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# Exponential stability of compactly coupled wave equations with delay terms in the boundary feedbacks

Salah-Eddine Rebiai and Fatima Zohra Sidi Ali

LTM, Department of Mathematics, Faculty of Sciences,  
University of Batna, 05000 Batna, Algeria

**Abstract.** We consider a linear system of compactly coupled wave equations with Neumann feedback controllers that contain delay terms. First, we prove under some assumptions that the closed-loop system generates a  $C_0$ -semigroup of contractions on an appropriate Hilbert space. Then, under further assumptions, we show that the closed-loop system is exponentially stable. This result is obtained by introducing a suitable energy function and by using an observability estimate.

**Keywords:** Coupled wave equations, time delays, boundary stabilization

## 1 Introduction

In [1] and [2], Datko et al presented examples of infinite-dimensional second-order systems that become unstable when arbitrary small time delays occur in the damping.

Xu et al established in [9] sufficient conditions that guarantee the exponential stability of the one-dimensional wave equation with a delay term in the boundary feedback. Nicaise and Pignotti [6] extended this result to the multi-dimensional wave equation with a delay term in the boundary or internal feedbacks. The same type of result was obtained by Nicaise and Rebiai [7] for the Schrödinger equation.

Motivated by the references [9], [6], [3] and [5], we investigate in this paper the problem of exponential stability for a linear system of compactly coupled wave equations with delay terms in the boundary feedbacks.

Let  $\Omega$  be an open bounded domain of  $\mathbb{R}^n$  with a boundary  $\Gamma$  of class  $C^2$  which consists of two non-empty parts  $\Gamma_1$  and  $\Gamma_2$  such that  $\overline{\Gamma_1} \cap \overline{\Gamma_2} = \emptyset$ . Furthermore, assume that there exists a real vector field  $h \in (C^2(\overline{\Omega}))^n$  such that:

(H.1) The Jacobian matrix  $J$  of  $h$  satisfies

$$\int_{\Omega} J(x)\zeta(x) \cdot \zeta(x) d\Omega \geq c \int_{\Omega} |\zeta(x)|^2 d\Omega,$$

for some constant  $c > 0$  and for all  $\zeta \in L^2(\Omega; \mathbb{R}^n)$ ,

(H.2)  $h(x) \cdot \nu(x) \leq 0$  on  $\Gamma_1$ ,

where  $\nu$  is the unit normal on  $\Gamma$  pointing towards the exterior of  $\Omega$ .

Consider the following coupled system of two wave equations with delay terms in the boundary conditions:

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \Delta u(x, t) + l(u(x, t) - v(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (1)$$

$$\frac{\partial^2 v(x, t)}{\partial t^2} - \Delta v(x, t) + l(v(x, t) - u(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \quad (2)$$

$$u(x, 0) = u_0(x), \frac{\partial u(x, 0)}{\partial t} = u_1(x) \quad \text{in } \Omega, \quad (3)$$

$$v(x, 0) = v_0(x), \frac{\partial v(x, 0)}{\partial t} = v_1(x) \quad \text{in } \Omega, \quad (4)$$

$$u(x, t) = v(x, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \quad (5)$$

$$\frac{\partial u(x, t)}{\partial \nu} = -\alpha_1 \frac{\partial u(x, t)}{\partial t} - \alpha_2 \frac{\partial u(x, t - \tau)}{\partial t} \quad \text{on } \Gamma_2 \times (0, +\infty), \quad (6)$$

$$\frac{\partial v(x, t)}{\partial \nu} = -\beta_1 \frac{\partial v(x, t)}{\partial t} - \beta_2 \frac{\partial v(x, t - \tau)}{\partial t} \quad \text{on } \Gamma_2 \times (0, +\infty), \quad (7)$$

$$\frac{\partial u(x, t - \tau)}{\partial t} = g(x, t - \tau) \quad \text{on } \Gamma_2 \times (0, \tau), \quad (8)$$

$$\frac{\partial v(x, t - \tau)}{\partial t} = h(x, t - \tau) \quad \text{on } \Gamma_2 \times (0, \tau). \quad (9)$$

Physically,  $u$  and  $v$  may represent the displacements of two vibrating objects measured from their equilibrium positions, the coupling terms  $\pm l(u - v)$  are the distributed springs linking the two vibrating objects.  $l, \alpha_1, \alpha_2, \beta_1, \beta_2$  are positive constants,  $\tau$  is the time delay,  $u_0, u_1, v_0, v_1, g$  and  $h$  are the initial data.

It is well known that in the absence of delay (*i.e.*  $\alpha_2 = \beta_2 = 0$ ), the solution of (1)-(9) with  $\alpha_1$  and  $\beta_1$  positive, decays exponentially to zero in the energy space  $H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega)$  (see [5] and [3]).

The purpose of this paper is to investigate the uniform exponential stability of system (1)–(9) in the case where all the boundary damping coefficients  $\alpha_1, \alpha_2, \beta_1$  and  $\beta_2$  are positive. To this end, assume as in [6] that

$$\alpha_1 > \alpha_2, \beta_1 > \beta_2 \quad (10)$$

and define the energy of a solution of (1) – (9) by

$$\begin{aligned} E(t) = & \frac{1}{2} \int_{\Omega} [|\nabla u(x, t)|^2 + \left| \frac{\partial u(x, t)}{\partial t} \right|^2 + |\nabla v(x, t)|^2 + \left| \frac{\partial v(x, t)}{\partial t} \right|^2 + \\ & l |u(x, t) - v(x, t)|^2] dx + \frac{1}{2} \int_{\Gamma_2} \int_0^1 [\mu \left| \frac{\partial u(x, t - \tau \rho)}{\partial t} \right|^2 + \\ & \xi \left| \frac{\partial v(x, t - \tau \rho)}{\partial t} \right|^2] d\rho d\Gamma \end{aligned} \quad (11)$$

where

$$\tau \alpha_2 < \mu < \tau(2\alpha_1 - \alpha_2) \quad (12)$$

and

$$\tau\beta_2 < \xi < \tau(2\beta_1 - \beta_2) \tag{13}$$

We show that if  $\{\Omega, \Gamma_1, \Gamma_2\}$  satisfies (H.1) and (H.2), then there is an exponential decay rate for  $E(t)$ . The proof of this result is based on Carleman estimates for a system of coupled nonconservative hyperbolic systems established by Lasieka and Triggiani in [4] and on compactness-uniqueness arguments.

The main result of this paper can be stated as follows.

**Theorem 1.** *Assume (H1), (H.2), (10), (12) and (13). Then there exist constants  $M \geq 1$  and  $\omega > 0$  such that*

$$E(t) \leq Me^{-\omega t} E(0).$$

Theorem 1 is proved in Section 3. In Section 2, we study the well-posedness of system (1) – (9) using semigroup theory.

## 2 Well-posedness of system (1) – (9)

Inspired from [6] and [7], we introduce the auxilliary variables

$$y(x, \rho, t) = \frac{\partial u(x, t - \tau\rho)}{\partial t}$$

$$z(x, \rho, t) = \frac{\partial v(x, t - \tau\rho)}{\partial t}$$

With these new unknowns, system (1) – (9) is equivalent to

$$\frac{\partial^2 u(x, t)}{\partial t^2} - \Delta u(x, t) + l(u(x, t) - v(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{14}$$

$$\frac{\partial y(x, \rho, t)}{\partial t} + \frac{1}{\tau} \frac{\partial y(x, \rho, t)}{\partial \rho} = 0 \quad \text{on } \Gamma_2 \times (0, 1) \times (0, +\infty), \tag{15}$$

$$\frac{\partial^2 v(x, t)}{\partial t^2} - \Delta v(x, t) + l(v(x, t) - u(x, t)) = 0 \quad \text{in } \Omega \times (0, +\infty), \tag{16}$$

$$\frac{\partial z(x, \rho, t)}{\partial t} + \frac{1}{\tau} \frac{\partial z(x, \rho, t)}{\partial \rho} = 0 \quad \text{on } \Gamma_2 \times (0, 1) \times (0, +\infty), \tag{17}$$

$$u(x, t) = v(x, t) = 0 \quad \text{on } \Gamma_1 \times (0, +\infty), \tag{18}$$

$$\frac{\partial u(x, t)}{\partial \nu} = -\alpha_1 \frac{\partial u(x, t)}{\partial t} - \alpha_2 y(x, 1, t) \quad \text{on } \Gamma_2 \times (0, +\infty), \tag{19}$$

$$\frac{\partial v(x, t)}{\partial \nu} = -\beta_1 \frac{\partial u(x, t)}{\partial t} - \beta_2 z(x, 1, t) \quad \text{on } \Gamma_2 \times (0, +\infty), \tag{20}$$

$$y(x, 0, t) = \frac{\partial u(x, t)}{\partial t}, z(x, 0, t) = \frac{\partial v(x, t)}{\partial t} \quad \text{on } \Gamma_2 \times (0, +\infty), \tag{21}$$

$$u(x, 0) = u_0(x), \frac{\partial u(x, 0)}{\partial t} = u_1(x) \quad \text{in } \Omega, \tag{22}$$

$$v(x, 0) = v_0(x), \frac{\partial v(x, 0)}{\partial t} = v_1(x) \quad \text{in } \Omega, \tag{23}$$

$$y(x, \rho, 0) = g(x, -\tau\rho), z(x, \rho, 0) = h(x, -\tau\rho) \quad \text{on } \Gamma_2 \times (0, 1). \tag{24}$$

Denote by  $\mathcal{H}$  the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2 \times L^2(0, 1)) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Omega) \times L^2(\Gamma_2 \times L^2(0, 1))$$

where

$$H^1_{\Gamma_1}(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_1\}$$

We equip  $\mathcal{H}$  with the inner product

$$\begin{aligned} \left\langle \begin{pmatrix} \zeta \\ \eta \\ \theta \\ \phi \\ \chi \\ \psi \end{pmatrix}; \begin{pmatrix} \tilde{\zeta} \\ \tilde{\eta} \\ \tilde{\theta} \\ \tilde{\phi} \\ \tilde{\chi} \\ \tilde{\psi} \end{pmatrix} \right\rangle &= \int_{\Omega} (\nabla \zeta(x) \cdot \nabla \tilde{\zeta}(x) + \eta(x)\tilde{\eta}(x)) \, dx + \\ &\mu \int_{\Gamma_2} \int_0^1 \theta(x, \rho)\tilde{\theta}(x, \rho) \, d\rho \, d\Gamma + \int_{\Omega} (\nabla \phi(x) \cdot \nabla \tilde{\phi}(x) + \chi(x)\tilde{\chi}(x)) \, dx + \\ &\xi \int_{\Gamma_2} \int_0^1 \psi(x, \rho)\tilde{\psi}(x, \rho) \, d\rho \, d\Gamma + l \int_{\Omega} (\zeta(x) - \phi(x))(\tilde{\zeta}(x) - \tilde{\phi}(x)) \, dx \end{aligned}$$

Define in  $\mathcal{H}$  a linear operator  $\mathcal{A}$  by

$$\begin{aligned} D(\mathcal{A}) &= \{(\zeta, \eta, \theta, \phi, \chi, \psi)^T \in H^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0, 1)) \times \\ &H^2(\Omega) \times H^1_{\Gamma_1}(\Omega) \times L^2(\Gamma_2 \times H^1(0, 1)); \frac{\partial \zeta}{\partial \nu} = -\alpha_1 \eta - \alpha_2 \theta(\cdot, 1), \\ &\eta = \theta(\cdot, 0) \text{ on } \Gamma_2; \frac{\partial \phi}{\partial \nu} = -\beta_1 \chi - \beta_2 \psi(\cdot, 1), \chi = \psi(\cdot, 0) \text{ on } \Gamma_2\} \end{aligned} \tag{25}$$

$$\mathcal{A}(\zeta, \eta, \theta, \phi, \chi, \psi)^T = (\eta, \Delta \zeta + l\phi - l\zeta, -\tau^{-1} \frac{\partial \theta}{\partial \rho}, \chi, \Delta \phi - l\phi + l\zeta, -\tau^{-1} \frac{\partial \psi}{\partial \rho})^T \tag{26}$$

Then we can rewrite (14) – (24) as an abstract Cauchy problem in  $\mathcal{H}$

$$\begin{cases} \frac{d}{dt} W(t) = \mathcal{A}W(t) \\ W(0) = W_0 \end{cases} \tag{27}$$

where

$$\begin{aligned} W(t) &= (u(x, t), \frac{\partial u(x, t)}{\partial t}, y(x, \rho, t), v(x, t), \frac{\partial v(x, t)}{\partial t}, z(x, \rho, t))^T, \\ \text{and } W_0 &= (u_0, u_1, g(\cdot, -\tau), v_0, v_1, h(\cdot, -\tau))^T \end{aligned}$$

We verify that  $\mathcal{A}$  is dissipative and that  $\lambda I - \mathcal{A}$  is onto for a fixed  $\lambda > 0$ . Thus, by the Lumer-Phillips Theorem (see for instance [8])  $\mathcal{A}$  generates a strongly continuous semigroup on  $\mathcal{H}$  and consequently we have

**Proposition 1.** For every  $W_0 \in \mathcal{H}$ , problem (27) has a unique solution  $W$  whose regularity depends on the the initial datum  $W_0$  as follows:

$$\begin{aligned} W(\cdot) &\in C([0, +\infty); \mathcal{H}) \text{ if } W_0 \in \mathcal{H}, \\ W(\cdot) &\in C^1([0, +\infty); \mathcal{H}) \cap C([0, +\infty); D(\mathcal{A})) \text{ if } W_0 \in D(\mathcal{A}). \end{aligned}$$

### 3 Proof of Theorem 1

We prove Theorem 1 for smooth initial data. The general case follows by a standard density argument.

We proceed in several steps.

**Step 1.**

Differentiating  $E(t)$  with respect to time, we obtain

$$\frac{d}{dt}E(t) \leq -k \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x, t)}{\partial t} \right|^2 + \left| \frac{\partial u(x, t - \tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x, t)}{\partial t} \right|^2 + \left| \frac{\partial v(x, t - \tau)}{\partial t} \right|^2 \right\} d\Gamma \tag{28}$$

where

$$k = \min \left\{ \alpha_1 - \frac{\alpha_2}{2} - \frac{\mu}{2\tau}, \frac{\mu}{2\tau} - \frac{\alpha_2}{2}, \beta_1 - \frac{\beta_2}{2} - \frac{\xi}{2\tau}, \frac{\xi}{2\tau} - \frac{\beta_2}{2} \right\}$$

**Step 2.**

We rewrite

$$E(t) = \mathcal{E}(t) + E_d(t)$$

where

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} \left\{ |\nabla u(x, t)|^2 + \left| \frac{\partial u(x, t)}{\partial t} \right|^2 + |\nabla v(x, t)|^2 + \left| \frac{\partial v(x, t)}{\partial t} \right|^2 + l |u(x, t) - v(x, t)|^2 \right\} dx$$

and

$$E_d(t) = \frac{1}{2} \int_{\Gamma_2} \int_0^1 \left\{ \mu \left| \frac{\partial u(x, t - \tau\rho)}{\partial t} \right|^2 + \xi \left| \frac{\partial v(x, t - \tau\rho)}{\partial t} \right|^2 \right\} d\rho d\Gamma$$

$E_d(t)$  can be rewritten via a change of variable as

$$E_d(t) = \frac{1}{2\tau} \int_t^{t+\tau} \int_{\Gamma_2} \left\{ \mu \left| \frac{\partial u(x, s - \tau)}{\partial t} \right|^2 + \xi \left| \frac{\partial v(x, s - \tau)}{\partial t} \right|^2 \right\} d\Gamma ds \tag{29}$$

From (29), we obtain (here and throughout the rest of the paper  $C$  is some positive constant different at different occurrences)

$$E_d(t) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x, s - \tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x, s - \tau)}{\partial t} \right|^2 \right\} d\Gamma ds \tag{30}$$

for  $0 \leq t + \tau \leq T$  and  $T$  large enough.

**Step 3.**

From Poincaré inequality and Proposition 3.5 of [4], we have for  $T$  sufficiently large and for any  $\epsilon > 0$

$$\begin{aligned} \mathcal{E}(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial \nu} \right|^2 + \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial \nu} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 \right\} d\Gamma dt + \\ C \{ \|u\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 + \|v\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 \} \end{aligned} \tag{31}$$

Inserting the boundary conditions (6) and (7) into (31), we obtain

$$\begin{aligned} \mathcal{E}(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt + \\ C \{ \|u\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 + \|v\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 \} \end{aligned} \tag{32}$$

**Step 4.**

Estimate (30) together with (32) yields

$$\begin{aligned} E(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt + \\ C \{ \|u\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 + \|v\|_{L^2(0,T;H^{1/2+\epsilon}(\Omega))}^2 \} \end{aligned} \tag{33}$$

**Step 5.**

We drop the lower order terms on the right-hand side of (33) by a compactness-uniqueness argument to obtain

$$E(0) \leq C \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt \tag{34}$$

**Step 6.**

From (28), we have

$$E(T) - E(0) \leq -k \int_0^T \int_{\Gamma_2} \left\{ \left| \frac{\partial u(x,t)}{\partial t} \right|^2 + \left| \frac{\partial u(x,t-\tau)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t)}{\partial t} \right|^2 + \left| \frac{\partial v(x,t-\tau)}{\partial t} \right|^2 \right\} d\Gamma dt$$

which together with (34) leads to

$$E(T) \leq \frac{Ck^{-1}}{1 + Ck^{-1}} E(0) \tag{35}$$

The desired conclusion follows now from (35).

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