldsymbol{x})\Big)$, is also a summable function of $\boldsymbol{x} \in \mathbb{R}^2$. Indeed,

$$[\partial_{x_3} P_{x_3}]_{|x_3=h}(\boldsymbol{x}) = \frac{1}{2\pi}\,\frac{|\boldsymbol{x}|^2 - 2h^2}{d_h(\boldsymbol{x})^5},$$

and it is hence negative inside the disk of radius $h\sqrt{2}$ and positive outside. By integrating separately on these domains and switching to polar coordinates, we get

$$\iint_{\mathbb{R}^2} \Big|[\partial_{x_3} P_{x_3}]_{|x_3=h}(\boldsymbol{x})\Big| \,\mathrm{d}\boldsymbol{x} = \int_0^{h\sqrt{2}} \frac{2h^2 - r^2}{(r^2+h^2)^{5/2}}\, r \,\mathrm{d}r + \int_{h\sqrt{2}}^{+\infty} \frac{r^2 - 2h^2}{(r^2+h^2)^{5/2}}\, r \,\mathrm{d}r.$$

Now, we observe that $\frac{(2h^2-r^2)r}{(r^2+h^2)^{5/2}} = \partial_r \frac{r^2}{(r^2+h^2)^{3/2}}$ and thus finally get

$$\Big\|[\partial_{x_3} P_{x_3}]_{|x_3=h}\Big\|_{L^1(\mathbb{R}^2)} = \frac{4}{3^{3/2}\, h}. \tag{3}$$

We consider the Fourier transform on $\mathcal{F}$ on $L^2(\mathbb{R}^2)$ such that, for $f \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$,

$$\hat{f}(\boldsymbol{\kappa}) = \mathcal{F}(f)(\boldsymbol{\kappa}) = \iint f(\boldsymbol{x})\, \mathrm{e}^{-2\mathrm{i}\pi\langle \boldsymbol{x}, \boldsymbol{\kappa}\rangle} \,\mathrm{d}\boldsymbol{x}.$$

We recall that it is an isometry from $L^2(\mathbb{R}^2)$ onto $L^2(\mathbb{R}^2)$ whose inverse $\mathcal{F}^{-1}$ satisfies $\mathcal{F}^{-1}(f)(x) = \mathcal{F}(f)(-x)$. It holds that $\mathcal{F}(P_h)(\boldsymbol{\kappa}) = \mathcal{F}^{-1}(P_h)(\boldsymbol{\kappa}) = \mathrm{e}^{-2\pi h|\boldsymbol{\kappa}|}$ (see, *e.g.*, [16, Ch. I, Cor. 1.27]). The operator $\mathcal{P}_h : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by $\mathcal{P}_h(g) = P_h \star g$ is the Poisson operator at height h. By construction it satisfies, for all $g \in L^2(\mathbb{R}^2)$,

$$\mathcal{F}(\mathcal{P}_h(g))(\boldsymbol{\kappa}) = \mathrm{e}^{-2\pi h|\boldsymbol{\kappa}|}\, \hat{g}(\boldsymbol{\kappa}) \qquad \text{for a.e. } \boldsymbol{\kappa} \in \mathbb{R}^2,$$

from which we see that it is a bounded contractive self-adjoint operator, and that

$$\forall h_1 > 0, \forall h_2 > 0, \forall g \in L^2(\mathbb{R}^2), \quad \mathcal{P}_{h_1+h_2}(g) = \mathcal{P}_{h_1}(\mathcal{P}_{h_2}(g)). \tag{4}$$

Finally, since $(\boldsymbol{x}, x_3) \mapsto P_{x_3}(\boldsymbol{x})$ is differentiable on $\mathbb{R}^3_+$ and its derivatives with respect to all three variables belong to $L^1(\mathbb{R}^2)$, we have $\mathcal{F}(\partial_{x_j} P_h)(\boldsymbol{\kappa}) = 2\mathrm{i}\pi\kappa_j \hat{P}_h(\boldsymbol{\kappa})$ for $j = 1, 2$ and $\mathcal{F}([\partial_{x_3} P_{x_3}]_{|x_3=h})(\boldsymbol{\kappa}) = [\partial_{x_3}\mathcal{F}(P_{x_3})(\boldsymbol{\kappa})]_{|x_3=h}$, whence

$$\mathcal{F}(\partial_{x_j} P_h \star g)(\boldsymbol{\kappa}) = 2\mathrm{i}\pi\kappa_j\, \mathrm{e}^{-2\pi h|\boldsymbol{\kappa}|}\, \hat{g}(\boldsymbol{\kappa}), \tag{5}$$

$$\mathcal{F}([\partial_{x_3} P_{x_3} \star g]_{|x_3=h})(\boldsymbol{\kappa}) = -2\pi|\boldsymbol{\kappa}|\, \mathrm{e}^{-2\pi h|\boldsymbol{\kappa}|}\, \hat{g}(\boldsymbol{\kappa}) \qquad \text{for a.e. } \boldsymbol{\kappa} \in \mathbb{R}^2. \tag{6}$$

Finally, observe that $\boldsymbol{\kappa} \mapsto 2\mathrm{i}\pi\kappa_j\,\mathrm{e}^{-2\pi h|\boldsymbol{\kappa}|}$ belongs to $L^\infty(\mathbb{R}^2)$ and its uniform norm is equal to $1/(\mathrm{e}h)$. The same result (with the same constant) also holds for $\boldsymbol{\kappa} \mapsto -2\pi|\boldsymbol{\kappa}|\,\mathrm{e}^{-2\pi h|\boldsymbol{\kappa}|}$. Using the fact that $\mathcal{F}$ is an isometry, we thus get

$$\begin{aligned} \|\partial_{x_j}\, P_h \star g\|_{L^2(\mathbb{R}^2)} &\leq \frac{C_1}{h}\, \|g\|_{L^2(\mathbb{R}^2)}, \quad (j = 1, 2). &(7)\\ \|[\partial_{x_3}\, P_{x_3} \star g]_{|x_3=h}\|_{L^2(\mathbb{R}^2)} &\leq \frac{C_1}{h}\, \|g\|_{L^2(\mathbb{R}^2)}, &(8)\end{aligned}$$

with $C_1 = 1/\mathrm{e}$. The constant $1/\mathrm{e}$ is optimal, as can be seen by taking $\hat{g}$ concentrated around $\boldsymbol{\kappa} = (0,\, 1/(2\pi h))$ or $\boldsymbol{\kappa} = (1/(2\pi h),\, 0)$, depending which case of Equations (5) and (6) is dealt with.

2.2.2 Riesz transforms

The Riesz transforms R_1 and R_2 are operators on $L^2(\mathbb{R}^2)$ that satisfy (see [16, Ch. 6, Sec. 2]), for any $g \in L^2(\mathbb{R}^2)$ and $j \in \{1, 2\}$,

$$\mathcal{F}(R_j\, g)(\boldsymbol{\kappa}) = -\mathrm{i}\, \frac{\kappa_j}{|\boldsymbol{\kappa}|}\, \hat{g}(\boldsymbol{\kappa}) \qquad \text{for a.e. } \boldsymbol{\kappa} \in \mathbb{R}^2. \tag{9}$$

It immediately follows that the adjoint of R_j is $-R_j$ and that the Riesz transforms satisfy the identity

$$R_1^2 + R_2^2 = -I, \tag{10}$$

where I denotes the identity operator on $L^2(\mathbb{R}^2)$. Using these remarks we get $\langle R_1\, g,\, R_1\, g\rangle + \langle R_2\, g,\, R_2\, g\rangle = \langle -R_1^2\, g,\, g\rangle + \langle -R_2^2\, g,\, g\rangle = \langle g,\, g\rangle$, that is

$$\forall g \in L^2(\mathbb{R}^2), \qquad \|R_1\, g\|^2_{L^2(\mathbb{R}^2)} + \|R_2\, g\|^2_{L^2(\mathbb{R}^2)} = \|g\|^2_{L^2(\mathbb{R}^2)}\,. \tag{11}$$

An important property of Riesz transforms (immediately obtained by passing to the Fourier domain) is that they map the vertical component of the gradient of the Poisson extension of a function to its horizontal components, namely,

$$R_j\, [\partial_{x_3}\, P_{x_3} \star g]_{|x_3=h} = \partial_{x_j}\, P_h \star g\,,\ j = 1, 2\,. \tag{12}$$

Conversely, the vertical component is obtained from the horizontal ones (see Equations (12) and (10)):

$$R_1\, (\partial_{x_1} \mathcal{P}_h\, g) + R_2\, (\partial_{x_2} \mathcal{P}_h\, g) = -[\partial_{x_3}\, P_{x_3} \star g]_{|x_3=h}. \tag{13}$$

Because Riesz transforms, Poisson extension and differentiation correspond to multipliers in the Fourier domain, they commute whenever their composition makes sense. More precisely, we have, for $j \in \{1, 2\}$ and $g \in L^2(\mathbb{R}^2)$,

$$\begin{aligned} R_1 R_2 &= R_2 R_1, &(14)\\ \mathcal{P}_h\, R_j &= R_j\, \mathcal{P}_h, &(15)\\ [\partial_{x_3}\, P_{x_3} \star R_j\, g]_{|x_3=h} &= R_j\, [\partial_{x_3}\, P_{x_3} \star g]_{|x_3=h}, &(16)\\ \partial_{x_j} \mathcal{P}_h R_j\, g &= R_j\, (\partial_{x_j} \mathcal{P}_h\, g). &(17)\end{aligned}$$

Applying Equation (12) and then using Equation (11), we obtain, for any $g \in L^2(\mathbb{R}^2)$:

$$\|\nabla_2\, P_h \star g\|_{[L^2(\mathbb{R}^2)]^2} = \|[\partial_{x_3} P_{x_3} \star g]_{x_3=h}\|_{L^2(\mathbb{R}^2)}. \tag{18}$$

Lemma 1. *Let Ω be open and non-empty in $\mathbb{R}^2$ and $g \in L^2(\mathbb{R}^2)$. If either $[\partial_{x_3} P_{x_3} \star g]_{|x_3=h}$, $\nabla_2 P_h \star g$ or $[\nabla_3 P_{x_3} \star g]_{|x_3=h}$ is identically zero on Ω, then $g = 0$ (hence all three functions are identically zero on all of $\mathbb{R}^2$).*

Proof. The function $(\boldsymbol{x}, x_3) \mapsto P_{x_3} \star g$ is harmonic on the upper half-space $\mathbb{R}^3_+$ as well as each component of its gradient. In particular these are real analytic on $\mathbb{R}^3_+$ (see, *e.g.*, [3, Thm 1.28]), and so is their restriction to the plane at height h. Now, if a real analytic function vanishes on the open and non-empty subset Ω, it must vanish on the whole plane (see, *e.g.*, [3, Thm 1.27]).

Therefore, if one of $[\partial_{x_3} P_{x_3} \star g]_{|x_3=h}$, $\nabla_2 P_h \star g$ or $[\nabla_3 P_{x_3} \star g]_{|x_3=h}$ is identically zero on Ω, it is zero on $\mathbb{R}^2$. Since $\|[\nabla_3 P_{x_3} \star g]_{x_3=h}\|^2_{L^2(\mathbb{R}^2)} = \|\nabla_2 P_h \star g\|^2_{L^2(\mathbb{R}^2)} + \|[\partial_{x_3} P_{x_3} \star g]_{x_3=h}\|^2_{L^2(\mathbb{R}^2)}$, and in view of Equation (18), the norm of any of them being zero implies that the other two are also zero.

Then, we have that $\nabla_2 P_h \star g = 0$, which means that $P_h \star g$ is a constant with respect to the variable $\boldsymbol{x}$ on $\mathbb{R}^2$. Since, moreover, $P_h \star g \in L^2(\mathbb{R}^2)$, this constant must be zero. Passing to the Fourier domain, it follows that $\mathrm{e}^{-2\pi h|\boldsymbol{\kappa}|}\, \hat{g}(\boldsymbol{\kappa}) = 0$ for a.e. $\boldsymbol{\kappa} \in \mathbb{R}^2$, whence $\hat{g} = 0$ and finally $g = 0$. $\square$

2.3 Properties of vector fields on $\mathbb{R}^2$

Introduce the spaces of solenoidal (divergence free) and irrotational 2-dimensional vector fields in $L^2(\mathbb{R}^2)$ such that

$$\begin{aligned} \mathrm{Sole}(L^2(\mathbb{R}^2)) &= \{\boldsymbol{g} = (g_1, g_2) \in [L^2(\mathbb{R}^2)]^2\,,\ \nabla_2 \cdot \boldsymbol{g} = 0\}\ , \\ \mathrm{Irrt}(L^2(\mathbb{R}^2)) &= \{\boldsymbol{g} = (g_1, g_2) \in [L^2(\mathbb{R}^2)]^2\,,\ \nabla_2 \times \boldsymbol{g} = 0\}\ , \end{aligned}$$

where $\nabla_2 \cdot \boldsymbol{g}$ and $\nabla_2 \times \boldsymbol{g}$ are taken in the sense of distributions. From [14, Ch. II, Sec. 6, Thm VI], members of $\mathrm{Irrt}(L^2(\mathbb{R}^2))$ are gradients of distributions with first (distributional) derivatives in $L^2(\mathbb{R}^2)$, hence belong to the so-called homogeneous Sobolev space of $\mathbb{R}^2$.

Next, for $\boldsymbol{g} = (g_1, g_2) \in [L^2(\mathbb{R}^2)]^2$, let $J(\boldsymbol{g}) = (-g_2, g_1) \in [L^2(\mathbb{R}^2)]^2$; the map J satisfies $J^2(\boldsymbol{g}) = -\boldsymbol{g}$ and is an isometry from $\mathrm{Irrt}(L^2(\mathbb{R}^2))$ onto $\mathrm{Sole}(L^2(\mathbb{R}^2))$, as can be directly checked from their definitions.

Hence, members of $\mathrm{Sole}(L^2(\mathbb{R}^2))$ are of the form $J(\nabla_2 \Phi)$ for some distribution Φ on $\mathbb{R}^2$ that has $L^2(\mathbb{R}^2)$ derivatives.

The Helmholtz-Hodge decomposition of 2-dimensional vector fields in $\mathbb{R}^2$ established in [10, Sec. 10.6] and recalled in [5, Sec. 2.3] states that:

$$[L^2(\mathbb{R}^2)]^2 = \mathrm{Sole}(L^2(\mathbb{R}^2)) \oplus \mathrm{Irrt}(L^2(\mathbb{R}^2))\,, \tag{19}$$

as an orthogonal sum in $[L^2(\mathbb{R}^2)]^2$.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and Lipschitz-smooth set. Restrictions to Ω of members of $\mathrm{Irrt}(L^2(\mathbb{R}^2))$ are then exactly the gradients of $W^{1,2}(\Omega)$-functions, as follows from [9, Thm 6.74]. Concerning properties of Sobolev spaces, see also [4, Sec. 2] and [1, 6, 7, 15, 17].

Hence, using J, restrictions of members of $\mathrm{Sole}(L^2(\mathbb{R}^2))$ to Ω also belong to $W^{1,2}(\Omega)$.

3 Operators related to magnetic quantities

Below, we introduce the (scalar) magnetic potential and the magnetic field in terms of Poisson and Riesz transforms. Then, using results from Sections 2.2 and 2.3, we establish some of their basic properties that will be of use to establish the results in Section 4.

3.1 Magnetic potential and field

If $\boldsymbol{g} = (g_1, g_2, g_3) \in [L^2(\mathbb{R}^2)]^3$ models a magnetization density on a thin plate, the magnetic scalar potential $\Lambda[\boldsymbol{g}]$ produced by the magnetization on the upper half-space $\mathbb{R}^3_+$ is given by (see [5, Thm 2.1])

$$\Lambda[\boldsymbol{g}](\boldsymbol{x}, x_3) = \frac{1}{2} P_{x_3} \star (R_1 g_1 + R_2 g_2 + g_3)(\boldsymbol{x}).$$

where the Poisson kernel P_{x_3} and the Riesz transforms R_j, $j = 1, 2$, were introduced in Section 2.2. Note that an equivalent integral expression with $\boldsymbol{t} = (t_1, t_2)$:

$$\begin{aligned}\Lambda[\boldsymbol{g}](\boldsymbol{x}, x_3) &= \frac{1}{4\pi} \iint_{\mathbb{R}^2} \left[\frac{g_1(\boldsymbol{t})(x_1 - t_1)}{d_{x_3}(\boldsymbol{x} - \boldsymbol{t})^3} + \frac{g_2(\boldsymbol{t})(x_2 - t_2)}{d_{x_3}(\boldsymbol{x} - \boldsymbol{t})^3} + \frac{g_3(\boldsymbol{t})\, x_3}{d_{x_3}(\boldsymbol{x} - \boldsymbol{t})^3} \right] \mathrm{d}\boldsymbol{t} \\ &= \frac{1}{4\pi} \iint_{\mathbb{R}^2} \frac{\langle \boldsymbol{g}(t),\, (x_1 - t_1,\, x_2 - t_2,\, x_3) \rangle}{d_{x_3}(\boldsymbol{x} - \boldsymbol{t})^3} \, \mathrm{d}\boldsymbol{t}.\end{aligned}$$

The corresponding magnetic field is $\boldsymbol{B}[\boldsymbol{g}] = -\mu_0 \nabla_3 \Lambda[\boldsymbol{g}]$ with $\mu_0 = 4\pi \times 10^{-7}$. We denote by $B_3[\boldsymbol{g}]$ the function of $L^2(\mathbb{R}^2)$ obtained by restricting its third component to the plane at height h, *i.e.*,

$$B_3[\boldsymbol{g}](\boldsymbol{x}) = -\frac{\mu_0}{2} \left[\partial_{x_3} P_{x_3} \star (R_1 g_1 + R_2 g_2 + g_3)(\boldsymbol{x}) \right]_{|x_3 = h}. \tag{20}$$

We are interested in the specific situation where the magnetization has bounded support and the vertical component of the magnetic field is measured on some bounded region of the plane at height h. In order to model this framework, we assume that $S \subset \mathbb{R}^2$ and $Q \subset \mathbb{R}^2$ are two open bounded connected Lipschitz-smooth sets and that the magnetization is of the form $\boldsymbol{g} = \widetilde{\boldsymbol{m}} = \boldsymbol{m} \vee 0$ where $\boldsymbol{m} \in [L^2(S)]^3$. Introduce now the operator $b_3 : [L^2(S)]^3 \to L^2(Q)$ defined by

$$b_3 : \boldsymbol{m} \mapsto B_3[\widetilde{\boldsymbol{m}}]_{|Q}.$$

3.2 Properties of the operator b_3

Using Equations (12) and (16) and the definition of B_3, we get

$$B_3[\boldsymbol{g}] = -\frac{\mu_0}{2} \Big(\partial_{x_1} P_h \star g_1 + \partial_{x_2} P_h \star g_2 + [\partial_{x_3} P_{x_3} \star g_3]_{|x_3 = h} \Big). \tag{21}$$

Hence from Equations (7) and (8), we have

$$\begin{aligned}\|B_3[\boldsymbol{g}]\|_{L^2(\mathbb{R}^2)} &\leq \frac{\mu_0}{2} \Big(\frac{C_1}{h} \|g_1\|_{L^2(\mathbb{R}^2)} + \frac{C_1}{h} \|g_2\|_{L^2(\mathbb{R}^2)} + \frac{C_1}{h} \|g_3\|_{L^2(\mathbb{R}^2)} \Big) \\ &\leq \frac{C_2}{h} \|\boldsymbol{g}\|_{[L^2(\mathbb{R}^2)]^3}\end{aligned}$$

for some constant C_2 (we may take $C_2 = \frac{\mu_0}{2} \sqrt{3}\, C_1$). Notice that, for any $\boldsymbol{m} \in [L^2(\mathbb{R}^2)]^3$, we have that $\|\boldsymbol{m}\|_{[L^2(S)]^3} = \|\widetilde{\boldsymbol{m}}\|_{[L^2(\mathbb{R}^2)]^3}$, consequently

$$\|b_3[\boldsymbol{m}]\|_{L^2(Q)} = \|B_3[\widetilde{\boldsymbol{m}}]_{|Q}\|_{L^2(Q)} \leq \|B_3[\widetilde{\boldsymbol{m}}]\|_{L^2(\mathbb{R}^2)} \leq \frac{C_2}{h} \|\boldsymbol{m}\|_{[L^2(S)]^3}, \tag{22}$$

which shows that b_3 is continuous $[L^2(S)]^3 \to L^2(Q)$.

3.3 Properties of the adjoint operator b_3^*

Let $\boldsymbol{m} = (m_1, m_2, m_3) \in \left[L^2(S)\right]^3$ and $\phi \in L^2(Q)$. Since $\widetilde{\phi}$ is zero outside Q, we have that $\langle b_3[\boldsymbol{m}], \phi\rangle_{L^2(Q)} = \langle B_3[\widetilde{\boldsymbol{m}}], \widetilde{\phi}\rangle_{L^2(\mathbb{R}^2)}$, and so by Equation (20):

$$\begin{aligned}\langle b_3[\boldsymbol{m}], \phi\rangle_{L^2(Q)} &= -\frac{\mu_0}{2}\,\langle [\partial_{x_3} P_{x_3} \star (R_1\,\widetilde{m}_1 + R_2\,\widetilde{m}_2 + \widetilde{m}_3)]_{|x_3=h}, \widetilde{\phi}\rangle_{L^2(\mathbb{R}^2)} \\ &= -\frac{\mu_0}{2}\left\langle \widetilde{\boldsymbol{m}}, \begin{pmatrix} -R_1\,[\partial_{x_3} P_{x_3} \star \widetilde{\phi}]_{|x_3=h} \\ -R_2\,[\partial_{x_3} P_{x_3} \star \widetilde{\phi}]_{|x_3=h} \\ [\partial_{x_3} P_{x_3} \star \widetilde{\phi}]_{|x_3=h}\end{pmatrix}\right\rangle_{[L^2(\mathbb{R}^2)]^3},\end{aligned}$$

the last equality being trivially verified by going over to the Fourier domain. Using Equation (12) to get rid of R_1 and R_2, we see that the adjoint operator of b_3 acts as $b_3^* : L^2(Q) \to \left[L^2(S)\right]^3$ given by:

$$b_3^*[\phi] = \frac{\mu_0}{2}\begin{pmatrix} \partial_{x_1}\,P_h \star \widetilde{\phi} \\ \partial_{x_2}\,P_h \star \widetilde{\phi} \\ -[\partial_{x_3} P_{x_3} \star \widetilde{\phi}]_{|x_3=h}\end{pmatrix}_{|S}. \tag{23}$$

Obviously

$$\|b_3^*[\phi]\|_{[L^2(S)]^3} \leq \left\| \frac{\mu_0}{2}\begin{pmatrix} \partial_{x_1}\,P_h \star \widetilde{\phi} \\ \partial_{x_2}\,P_h \star \widetilde{\phi} \\ -[\partial_{x_3} P_{x_3} \star \widetilde{\phi}]_{|x_3=h}\end{pmatrix}\right\|_{[L^2(\mathbb{R}^2)]^3},$$

therefore by Equations (8), (18) and since $\|\widetilde{\phi}\|_{L^2(\mathbb{R}^2)} = \|\phi\|_{L^2(Q)}$, we get that

$$\|b_3^*[\phi]\|_{[L^2(S)]^3} \leq \frac{C_3}{h}\,\|\phi\|_{L^2(Q)}$$

for some constant C_3 (we may take $C_3 = \frac{\mu_0}{2}\sqrt{2}\,C_1$). This gives a bound on the norm of the operator b_3^*. Notice that, since the norm of b_3 and b_3^* are equal [6, Chap. 2.6, Rem. 16], this is simply a reformulation of Equation (22) with a better constant.

Next, we remark that b_3^* is injective: indeed, if $\phi \in \operatorname{Ker} b_3^*$, Equation (23) implies that $\nabla_2\,P_h \star \widetilde{\phi}$ is identically 0 on S, which, according to Lemma 1, implies that $\phi = 0$. This entails that b_3 has dense range in $L^2(Q)$ because of the standard orthogonal decomposition $L^2(Q) = \operatorname{Ker} b_3^* \oplus \overline{\operatorname{Ran} b_3}$.

4 Magnetizations and moments recovery

We will now make use of the results in Section 3 to analyze certain aspects of inverse recovery problems for $\boldsymbol{m}$ and $\langle \boldsymbol{m}\rangle$. We first characterize in Section 4.1 the kernel $\operatorname{Ker} b_3$ of b_3, which consists by definition of silent (non-identifiable) sources compactly supported on $\overline{S}$. The fact that $\operatorname{Ker} b_3 \neq 0$ leads to non-uniqueness of $\boldsymbol{m}$ when trying to invert $\boldsymbol{m} \to b_3[\boldsymbol{m}]$. Subsequently, we establish in Section 4.2 the existence and uniqueness of the magnetization $\boldsymbol{m}_S$ of minimum $[L^2(S)]^3$-norm among all magnetizations $\boldsymbol{m}_e$ supported on $\overline{S}$ which are equivalent to a given $\boldsymbol{m}$ (*i.e.* such that $\boldsymbol{m}_e - \boldsymbol{m} \in \operatorname{Ker} b_3$). Thus, uniqueness can be enforced by adding a minimum norm constraint in the identification problem. We turn in Section 4.3 to moment recovery problems. This (much simpler) inversion problem is again ill-posed, this time because the solution is unstable with respect to measurement errors. This calls for regularization techniques and raises some issues in approximation by harmonic gradients that we comment upon in Section 5.

Hereafter $S \subset \mathbb{R}^2$ and $Q \subset \mathbb{R}^2$ are bounded connected Lipschitz-smooth open sets.

4.1 Silent source terms

By our assumption that S is Lipschitz-smooth, traces of functions in $W^{1,2}(S)$ on the boundary ∂S are well-defined in the fractional Sobolev space $W^{1/2,2}(\partial S)$, see [7, Cor. 2.2.3]. The closure in $W^{1,2}(S)$ of smooth functions compactly supported in S is denoted as $W_0^{1,2}(S)$, and it coincides with the subspace of functions with vanishing trace on ∂S. It is well-known that every function in $W^{1,2}(S)$ extends (in many different ways) to a function belonging to $W^{1,2}(\mathbb{R}^2)$; an equivalent characterization of functions in $W_0^{1,2}(S)$ is that the extension by zero does the job, see [7, Ch. 2] and [4, Sec. 2] or [1, 6, 7, 15, 17].

We denote by $\mathcal{D}_S$ the following set:

$$\mathcal{D}_S = \left\{(-\partial_{x_2}\psi,\, \partial_{x_1}\psi,\, 0)\ ,\ \psi \in W_0^{1,2}(S)\right\}. \tag{24}$$

Accordingly, we denote by $\mathcal{D}_{\mathbb{R}^2}$ the set

$$\mathcal{D}_{\mathbb{R}^2} = \left\{(-\partial_{x_2}\psi,\, \partial_{x_1}\psi,\, 0)\ ,\ \psi \in W_0^{1,2}(\mathbb{R}^2)\right\}.$$

Since $W_0^{1,2}(\mathbb{R}^2) = W^{1,2}(\mathbb{R}^2)$ (see [6, Ch. 9.4]) another expression for $\mathcal{D}_{\mathbb{R}^2}$ is simply $\mathcal{D}_{\mathbb{R}^2} = \{(\boldsymbol{s},\, 0)\ ,\ \boldsymbol{s} \in \mathrm{Sole}(L^2(\mathbb{R}^2))\}$.

The kernel $\mathrm{Ker}\, b_3 \subset \left[L^2(S)\right]^3$ can now be characterized as follows.

Proposition 2. *It holds that $\mathrm{Ker}\, b_3 = \mathcal{D}_S$.*

Proof: Let $\boldsymbol{m} \in \mathrm{Ker}\, b_3$. Together with the definitions of b_3 and Λ, Lemma 1 implies that $\widetilde{\boldsymbol{m}} = \boldsymbol{m} \vee 0 \in \mathrm{Ker}\,\Lambda = \mathrm{Ker}\,\boldsymbol{B} = \mathrm{Ker}\, B_3$. Because such magnetizations $\widetilde{\boldsymbol{m}}$ are supported on $\overline{S} \neq \mathbb{R}^2$, [5, Thm 2.3, iv)] implies that $\widetilde{\boldsymbol{m}} \in \mathrm{Ker}\,\Lambda$ if, and only if, $\widetilde{\boldsymbol{m}} \in \mathcal{D}_{\mathbb{R}^2}$.

Members $\boldsymbol{m}$ of $\mathrm{Ker}\, b_3$ are thus restrictions to S of horizontal vector fields $\widetilde{\boldsymbol{m}}$ in $\mathcal{D}_{\mathbb{R}^2}$ which are supported on $\overline{S}$.

Put $\widetilde{\boldsymbol{m}} = (\boldsymbol{s}, 0)$ with $\boldsymbol{s} \in \mathrm{Sole}(L^2(\mathbb{R}^2))$ supported on $\overline{S}$. We get from Section 2.3 that $\boldsymbol{s} = J(\nabla_2 \Phi)$ for some distribution Φ on $\mathbb{R}^2$ that admits $L^2(\mathbb{R}^2)$ derivatives, and

$$\boldsymbol{s}_{|S} = J(\nabla_2 f) = (-\partial_{x_2} f,\, \partial_{x_1} f)\,,\ \text{for } f = \Phi_{|S} \in W^{1,2}(S)\,. \tag{25}$$

Because $\boldsymbol{s}$ is supported on $\overline{S}$, so is $\nabla_2\Phi$ and therefore $\Phi = c$ is constant on $\mathbb{R}^2 \setminus \overline{S}$. Hence its trace on ∂S is also constant, and it is also the trace of f.

Observe now, using J, that normal components of traces on ∂S of restrictions to S of members of $\mathrm{Sole}(L^2(\mathbb{R}^2))$ are tangential derivatives of functions in $W^{1/2,2}(\partial S)$ (that act as distributions on Lipschitz-smooth functions on ∂S). Hence, $f - c \in W_0^{1,2}(S)$ (or equivalently $(f-c) \vee 0 \in W^{1,2}(\mathbb{R}^2)$). Finally, $\widetilde{\boldsymbol{m}} = (J(\nabla_2(\Phi - c)), 0)$ and $\boldsymbol{m} = (J(\nabla_2(f-c)), 0)$, which proves that $\mathrm{Ker}\, b_3 \subset \mathcal{D}_S$. The converse inclusion is easily obtained. $\square$

Remark 3. *For any element $(\boldsymbol{s}\,, 0)$ of $\mathcal{D}_S$, $\boldsymbol{s}$ coincides with the restriction to S of a member of Sole$(L^2(\mathbb{R}^2))$ supported on $\overline{S}$ and tangent to ∂S (although the tangential component is generally not defined, this wording means that the (well-defined) normal component on ∂S is identically zero).*

Indeed, it holds from Proposition 2 that $\boldsymbol{s} = J(\nabla_2\psi)$ for some $\psi \in W_0^{1,2}(S)$. Because the trace $\psi_{|\partial S}$ on ∂S vanishes, so does its tangential derivative there. Thus, the normal component of $\boldsymbol{s}$ on ∂S vanishes.

Actually, the converse also holds: the restriction to S of a member of Sole$(L^2(\mathbb{R}^2))$ which is supported on $\overline{S}$ and tangent to ∂S is such that $(\boldsymbol{s}\,, 0)$ belongs to $\mathcal{D}_S$.

We call S-silent magnetizations those $\boldsymbol{m} \in \left[L^2(S)\right]^3$ that belong to $\operatorname{Ker} b_3 = \mathcal{D}_S$ (silent for b_3). Similarly, $\mathbb{R}^2$-silent magnetizations are members of $\mathcal{D}_{\mathbb{R}^2}$ (silent for $\boldsymbol{B}$, B_3 or Λ). We also say that two magnetizations are S-equivalent (respectively $\mathbb{R}^2$-equivalent) if their difference is S-silent (resp. $\mathbb{R}^2$-silent), that is, if it belongs to $\mathcal{D}_S$ (resp. $\mathcal{D}_{\mathbb{R}^2}$).

With this terminology, Proposition 2 can be rephrased as follows. Given a magnetization $\boldsymbol{m} \in \left[L^2(S)\right]^3$, those magnetizations $\boldsymbol{m}_s \in \left[L^2(S)\right]^3$ such that $b_3[\boldsymbol{m}_s] = b_3[\boldsymbol{m}]$ (*i.e.* which are S-equivalent to $\boldsymbol{m}$) are given by $\boldsymbol{m}_s = \boldsymbol{m} + (\boldsymbol{d}, 0)$ with $\boldsymbol{d} \in \mathcal{D}_S$ (the S-silent magnetizations).

Also, given $\widetilde{\boldsymbol{m}} \in \left[L^2(\mathbb{R}^2)\right]^3$ supported on the compact set $\overline{S} \subset \mathbb{R}^2$, those $\widetilde{\boldsymbol{m}}_s$ that are supported on $\overline{S}$ and S-equivalent to $\widetilde{\boldsymbol{m}}$ are given by $\widetilde{\boldsymbol{m}}_s = \widetilde{\boldsymbol{m}} + (\boldsymbol{d} \vee 0, 0) = \widetilde{\boldsymbol{m}} + (\widetilde{\boldsymbol{d}}, 0)$ with $\boldsymbol{d} \in \mathcal{D}_S$, see also [5, Prop. 2.9].

As a consequence of Proposition 2, we have the next Lemma.

Lemma 4.

$$\overline{Ran\, b_3^*} = \overline{b_3^* \left[W_0^{1,2}(Q)\right]} = \mathcal{D}_S^\perp \subset \left[L^2(S)\right]^3 \text{ and } \mathcal{D}_S^\perp = \nabla_2 \left[W^{1,2}(S)\right] \times L^2(S)\,.$$

Proof: We saw in Section 3.2 that b_3^* is a continuous operator $L^2(Q) \to \left[L^2(S)\right]^3$. Hence, the following orthogonal decomposition holds:

$$\left[L^2(S)\right]^3 = \operatorname{Ker} b_3 \oplus \overline{\operatorname{Ran} b_3^*}\,,$$

so that $\overline{\operatorname{Ran} b_3^*} = \mathcal{D}_S^\perp \subset \left[L^2(S)\right]^3$, $\mathcal{D}_S^\perp$ being the orthogonal complement to $\mathcal{D}_S$ in $\left[L^2(S)\right]^3$ and

$$\left[L^2(S)\right]^3 = \mathcal{D}_S \oplus \mathcal{D}_S^\perp\,. \tag{26}$$

Moreover, the set of smooth functions with compact support in Q is dense in $L^2(Q)$ ([6, Cor. 4.23]) whence $W_0^{1,2}(Q)$ is dense in $L^2(Q)$. Together with the fact that b_3^* is continuous, this grants us that $\overline{\operatorname{Ran} b_3^*} = \overline{b_3^* \left[W_0^{1,2}(Q)\right]}$ which proves the first statement.

We turn to the characterization of $\mathcal{D}_S^\perp$. Because functions in $W^{1,2}(S)$ extend to functions in $W^{1,2}(\mathbb{R}^2)$ while members of $\mathcal{D}_S$ extend by zero to members of $\mathcal{D}_{\mathbb{R}^2}$ (see Remark 3), the fact that $\nabla_2 \left[W^{1,2}(S)\right] \times L^2(S) \subset \mathcal{D}_S^\perp$ follows from the orthogonal character of the Hodge decomposition (19). Conversely, let $(\boldsymbol{w}, w) \in \mathcal{D}_S^\perp$. By definition of $\mathcal{D}_S$, this is tantamount to say that w is arbitrary in $L^2(S)$ and that $\boldsymbol{w}$ is orthogonal to $J(\nabla_2 \psi)$ for every $\psi \in W_0^{1,2}(S)$. To show that $\nabla_2 \left[W^{1,2}(S)\right] \times L^2(S) \supset \mathcal{D}_S^\perp$, it is therefore enough to verify that a vector field in $[L^2(\mathbb{R}^2)]^2$ which is orthogonal to all vector fields of the form $(-\partial_{x_2}\phi, \partial_{x_1}\phi)$, where ϕ is smooth and compactly supported in S, must be a gradient. However, this orthogonality property means precisely that the vector field under consideration satisfies the distributional Schwarz rule, therefore it must be the gradient of a distribution and the latter belongs to $W^{1,2}(S)$ because it has L^2 derivatives. This completes the proof. $\square$

4.2 Equivalent source terms of minimal norm

In view of Proposition 2, it is natural to look for an S-equivalent magnetization $\boldsymbol{m}_S$ to $\boldsymbol{m}$ in $[L^2(S)]^3$ that has minimal $L^2(S)$-norm. Observe from [5, Cor. 2.4] that the $\mathbb{R}^2$-equivalent magnetization to $\widetilde{\boldsymbol{m}} = \boldsymbol{m} \vee 0 \in [L^2(\mathbb{R}^2)]^3$ which has minimum $L^2(\mathbb{R}^2)$-norm is not compactly supported when $\boldsymbol{m}$ is not S-silent. In particular, its restriction to S cannot furnish the field we are presently looking for.

Due to the Hardy-Hodge orthogonal decomposition of 3-dimensional vector fields in $\mathbb{R}^2$ provided by [5, Thm 2.2], there exists an orthogonal projection $P_{\mathcal{D}_{\mathbb{R}^2}} : [L^2(\mathbb{R}^2)]^3 \to \mathcal{D}_{\mathbb{R}^2}$ which acts on $\boldsymbol{g} = (g_1, g_2, g_3) \in [L^2(\mathbb{R}^2)]^3$ through Riesz transforms (see Section 2.2.2) as

$$P_{\mathcal{D}_{\mathbb{R}^2}} \, \boldsymbol{g} = (-R_2 \, d \, , \; R_1 \, d \, , 0) \; , \text{ with } d = R_2 \, g_1 - R_1 \, g_2 \, .$$

Following Equation (26), we let $P_{\mathcal{D}_S}$ be the orthogonal projection $\left[L^2(S)\right]^3 \to \mathcal{D}_S$.

Proposition 5. *Let $S \subset \mathbb{R}^2$ be an open bounded connected Lipschitz-smooth set and $\boldsymbol{m} \in [L^2(S)]^3$. Write $(\boldsymbol{s}, 0) = P_{\mathcal{D}_{\mathbb{R}^2}} \, \widetilde{\boldsymbol{m}}$.*

(i) The magnetization $\boldsymbol{m}_S \in [L^2(S)]^3$ which is S-equivalent to $\boldsymbol{m}$ and has minimum $L^2(S)$-norm is equal to

$$\boldsymbol{m}_S = \boldsymbol{m} + (\boldsymbol{h} - \boldsymbol{s}, 0) \, ,$$

where $(\boldsymbol{h}, 0) \in \mathcal{D}_{\mathbb{R}^2}$ is given as follows: $\boldsymbol{h}_{|_S}$ is the unique gradient of a harmonic function in S with normal component on ∂S equal to that of $\boldsymbol{s}_{|_S}$, and $\boldsymbol{h}_{|_{\mathbb{R}^2 \setminus S}} = \boldsymbol{s}_{|_{\mathbb{R}^2 \setminus S}}$.

(ii) The magnetization $\boldsymbol{m}$ has minimal $L^2(S)$-norm among S-equivalent magnetizations in $[L^2(S)]^3$ if and only if the restriction $\boldsymbol{s}_{|_S}$ to S of $\boldsymbol{s}$ is the gradient of a harmonic function in S.

Proof: Let us establish point (i) from which point (ii) directly follows.

For any $(\boldsymbol{d}, 0) \in \mathcal{D}_S$, and from Hardy-Hodge orthogonal decomposition [5, Thm 2.2],

$$\begin{aligned} \|\boldsymbol{m} - (\boldsymbol{d}, 0)\|^2_{[L^2(S)]^3} &= \|\widetilde{\boldsymbol{m}} - (\widetilde{\boldsymbol{d}}, 0)\|^2_{[L^2(\mathbb{R}^2)]^3} \\ &= \|\widetilde{\boldsymbol{m}} - (\boldsymbol{s}, 0)\|^2_{[L^2(\mathbb{R}^2)]^3} + \|\boldsymbol{s} - \widetilde{\boldsymbol{d}}\|^2_{[L^2(\mathbb{R}^2)]^2} \\ &= \|\widetilde{\boldsymbol{m}} - (\boldsymbol{s}, 0)\|^2_{[L^2(\mathbb{R}^2)]^3} + \|\boldsymbol{s}_{|_{\mathbb{R}^2 \setminus S}}\|^2_{[L^2(\mathbb{R}^2 \setminus S)]^2} + \|\boldsymbol{s}_{|_S} - \boldsymbol{d}\|^2_{[L^2(S)]^2} . \end{aligned}$$

The last term above is the one to be minimized among $\boldsymbol{d} \in \mathcal{D}_S$. From Equation (26) and the characteristic property of the orthogonal projection [6], we get with $\boldsymbol{d}_S = P_{\mathcal{D}_S}[\boldsymbol{s}_{|_S}]$ that:

$$\langle \boldsymbol{s}_{|_S} - \boldsymbol{d}_S \, , \boldsymbol{w} \rangle_{[L^2(S)]^2} = 0 \, , \; \forall (\boldsymbol{w}, 0) \in \mathcal{D}_S \, . \tag{27}$$

Note that in general $\boldsymbol{d}_S \neq \boldsymbol{s}_{|_S}$ since $\boldsymbol{s}$ may not be compactly supported on $\overline{S}$.

Because $(\boldsymbol{s}, 0) \in \mathcal{D}_{\mathbb{R}^2}$ whence $\boldsymbol{s} \in \mathrm{Sole}(L^2(\mathbb{R}^2))$ and since S is open and bounded, we get from properties in Section 2.3 that there exists a function $f \in W^{1,2}(S)$ that coincides with the restriction to S of a distribution Φ on $\mathbb{R}^2$ which admits $L^2(\mathbb{R}^2)$ derivatives ($f = \Phi_{|_S}$), such that Equation (25) is satisfied: $\boldsymbol{s}_{|_S} = J(\nabla_2 \, f)$.

From Proposition 2 it also holds that:

$$\boldsymbol{d}_S = (-\partial_{x_2} \, g_S, \, \partial_{x_1} \, g_S) \, , \boldsymbol{w} = (-\partial_{x_2} \, g, \, \partial_{x_1} \, g) \, .$$

for some $g_S \, , \; g \in W^{1,2}_0(S)$. In view of Equation (27), they must satisfy

$$\langle \nabla_2 (f - g_S) \, , \nabla_2 g \rangle_{[L^2(S)]^2} = 0 \, , \; \forall g \in W^{1,2}_0(S) \, .$$

Green formula is then to the effect that

$$\langle \Delta_2 (f - g_S) \, , g \rangle_{[L^2(S)]^2} = 0 \, , \; \forall g \in W^{1,2}_0(S) \, ,$$

hence $f - g_S$ is a harmonic distribution on S and therefore a harmonic function by Weyl's lemma. Now, we have on $\mathbb{R}^2$:

$$\boldsymbol{s} - \widetilde{\boldsymbol{d}}_S = (-\partial_{x_2} \, (\Phi - \widetilde{g}_S), \, \partial_{x_1} \, (\Phi - \widetilde{g}_S)) = J[\nabla_2 (\Phi - \widetilde{g}_S)] \, .$$

Let $\boldsymbol{j} = J[\nabla_2(\Phi - \widetilde{g}_S)]$. As a gradient, $\nabla_2(\Phi - \widetilde{g}_S) \in \mathrm{Irrt}(L^2(\mathbb{R}^2))$ so that $\boldsymbol{j} \in \mathrm{Sole}(L^2(\mathbb{R}^2))$ by properties of J given in Section 2.3.

We also have that $\nabla_2 \times \boldsymbol{j}_{|_S} = 0$ on S, because $\nabla_2 \times \boldsymbol{j}_{|_S} = \Delta_2 (\Phi - \widetilde{g}_S)_{|_S} = 0$, since $(\Phi - \widetilde{g}_S)_{|_S} = f - g_S$ is harmonic there.

Therefore $\boldsymbol{j}_{|_S}$ is equal to the gradient $\nabla_2 u$ of a function $u \in W^{1,2}(S)$. Since $\boldsymbol{j}$ is also divergence free (as an element of $\mathrm{Sole}(L^2(\mathbb{R}^2))$), it follows that u is harmonic in S.

Finally, $\boldsymbol{s}_{|_S} - \boldsymbol{d}_S = \boldsymbol{j}_{|_S} = \nabla_2 u$ and $\boldsymbol{h} = \boldsymbol{s} - \widetilde{\boldsymbol{d}}_S$ is the function we are looking for: indeed $\boldsymbol{h}_{|_S} = \nabla_2 u$, and on ∂S the normal components of $\boldsymbol{h}_{|_S}$ and $\boldsymbol{s}_{|_S}$ coincide (for the normal component of $\boldsymbol{d}_S$ on ∂S vanishes by Remark 3) and are equal to $\partial_n u$. Moreover, by construction, $\boldsymbol{h}$ and $\boldsymbol{s}$ coincide off $\overline{S}$. $\square$

Remark 6. *The above function u is harmonic on S with normal derivative $\partial_n u$ equal to the normal component $\boldsymbol{s}_n$ of the trace of $\boldsymbol{s}_{|_S}$ on ∂S. Because $\nabla_2 \cdot \boldsymbol{s} = 0$, the divergence formula (in the distributional sense) applies on S to the effect that $\boldsymbol{s}_n$ has vanishing mean there (even if $\boldsymbol{s}_n$ does not identically vanish on ∂S which may happen if $\boldsymbol{s}$ is not supported in $\overline{S}$, see Remark 3). Thus, u is uniquely defined up to an additive constant as a solution to a Neumann problem for the Laplace equation in S whose boundary data have mean zero [7]. Actually, u and $f - g_S$ are associated conjugate harmonic functions on S, see [15, Ch. II, Sec. 4]. Indeed, they satisfy the Cauchy-Riemann equations as follows easily from properties of J.*

4.3 Moment recovery issues

We now discuss moment recovery problems. Being given $b_3[\boldsymbol{m}] \in L^2(Q)$ for some unknown $\boldsymbol{m} \in [L^2(S)]^3$, we want to recover its net moment as a vector in $\mathbb{R}^3$ defined by:

$$\langle \boldsymbol{m} \rangle = (\langle m_1 \rangle \,,\, \langle m_2 \rangle \,,\, \langle m_3 \rangle) \,, \text{ with } \langle m_i \rangle = \iint_S m_i(\boldsymbol{t}) \, \mathrm{d}\boldsymbol{t} \,.$$

Note that $m_i \in L^2(S) \subset L^1(S)$ implies that the i^{th} components $\langle m_i \rangle$ of the net moment of $\boldsymbol{m}$ (the net moments of m_i for $i = 1, 2, 3$) are finite quantities.

Let $\boldsymbol{e}_1 = (\chi_S,\, 0,\, 0)$, $\boldsymbol{e}_2 = (0,\, \chi_S,\, 0)$, $\boldsymbol{e}_3 = (0,\, 0,\, \chi_S)$, where χ_S denotes the characteristic function of S. We obviously get from the definition above that

$$\langle m_i \rangle = \langle \boldsymbol{m} \,,\, \boldsymbol{e}_i \rangle_{[L^2(S)]^3} \,. \tag{28}$$

Because members of $\mathcal{D}_S$ have vanishing net moment (see point (*ii*) in Lemma 7 below), all magnetizations equivalent to a given $\boldsymbol{m}$ have the same moment, therefore the latter is uniquely and linearly defined by $b_3[\boldsymbol{m}]$. We then raise the natural issue as to whether the computation of the moment is a continuous (necessarily linear) operation on $b_3[\boldsymbol{m}]$. Equivalently, by the Hahn-Banach theorem, we ask if there exists $\phi \in L^2(Q)$ such that the quantity

$$\langle b_3 [\boldsymbol{m}] \,,\, \phi \rangle_{L^2(Q)} - \langle m_i \rangle = \langle b_3 [\boldsymbol{m}] \,,\, \phi \rangle_{L^2(Q)} - \langle \boldsymbol{m} \,,\, \boldsymbol{e}_i \rangle_{[L^2(S)]^3} = \langle \boldsymbol{m} \,,\, b_3^* [\phi] - \boldsymbol{e}_i \rangle_{[L^2(S)]^3} \tag{29}$$

vanishes, for all $\boldsymbol{m} \in \left[L^2(S)\right]^3$. The answer is no, as can be seen from the next result (recall that $\mathrm{Ran}\, b_3^* = b_3^* \left[L^2(Q)\right] \subset \left[L^2(S)\right]^3$).

Lemma 7. *The following three statements hold true.*

(*i*) $\boldsymbol{e}_i \in \mathcal{D}_S^{\perp} = \overline{b_3^* \left[W_0^{1,2}(Q)\right]} = \overline{\mathrm{Ran}\, b_3^*}$ *for* $i \in \{1, 2, 3\}$.

(*ii*) *Members of $\mathcal{D}_S$ possess vanishing net moment.*

(iii) However, $\boldsymbol{e}_i \notin Ran\, b_3^*$ for $i \in \{1,2,3\}$.

Proof: Let us denote by $\Pi_j : S \to \mathbb{R}$ $(j = 1,2)$ the projection onto the j-th component : $\Pi_j(\boldsymbol{x}) = x_j$. We observe first that $\boldsymbol{e}_1 = (\nabla_2(\Pi_1), 0)$ and $\boldsymbol{e}_2 = (\nabla_2(\Pi_2), 0)$ while $\boldsymbol{e}_3 = (\nabla_2(0), \chi_S)$ on S, hence $\boldsymbol{e}_i \in \nabla_2\left[W^{1,2}(S)\right] \times L^2(S)$ for $i = 1,2,3$. Point (i) therefore follows from Lemma 4. Now, since the $\boldsymbol{e}_i$ are orthogonal to $\mathcal{D}_S$ in $\left[L^2(S)\right]^3$, point (ii) follows from Equation (28).

To establish point (iii), assume that there exists $\phi \in L^2(Q)$ such that $b_3^*[\phi] = \boldsymbol{e}_i$ on S. Hence, (23) is to the effect that $\left[\nabla_3\, P_{x_3} \star \widetilde{\phi}\right]_{|x_3=h} = (\boldsymbol{e}_{i,T}\,, -e_{i,n})$ on S, if we put $\boldsymbol{e}_{i,T}$ and $e_{i,n}$ for the tangential and normal components of $\boldsymbol{e}_i$. Due to the definition of $\boldsymbol{e}_i$, we then have that either $\left[\partial_{x_3}\, P_{x_3} \star \widetilde{\phi}\right]_{|x_3=h} = 0$ if $i = 1,2$ or $\nabla_2\, P_h \star \widetilde{\phi} = 0$ if $i = 3$. In any case Lemma 1 implies that $\phi = 0$ which entails that $\boldsymbol{e}_i = 0$, a contradiction. $\square$

Lemma 7 point (iii) is to the effect that there is no $\phi \in L^2(Q)$ for which the quantity in (29) vanishes for all magnetizations $\boldsymbol{m} \in \left[L^2(S)\right]^3$. However, this quantity can be made arbitrarily small by approximating $\boldsymbol{e}_i$ by some $b_3^*[\phi]$ with $\phi \in W_0^{1,2}(Q) \subset L^2(Q)$. This we now investigate.

Lemma 8. *Let $\boldsymbol{e} \in \overline{Ran\, b_3^*} \subset \left[L^2(S)\right]^3$. It holds that*

$$\inf_{\phi \in W_0^{1,2}(Q)} \|b_3^*[\phi] - \boldsymbol{e}\|_{[L^2(S)]^3} = 0\,.$$

Whenever $\phi_n \in W_0^{1,2}(Q)$ is such that $\|b_3^*[\phi_n] - \boldsymbol{e}\|_{[L^2(S)]^3} \to 0$ as $n \to \infty$, then either $\boldsymbol{e} \in b_3^*\left[W_0^{1,2}(Q)\right]$ or $\|\phi_n\|_{W^{1,2}(Q)} \to \infty$.

Note that $\boldsymbol{e} \in b_3^*\left[W_0^{1,2}(Q)\right]$ is the only case where the above inf is reached.

Proof: The first assertion holds by density, see Lemma 4. To prove the second one, assume that $\|\phi_n\|_{W^{1,2}(Q)}$ is bounded. There exists a sub-sequence of $(\phi_n)_n$ that weakly converges in $W^{1,2}(Q)$. Since the injection $W^{1,2}(Q) \subset L^2(Q)$ is compact (by the Rellich-Kondrachov theorem [6, Thm 9.16]), the sub-sequence actually strongly converges in $L^2(Q)$. Its limit ϕ lies in $W_0^{1,2}(Q)$ (indeed, $W_0^{1,2}(Q)$ is closed hence weakly closed in $W^{1,2}(Q)$, see e.g. [6, Thm 3.7]). By continuity, $b_3^*\,[\phi_n]$ then converges to $b_3^*\,[\phi]$ and $\boldsymbol{e} = b_3^*\,[\phi]$. $\square$

Following Section 3.3, observe that for $\phi \in W_0^{1,2}(Q)$, then $\widetilde{\phi} \in W^{1,2}(\mathbb{R}^2)$ and, using Equation (13) together with Equation (23), we get:

$$b_3^*[\phi] = \frac{\mu_0}{2}\begin{pmatrix} P_h \star \left(\partial_{x_1}\widetilde{\phi}\right) \\ P_h \star \left(\partial_{x_2}\widetilde{\phi}\right) \\ P_h \star \left(R_1\,\partial_{x_1}\widetilde{\phi} + R_2\,\partial_{x_2}\widetilde{\phi}\right)\end{pmatrix}_{|S}.$$

Finally, Lemma 8 implies that the quantity in Equation (29), whose modulus satisfies

$$\left|\langle b_3\,[\boldsymbol{m}]\,,\,\phi\rangle_{L^2(Q)} - \langle m_i\rangle\right| \le \|b_3^*\,[\phi] - \boldsymbol{e}_i\|_{[L^2(S)]^3}\,\|\boldsymbol{m}\|_{[L^2(S)]^3}\,,$$

can be made arbitrarily small for every $\boldsymbol{m} \in \left[L^2(S)\right]^3$ of unit norm, say, at the expense of making $\|\nabla_2\,\phi\|_{[L^2(Q)]^2}$ arbitrary large.

5 Conclusion

Below, we recap our main results from Section 4 and we point at pending issues regarding magnetization and net moment recovery,

For $\boldsymbol{m} \in [L^2(S)]^3$, Proposition 5 asserts that there exists a unique $\boldsymbol{m}_S \in [L^2(S)]^3$ equivalent to $\boldsymbol{m}$ and of minimal $L^2(S)$-norm, *i.e.*, such that $\boldsymbol{m} - \boldsymbol{m}_S \in \mathcal{D}_S$ and

$$\|\boldsymbol{m}_S\|_{[L^2(S)]^3} = \min \left\{ \|\boldsymbol{d}\|_{[L^2(S)]^3} \ , \ \boldsymbol{d} \in [L^2(S)]^3 \, , \ b_3[\boldsymbol{d}] = b_3[\boldsymbol{m}] \text{ in } L^2(Q) \right\} . \tag{30}$$

If a magnetization $\boldsymbol{m}$ is obtained that reproduces the measurements up to reasonable accuracy, but which is irregular for instance because it was obtained by fitting some over parametrized approximant to the data, solving Equation (30) may be a way to smooth out the recovery. From a practical point of view, though, measurements of $b_3[\boldsymbol{m}]$ will always be corrupted by errors. This calls for a regularized formulation of Equation (30) in order to secure reasonably stable numerical schemes. Such a formulation may consist in relaxing the equality between $b_3[\boldsymbol{d}]$ and $b_3[\boldsymbol{m}]$, say to within $\varepsilon > 0$, and to seek $\boldsymbol{d}$ achieving

$$\min \left\{ \|\boldsymbol{d}\|_{[L^2(S)]^3} \ , \ \boldsymbol{d} \in [L^2(S)]^3 \, , \ \|b_3[\boldsymbol{m}] - b_3[\boldsymbol{d}]\|_{L^2(Q)} \leq \varepsilon \right\} .$$

This is a bounded extremal problem on planar subsets of $\mathbb{R}^3$ akin to those studied in [2]. The latter usually show good stability properties in their resolution, at least under some smoothness assumptions. However, this might not be sufficient here to ensure stability of the recovery. Indeed, we numerically observed the existence of sources $\boldsymbol{d}$ such that $b_3[\boldsymbol{d}] - b_3[\boldsymbol{m}] \neq 0$ has small $L^2(Q)$-norm but $\boldsymbol{d} - \boldsymbol{m}$ does not seem to be close to $\mathcal{D}_S$.

Additional information on the unknown magnetization $\boldsymbol{m}$ are thus likely to be required. An assumption of special interest to geophysicists is unidirectionality of $\boldsymbol{m}$ (see [5]). In this case, Fourier techniques have been applied with some success [13]. To apply such methods, however, a preliminary estimation of the net moment $\langle \boldsymbol{m} \rangle \in \mathbb{R}^3$ is very helpful so as to get the direction right.

Regarding this moment recovery issue, our results in Section 4.3 lay ground for a numerical procedure: the idea is to determine functions $\phi \in L^2(Q)$ against which the scalar product with $b_3[\boldsymbol{m}]$ in $L^2(Q)$ approximates the moment $\langle \boldsymbol{m} \rangle$ with prescribed accuracy, given a bound on $\|\boldsymbol{m}\|_{[L^2(S)]^3}$. According to Lemma 8, this is possible at the cost of making the $W^{1,2}(Q)$-norm of ϕ large enough, thus indicating that a trade-off must be found between the accuracy of the linear estimator for the moment provided to us by ϕ, and the size of the derivative of ϕ which accounts in a sense for the stability of this estimator, *i.e.* its sensitivity with respect to small changes in the data. One may for instance raise a best constrained approximation problem like

$$\min \left\{ \|b_3^* \, [\phi] - \boldsymbol{e}_i\|_{[L^2(S)]^3} \ , \ \phi \in W_0^{1,2}(Q) \, , \ \|\nabla_2 \, \phi\|_{[L^2(Q)]^2} \leq M \right\} ,$$

for some $M > 0$. Also, the $\boldsymbol{e}_i$ we used so far are rather simple (constant) functions, but of course the problem still makes sense if they are replaced by other functions in $\left[L^2(S)\right]^3$ aiming for example at estimating local moments.

From the point of view of function theory, a natural prospect is to generalize Propositions 2 and 5 to spaces $[L^p(S)]^3$ for $1 < p < \infty$. Indeed, the results of [5, Sec. 2.3], on which the present work rests, are stated there in $L^p(\mathbb{R}^2)$ for arbitrary $p \in (1, \infty)$. A more important extension of the present work, though, would be to consider recovery issues for magnetizations which are $[L^1(S)]^3$-functions or even measures. Because Riesz transforms of summable functions (resp. measures) need no longer be summable (resp. measures), this appears to require a substantial deepening of the theory. The latter, however, seems worthy because, from a physical point of view, the L^1 norm of $\boldsymbol{m}$ makes probably good sense as it represents the total magnetic capacity of a given sample. Moreover, this issue connects to sparse identification which is especially popular today.

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