

Exponential stability of compactly coupled wave equations with delay terms in the boundary feedbacks

Salah-Eddine Rebiai and Fatima Zohra Sidi Ali

LTM, Department of Mathematics, Faculty of Sciences,
University of Batna, 05000 Batna, Algeria

Abstract. We consider a linear system of compactly coupled wave equations with Neumann feedback controllers that contain delay terms. First, we prove under some assumptions that the closed-loop system generates a C_0 -semigroup of contractions on an appropriate Hilbert space. Then, under further assumptions, we show that the closed-loop system is exponentially stable. This result is obtained by introducing a suitable energy function and by using an observability estimate.

Keywords: Coupled wave equations, time delays, boundary stabilization

1 Introduction

In [1] and [2], Datko et al presented examples of infinite-dimensional second-order systems that become unstable when arbitrary small time delays occur in the damping.

Xu et al established in [9] sufficient conditions that guarantee the exponential stability of the one-dimensional wave equation with a delay term in the boundary feedback. Nicaise and Pignotti [6] extended this result to the multi-dimensional wave equation with a delay term in the boundary or internal feedbacks. The same type of result was obtained by Nicaise and Rebiai [7] for the Schrödinger equation.

Motivated by the references [9], [6], [3] and [5], we investigate in this paper the problem of exponential stability for a linear system of compactly coupled wave equations with delay terms in the boundary feedbacks.

Let Ω be an open bounded domain of \mathbb{R}^n with a boundary Γ of class C^2 which consists of two non-empty parts Γ_1 and Γ_2 such that $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$. Furthermore, assume that there exists a real vector field $h \in (C^2(\overline{\Omega}))^n$ such that:

(H.1) The Jacobian matrix J of h satisfies

$$\int_{\Omega} J(x)\zeta(x) \cdot \zeta(x) d\Omega \geq c \int_{\Omega} |\zeta(x)|^2 d\Omega,$$

for some constant $c > 0$ and for all $\zeta \in L^2(\Omega; \mathbb{R}^n)$,

(H.2) $h(x) \cdot \nu(x) \leq 0$ on Γ_1 ,

where ν is the unit normal on Γ pointing towards the exterior of Ω .

Consider the following coupled system of two wave equations with delay terms in the boundary conditions:

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \Delta u(x, t) + l(u(x, t) - v(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1)$$

$$\frac{\partial^2 v(x, t)}{\partial t^2} - \Delta v(x, t) + l(v(x, t) - u(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (2)$$

$$u(x, 0) = u_0(x), \frac{\partial u(x, 0)}{\partial t} = u_1(x) \quad \text{in } \Omega, \quad (3)$$

$$v(x, 0) = v_0(x), \frac{\partial v(x, 0)}{\partial t} = v_1(x) \quad \text{in } \Omega, \quad (4)$$

$$u(x, t) = v(x, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (5)$$

$$\frac{\partial u(x, t)}{\partial \nu} = -\alpha_1 \frac{\partial u(x, t)}{\partial t} - \alpha_2 \frac{\partial u(x, t - \tau)}{\partial t} \quad \text{on } \Gamma_2 \times (0, +\infty), \quad (6)$$

$$\frac{\partial v(x, t)}{\partial \nu} = -\beta_1 \frac{\partial v(x, t)}{\partial t} - \beta_2 \frac{\partial v(x, t - \tau)}{\partial t} \quad \text{on } \Gamma_2 \times (0, +\infty), \quad (7)$$

$$\frac{\partial u(x, t - \tau)}{\partial t} = g(x, t - \tau) \quad \text{on } \Gamma_2 \times (0, \tau), \quad (8)$$

$$\frac{\partial v(x, t - \tau)}{\partial t} = h(x, t - \tau) \quad \text{on } \Gamma_2 \times (0, \tau). \quad (9)$$

Physically, u and v may represent the displacements of two vibrating objects measured from their equilibrium positions, the coupling terms $\pm l(u - v)$ are the distributed springs linking the two vibrating objects. $l, \alpha_1, \alpha_2, \beta_1, \beta_2$ are positive constants, τ is the time delay, u_0, u_1, v_0, v_1, g and h are the initial data.

It is well known that in the absence of delay (*i.e.* $\alpha_2 = \beta_2 = 0$), the solution of (1)-(9) with α_1 and β_1 positive, decays exponentially to zero in the energy space $H_{\Gamma_1}^1(\Omega) \times L^2(\Omega) \times H_{\Gamma_1}^1(\Omega) \times L^2(\Omega)$ (see [5] and [3]).

The purpose of this paper is to investigate the uniform exponential stability of system (1)–(9) in the case where all the boundary damping coefficients $\alpha_1, \alpha_2, \beta_1$ and β_2 are positive. To this end, assume as in [6] that

$$\alpha_1 > \alpha_2, \beta_1 > \beta_2 \quad (10)$$

and define the energy of a solution of (1) – (9) by

$$\begin{aligned} E(t) = & \frac{1}{2} \int_{\Omega} [|\nabla u(x, t)|^2 + \left| \frac{\partial u(x, t)}{\partial t} \right|^2 + |\nabla v(x, t)|^2 + \left| \frac{\partial v(x, t)}{\partial t} \right|^2 + \\ & l |u(x, t) - v(x, t)|^2] dx + \frac{1}{2} \int_{\Gamma_2} \int_0^1 [\mu \left| \frac{\partial u(x, t - \tau \rho)}{\partial t} \right|^2 + \\ & \xi \left| \frac{\partial v(x, t - \tau \rho)}{\partial t} \right|^2] d\rho d\Gamma \end{aligned} \quad (11)$$

where

$$\tau \alpha_2 < \mu < \tau(2\alpha_1 - \alpha_2) \quad (12)$$

and

$$\tau\beta_2 < \xi < \tau(2\beta_1 - \beta_2) \tag{13}$$

We show that if $\{\Omega, \Gamma_1, \Gamma_2\}$ satisfies (H.1) and (H.2), then there is an exponential decay rate for $E(t)$. The proof of this result is based on Carleman estimates for a system of coupled nonconservative hyperbolic systems established by Lasieka and Triggiani in [4] and on compactness-uniqueness arguments.

The main result of this paper can be stated as follows.

Theorem 1. *Assume (H1), (H.2), (10), (12) and (13). Then there exist constants $M \geq 1$ and $\omega > 0$ such that*

$$E(t) \leq Me^{-\omega t} E(0).$$

Theorem 1 is proved in Section 3. In Section 2, we study the well-posedness of system (1) – (9) using semigroup theory.

2 Well-posedness of system (1) – (9)

Inspired from [6] and [7], we introduce the auxilliary variables

$$y(x, \rho, t) = \frac{\partial u(x, t - \tau\rho)}{\partial t}$$

$$z(x, \rho, t) = \frac{\partial v(x, t - \tau\rho)}{\partial t}$$

With these new unknowns, system (1) – (9) is equivalent to

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \Delta u(x, t) + l(u(x, t) - v(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{14}$$

$$\frac{\partial y(x, \rho, t)}{\partial t} + \frac{1}{\tau} \frac{\partial y(x, \rho, t)}{\partial \rho} = 0 \quad \text{on } \Gamma_2 \times (0, 1) \times (0, +\infty), \tag{15}$$

$$\frac{\partial^2 v(x, t)}{\partial t^2} - \Delta v(x, t) + l(v(x, t) - u(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{16}$$

$$\frac{\partial z(x, \rho, t)}{\partial t} + \frac{1}{\tau} \frac{\partial z(x, \rho, t)}{\partial \rho} = 0 \quad \text{on } \Gamma_2 \times (0, 1) \times (0, +\infty), \tag{17}$$

$$u(x, t) = v(x, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \tag{18}$$

$$\frac{\partial u(x, t)}{\partial \nu} = -\alpha_1 \frac{\partial u(x, t)}{\partial t} - \alpha_2 y(x, 1, t) \quad \text{on } \Gamma_2 \times (0, +\infty), \tag{19}$$

$$\frac{\partial v(x, t)}{\partial \nu} = -\beta_1 \frac{\partial u(x, t)}{\partial t} - \beta_2 z(x, 1, t) \quad \text{on } \Gamma_2 \times (0, +\infty), \tag{20}$$

$$y(x, 0, t) = \frac{\partial u(x, t)}{\partial t}, z(x, 0, t) = \frac{\partial v(x, t)}{\partial t} \quad \text{on } \Gamma_2 \times (0, +\infty), \tag{21}$$

$$u(x, 0) = u_0(x), \frac{\partial u(x, 0)}{\partial t} = u_1(x) \quad \text{in } \Omega, \tag{22}$$

$$v(x, 0) = v_0(x), \frac{\partial v(x, 0)}{\partial t} = v_1(x) \quad \text{in } \Omega, \tag{23}$$

$$y(x, \rho, 0) = g(x, -\tau\rho), z(x, \rho, 0) = h(x, -\tau\rho) \quad \text{on } \Gamma_2 \times (0, 1). \tag{24}$$

Denote by \mathcal{H} the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2 \times L^2(0, 1)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2 \times L^2(0, 1))$$

where

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$$

We equip \mathcal{H} with the inner product

$$\begin{aligned} \left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix}; \begin{pmatrix} \tilde{\zeta} \\ \tilde{\eta} \\ \tilde{\theta} \\ \tilde{\phi} \\ \tilde{\chi} \\ \tilde{\psi} \end{pmatrix} \right\rangle &= \int_{\Omega} (\nabla \zeta(x) \cdot \nabla \tilde{\zeta}(x) + \eta(x) \tilde{\eta}(x)) \, dx + \\ &\mu \int_{\Gamma_2} \int_0^1 \theta(x, \rho) \tilde{\theta}(x, \rho) \, d\rho \, d\Gamma + \int_{\Omega} (\nabla \phi(x) \cdot \nabla \tilde{\phi}(x) + \chi(x) \tilde{\chi}(x)) \, dx + \\ &\xi \int_{\Gamma_2} \int_0^1 \psi(x, \rho) \tilde{\psi}(x, \rho) \, d\rho \, d\Gamma + l \int_{\Omega} (\zeta(x) - \phi(x)) (\tilde{\zeta}(x) - \tilde{\phi}(x)) \, dx \end{aligned}$$

Define in \mathcal{H} a linear operator \mathcal{A} by

$$\begin{aligned} D(\mathcal{A}) &= \{(\zeta, \eta, \theta, \phi, \chi, \psi)^T \in H^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0, 1)) \times \\ &H^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0, 1)); \frac{\partial \zeta}{\partial \nu} = -\alpha_1 \eta - \alpha_2 \theta(\cdot, 1), \\ &\eta = \theta(\cdot, 0) \text{ on } \Gamma_2; \frac{\partial \phi}{\partial \nu} = -\beta_1 \chi - \beta_2 \psi(\cdot, 1), \chi = \psi(\cdot, 0) \text{ on } \Gamma_2\} \end{aligned} \tag{25}$$

$$\mathcal{A}(\zeta, \eta, \theta, \phi, \chi, \psi)^T = (\eta, \Delta \zeta + l\phi - l\zeta, -\tau^{-1} \frac{\partial \theta}{\partial \rho}, \chi, \Delta \phi - l\phi + l\zeta, -\tau^{-1} \frac{\partial \psi}{\partial \rho})^T \tag{26}$$

Then we can rewrite (14) – (24) as an abstract Cauchy problem in \mathcal{H}

$$\begin{cases} \frac{d}{dt} W(t) = \mathcal{A}W(t) \\ W(0) = W_0 \end{cases} \tag{27}$$

where

$$\begin{aligned} W(t) &= (u(x, t), \frac{\partial u(x, t)}{\partial t}, y(x, \rho, t), v(x, t), \frac{\partial v(x, t)}{\partial t}, z(x, \rho, t))^T, \\ \text{and } W_0 &= (u_0, u_1, g(\cdot, -\tau), v_0, v_1, h(\cdot, -\tau))^T \end{aligned}$$

We verify that \mathcal{A} is dissipative and that $\lambda I - \mathcal{A}$ is onto for a fixed $\lambda > 0$. Thus, by the Lumer-Phillips Theorem (see for instance [8]) \mathcal{A} generates a strongly continuous semigroup on \mathcal{H} and consequently we have

Proposition 1. For every $W_0 \in \mathcal{H}$, problem (27) has a unique solution W whose regularity depends on the the initial datum W_0 as follows:

$$\begin{aligned} W(\cdot) &\in C([0, +\infty); \mathcal{H}) \text{ if } W_0 \in \mathcal{H}, \\ W(\cdot) &\in C^1([0, +\infty); \mathcal{H}) \cap C([0, +\infty); D(\mathcal{A})) \text{ if } W_0 \in D(\mathcal{A}). \end{aligned}$$

3 Proof of Theorem 1

We prove Theorem 1 for smooth initial data. The general case follows by a standard density argument.

We proceed in several steps.

Step 1.

Differentiating $E(t)$ with respect to time, we obtain

$$\frac{d}{dt}E(t) \leq -k \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x, t)}{\partial t} \right|^2 + \left| \frac{\partial u(x, t - \tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x, t)}{\partial t} \right|^2 + \left| \frac{\partial v(x, t - \tau)}{\partial t} \right|^2 \right\} d\Gamma \tag{28}$$

where

$$k = \min \left\{ \alpha_1 - \frac{\alpha_2}{2} - \frac{\mu}{2\tau}, \frac{\mu}{2\tau} - \frac{\alpha_2}{2}, \beta_1 - \frac{\beta_2}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{\beta_2}{2} \right\}$$

Step 2.

We rewrite

$$E(t) = \mathcal{E}(t) + E_d(t)$$

where

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} \left\{ |\nabla u(x, t)|^2 + \left| \frac{\partial u(x, t)}{\partial t} \right|^2 + |\nabla v(x, t)|^2 + \left| \frac{\partial v(x, t)}{\partial t} \right|^2 + l |u(x, t) - v(x, t)|^2 \right\} dx$$

and

$$E_d(t) = \frac{1}{2} \int_{\Gamma_2} \int_0^1 \left\{ \mu \left| \frac{\partial u(x, t - \tau\rho)}{\partial t} \right|^2 + \xi \left| \frac{\partial v(x, t - \tau\rho)}{\partial t} \right|^2 \right\} d\rho d\Gamma$$

$E_d(t)$ can be rewritten via a change of variable as

$$E_d(t) = \frac{1}{2\tau} \int_t^{t+\tau} \int_{\Gamma_2} \left\{ \mu \left| \frac{\partial u(x, s - \tau)}{\partial t} \right|^2 + \xi \left| \frac{\partial v(x, s - \tau)}{\partial t} \right|^2 \right\} d\Gamma ds \tag{29}$$

From (29), we obtain (here and throughout the rest of the paper C is some positive constant different at different occurrences)

$$E_d(t) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x, s - \tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x, s - \tau)}{\partial t} \right|^2 \right\} d\Gamma ds \tag{30}$$

for $0 \leq t + \tau \leq T$ and T large enough.

Step 3.

From Poincaré inequality and Proposition 3.5 of [4], we have for T sufficiently large and for any $\epsilon > 0$

$$\begin{aligned} \mathcal{E}(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial \nu} \right|^2 + \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial \nu} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 \right\} d\Gamma dt + \\ C \{ \|u\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 + \|v\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 \} \end{aligned} \tag{31}$$

Inserting the boundary conditions (6) and (7) into (31), we obtain

$$\begin{aligned} \mathcal{E}(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt + \\ C \{ \|u\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 + \|v\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 \} \end{aligned} \tag{32}$$

Step 4.

Estimate (30) together with (32) yields

$$\begin{aligned} E(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt + \\ C \{ \|u\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 + \|v\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 \} \end{aligned} \tag{33}$$

Step 5.

We drop the lower order terms on the right-hand side of (33) by a compactness-uniqueness argument to obtain

$$E(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt \tag{34}$$

Step 6.

From (28), we have

$$E(T) - E(0) \leq -k \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt$$

which together with (34) leads to

$$E(T) \leq \frac{Ck^{-1}}{1 + Ck^{-1}} E(0) \tag{35}$$

The desired conclusion follows now from (35).

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