HAL
archives-ouvertes.fr

Exponential Stability of Compactly Coupled Wave Equations with Delay Terms in the Boundary Feedbacks

Salah-Eddine Rebiai, Fatima Sidi Ali

To cite this version:

Salah-Eddine Rebiai, Fatima Sidi Ali. Exponential Stability of Compactly Coupled Wave Equations with Delay Terms in the Boundary Feedbacks. Christian Pötzsche; Clemens Heuberger; Barbara Kaltenbacher; Franz Rendl. 26th Conference on System Modeling and Optimization (CSMO), Sep 2013, Klagenfurt, Austria. Springer Berlin Heidelberg, IFIP Advances in Information and Communication Technology, AICT-443, pp.278-284, 2014, System Modeling and Optimization. <10.1007/978-3-662-45504-3_27>. <hal-01286436>

HAL Id: hal-01286436
https://hal.inria.fr/hal-01286436

Submitted on 10 Mar 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution| 4.0 International License

Exponential stability of compactly coupled wave equations with delay terms in the boundary feedbacks

Salah-Eddine Rebiai and Fatima Zohra Sidi Ali

LTM, Department of Mathematics, Faculty of Sciences,
University of Batna, 05000 Batna, Algeria

Abstract. We consider a linear system of compactly coupled wave equations with Neumann feedback controllers that contain delay terms. First, we prove under some assumptions that the closed-loop system generates a C_0-semigroup of contractions on an appropriate Hilbert space. Then, under further assumptions, we show that the closed-loop system is exponentially stable. This result is obtained by introducing a suitable energy function and by using an observability estimate.

Keywords: Coupled wave equations, time delays, boundary stabilization

1 Introduction

In [1] and [2], Datko et al presented examples of infinite-dimensional second-order systems that become unstable when arbitrary small time delays occur in the damping.

Xu et al established in [9] sufficient conditions that guarantee the exponential stability of the one-dimensional wave equation with a delay term in the boundary feedback. Nicaise and Pignotti [6] extended this result to the multi-dimensional wave equation with a delay term in the boundary or internal feedbacks. The same type of result was obtained by Nicaise and Rebiai [7] for the Schrödinger equation.

Motivated by the references [9], [6], [3] and [5], we investigate in this paper the problem of exponential stability for a linear system of compactly coupled wave equations with delay terms in the boundary feedbacks.

Let Ω be an open bounded domain of $\mathbb{R}^n$ with a boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. Furthermore, assume that there exists a real vector field $h \in (C^2(\overline{\Omega}))^n$ such that:
$(H.1)$ The Jacobian matrix J of h satisfies

$$\int_\Omega J(x)\zeta(x).\zeta(x)d\Omega \geq c \int_\Omega |\zeta(x)|^2 \, d\Omega,$$

for some constant $c > 0$ and for all $\zeta \in L^2(\Omega; \mathbb{R}^n)$,
$(H.2)$ $h(x).\nu(x) \leq 0$ on Γ_1,
where ν is the unit normal on Γ pointing towards the exterior of Ω.

Consider the following coupled system of two wave equations with delay terms in the boundary conditions:

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) + l(u(x,t) - v(x,t)) = 0 \qquad \text{in } \Omega \times (0,+\infty), \tag{1}$$

$$\frac{\partial^2 v(x,t)}{\partial t^2} - \Delta v(x,t) + l(v(x,t) - u(x,t)) = 0 \qquad \text{in } \Omega \times (0,+\infty), \tag{2}$$

$$u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = u_1(x) \qquad \text{in } \Omega, \tag{3}$$

$$v(x,0) = v_0(x), \frac{\partial v(x,0)}{\partial t} = v_1(x) \qquad \text{in } \Omega, \tag{4}$$

$$u(x,t) = v(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0,+\infty), \tag{5}$$

$$\frac{\partial u(x,t)}{\partial \nu} = -\alpha_1 \frac{\partial u(x,t)}{\partial t} - \alpha_2 \frac{\partial u(x,t-\tau)}{\partial t} \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{6}$$

$$\frac{\partial v(x,t)}{\partial \nu} = -\beta_1 \frac{\partial v(x,t)}{\partial t} - \beta_2 \frac{\partial v(x,t-\tau)}{\partial t} \qquad \text{on } \Gamma_2 \times (0,+\infty), \tag{7}$$

$$\frac{\partial u(x,t-\tau)}{\partial t} = g(x,t-\tau) \qquad \text{on } \Gamma_2 \times (0,\tau), \tag{8}$$

$$\frac{\partial v(x,t-\tau)}{\partial t} = h(x,t-\tau) \qquad \text{on } \Gamma_2 \times (0,\tau). \tag{9}$$

Physically, u and v may represent the displacements of two vibratings objects measured from their equilibrium positions, the coupling terms $\pm l(u-v)$ are the distributed springs linking the two vibrating objects. $l, \alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants, τ is the time delay, u_0, u_1, v_0, v_1, g and h are the initial data.
It is well known that in the absence of delay (*i.e.* $\alpha_2 = \beta_2 = 0$), the solution of (1)-(9) with α_1 and β_1 positive, decays exponentially to zero in the energy space $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$ (see [5] and [3]).
The purpose of this paper is to investigate the uniform exponential stability of system $(1)-(9)$ in the case where all the boundary damping coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2 are positive. To this end, assume as in [6] that

$$\alpha_1 > \alpha_2, \beta_1 > \beta_2 \tag{10}$$

and define the energy of a solution of $(1) - (9)$ by

$$E(t) = \frac{1}{2}\int_\Omega [|\nabla u(x,t)|^2 + \left|\frac{\partial u(x,t)}{\partial t}\right|^2 + |\nabla v(x,t)|^2 + \left|\frac{\partial v(x,t)}{\partial t}\right|^2 + l\,|u(x,t) - v(x,t)|^2\,]dx + \frac{1}{2}\int_{\Gamma_2}\int_0^1 [\mu \left|\frac{\partial u(x,t-\tau\rho)}{\partial t}\right|^2 + \xi \left|\frac{\partial v(x,t-\tau\rho)}{\partial t}\right|^2]d\rho\, d\Gamma \tag{11}$$

where

$$\tau\alpha_2 < \mu < \tau(2\alpha_1 - \alpha_2) \tag{12}$$

and

$$\tau\beta_2 < \xi < \tau(2\beta_1 - \beta_2) \tag{13}$$

We show that if $\{\Omega, \Gamma_1, \Gamma_2\}$ satisfies $(H.1)$ and $(H.2)$, then there is an exponential decay rate for $E(t)$.The proof of this result is based on Carleman estimates for a system of coupled nonconservative hyperbolic systems established by Lasieka and Triggiani in [4] and on compactness-uniqueness arguments.
The main result of this paper can be stated as follows.

Theorem 1. *Assume* $(H1)$, $(H.2)$, $(10), (12)$ *and* (13). *Then there exist constants* $M \geq 1$ *and* $\omega > 0$ *such that*

$$E(t) \leq Me^{-\omega t}E(0).$$

Theorem 1 is proved in Section 3. In Section 2, we study the well-posedness of system $(1)-(9)$ using semigroup theory.

2 Well-posedness of system (1) − (9)

Inspired from [6] and [7], we introduce the auxilliary variables

$$y(x,\rho,t) = \frac{\partial u(x, t-\tau\rho)}{\partial t}$$
$$z(x,\rho,t) = \frac{\partial v(x, t-\tau\rho)}{\partial t}$$

With these new unknowns, system $(1)-(9)$ is equivalent to

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) + l(u(x,t) - v(x,t)) = 0 \qquad \text{in } \Omega \times (0, +\infty), \tag{14}$$
$$\frac{\partial y(x,\rho,t)}{\partial t} + \frac{1}{\tau}\frac{\partial y(x,\rho,t)}{\partial \rho} = 0 \qquad \text{on } \Gamma_2 \times (0,1) \times (0, +\infty), \tag{15}$$
$$\frac{\partial^2 v(x,t)}{\partial t^2} - \Delta v(x,t) + l(v(x,t) - u(x,t)) = 0 \qquad \text{in } \Omega \times (0, +\infty), \tag{16}$$
$$\frac{\partial z(x,\rho,t)}{\partial t} + \frac{1}{\tau}\frac{\partial z(x,\rho,t)}{\partial \rho} = 0 \qquad \text{on } \Gamma_2 \times (0,1) \times (0, +\infty), \tag{17}$$
$$u(x,t) = v(x,t) = 0 \qquad \text{on } \Gamma_1 \times (0, +\infty), \tag{18}$$
$$\frac{\partial u(x,t)}{\partial \nu} = -\alpha_1 \frac{\partial u(x,t)}{\partial t} - \alpha_2 y(x,1,t) \qquad \text{on } \Gamma_2 \times (0, +\infty), \tag{19}$$
$$\frac{\partial v(x,t)}{\partial \nu} = -\beta_1 \frac{\partial u(x,t)}{\partial t} - \beta_2 z(x,1,t) \qquad \text{on } \Gamma_2 \times (0, +\infty), \tag{20}$$
$$y(x,0,t) = \frac{\partial u(x,t)}{\partial t}, z(x,0,t) = \frac{\partial v(x,t)}{\partial t} \qquad \text{on } \Gamma_2 \times (0, +\infty), \tag{21}$$
$$u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = u_1(x) \qquad \text{in } \Omega, \tag{22}$$
$$v(x,0) = v_0(x), \frac{\partial v(x,0)}{\partial t} = v_1(x) \qquad \text{in } \Omega, \tag{23}$$
$$y(x,\rho,0) = g(x, -\tau\rho), \; z(x,\rho,0) = h(x, -\tau\rho) \qquad \text{on } \Gamma_2 \times (0,1). \tag{24}$$

Denote by $\mathcal{H}$ the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2 \times L^2(0,1)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2 \times L^2(0,1))$$

where

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$$

We equip $\mathcal{H}$ with the inner product

$$\left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix} ; \begin{pmatrix} \widetilde{\zeta} \\ \widetilde{\eta} \\ \widetilde{\theta} \\ \widetilde{\phi} \\ \widetilde{\chi} \\ \widetilde{\psi} \end{pmatrix} \right\rangle = \int_\Omega (\nabla\zeta(x).\nabla\widetilde{\zeta}(x) + \eta(x)\widetilde{\eta}(x))\, dx +$$

$$\mu \int_{\Gamma_2} \int_0^1 \theta(x,\rho)\widetilde{\theta}(x,\rho) d\rho\, d\Gamma + \int_\Omega (\nabla\phi(x).\nabla\widetilde{\phi}(x) + \chi(x)\widetilde{\chi}(x))\, dx +$$

$$\xi \int_{\Gamma_2} \int_0^1 \psi(x,\rho)\widetilde{\psi}(x,\rho) d\rho\, d\Gamma + l \int_\Omega (\zeta(x) - \phi(x))(\widetilde{\zeta}(x) - \widetilde{\phi}(x)) dx$$

Define in $\mathcal{H}$ a linear operator $\mathcal{A}$ by

$$D(\mathcal{A}) = \{(\zeta,\eta,\theta,\phi,\chi,\psi)^T \in H^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0,1)) \times$$
$$H^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0,1)); \frac{\partial\zeta}{\partial\nu} = -\alpha_1\eta - \alpha_2\theta(.,1),$$
$$\eta = \theta(.,0) \text{ on } \Gamma_2;\ \frac{\partial\phi}{\partial\nu} = -\beta_1\chi - \beta_2\psi(.,1), \chi = \psi(.,0) \text{ on } \Gamma_2\} \tag{25}$$
$$A(\zeta,\eta,\theta,\phi,\chi,\psi)^T = (\eta, \Delta\zeta + l\phi - l\zeta, -\tau^{-1}\frac{\partial\theta}{\partial\rho}, \chi, \Delta\phi - l\phi + l\zeta, -\tau^{-1}\frac{\partial\psi}{\partial\rho})^T \tag{26}$$

Then we can rewrite $(14) - (24)$ as an abstract Cauchy problem in $\mathcal{H}$

$$\begin{cases} \frac{d}{dt}W(t) = \mathcal{A}W(t) \\ W(0) = W_0 \end{cases} \tag{27}$$

where

$$W(t) = (u(x,t), \frac{\partial u(x,t)}{\partial t}, y(x,\rho,t), v(x,t), \frac{\partial v(x,t)}{\partial t}, z(x,\rho,t))^T,$$
$$\text{and } W_0 = (u_0, u_1, g(.,-.\tau), v_0, v_1, h(.,-.\tau))^T$$

We verify that $\mathcal{A}$ is dissipative and that $\lambda I - \mathcal{A}$ is onto for a fixed $\lambda > 0$. Thus, by the Lumer-Phillips Theorem (see for instance [8]) $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{H}$ and consequently we have

Proposition 1. *For every $W_0 \in \mathcal{H}$, problem (27) has a unique solution W whose regularity depends on the the initial datum W_0 as follows:*

$$W(.) \in C([0,+\infty);\mathcal{H}) \text{ if } W_0 \in \mathcal{H},$$
$$W(.) \in C^1([0,+\infty);\mathcal{H}) \cap C([0,+\infty);D(\mathcal{A})) \text{ if } W_0 \in D(\mathcal{A}).$$

3 Proof of Theorem 1

We prove Theorem 1 for smooth initial data. The general case follows by a standard density argument.
We proceed in several steps.
Step 1.
Differentiating $E(t)$ with respect to time, we obtain

$$\frac{d}{dt}E(t) \leq -k\int_{\Gamma_2}\{\left|\frac{\partial u(x,t)}{\partial t}\right|^2+\left|\frac{\partial u(x,t-\tau)}{\partial t}\right|^2+\left|\frac{\partial v(x,t)}{\partial t}\right|^2+\left|\frac{\partial v(x,t-\tau)}{\partial t}\right|^2\}d\Gamma \tag{28}$$

where

$$k=\min\{\alpha_1-\frac{\alpha_2}{2}-\frac{\mu}{2\tau},\frac{\mu}{2\tau}-\frac{\alpha_2}{2},\beta_1-\frac{\beta_2}{2}-\frac{\xi}{2\tau},\frac{\xi}{2\tau}-\frac{\beta_2}{2}\}$$

Step 2.
We rewrite

$$E(t)=\mathcal{E}(t)+E_d(t)$$

where

$$\mathcal{E}(t)=\frac{1}{2}\int_{\Omega}\{|\nabla u(x,t)|^2+\left|\frac{\partial u(x,t)}{\partial t}\right|^2+|\nabla v(x,t)|^2+\left|\frac{\partial v(x,t)}{\partial t}\right|^2+l\,|u(x,t)-v(x,t)|^2\}dx$$

and

$$E_d(t)=\frac{1}{2}\int_{\Gamma_2}\int_0^1\{\mu\left|\frac{\partial u(x,t-\tau\rho)}{\partial t}\right|^2+\xi\left|\frac{\partial v(x,t-\tau\rho)}{\partial t}\right|^2\}d\rho d\Gamma$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(t)=\frac{1}{2\tau}\int_t^{t+\tau}\int_{\Gamma_2}\{\mu\left|\frac{\partial u(x,s-\tau)}{\partial t}\right|^2+\xi\left|\frac{\partial v(x,s-\tau)}{\partial t}\right|^2\}d\Gamma ds \tag{29}$$

From (29), we obtain (here and throughout the rest of the paper C is some positive constant different at different occurences)

$$E_d(t) \leq C\int_0^T\int_{\Gamma_2}\{\left|\frac{\partial u(x,s-\tau)}{\partial t}\right|^2+\left|\frac{\partial v(x,s-\tau)}{\partial t}\right|^2\}d\Gamma ds \tag{30}$$

for $0 \leq t+\tau \leq T$ and T large enough.

Step 3.
From Poincaré inequality and Proposition 3.5 of [4], we have for T sufficiently large and for any $\epsilon > 0$

$$\mathcal{E}(0) \le C\int_0^T \int_{\Gamma_2} \{\left|\frac{\partial u(x,t)}{\partial \nu}\right|^2 + \left|\frac{\partial u(x,t)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t)}{\partial \nu}\right|^2 + \left|\frac{\partial v(x,t)}{\partial t}\right|^2\} d\Gamma dt + C\{\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}\} \tag{31}$$

Inserting the boundary conditions (6) and (7) into (31), we obtain

$$\mathcal{E}(0) \le C\int_0^T \int_{\Gamma_2} \{\left|\frac{\partial u(x,t)}{\partial t}\right|^2 + \left|\frac{\partial u(x,t-\tau)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t-\tau)}{\partial t}\right|^2\} d\Gamma dt + C\{\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}\} \tag{32}$$

Step 4.
Estimate (30) together with (32) yields

$$E(0) \le C\int_0^T \int_{\Gamma_2} \{\left|\frac{\partial u(x,t)}{\partial t}\right|^2 + \left|\frac{\partial u(x,t-\tau)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t-\tau)}{\partial t}\right|^2\} d\Gamma dt + C\{\|u\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))} + \|v\|^2_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}\} \tag{33}$$

Step 5.
We drop the lower order terms on the right-hand side of (33) by a compactness-uniqueness argument to obtain

$$E(0) \le C\int_0^T \int_{\Gamma_2} \{\left|\frac{\partial u(x,t)}{\partial t}\right|^2 + \left|\frac{\partial u(x,t-\tau)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t-\tau)}{\partial t}\right|^2\} d\Gamma dt \tag{34}$$

Step 6.
From (28), we have

$$E(T) - E(0) \le -k\int_0^T \int_{\Gamma_2} \{\left|\frac{\partial u(x,t)}{\partial t}\right|^2 + \left|\frac{\partial u(x,t-\tau)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t)}{\partial t}\right|^2 + \left|\frac{\partial v(x,t-\tau)}{\partial t}\right|^2\} d\Gamma dt$$

which together with (34) leads to

$$E(T) \le \frac{Ck^{-1}}{1 + Ck^{-1}} E(0) \tag{35}$$

The desired conclusion follows now from (35).

References

1. Datko, R., Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks, SIAM J. Control Optim. 26, 697–713 (1988)

2. Datko, R., Lagnese, J., Polis, M.P., An example on the effect of time delays in boundary feedback stabilization of wave equations, SIAM J. Control Optim. 24, 152–156 (1986)
3. Komornik, V., Rao, B., Boundary stabilization of compactly wave equations, Asymptotic Analysis. 14, 339–359 (1997)
4. Lasiecka, I., Triggiani, R., Carleman estimates and exact boundary controllability for a system of coupled non-conservative second-order hyperbolic equations, in "Lecture Notes in Pure and Applied Mathematics", vol. 188, pp. 215–243, Marcel Dekker, New York (1997)
5. Najafi, M., Sarhangi, G.R., Wang, H., Stabilizability of coupled wave equations in parallel under various boundary conditions, IEEE Trans. Automat. Control 42, 1308–1312 (1997)
6. Nicaise, S., Pignotti, C., Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim. 45, 1561-1585 (2006)
7. Nicaise, S., Rebiai, S., Stabilization of the Schrödinger equation with a delay term in boundary feedback or internal feedback, Portugal. Math. 68, 19–39 (2011)
8. Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983).
9. Xu, G.Q., Yung, S.P., Li, L.K., Stabilization of wave systems with input delay in the boundary control, ESAIM Control Optim. Calc. Var. 12, 770–785 (2006)