Sensitivity of the Solution Set to Second Order Evolution Inclusions

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Abstract. In this note we study second order evolution inclusions in the framework of evolution triple of spaces. The existence of mild solutions (i.e. trajectory-selection pairs) to the inclusion, and the upper and lower semicontinuity properties of the solution set with respect to a parameter are established.

Keywords: evolution inclusion, Kuratowski convergence, upper semicontinuity, lower semicontinuity.

1 Introduction and preliminaries

In this paper we investigate a class of systems described by abstract second order evolution equations with multivalued right hand side. We consider Problem (P) of the form

$$\begin{cases} \ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) \in F(t, x(t), \dot{x}(t)) & \text{a.e. } t \in (0, T), \\ x(0) = x_0, \quad \dot{x}(0) = x_1 \end{cases}$$

and the following sequence of Problems $(\mathrm{P})_{\mathrm{n}}$, $n \in \mathbb{N}$, that can be regarded as the perturbed ones

$$\begin{cases} \ddot{x}(t) + A_n(t, \dot{x}(t)) + B_n x(t) \in F_n(t, x(t), \dot{x}(t)) & \text{a.e. } t \in (0, T), \\ x(0) = x_0^n, \quad \dot{x}(0) = x_1^n. \end{cases}$$

The goal is to establish the lower and upper semicontinuity properties of the solution set to Problem (P) with respect to the parameter $n \in \mathbb{N}$. The main result concerns the Kuratowski convergence of the sequence of solution sets to

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Problem $(\mathrm{P})_\mathrm{n}$ to that of Problem (P). Evolution inclusions of second order and their applications have been considered in several papers, see e.g. [6–9] and the references therein.

We introduce below the notation and preliminary material needed in the next sections. For a Banach space X, we indicate by w–X, s–X the space X equipped with the weak and the strong (norm) topology, respectively. Let (Ω, Σ, μ) be a measure space. A multifunction F defined on Ω with values in the space 2^X of all nonempty subsets of X is called measurable if $F^-(E) = \{\omega \in \Omega \mid F(\omega) \cap E \neq \emptyset\} \in \Sigma$ for every closed set $E \subset X$. It is called graph measurable if $GrF = \{(\omega, x) \in \Omega \times X \mid x \in F(\omega)\} \in \Sigma \times \mathcal{B}(X)$ where $\mathcal{B}(X)$ is the family of all Borel subsets of X. We denote by S_F^r, $1 \leq r \leq \infty$, the set of all selectors of F that belong to $L^r(\Omega; X)$, i.e., $S_F^r = \{f \in L^r(\Omega; X) \mid f(\omega) \in F(\omega)\ \mu \text{ a.e.}\}$. The symbol $\mathcal{P}_{f(c)}(X)$ stands for the family of all closed, (convex) subsets of 2^X. On $\mathcal{P}_f(X)$ we define the Hausdorff metric, by setting $h(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\}$. We also write $|A| = \sup\{|a| \mid a \in A\}$.

Given $\{S_n, S\}_{n\in\mathbb{N}} \subset 2^Z$, we define (see e.g. [3]) the sequential Kuratowski lower and upper limits respectively by $\tau_Z\text{--}\liminf S_n = \{z \in Z \mid \exists\, z_n \in S_n, z_n \to z \text{ in } \tau_Z\text{--}Z, \text{ as } n \to +\infty\}$ and $\tau_Z\text{--}\limsup S_n = \{z \in Z \mid \exists\, \{n_\nu\}, z_{n_\nu} \in S_{n_\nu}, z_{n_\nu} \to z \text{ in } \tau_Z\text{--}Z, \text{ as } \nu \to +\infty\}$. We say that S_n converge to S in the Kuratowski sense (denoted by $S_n \xrightarrow{K} S$) if and only if $\tau_Z\text{--}\limsup S_n \subset S \subset \tau_Z\text{--}\liminf S_n$.

Let (Y, τ_Y) and (Z, τ_Z) be Hausdorff topological spaces. A multifunction $G\colon Y \to 2^Z$ is said to be $(\tau_Y\text{--}\tau_Z)$ upper semicontinuous (usc) (respectively lower semicontinuous (lsc)) (cf. [2], Section 4.7 of [3]), if for every $C \subset Z$ closed in τ_Z topology, $G^-(C)$ (respectively, $G^+(C) = \{y \in Y \mid G(y) \subset C\}$) is closed in τ_Y topology in Y. The definition of lsc is equivalent to saying that if $y_n \to y$ in τ_Y–Y, then $G(y) \subset \tau_Z\text{--}\liminf G(y_n)$. For a sequence of multifunctions G, $G_n\colon Y \to 2^Z$, we write

$$K(\tau_Y, \tau_Z) \limsup_{n\to+\infty, y\to\tilde{y}} G_n(y) \subset G(\tilde{y})$$

if $\tau_Z\text{--}\limsup G_n(y_n) \subset G(y)$ for every $y_n \to y$ in τ_Y–Y. Similar notation is used for $\tau_Z\text{--}\liminf$.

Let H be a separable Hilbert space and V be a reflexive Banach space which is densely, continuously and compactly embedded in H. Identifying H with its dual H^*, we have the Gelfand triple $V \subset H \subset V^*$, where V^* is the dual of V. Let $\langle\cdot,\cdot\rangle$ be the duality of V and V^* as well as the inner product on H, let $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_{V^*}$ denote the norms in V, H and V^*, respectively. For $T > 0$ and $2 \leq p < +\infty$, we introduce the following spaces $\mathcal{V} = L^p(0,T;V)$, $\mathcal{H} = L^p(0,T;H)$, $\mathcal{H}^* = L^q(0,T;H)$, $\mathcal{V}^* = L^q(0,T;V^*)$, where $1/p + 1/q = 1$, $1 < q \leq 2$, and $\mathcal{W} = \{w \in \mathcal{V} \mid w' \in \mathcal{V}^*\}$. The derivative is understood in the sense of vector valued distributions. Clearly $\mathcal{W} \subset \mathcal{V} \subset \mathcal{H} \subset \mathcal{H}^* \subset \mathcal{V}^*$. The pairing of $\mathcal{V}$ and $\mathcal{V}^*$ and the duality between $\mathcal{H}$ and $\mathcal{H}^*$ are denoted by $\langle\langle f, v\rangle\rangle = \int_0^T \langle f(s), v(s)\rangle\, ds$. It is well known that the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous. Since $V \subset H$ compactly we know that the embedding $\mathcal{W} \subset \mathcal{H}$ is also compact. Finally, the class of linear bounded operators from V into V^* is denoted by $\mathcal{L}(V, V^*)$. For additional details on the material, we refer to [3, 11].

2 Results on evolution equations

In this section we investigate the existence, uniqueness and continuous dependence of solutions on the data for an evolution equation of second order. We consider the following problem

$$\text{(E)} \qquad \begin{cases} \ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t) \quad \text{a.e. } t \in (0,T), \\ x(0) = x_0, \quad \dot{x}(0) = x_1. \end{cases}$$

A function $x \in C(0,T;V)$ is called a solution to the problem (E) if and only if $\dot{x} \in \mathcal{W}$ and (E) is satisfied.

We will need the following hypotheses.

$\underline{H(A)}$: $A\colon (0,T) \times V \to V^*$ is an operator such that

(1) $t \mapsto A(t,v)$ is measurable, for every $v \in V$,

(2) $v \mapsto A(t,v)$ is monotone and hemicontinuous, a.e. $t \in (0,T)$,

(3) $\langle A(t,v), v\rangle \geq c\,\|v\|^p - d\,|v|^2$ a.e. for all $v \in V$ with $c > 0$ and $d \geq 0$,

(4) $\|A(t,v)\|_{V^*} \leq a(t) + b\,\|v\|^{p-1}$ for all $v \in V$, a.e. $t \in (0,T)$ with $a \in L^q_+(0,T)$ and $b > 0$.

$\underline{H(B)}$: $B \in \mathcal{L}(V,V^*)$ is symmetric and coercive (i.e., $\langle Bv, v\rangle \geq m\,\|v\|^2$ for all $v \in V$ with $m > 0$).

$\underline{(H_0)}$: $x_0 \in V,\ x_1 \in H$.

The proof of the following result follows from the standard application of the Galerkin method and can be found in [1, 5, 6].

Proposition 1. *Under hypotheses $H(A)$, $H(B)$, (H_0) and $f \in \mathcal{H}^*$, the problem (E) admits a unique solution which satisfies $x \in C(0,T;V)$, $\dot{x} \in \mathcal{W}$, and the following estimate*

$$\|x(t)\|^2 + |\dot{x}(t)|^2 + \|\dot{x}\|^2_{\mathcal{W}} \leq C\left(1 + \|x_0\|^2 + |x_1|^2 + \|B\|^2_{\mathcal{L}(V,V^*)} + \|f\|^q_{\mathcal{H}^*}\right)$$

for all $t \in [0,T]$ with $C > 0$.

We present now a result on the continuous dependence of solutions to the problem

$$\text{(E)}_n \qquad \begin{cases} \ddot{x}(t) + A_n(t, \dot{x}(t)) + B_n x(t) = f_n(t) \quad \text{a.e. } t \in (0,T), \\ x(0) = x_0^n, \quad \dot{x}(0) = x_1^n. \end{cases}$$

on the data. We will need the following assumptions.

$\underline{H(A)_1}$: $A\colon (0,T) \times V \to V^*$ is such that $H(A)$ holds, $A_n\colon (0,T) \times V \to V^*$ satisfy $H(A)(1)(2)(3)$ uniformly with respect to $n \in \mathbb{N}$ and the condition

$$\|A_n(t,v)\|_{V^*} \le a_n(t) + b\,\|v\|^{p-1} \quad \text{for all } v \in V, \ \text{ a.e. } t \in (0,T)$$
with $a_n \in L^q_+(0,T)$, $\sup_{n\in\mathbb{N}} ||a_n||_{L^q} < +\infty$, $b > 0$ and
$$A_n(\cdot, w(\cdot)) \to A(\cdot, w(\cdot)) \text{ in } s\text{–}\mathcal{V}^* \text{ for all } w \in \mathcal{V} \cap L^\infty(0,T;H).$$

$\underline{H(B)_1}$: $B_n \in \mathcal{L}(V,V^*)$ satisfy $H(B)$ uniformly with respect to $n \in \mathbb{N}$ and $B_n \to B$ in $\mathcal{L}(V,V^*)$.

$\underline{(H_0)_1}$: $x_0^n, x_0 \in V$, $x_1^n, x_1 \in H$, $x_0^n \to x_0$ in s–V and $x_1^n \to x_1$ in s–H.

For every $n \in \mathbb{N}$, let x_n be a solution of the problem $(\mathrm{E})_n$ and let x be a solution of the problem (E). We have

Proposition 2. *If hypotheses $H(A)_1$, $H(B)_1$, $(H_0)_1$ hold, $f_n \in \mathcal{H}^*$, $f_n \to f$ weakly in $\mathcal{H}^*$, then the sequence $\{(x_n, \dot{x}_n)\}$ converges to $(x, \dot{x})$ in $C(0,T;V\times H)$, as $n \to +\infty$.*

Proof. By Proposition 1 we know that, for every $n \in \mathbb{N}$, the problem $(\mathrm{E})_n$ has the unique solution $x_n \in C(0,T;V)$ such that $\dot{x}_n \in \mathcal{W}$. From $(\mathrm{E})_n$ and (E), we have

$$\langle \ddot{x}_n(s) - \ddot{x}(s), \dot{x}_n(s) - \dot{x}(s)\rangle + \langle A_n(s,\dot{x}_n(s)) - A(s,\dot{x}(s)), \dot{x}_n(s) - \dot{x}(s)\rangle +$$
$$+\langle B_n x_n(s) - Bx(s), \dot{x}_n(s) - \dot{x}(s)\rangle = \langle f_n(s) - f(s), \dot{x}_n(s) - \dot{x}(s)\rangle \ \text{ a.e.}$$

for every $n \in \mathbb{N}$. Integrating this equality and using the monotonicity of $A_n(s,\cdot)$, we get

$$|\dot{x}_n(t) - \dot{x}(t)|^2 - |\dot{x}_n(0) - \dot{x}(0)|^2 + 2\int_0^t \langle A_n(s,\dot{x}(s)) - A(s,\dot{x}(s)), \dot{x}_n(s) - \dot{x}(s)\rangle\, ds +$$
$$+2\int_0^t \langle B_n x_n(s) - Bx_n(s), \dot{x}_n(s) - \dot{x}(s)\rangle\, ds + \langle Bx_n(t) - Bx(t), x_n(t) - x(t)\rangle -$$
$$-\langle Bx_n(0) - Bx(0), x_n(0) - x(0)\rangle \le 2\int_0^t \langle f_n(s) - f(s), \dot{x}_n(s) - \dot{x}(s)\rangle\, ds$$

for all $t \in [0,T]$. Hence using $H(B)_1$ and applying the Hölder inequality, we obtain

$$|\dot{x}_n(t) - \dot{x}(t)|^2 + m\,\|x_n(t) - x(t)\|^2 \le \|B\|\,\|x_0^n - x_0\| + |x_1^n - x_1|^2 + \tag{1}$$
$$+2\,\|\widehat{A}_n(\dot{x}) - \widehat{A}(\dot{x})\|_{\mathcal{V}^*} \|\dot{x}_n - \dot{x}\|_{\mathcal{V}} + \widetilde{C}\,\|B_n - B\|\,\|x_n\|_{\mathcal{V}}\,\|\dot{x}_n - \dot{x}\|_{\mathcal{V}} +$$
$$+2\,\langle\langle f_n - f, \dot{x}_n - \dot{x}\rangle\rangle$$

for all $t \in [0,T]$, where $\widehat{A}_n$ and $\widehat{A}$ are the Nemitsky operators corresponding to A_n and A, respectively, and $\widetilde{C}$ is a positive constant independent of n. On the other hand, due to $H(A)_1$, $H(B)_1$ and $(H_0)_1$, from Proposition 1, we have

$$\|x_n(t)\|^2 + |\dot{x}_n(t)|^2 + \|\dot{x}_n\|^2_{\mathcal{W}} \le C\left(1 + \|x_0^n\|^2 + |x_1^n|^2 + \|B_n\|^2 + \|f_n\|^q_{\mathcal{H}^*}\right). \tag{2}$$

Hence, it follows that $\{\dot{x}_n\}$ lies in a bounded subset of $\mathcal{W}$. Thus, up to a subsequence, $\dot{x}_n$ converges weakly in $\mathcal{W}$ and (since $\mathcal{W} \subset \mathcal{H}$ compactly) strongly in $\mathcal{H}$. So we have

$$\lim_{n\to+\infty} \langle\langle f_n - f, \dot{x}_n - \dot{x}\rangle\rangle = 0. \tag{3}$$

Using the assumptions, (2) and (3), from (1), we get $(x_n(t), \dot{x}_n(t)) \to (x(t), \dot{x}(t))$ in s–$(V \times H)$ for all $t \in [0, T]$, as $n \to +\infty$. Since the solution to (E) is unique, we deduce that the whole sequence $\{(x_n, \dot{x}_n)\}$ converges to $(x, \dot{x})$ in $C(0, T; V \times H)$. The proof is completed. □

In the sequel, we make use of the solution map $r\colon \mathcal{H}^* \to C(0, T; V) \times \mathcal{W}$ for (E) defined by $r(f) = (x, \dot{x})$, where x (respectively $\dot{x}$) is the solution (and its derivative, respectively) to (E). By Proposition 1 this map is well defined and Proposition 2 implies the following result.

Corollary 1. *Under hypotheses $H(A)$, $H(B)$ and (H_0), the solution map r for the problem (E) is continuous from w–$\mathcal{H}^*$ into $C(0, T; V \times H)$.*

3 Existence result for inclusions

In this section we study the existence of solutions to Problem (P). We start with the following

Definition 1. *A couple $(x, f) \in C(0, T; V) \times \mathcal{H}^*$ is called a mild solution to Problem* (P) *if and only if x is a solution to the evolution equation (E) and $f(\cdot) \in S^q_{F(\cdot, x(\cdot), \dot{x}(\cdot))}$.*

Prior to the existence theorem, we state the a priori bound on the solution to the evolution inclusion. We need the following hypotheses.

$\underline{H(F)}$: $F\colon (0, T) \times H \times H \to \mathcal{P}_{fc}(H)$ is a multifunction such that

(1) F is graph measurable,

(2) $GrF(t, \cdot, \cdot)$ is sequentially closed in $H \times H \times (w$–$H)$, a.e. $t \in (0, T)$,

(3) $|F(t, x, y)| \le a_1(t) + b_1|x|^{2/q} + c_1|y|^{2/q}$, a.e. $t \in (0, T)$, where $a_1 \in L^q_+(0, T)$ and $b_1, c_1 > 0$.

Lemma 1. *Assume $H(A)$, $H(B)$, $H(F)$ and (H_0). If (x, f) is a mild solution to Problem* (P)*, then $(x, \dot{x}, f)$ lies in a bounded set of $(L^\infty(0, T; V) \cap W^{1,\infty}(0, T; H)) \times \mathcal{W} \times \mathcal{H}^*$.*

In the proof of the next result, we follow methods used in [4, 10].

Theorem 1. *If hypotheses $H(A)$, $H(B)$, $H(F)$ and (H_0) hold, then Problem* (P) *admits a mild solution.*

Proof. From Lemma 1, it is clear that every solution to Problem (P) satisfies

$$|x(t)| \le M_1, \qquad |\dot{x}(t)| \le M_2, \tag{4}$$

for all $t \in (0,T)$ with positive constants M_1, M_2. We define multifunction $\widehat{F}\colon (0,T) \times H \times H \to \mathcal{P}_{fc}(H)$ by $\widehat{F}(t,x,y) = F(t,p(x,y))$, where the map $p\colon H \times H \to B(0,M_1) \times B(0,M_2)$ is as follows

$$p(x,y) = \begin{cases} (x,y) & \text{if } |x| \le M_1 \text{ and } |y| \le M_2, \\ ((M_1 x/|x|),(M_2 y/|y|)) & \text{if } |x| > M_1 \text{ and } |y| > M_2, \\ ((M_1 x/|x|), y) & \text{if } |x| > M_1 \text{ and } |y| \le M_2, \\ (x,(M_2 y/|y|)) & \text{if } |x| \le M_1 \text{ and } |y| > M_2. \end{cases}$$

Since the map p is Lipschitz continuous, from the properties of F, we deduce that $\widehat{F}$ satisfies $H(F)(1)(2)$. Furthermore, we note that $|\widehat{F}(t,x,y)| \le \widetilde{a}_1(t)$ a.e. $t \in (0,T)$, where $\widetilde{a}_1 \in L^q_+(0,T)$ is given by $\widetilde{a}_1(t) = a_1(t) + b_1 M_1^{2/q} + c_1 M_2^{2/q}$.

We define $\mathcal{Z} = \{f \in \mathcal{H}^* \mid |f(t)| \le \widetilde{a}_1(t) \text{ a.e. } t \in (0,T)\}$ and a multifunction $\mathcal{R}$ on $\mathcal{Z}$ by

$$\mathcal{R}(f) = S^1_{\widehat{F}(\cdot,r(f)(\cdot))} = \left\{ f \in L^1(0,T;H) \mid f(t) \in \widehat{F}(t,r(f)(t)) \ \text{ a.e. } t \in (0,T) \right\}$$

(recall that $r(\cdot)$ is the solution map for the equation (E)). Since $\widehat{F}$ is graph measurable and L^1 integrably bounded, using the Aumann selection theorem (see Theorem 4.3.7 of [3]), we have $\mathcal{R}(f) \neq \emptyset$ for $f \in \mathcal{Z}$. Moreover, because $\widehat{F}$ is $\mathcal{P}_{fc}(H)$-valued and $|\widehat{F}(t,r(f)(t))| \le \widetilde{a}_1(t)$ a.e. $t \in (0,T)$, we obtain that $\mathcal{R}\colon \mathcal{Z} \to \mathcal{P}_{fc}(\mathcal{Z})$.

We will show that $\mathcal{R}$ is (w–$\mathcal{H}^*$)×(w–$\mathcal{H}^*$) usc on $\mathcal{Z}$. Since $\mathcal{Z}$ is compact in w–$\mathcal{H}^*$, it suffices to prove (see Chapter I of [2], Section 4.1 of [3]) that $Gr\mathcal{R}$ is weakly–weakly closed in $\mathcal{Z} \times \mathcal{Z}$. Let $(f_n, z_n) \in Gr\mathcal{R}$, $f_n \to f$ and $z_n \to z$ both in w–$\mathcal{H}^*$. By Corollary 1, we know that $r(f_n)(t) \to r(f)(t)$ in $(s\text{–}H) \times (s\text{–}H)$ for all $t \in [0,T]$. Since $\widehat{F}$ satisfies $H(F)(1)(2)$, we deduce that $w\text{–}\limsup \widehat{F}(t,r(f_n)(t)) \subset \widehat{F}(t,r(f)(t))$ a.e. $t \in (0,T)$. Using Theorem 4.7.51 of [3], we obtain

$$w\text{–}\limsup \mathcal{R}(f_n) = w\text{–}\limsup S^1_{\widehat{F}(\cdot,r(f_n)(\cdot))} \subset$$

$$\subset S^1_{w-\limsup \widehat{F}(\cdot,r(f_n)(\cdot))} \subset S^1_{\widehat{F}(\cdot,r(f)(\cdot))} = \mathcal{R}(f).$$

From these inclusions we have $(f,z) \in Gr\mathcal{R}$. This means that $Gr\mathcal{R}$ is closed in $(w\text{–}\mathcal{Z}) \times (w\text{–}\mathcal{Z})$ and proves that $\mathcal{R}$ is weakly–weakly usc on $\mathcal{Z}$.

We apply the well known Kakutani-KyFan fixed point theorem for set-valued mappings (see Chapter I.12 of [2]) to the multifunction $\mathcal{R}$. We deduce that there exists $f^* \in \mathcal{Z}$ such that $f^* \in \mathcal{R}(f^*)$. The corresponding pair $(x^*, \dot{x}^*) = r(f^*)$ is a solution to Problem (P) with F replaced by $\widehat{F}$. However, the same estimates as in Lemma 1 (cf. also (4)), imply that $|x^*(t)| \le M_1$, $|\dot{x}^*(t)| \le M_2$ for every

$t \in (0,T)$. Thus $\widehat{F}(t, x^*(t), \dot{x}^*(t)) = F(t, x^*(t), \dot{x}^*(t))$ for a.e. $t \in (0,T)$, which means that (x^*, f^*) is a mild solution to Problem (P). This completes the proof of the theorem. $\square$

Corollary 2. *If $F(t,u,v) = \{f(t,u,v)\}$, where $f\colon (0,T) \times H \times H \to H$ is a function measurable in t, continuous in (u,v) and*

$$|f(t,u,v)| \le a_1(t) + b_1|u|^{2/q} + c_1|v|^{2/q} \quad \text{a.e. } t \in (0,T) \tag{5}$$

for all $u, v \in H$, then Theorem 1 ensures that the Cauchy problem for the nonlinear equation $\ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t, x(t), \dot{x}(t))$ has at least one solution.

Let S be the set of mild solutions to Problem (P) and let

$$\mathcal{M} = \{(x, \dot{x}, f) \in C(0,T; V \times H) \times \mathcal{H}^* \mid (x,f) \in S\}.$$

Corollary 3. *Under the hypotheses of Theorem 1, the set $\mathcal{M}$ is nonempty, compact subset of $C(0,T;V \times H) \times (w\text{–}\mathcal{H}^*)$.*

Proof. The nonemptiness of $\mathcal{M}$ follows from Theorem 1. Let $\{(x_k, \dot{x}_k, f_k)\}_{k\in\mathbb{N}} \subset \mathcal{M}$. We will show that this sequence has a subsequence which converges in an appropriate topology to an element of $\mathcal{M}$. By the definition, x_k satisfies the evolution equation (E) with the right-hand side f_k and $f_k(\cdot) \in S^q_{F(\cdot, x_k(\cdot), \dot{x}_k(\cdot))}$. From Lemma 1, we obtain in particular that f_k remains in a bounded subset of $\mathcal{H}^*$. So after a possible passing to subsequence, we have $f_k \to f$ weakly in $\mathcal{H}^*$, as $k \to +\infty$, with $f \in \mathcal{H}^*$. Corollary 1 says that $r(f_k) \to r(f)$ in $C(0,T; V \times H)$, where $r(f) = (x, \dot{x})$ is a solution to (E). In order to conclude the proof, it suffices to show that f is a selection for $F(\cdot, x(\cdot), \dot{x}(\cdot))$. From Theorem 4.7.44 of [3], we have

$$f(t) \in \overline{co}\ w\text{–}\limsup\{f_k(t)\}_{k\ge 1} \subset \overline{co}\ w\text{–}\limsup F(t, x_k(t), \dot{x}_k(t))$$

a.e. $t \in (0,T)$. Since $(x_k(t), \dot{x}_k(t)) \to (x(t), \dot{x}(t))$ in $s\text{–}(H \times H)$ for all $t \in [0,T]$, from $H(F)(1)$ (2), we easily deduce that $w\text{–}\limsup F(t, x_k(t), \dot{x}_k(t)) \subset F(t, x(t), \dot{x}(t))$ a.e. $t \in (0,T)$. Hence, we get $f(t) \in F(t, x(t), \dot{x}(t))$ a.e. $t \in (0,T)$. So we have obtained $(x, \dot{x}, f) \in \mathcal{M}$ which completes the proof. $\square$

4 Upper semicontinuity property of the solution set

Consider now a sequence of evolution inclusions Problem $(\mathrm{P})_\mathrm{n}$. Let us denote by S_n the set of mild solutions to Problem $(\mathrm{P})_\mathrm{n}$, i.e., $S_n = \{(x,f) \in C(0,T;V) \times \mathcal{H}^* \mid (x,f)$ is a mild solution to Problem $(\mathrm{P})_\mathrm{n}\}$.

Theorem 2. *Suppose that hypotheses $H(A)_1$, $H(B)_1$, $(H_0)_1$ hold, F, $F_n\colon (0,T) \times H \times H \to \mathcal{P}_{fc}(H)$ are multifunctions satisfying $H(F)$ uniformly with respect to $n \in \mathbb{N}$ and*

$$K\,(s\text{–}(H \times H) \times (w\text{–}H)) \limsup_{n\to+\infty, (u,v)\to(\tilde{u},\tilde{v})} F_n(t,u,v) \subset F(t, \tilde{u}, \tilde{v}) \quad \text{a.e.} \tag{6}$$

If $(x_n, f_n) \in S_n$, $n \in \mathbb{N}$ and $f_n \to f$ in $w\text{–}\mathcal{H}^*$, then $(x,f) \in S$.

Proof. From Theorem 1 we know that S_n, $S \neq \emptyset$. Let $(x_n, f_n) \in S_n$ for $n \in \mathbb{N}$ and $f_n \to f$ weakly in $\mathcal{H}^*$. By Proposition 2, we infer that $(x_n, \dot{x}_n)$ converges in $C(0,T;V \times H)$ to $(x, \dot{x})$, as $n \to +\infty$, where x is a solution to the equation (E) (corresponding to the right hand side f). It remains to prove that $f(\cdot) \in S^q_{F(\cdot,x(\cdot),\dot{x}(\cdot))}$. From Theorem 4.7.44 of [3], we have

$$f(t) \in \overline{co}\ w\text{–}\limsup\{f_n(t)\}_{n\in\mathbb{N}} \subset \overline{co}\ w\text{–}\limsup F_n(t, x_n(t), \dot{x}_n(t))$$

a.e. $t \in (0,T)$ and by (6) we obtain

$$w\text{–}\limsup F_n(t, x_n(t), \dot{x}_n(t)) \subset F(t, x(t), \dot{x}(t)) \quad \text{a.e. } t \in (0,T).$$

This facts imply $f(t) \in F(t, x(t), \dot{x}(t))$ a.e. $t \in (0,T)$. Hence (x, f) is a mild solution to Problem (P) which concludes the proof. □

We introduce the sets $\mathcal{M}_n = \{(x, \dot{x}, f) \in C(0,T;V \times H) \times \mathcal{H}^* \mid (x, f) \in S_n\}$ for every $n \in \mathbb{N}$. We have the following upper semicontinuity property.

Corollary 4. *If hypotheses of Theorem 2 hold, then* $\limsup \mathcal{M}_n \subset \mathcal{M}$*, where the upper limit is taken in* $C(0,T;V \times H) \times (w\text{–}\mathcal{H}^*)$ *topology.*

5 Lower semicontinuity property of the solution set

In order to state a result on lower semicontinuity of the set of mild solutions, we admit the following stronger assumption on the multivalued term.

$\underline{H(F)_1}$: F, $F_n \colon (0,T) \times H \times H \to \mathcal{P}_{fc}(H)$ are multifunctions satisfying uniformly with respect to $n \in \mathbb{N}$ the conditions

(1) $F(\cdot, u, v)$ is measurable, for all $u, v \in H$,

(2) $F(t, \cdot, \cdot)$ is h-continuous, a.e. $t \in (0,T)$,

(3) $H(F)(3)$ holds and

$$h(F_n(t, u_1, v_1), F(t, u_2, v_2)) \le \alpha_n(t)\,(|u_1 - u_2| + |v_1 - v_2|) + \beta_n(t) \tag{7}$$

a.e. $t \in (0,T)$, with $\alpha_n \in L^1_+(0,T)$, $\alpha(t) = \sup_{n\in\mathbb{N}} \alpha_n(t) \in L^1_+(0,T)$ and $\beta_n \to 0$ in $L^2(0,T)$, as $n \to +\infty$.

Remark 1. The estimate (7) holds, for instance, if we suppose that

(a) $h(F_n(t, u_1, v_1), F_n(t, u_2, v_2)) \le \alpha_n(t)\,(|u_1 - u_2| + |v_1 - v_2|)$ a.e., for every $n \in \mathbb{N}$, $u_1, u_2, v_1, v_2 \in H$,

(b) $F_n(t, u, v) \to F(t, u, v)$ in the Hausdorff metric, for all $u, v \in H$, a.e. t.

Theorem 3. *If hypotheses* $H(A)_1$*,* $H(B)_1$*,* $H(F)_1$ *and* $(H_0)_1$ *hold, then* $\mathcal{M} \subset \liminf \mathcal{M}_n$*, where the lower limit is taken in* $C(0,T;V \times H) \times (s\text{–}\mathcal{H}^*)$ *topology.*

Proof. Let $(x, \dot{x}, f) \in \mathcal{M}$. We have to find $(x_n, \dot{x}_n, f_n) \in \mathcal{M}_n$ such that

$$(x_n, \dot{x}_n) \to (x, \dot{x}) \ \text{ in } \ C(0,T; V \times H), \tag{8}$$

$$f_n \to f \ \text{ in } \ s\text{–}\mathcal{H}^*. \tag{9}$$

Define $f_n(t,u,v) = \mathrm{proj}(f(t), F_n(t,u,v))$ for $n \in \mathbb{N}$, where $\mathrm{proj}(a, \mathcal{A})$ denotes the projection of point a onto the set $\mathcal{A}$. Due to Lemma α of [10], we have that f_n is measurable in t and continuous in (u,v). Moreover, $f_n(t,u,v) \in F_n(t,u,v)$ and $H(F)(3)$ implies that f_n satisfies the growth condition (5). Therefore applying Corollary 2, we obtain that for every $n \in \mathbb{N}$, the problem

$$\begin{cases} \ddot{x}(t) + A_n(t, \dot{x}(t)) + B_n x(t) = \{f_n(t, x(t), \dot{x}(t))\} \ \text{ a.e. } t \in (0,T), \\ x(0) = x_0, \quad \dot{x}(0) = x_1 \end{cases}$$

possesses a solution $x_n \in C(0,T;V)$ with $\dot{x}_n \in \mathcal{W}$. From the equality

$$\langle \ddot{x}_n(s) - \ddot{x}(s), \dot{x}_n(s) - \dot{x}(s) \rangle + \langle A_n(s, \dot{x}_n(s)) - A(s, \dot{x}(s)), \dot{x}_n(s) - \dot{x}(s) \rangle +$$

$$+ \langle B_n x_n(s) - Bx(s), \dot{x}_n(s) - \dot{x}(s) \rangle = \langle f_n(s, x_n(s), \dot{x}_n(s)) - f(s), \dot{x}_n(s) - \dot{x}(s) \rangle$$

a.e. $s \in (0,T)$, by integrating by parts, using $H(A)_1$, $H(B)_1$, similarly as in the proof of Proposition 2, we obtain

$$|\dot{x}_n(t) - \dot{x}(t)|^2 + m\, \|x_n(t) - x(t)\|^2 \le \sigma_n +$$

$$+ 2 \int_0^t |f_n(s, x_n(s), \dot{x}_n(s)) - f(s)| |\dot{x}_n(s) - \dot{x}(s)| \, ds$$

for all $t \in [0,T]$, where $\sigma_n = 2\, \|\widehat{A}_n(\dot{x}) - \widehat{A}(\dot{x})\|_{\mathcal{V}^*} \|\dot{x}_n - \dot{x}\|_{\mathcal{V}} + C\, \|B_n - B\|\, \|x_n\|_{\mathcal{V}}$ $\|\dot{x}_n - \dot{x}\|_{\mathcal{V}}$ and $C > 0$. Taking into account that

$$\begin{aligned} |f_n(s, x_n(s), \dot{x}_n(s)) - f(s)| &= d\left(f(s), F_n(s, x_n(s), \dot{x}_n(s))\right) \le \\ &\le h\left(F(s, x(s), \dot{x}(s)), F_n(s, x_n(s), \dot{x}_n(s))\right) \le \\ &\le \alpha_n(s)\left(|x_n(s) - x(s)| + |\dot{x}_n(s) - \dot{x}(s)|\right) + \beta_n(s) \ \text{ a.e. } s \in (0,T), \end{aligned} \tag{10}$$

we have

$$|\dot{x}_n(t) - \dot{x}(t)|^2 + m\, \|x_n(t) - x(t)\|^2 \le \sigma_n + 2 \int_0^t \alpha(s) |\dot{x}_n(s) - \dot{x}(s)|^2 \, ds +$$

$$+ 2 \int_0^t \alpha(s) |x_n(s) - x(s)|\, |\dot{x}_n(s) - \dot{x}(s)|^2 \, ds + 2 \int_0^t \beta_n(s) |\dot{x}_n(s) - \dot{x}(s)| \, ds$$

for all $t \in [0,T]$. Applying the inequality $2ab \le a^2 + b^2$, $a, b > 0$ to the last two integrals on the right hand side and using the fact that $|\cdot| \le \gamma \|\cdot\|$ with $\gamma > 0$, we have

$$|\dot{x}_n(t) - \dot{x}(t)|^2 + m\, \|x_n(t) - x(t)\|^2 \le \sigma_n + \|\beta_n\|^2_{L^2(0,T)} +$$

$$+\int_0^t \left[(3\alpha(s)+1)\,|\dot{x}_n(s)-\dot{x}(s)|^2+\alpha(s)\gamma^2\|x_n(s)-x(s)\|^2\right]\,ds$$

for all $t \in [0,T]$. Invoking the Gronwall inequality, we get

$$|\dot{x}_n(t)-\dot{x}(t)|^2+\|x_n(t)-x(t)\|^2 \le C\left(\sigma_n+\|\beta_n\|^2_{L^2}\right) \quad \text{for all } t\in(0,T),$$

where C is a positive constant independent of n. From Lemma 1, $H(A)_1$ and $H(B)_1$, we infer that $\lim \sigma_n = 0$. Hence, we have shown (8).

In order to prove (9), by using (10), we write

$$\int_0^T |f_n(s,x_n(s),\dot{x}_n(s))-f(s)|^q\,ds \le$$

$$\le 2^{q-1}\int_0^T (\alpha(s))^q(|x_n(s)-x(s)|+|\dot{x}_n(s)-\dot{x}(s)|)^q\,ds+2^{q-1}\|\beta_n\|^2_{L^2}.$$

In view of (8) and the convergence $\beta_n \to 0$ in $L^2(0,T)$, we easily get (9). This completes the proof. □

Corollary 5. *If hypotheses of Theorem 3 hold, then* $\mathcal{M}_n \xrightarrow{K} \mathcal{M}$ *in* $C(0,T;V\times H)\times(s\text{–}\mathcal{H}^*)$ *topology. This follows from Theorem 3 and the fact that Corollary 4 implies* $\limsup \mathcal{M}_n \subset \mathcal{M}$ *in this topology.*

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