Towards Intrinsic Almost Global Consensus on the n-Sphere

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***Abstract*— This paper concerns the global level convergence properties of a continuous consensus protocol for multi-agent systems that evolve in continuous time on the n-sphere. The feedback is intrinsic to the n-sphere since it does not rely on the use of local coordinates obtained through a parametrization. It is shown that, for any connected undirected graph topology and all $n \in \mathbb{N}\backslash\{1\}$, the protocol yields convergence that is akin to almost global consensus in a weak sense. Simulation results suggest that almost global consensus hold. This results is of interest in the context of consensus on Riemannian manifolds since it differs from what is known with regard to 1-sphere and on $\mathsf{SO}(3)$ where more advanced intrinsic consensus protocols are required in order to generate equivalent results.**

I. Introduction

Consider a system consisting of N agents, each of which is equipped with some limited communication and sensing capabilities. The goal is for all agents to converge to the same state, *i.e.*, to reach a consensus. This type of problem is widely studied in the literature, see *e.g.*, [1] and the references therein. As the field of networked and multi-agent system has matured, research focus shifts from linear dynamics to more realistic models such as switched and highly nonlinear systems including those featured in the attitude synchronization problem [2]–[6]. Research on attitude synchronization is motivated by applications such as satellite formation flying [7], [8], cooperative robotic manipulation [9], multi-camera networks [10], and distributed rotation averaging [11], [12].

The reduced attitude is a property of objects that for various reasons, such as task redundancy, cylindrical symmetry, or actuator failure, lack one degree of rotational freedom in three-dimensional space and whose orientation corresponds to a pointing direction with no regard for the rotation about said axis of pointing [13]. The reduced attitude synchronization problem is equivalent to the consensus problem on the sphere. The problem of cooperative control on the n-sphere, $\mathcal{S}^n$, has received some attention in the literature [14]–[16], but comparatively less than the full attitude synchronization problem. Applications aside from reduced attitude synchronization include planetary scale mobile sensing networks [17] and systems consisting of periodic oscillators described by so-called Kuramoto models [18].

The problem of almost global consensus has been studied on $\mathcal{S}^1$ [15], on $\mathsf{SO}(3)$ [19], and on $\mathcal{S}^n$ in the special case of a complete graph [14], [16]. The work [19] apply an optimization based method to characterize the stability of all equilibria for a particular discrete-time consensus protocol. Their result is akin to almost global consensus over any connected graph topology. The algorithm uses a particular reshaping function which depends on a constant parameter that must be bounded below. However, the bound depends on the graph topology which is unknown to the agents. Moreover, the overall convergence speed of the algorithm decreases with increasing values of the parameter. By shifting consideration from $\mathsf{SO}(3)$ to the n-sphere, this paper uses the direct method of Lyapunov to establish stability results of the same type as those in [19] for a basic consensus protocol that does not require the use of any reshaping function.

The work [19] divides the literature on attitude consensus into two categories: extrinsic and intrinsic algorithms. An algorithm is said to be extrinsic if it makes use of a parametrization that embeds $\mathsf{SO}(3)$ in some Euclidean space. There are algorithms in this class that provide consensus on a global level. For the second category of algorithms that work with $\mathsf{SO}(3)$ directly, global level results had not been obtained prior to the publication of [19]. Much of previous work on concerning intrinsic cooperative control on the n-sphere only regards the case of a complete graph [14], [16]. Some preliminary results regarding a class of of non-trivial topologies, essentially the case of cycle graphs, is found in the main authors previous work [20]–[22]. The gossip algorithm of [15] achieves consensus on $\mathcal{S}^n$ with probability 1 for a class of digraphs, but does not apply to mechanical systems.

The 2-sphere is akin to a subset of $\mathsf{SO}(3)$, and as such many results obtained for $\mathsf{SO}(3)$ also applies to $\mathcal{S}^2$. Special cases sometimes allow for stronger results. The findings of this paper indicates that the conditions for achieving almost global consensus are, in a certain sense, more favorable on the n-sphere for $n \geq 2$ than the particularization of previously known results concerning the 1-sphere and $\mathsf{SO}(3)$ would imply. A basic intrinsic consensus protocol over connected, undirected graph topologies render all equilibria except the consensus point unstable on $\mathcal{S}^2$ whereas simulation results indicate that some of the corresponding equilibria for certain graph topologies are asymptotically stable on the 1-sphere [15] and $\mathsf{SO}(3)$ [19].

The main contribution of this paper is to provide a convergence results on a global level for all connected, undirected graph topologies, *i.e.*, for a larger class of topologies than what has previously been found on the n-sphere [5], [14], [16], [19], [20], [22]. Unlike [19], the control law does not require the use of a reshaping function that depends on the graph topology, *i.e.*, on unavailable information. This result is conjectured in [14], [20], but may be considered unexpected

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since it is qualitatively different from what is known to hold with regard to discrete-time consensus on Riemannian manifolds for the Lie groups $\mathcal{S}^1$ and $\mathsf{SO}(3)$ of which $\mathcal{S}^2$ may be considered a subset [15], [19].

II. Problem Description

The following notation is used in this paper. The inner product and outer product of $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ are denoted by $\langle \mathbf{x},\mathbf{y}\rangle$ and $\mathbf{x}\otimes\mathbf{y}$ respectively. The Euclidean norm is used for vectors and the Frobenius norm is used for matrices. The special orthogonal group is $\mathsf{SO}(n) = \{\mathbf{R}\in\mathbb{R}^{n\times n}\,|\,\mathbf{R}^{-1} = \mathbf{R}^\top, \det\mathbf{R} = 1\}$. The Lie algebra of $\mathsf{SO}(n)$ is $\mathsf{so}(n) = \{\mathbf{S}\in\mathbb{R}^{n\times n}\,|\,\mathbf{S}^\top = -\mathbf{S}\}$. The n-sphere is denoted by $\mathcal{S}^n = \{\mathbf{x}\in\mathbb{R}^{n+1}\,|\,\|\mathbf{x}\| = 1\}$, where $n\in\mathbb{N}$. We do not consider the trivial 0-sphere $\mathcal{S}^0 = \{-1,1\}$. A graph is a pair $\mathcal{G} = (\mathcal{V},\mathcal{E})$ where $\mathcal{V}$ is the node set and $\mathcal{E}\subset\mathcal{V}\times\mathcal{V}$ is the edge set.

An equilibrium manifold is a set of equilibria of a system that also constitutes a manifold. The concepts of stability of an equilibrium can be extended to sets and hence to manifolds [23]–[25]. This paper uses the following terminology to describe stability properties of manifolds.

Definition 1. *An equilibrium manifold is said to be maximal if it is connected and not a strict subset of any other connected equilibrium manifold.*

Definition 2. *A maximal equilibrium manifold is said to be uniquely stable if it is stable and no other maximal equilibrium manifold is stable, uniquely attractive if it is attractive and no other maximal equilibrium manifold is attractive, and uniquely asymptotically stable if it is both uniquely stable and uniquely attractive.*

A. Distributed Control Design on the n-Sphere

Consider a multi-agent system where each agent corresponds to an index $i\in\mathcal{V}$. The agents are capable of limited pairwise communication and local sensing. The topology of the communication network is described by an undirected connected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1,2,\ldots,N\}$, and $(i,j)\in\mathcal{E}$ implies that agent i and j can communicate, or equivalently that agent i and j can sense the so-called local or relative information regarding the displacement of their states.

Control is assumed to be based on relative information and to be carried out on a kinematic level. The information $\mathcal{I}_{ij}$ that agent i has access to regarding its neighbor agent j includes

$$\operatorname{span}\{(\mathbf{I}-\mathbf{X}_i)(\mathbf{x}_j-\mathbf{x}_i)\} = \operatorname{span}\{\mathbf{x}_j - \langle\mathbf{x}_j,\mathbf{x}_i\rangle\mathbf{x}_i\}, \quad (1)$$

where $\mathbf{X}_i = \mathbf{x}_i\otimes\mathbf{x}_i$ and $\mathbf{I}-\mathbf{X}_i : \mathbb{R}^{n+1}\to\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$ is a projection. The set of neighbors of agent i is denoted $\mathcal{N}_i = \{j\in\mathcal{V}\,|\,(i,j)\in\mathcal{E}\}$. The space of relative information known to any agent $i\in\mathcal{V}$ is $\cup_{j\in\mathcal{N}_i}\mathcal{I}_{ij}$. The subset of $\mathcal{I}_{ij}$ given by (1) corresponds to the customary relative information in linear spaces $\mathcal{I}_{ij}\supset\operatorname{span}\{\mathbf{x}_j-\mathbf{x}_i\}$ projected on $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$. An agent can calculate this aspect of $\mathcal{I}_{ij}$ based on local sensing since all it needs to discern is the direction towards its neighbor along its tangent space.

System 3. *The system is given by N agents, an undirected and connected graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$, agent states $\mathbf{x}_i\in\mathcal{S}^n$, where $n\in\mathbb{N}\backslash\{1\}$, and dynamics*

$$\dot{\mathbf{x}}_i = \mathbf{u}_i - \langle\mathbf{u}_i,\mathbf{x}_i\rangle\mathbf{x}_i = (\mathbf{I}-\mathbf{X}_i)\mathbf{u}_i, \quad (2)$$

where $\mathbf{u}_i : \prod_{j\in\mathcal{N}_i}\mathcal{I}_{ij}\to\mathbb{R}^{n+1}$ is the input signal of agent i and $\mathbf{X}_i = \mathbf{x}_i\otimes\mathbf{x}_i$ for all $i\in\mathcal{V}$.

Note that the dynamics (2) projects the input $\mathbf{u}_i$ on the space of relative information. While some agent $i\in\mathcal{V}$ may not be able to calculate some $\mathbf{u}_i\in\mathcal{I}_{ij}$ based on the information (1) obtained from all its neighbors, that agent may still be able to calculate an input $\mathbf{v}_i$ whose projection on $\mathsf{T}_{\mathbf{x}_i}\mathcal{S}^n$ is identical to that of $\mathbf{u}_i$. This holds for inputs that belongs to $\operatorname{span}\{\mathbf{x}_j\,|\,j\in\mathcal{N}_i\}$, and in particular for elements of the positive cone $\operatorname{pos}\{\mathbf{x}_j\,|\,j\in\mathcal{N}_i\}$. Intuitively speaking, it is reasonable to assume that agent i should be able to move towards any point in $\operatorname{pos}\{\mathbf{x}_j\,|\,j\in\mathcal{N}_i\}$, *i.e.*, that $\operatorname{pos}\{\mathbf{x}_j\,|\,j\in\mathcal{N}_i\}\subset\mathcal{I}_{ij}$.

All vectors in this section are defined in the world frame $\mathcal{W}$. To implement the control law in a distributed fashion, $\mathbf{u}_i$ must be transfered to the body frame $\mathcal{B}_i$ of agent i. Let $[\mathbf{x}]_\mathcal{F}$ denote that the coordinates of vector $\mathbf{x}$ are given with respect to the $\mathcal{F}$. Suppose $\mathcal{B}_i$ is related to $\mathcal{W}$ by means of a rotation $\mathbf{R}_i : [\mathbf{v}]_\mathcal{W}\mapsto[\mathbf{v}]_{\mathcal{B}_i}$. The control law in $\mathcal{W}$ is given by $[\mathbf{u}_i]_\mathcal{W} = \sum_{j\in\mathcal{N}_i} w_{ij}[\mathbf{x}_j]_\mathcal{W}$. Hence $[\mathbf{u}_i]_{\mathcal{B}_i} = \sum_{j\in\mathcal{N}_i} w_{ij}\mathbf{R}_i[\mathbf{x}_j]_\mathcal{W} = \sum_{j\in\mathcal{N}_i} w_{ij}[\mathbf{x}_j]_{\mathcal{B}_i}$. Moreover,

$$\begin{aligned}[\dot{\mathbf{x}}_i]_{\mathcal{B}_i} &= \mathbf{R}_i[\mathbf{u}_i]_\mathcal{W} + \langle[\mathbf{u}_i]_\mathcal{W},\mathbf{R}_i^\top\mathbf{R}_i[\mathbf{x}_i]_\mathcal{W}\rangle\mathbf{R}_i[\mathbf{x}_i]_\mathcal{W}\\ &= [\mathbf{u}_i]_{\mathcal{B}_i} + \langle[\mathbf{u}_i]_{\mathcal{B}_i},[\mathbf{x}_i]_{\mathcal{B}_i}\rangle[\mathbf{x}_i]_{\mathcal{B}_i}, \quad (3)\end{aligned}$$

due to inner products being invariant under orthonormal changes of coordinates. From the perspective of stability analysis, (3) is the same as equation (2).

The problem of multi-agent consensus on the n-sphere concerns the design of distributed control protocols $\{\mathbf{u}_i\}_{i=1}^N$ based on relative information, as discussed in the above paragraphs, that stabilize the consensus manifold

$$\mathcal{C} = \{\{\mathbf{x}_i\}_{i=1}^N\in(\mathcal{S}^n)^N\,|\,\mathbf{x}_i = \mathbf{x}_j,\,\forall\, i,j\in\mathcal{V}\} \quad (4)$$

of System 3. If all agents converge to one point on the n-sphere, then they are said to reach consensus. For all connected graphs, it can easily be shown to suffice that the states of any pair of neighboring agents are equal.

B. Problem Statement

This paper concerns the following aspects of control design and stability analysis for a basic consensus protocol on the n-sphere. By basic we mean that the algorithm should be as simple as possible.

Problem 4. *Find a basic, intrinsic consensus protocol, i.e., input signals $\mathbf{u}_i : \prod_{j\in\mathcal{N}_i}\mathcal{I}_{ij}\to\mathbb{R}^{n+1}$ for all $i\in\mathcal{V}$, for System 3 such that the consensus manifold is uniquely asymptotically stable.*

Problem 4 concerns the global behavior of the system. Under certain assumptions regarding the connectivity of $\mathcal{G}$, local consensus on $\mathsf{SO}(3)$ can be established with the region of attraction being the largest geodesically convex sets on $\mathcal{S}^n$, *i.e.*, open hemispheres. See for example [5] in the case of an undirected graph and [6] in the case of a directed and time-varying graph. A global stability result for discrete-time consensus on $\mathsf{SO}(3)$ is provided in [19]. Almost global asymptotical stability of the consensus manifold on the n-sphere is known to hold when the graph is a tree [5] or is complete in the case of first- and second-order models [14], [16]. The author of [14] conjectures that global stability also holds for a larger class of topologies whereas [15] provides counter-examples of basic consensus protocols that fail to generate consensus on $\mathcal{S}^1$.

Remark 5. *It is not possible to achieve global consensus on $\mathcal{S}^n$ by means of a continuous feedback due to topological constraints [26]. It is however possible to achieve almost global asymptotical stability [15]. The paper [19] argues that adding a small perturbation to the feedback of a uniquely asymptotically stable system yields almost global consensus. However, that is not true in general. To prove almost global consensus is difficult since basic tools such as the Hartman-Grobman theorem or stable-unstable manifold theorems are unavailable due to the equilibria being nonhyperbolic [27]. Instead, center manifold theory for equilibrium manifolds may be required. That is the topic of future work whereas this paper deals with unique asymptotical stability.*

III. Stability of Desired Equilibria

This section concerns System 3 governed by Algorithm 6 below which provides a continuous protocol for continuous-time consensus on the n-sphere. It could be argued that Algorithm 6 is the most basic conceivable feedback for this problem, and actually corresponds to the weighted linear consensus protocols given by $\mathbf{u}_i = \sum_{i\in\mathcal{N}_i} w_{ij}(\mathbf{x}_j - \mathbf{x}_i)$ where $w_{ij}\in(0,\infty)$, $w_{ij}=w_{ji}$ for all $(i,j)\in\mathcal{E}$ since the dynamics on the n-sphere project an input to the orthogonal complement of the state of each agent.

Algorithm 6. *The feedback is given by $\mathbf{u}_i = \sum_{j\in\mathcal{N}_i} w_{ij}\mathbf{x}_j$, where $w_{ij}\in(0,\infty)$ and $w_{ij}=w_{ji}$ for all $(i,j)\in\mathcal{E}$.*

Algorithm 6 can be derived by taking the gradient of the potential function

$$V = \tfrac{1}{2}\sum_{(i,j)\in\mathcal{E}} w_{ij}\|\mathbf{x}_i-\mathbf{x}_j\|^2 = \sum_{(i,j)\in\mathcal{E}} w_{ij}(1-\cos\vartheta_{ij}), \quad (5)$$

where ϑ_{ij} is the angle between $\mathbf{x}_i$ and $\mathbf{x}_j$. It is possible to work with more general gains than the constant weights of Algorithm 6, but we prefer constants for ease of notation. Proposition 7 implies that there are no limits cycles in System 3 under Algorithm 6. If the graph in System 3 under Algorithm 6 were directed, then there would exist examples of topologies that result in limit cycles.

Proposition 7. *System 3 converges to an equilibrium. The following equilibrium configurations exist:*

$$(\mathbf{x}_i,\mathbf{u}_i)\in\left\{\left(-\frac{\mathbf{u}_i}{\|\mathbf{u}_i\|},\mathbf{u}_i\right),\left(\frac{\mathbf{u}_i}{\|\mathbf{u}_i\|},\mathbf{u}_i\right),(\mathbf{x}_i,\mathbf{0})\right\},$$

where $\mathbf{u}_i=\sum_{j\in\mathcal{N}_i} w_{ij}\mathbf{x}_j$ for all $i\in\mathcal{V}$.

Proof. Consider the potential function (5). It holds that

$$\begin{aligned}
\dot V &= \sum_{(i,j)\in\mathcal{E}} w_{ij}\langle \mathbf{x}_i-\mathbf{x}_j, \dot{\mathbf{x}}_i-\dot{\mathbf{x}}_j\rangle\\
&= \sum_{(i,j)\in\mathcal{E}} w_{ij}\langle \mathbf{x}_i-\mathbf{x}_j, (\mathbf{I}-\mathbf{X}_i)\mathbf{u}_i-(\mathbf{I}-\mathbf{X}_j)\mathbf{u}_j\rangle\\
&= -\sum_{(i,j)\in\mathcal{E}} w_{ij}[\langle \mathbf{x}_i,(\mathbf{I}-\mathbf{X}_j)\mathbf{u}_j\rangle+\langle \mathbf{x}_j,(\mathbf{I}-\mathbf{X}_i)\mathbf{u}_i\rangle]\\
&= -\sum_{i\in V}\sum_{j\in\mathcal{N}_i} w_{ij}\langle \mathbf{x}_j,(\mathbf{I}-\mathbf{X}_i)\mathbf{u}_i\rangle\\
&= -\sum_{i\in V}\left\langle\sum_{j\in\mathcal{N}_i} w_{ij}\mathbf{x}_j,(\mathbf{I}-\mathbf{X}_i)\mathbf{u}_i\right\rangle\\
&= -\sum_{i\in V}\langle \mathbf{u}_i,(\mathbf{I}-\mathbf{X}_i)\mathbf{u}_i\rangle = -\sum_{i\in V}\|\mathbf{u}_i\|^2-\langle\mathbf{u}_i,\mathbf{x}_i\rangle^2.
\end{aligned}$$

System 3 converges to the set $\{\{\mathbf{x}_i\}_{i=1}^N \,|\, \mathbf{u}_i \parallel \mathbf{x}_i\}$ by LaSalle's theorem, *i.e.*, the input and state of each agent align up to sign. This implies $\dot{\mathbf{x}}_i=\mathbf{0}$ for all $i\in\mathcal{V}$, *i.e.*, that the system is at an equilibrium by inspection of (2). □

The following result, Theorem 8, may be considered as one of the many known facts concerning consensus on convex subsets of manifolds [5]. To solve Problem 4 this paper provides a companion to Theorem 8 that regards all equilibrium manifolds of System 3 under Algorithm 6.

Theorem 8. *Consider System 3 under Algorithm 6. The system reaches consensus asymptotically if and only if there is some finite time such that all agents belong to an open hemisphere.*

Remark 9. *A proof of Theorem 8—or generalizations of this result such as local consensus over switched directed graph topologies—can be obtained by following lines of reasoning that is found in many works within the consensus literature, and is therefore omitted here.*

IV. Instability of Undesired Equilibria

The global behavior of the system is determined by the stability of the equilibrium configurations given in Proposition 7. Proposition 11 below establishes that any equilibrium where $\mathbf{x}_i = -\mathbf{u}_i/\|\mathbf{u}_i\|$ or $\mathbf{u}_i=\mathbf{0}$ for some i is unstable. Set all weights in Algorithm 6 to be equal. An example of $\mathbf{x}_i = -\mathbf{u}_i/\|\mathbf{u}_i\|$ is given by a tetrahedron formation with a tetrahedral graph, *i.e.*, the complete graph on four nodes. An example of $\mathbf{x}_i=\mathbf{0}$ for all $i\in\mathcal{V}$ is provided by the octahedral graph, see Figure 1. The case of $\mathbf{x}_i=\frac{\mathbf{u}_i}{\|\mathbf{u}_i\|}$ for all $i\in\mathcal{V}$ poses a more difficult challenge since it includes the consensus manifold as a special case. The difficulty is overcome by the indirect method of Lyapunov. As such, we study the signs of the real part of the linearization of System

3 under Algorithm 6 in order to establish the instability of all undesired equilibria.

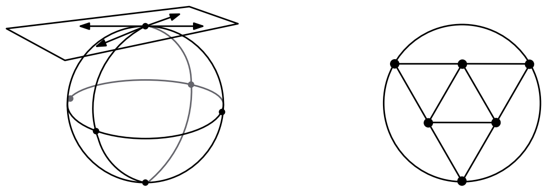


Fig. 1. An unstable equilibrium of a system on $\mathcal{S}^2$ (left) with an octahedral graph (right).

Proposition 10. *The $(n+1)\times(n+1)$ blocks of the $N(n+1)\times N(n+1)$ matrix $\mathbf{A}$ that describes the linearized dynamics of System 3 under Algorithm 6 are given by*

$$\mathbf{A}_{ij} = \begin{cases} -(\mathbf{x}_i \otimes \mathbf{u}_i + \langle \mathbf{u}_i, \mathbf{x}_i \rangle\, \mathbf{I})(\mathbf{I} - \mathbf{X}_i) & \text{if } j = i, \\ w_{ij}(\mathbf{I} - \mathbf{X}_i)(\mathbf{I} - \mathbf{X}_j) & \text{if } j \neq i, \end{cases}$$

for $(i,j) \in \mathcal{E}$ and $\mathbf{A}_{ij} = \mathbf{0}$ otherwise. The matrix $\mathbf{A}$ is symmetric.

Proof. For systems evolving on manifolds, a perturbation technique is used to obtain the linearized dynamics. Let $\mathbf{x}_i$ for all $i \in \mathcal{V}$ be a solution to (2). Consider a perturbed solution $\mathbf{x}_i(\varepsilon, \mathbf{v}_i)$ given by

$$\mathbf{x}_i(\varepsilon, \mathbf{v}_i) = \frac{\mathbf{x}_i + \varepsilon \mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon \mathbf{v}_i\|},$$

where $\mathbf{v}_i$ is a smooth function. The perturbed solution satisfies the differential equation

$$\dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i) = \mathbf{u}_i(\varepsilon, \mathbf{v}_i) - \langle \mathbf{u}_i(\varepsilon, \mathbf{v}_i), \mathbf{x}_i(\varepsilon, \mathbf{v}_i)\rangle \mathbf{x}_i(\varepsilon, \mathbf{v}_i).$$

The linearized dynamics on $\mathcal{S}^n$ can be derived by studying the linear effect of $\mathbf{v}_i$ on $\dot{\mathbf{x}}_i(\varepsilon, \mathbf{x}_i)$. Define

$$\begin{aligned} \mathbf{w}_i = \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{x}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \frac{\mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon \mathbf{v}_i\|}\Bigg|_{\varepsilon=0} - \\ \frac{\mathbf{x}_i + \varepsilon \mathbf{v}_i}{\|\mathbf{x}_i + \varepsilon \mathbf{v}_i\|^3}\langle \mathbf{x}_i, \mathbf{v}_i\rangle\Bigg|_{\varepsilon=0} \\ = \mathbf{v}_i - \mathbf{x}_i \otimes \mathbf{x}_i \mathbf{v}_i = (\mathbf{I} - \mathbf{X}_i)\mathbf{v}_i, \end{aligned} \tag{6}$$

where $\mathbf{X}_i = \mathbf{x}_i \otimes \mathbf{x}_i$. The matrix $\mathbf{I} - \mathbf{X}_i$ projects onto the tangent space $\mathcal{T}_{\mathbf{x}_i}\mathcal{S}^n$ where $\mathbf{w}_i$ lives. Note that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{X}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i$$

by the product rule. Then

$$\begin{aligned} \dot{\mathbf{w}}_i &= \tfrac{\mathrm{d}^2}{\mathrm{d}t\mathrm{d}\varepsilon}\mathbf{x}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} = \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\dot{\mathbf{x}}_i(\varepsilon, \mathbf{v}_i)|_{\varepsilon=0} \\ &= \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i)) \sum_{j\in\mathcal{N}_i} w_{ij}\mathbf{x}_j(\varepsilon, \mathbf{v}_j) \\ &= -\left[\tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{X}_i(\varepsilon, \mathbf{v}_i)\right] \sum_{j\in\mathcal{N}_i} \mathbf{x}_j(\varepsilon, \mathbf{v}_j)\big|_{\varepsilon=0} + \\ &\quad (\mathbf{I} - \mathbf{X}_i(\varepsilon, \mathbf{v}_i)) \sum_{j\in\mathcal{N}_i} \tfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{x}_j(\varepsilon, \mathbf{v}_j)\big|_{\varepsilon=0} \\ &= -(\mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i) \sum_{j\in\mathcal{N}_i} \mathbf{x}_j + (\mathbf{I} - \mathbf{X}_i) \sum_{j\in\mathcal{N}_i} \mathbf{w}_j \\ &= -(\mathbf{w}_i \otimes \mathbf{x}_i + \mathbf{x}_i \otimes \mathbf{w}_i)\mathbf{u}_i + (\mathbf{I} - \mathbf{X}_i) \sum_{j\in\mathcal{N}_i} \mathbf{w}_j \\ &= -(\langle \mathbf{u}_i, \mathbf{x}_i\rangle \mathbf{I} + \mathbf{x}_i \otimes \mathbf{u}_i)\mathbf{w}_i + (\mathbf{I} - \mathbf{X}_i) \sum_{j\in\mathcal{N}_i} \mathbf{w}_j \end{aligned}$$

where the relation $\langle \mathbf{x}, \mathbf{y}\rangle \mathbf{z} = (\mathbf{z} \otimes \mathbf{x})\,\mathbf{y} = (\mathbf{z} \otimes \mathbf{y})\mathbf{x}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^{n+1}$ and (6) are used.

The vector $\mathbf{w} = [\mathbf{w}_1^\top \ldots \mathbf{w}_N^\top]^\top$ has $N(n+1)$ components whereas the linearized system actually evolves on an Nn-dimensional space that lies embedded in $\mathbb{R}^{N(n+1)}$. The dimension reduction is given implicitly by the definition of $\mathbf{w}_i$ which requires $\mathbf{w}_i \in \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^{n-1}$. This constraint can be removed by using variables that are premultiplied by the projection matrices $\mathbf{I} - \mathbf{X}_i : \mathbb{R}^{n+1} \to \mathsf{T}_{\mathbf{x}_i}\mathcal{S}^{n-1}$, *i.e.*, the variables $\mathbf{v}_i$, whereby the matrix $\mathbf{A}$ is obtained.

It remains to show that $\mathbf{A}$ is symmetric. Write

$$\mathbf{A}_{ii} = -(\mathbf{x}_i \otimes \mathbf{u}_i + \mathbf{u}_i \otimes \mathbf{x}_i + \langle \mathbf{u}_i, \mathbf{x}_i\rangle\, \mathbf{I})(\mathbf{I} - \mathbf{X}_i)$$

which is clearly symmetric. Moreover,

$$\begin{aligned} \mathbf{A}_{ji}^\top - \mathbf{A}_{ij} = w_{ji}[(\mathbf{I} - \mathbf{X}_j)(\mathbf{I} - \mathbf{X}_i)]^\top - \\ w_{ij}(\mathbf{I} - \mathbf{X}_i)(\mathbf{I} - \mathbf{X}_j) = \mathbf{0}, \end{aligned}$$

since $w_{ij} = w_{ji}$ for all $(i,j) \in \mathcal{E}$. □

Proposition 11. *Any equilibrium $\{\mathbf{x}_i\}_{i=1}^N \notin \mathcal{C}$ of System 3 under Algorithm 6 where $n \in \mathbb{N}$ and either $\mathbf{x}_i = {-\mathbf{u}_i}/{\|\mathbf{u}_i\|}$ or $\mathbf{u}_i = \mathbf{0}$ for some $i \in \mathcal{V}$ is unstable.*

Proof. The proof makes use of the linearization provided by Proposition 10. The Courant-Fischer-Weyl min-max principle bounds the range of the Rayleigh quotient of a a symmetric matrix by its minimum and maximum eigenvalues [28]. Recall that if $\mathbf{A}$ has a positive eigenvalue at an equilibrium, then that equilibrium is unstable by the indirect method of Lyapunov [29].

In the case of $\mathbf{x}_i = {-\mathbf{u}_i}/{\|\mathbf{u}_i\|}$ for some $i \in \mathcal{V}$, note that the diagonal block of the linearization matrix $\mathbf{A}$ is given by $\mathbf{A}_{ii} = \|\mathbf{u}_i\|(\mathbf{I} - \mathbf{X}_i)$, which is a positive semidefinite, nonzero matrix. There is hence an eigenpair $(\lambda, \mathbf{u})$ of $\mathbf{A}_{ii}$ with $\lambda \in (0, \infty)$, $\mathbf{u} \in \mathcal{S}^n$. Let $\mathbf{v} \in \mathbb{R}^{N(n+1)}$ denote a vector composed of the blocks $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbb{R}^{n+1}$. Set $\mathbf{v}_i = \mathbf{u}$ and $\mathbf{v}_j = \mathbf{0}$ for all $j \notin \mathcal{V}/\{i\}$ whereby $\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle = \lambda$. The largest eigenvalue of $\mathbf{A}$ is hence bounded below by a positive number.

In the case of $\mathbf{u}_i = \mathbf{0}$ and $n \geq 2$, take a $j \in \mathcal{N}_i$ and set $\mathbf{v}_j = \varepsilon \mathbf{u}$ for a $\mathbf{u} \in \mathcal{S}^n$ such that $\langle \mathbf{u}, \mathbf{x}_i\rangle = 0$, $\langle \mathbf{u}, \mathbf{x}_j\rangle = 0$ and an $\varepsilon \in (0, \infty)$. Set $\mathbf{v}_i = \mathbf{u}$. Then $\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{A}_{jj}\mathbf{u}\rangle\varepsilon^2 + 2w_{ij}\varepsilon$, which is positive given that ε is sufficiently small.

In the case of $\mathbf{u}_i = \mathbf{0}$ and $n = 1$, if possible, take a $j \in \mathcal{N}_i$ and set $\mathbf{v}_j = \varepsilon \mathbf{u}$ for a $\mathbf{u} \in \mathcal{S}^n$ such that $\langle \mathbf{u}, \mathbf{x}_i\rangle = 0$, $\langle \mathbf{u}, \mathbf{x}_j\rangle \in (-1, 1)$ and an $\varepsilon \in (0, \infty)$. Then $\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{A}_{jj}\mathbf{u}\rangle\varepsilon^2 + 2w_{ij}(1 - \langle \mathbf{u}, \mathbf{x}_j\rangle^2)\varepsilon$, which is positive given that ε is sufficiently small. If this is not possible, it is because $\langle \mathbf{x}_i, \mathbf{x}_j\rangle = 0$, $\forall\, j \in \mathcal{N}_i$. Then $\mathbf{x}_j \in \{-\mathbf{R}\mathbf{x}_i, \mathbf{R}\mathbf{x}_i\}$ where $\mathbf{R}$ is a rotation of ${\pi}/{2}$ radians. It follows that $\mathbf{X}_j = \mathbf{X}_i$ whereby $\mathbf{A}_{ij} = w_{ij}(\mathbf{I} - \mathbf{X}_i)$. Take a $\mathbf{u} \in \mathcal{S}^n$ such that $\langle \mathbf{u}, \mathbf{x}_i\rangle \in$

$(-1,1)$ and set $\mathbf{v}_j = \varepsilon\mathbf{u}$, $\mathbf{v}_i = \mathbf{u}$ for some $\varepsilon \in (0,\infty)$. Then $\langle \mathbf{v}, \mathbf{A}\mathbf{v}\rangle = \langle \mathbf{u}, \mathbf{A}_{jj}\mathbf{u}\rangle\varepsilon^2 + 2w_{ij}(1-\langle \mathbf{u}, \mathbf{x}_i\rangle^2)\varepsilon$, which is positive given that ε is sufficiently small. $\square$

The following basic idea from [20], [22] forms the intuition behind our next result. Consider an equilibrium such that all agents belong to the intersection of $\mathcal{S}^n$ and a hyperplane in $\mathbb{R}^n$. Perturb all agents into an open hemisphere by an arbitrarily small movement along a direction orthogonal to the hyperplane. Theorem 8 then yields asymptotical consensus. The idea of perturbing all agents in one direction is the basis of the next result. The effect of such a perturbation depends of the position of an agent, *e.g.*, if all agents are perturbed along the direction $\mathbf{x}_i$, then agent i is invariant whereas the other agents move towards it.

Proposition 12. *Any equilibrium $\{\mathbf{x}_i\}_{i=1}^N \notin \mathcal{C}$ of System 3 where $n \in \mathbb{N}\backslash\{1\}$ under Algorithm 6 is unstable.*

Proof. The proof continues where that of Proposition 11 left off. It remains to consider the case of $\mathbf{x}_i = {}^{\mathbf{u}_i}/_{\|\mathbf{u}_i\|}$. Then $\mathbf{u}_i = \|\mathbf{u}_i\|\mathbf{x}_i$ and hence

$$\mathbf{A}_{ij} = \begin{cases} -\|\mathbf{u}_i\|(\mathbf{I}-\mathbf{X}_i) & \text{if } j = i, \\ w_{ij}(\mathbf{I}-\mathbf{X}_i)(\mathbf{I}-\mathbf{X}_j) & \text{if } j \neq i, \end{cases}$$

for $(i,j) \in \mathcal{E}$ and $\mathbf{A}_{ij} = \mathbf{0}$ otherwise. The matrix $\mathbf{A}$ is symmetric at all equilibria by Proposition 10.

Let $\mathbf{z} = [\mathbf{y}^\top \ldots \mathbf{y}^\top]^\top$ and consider

$$\begin{aligned}\langle \mathbf{z}, \mathbf{A}\mathbf{z}\rangle &= \sum_{i\in\mathcal{V}}\langle \mathbf{y}, \mathbf{A}_{ii}\mathbf{y}\rangle + \sum_{j\in\mathcal{N}_i}\langle \mathbf{y}, \mathbf{A}_{ij}\mathbf{y}\rangle \\ &= \left\langle \mathbf{y}, \left(\sum_{i\in\mathcal{V}}\mathbf{A}_{ii} + \sum_{j\in\mathcal{N}_i}\mathbf{A}_{ij}\right)\mathbf{y}\right\rangle\end{aligned}$$

Denote $\mathbf{B} = \sum_{i\in\mathcal{V}}\mathbf{A}_{ii} + \sum_{j\in\mathcal{N}_i}\mathbf{A}_{ij}$. The matrix $\mathbf{B}$ is symmetric due to $\mathbf{A}$ being symmetric whereby $\sigma(\mathbf{B}) \subset \mathbb{R}$ by the spectral theorem. If $(\lambda, \mathbf{v})$ is an eigenpair of $\mathbf{B}$, then $(\lambda, [\mathbf{v}^\top \ldots \mathbf{v}^\top]^\top)$ is an eigenpair of $\mathbf{A}$.

Let us prove that $\mathbf{B}$ has a positive eigenvalue, which in turn will imply that $\mathbf{A}$ has a positive eigenvalue. Consider

$$\begin{aligned}\operatorname{tr}\mathbf{B} =& -\sum_{i\in\mathcal{V}}\|\mathbf{u}_i\|(n+1-\operatorname{tr}\mathbf{X}_i) + \\ &\sum_{j\in\mathcal{N}_i} w_{ij}(n+1-\operatorname{tr}\mathbf{X}_i - \operatorname{tr}\mathbf{X}_j + \operatorname{tr}\mathbf{X}_i\mathbf{X}_j) \\ =& -\sum_{i\in\mathcal{V}} n\|\mathbf{u}_i\| + \sum_{j\in\mathcal{N}_i} w_{ij}(n-1+\langle \mathbf{x}_i, \mathbf{x}_j\rangle^2) \\ =&\, n\left(\sum_{i\in\mathcal{V}} -\|\mathbf{u}_i\| + \sum_{j\in\mathcal{N}_i} w_{ij}\right) - \\ &\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} w_{ij}(1-\cos^2\vartheta_{ij}) \\ =&\, n\left(\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} w_{ij}(1-\cos\vartheta_{ij})\right) - \\ &\sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} w_{ij}(1+\cos\vartheta_{ij})(1-\cos\vartheta_{ij}) \\ =& \sum_{i\in\mathcal{V}}\sum_{j\in\mathcal{N}_i} w_{ij}(n-1-\cos\theta_{ij})(1-\cos\vartheta_{ij})\end{aligned}$$

where we used that $\operatorname{tr}\mathbf{X}_i = \|\mathbf{x}_i\|^2 = 1$ and $\|\mathbf{u}_i\| = \langle \mathbf{x}_i, \mathbf{u}_i\rangle = \sum_{j\in\mathcal{N}_i}\langle \mathbf{x}_i, w_{ij}\mathbf{x}_j\rangle = \sum_{j\in\mathcal{N}_i} w_{ij}\cos\vartheta_{ij}$. Since $\operatorname{tr}\mathbf{B} \geq 0$ for all $n \in \mathbb{N}\backslash\{1\}$ with strict inequality unless $\theta_{ij} = 0$ for all $(i,j) \in \mathcal{E}$, *i.e.*, unless $\{\mathbf{x}_i\}_{i=1}^N \in \mathcal{C}$, it follows that $\mathbf{B}$ has an eigenvalue $\lambda \in (0,\infty)$. $\square$

Remark 13. *The case when the angle ϑ_i between at least one agent and the average of its neighbors, defined by $\vartheta_i = \arccos {}^{\langle \mathbf{u}_i, \mathbf{x}_i\rangle}/_{\|\mathbf{u}_i\|}$, belongs to $[{}^{\pi}/_{2}, \pi]$ turns out to be, roughly speaking, more unstable than the case of $\vartheta_i \in [0, {}^{\pi}/_{2})$ for all $(i,j) \in \mathcal{E}$ since instability holds for $n \in \mathbb{N}$ rather than for $n \in \mathbb{N}\backslash\{1\}$, see also [15].*

Proposition 14. *Any connected equilibrium manifold $\mathcal{M} \not\subset \mathcal{C}$ of System 3 where $n \in \mathbb{N}\backslash\{1\}$ under Algorithm 6 is unattractive.*

Proof. Consider an equilibrium $\{\mathbf{x}_i\}_{i=1}^N \in \mathcal{M}$. Since $\{\mathbf{x}_i\}_{i=1}^N$ is unstable by Proposition 12, there must be a smooth trajectory $\mathcal{T} : (0,\infty) \to (\mathcal{S}^n)^N$ with $\mathcal{T}(0) \notin \mathcal{M}$ such that, for some $\varepsilon \in (0,\infty)$, there is no $\delta \in (0,\infty)$ that would make $d(\mathcal{T}(0), \{\mathbf{x}_i\}_{i=1}^N) < \delta$ imply $\sup_{t\in(0,\infty)} d(\mathcal{T}(t), \{\mathbf{x}_i\}_{i=1}^N) < \varepsilon$. Suppose that the trajectory returns to the manifold, *i.e.*, $\lim_{t\to\infty}\mathcal{T}(t) \in \mathcal{M}$. The potential function V given by (5) decreases along every trajectory of the system but is constant over each connected equilibrium manifold. If the value of V is lower on some point on $\mathcal{T}$ than it is on $\mathcal{M}$, then $\mathcal{T}$ cannot return to $\mathcal{M}$. If V is higher on some point on $\mathcal{T}$ than on $\mathcal{M}$, then it increases even higher backwards in time and by moving $\mathcal{T}(0)$ arbitrarily close to $\mathcal{M}$, the continuity of V is contradicted. Finally, if V is constant on $\mathcal{T}$ and $\lim_{t\to\infty}\mathcal{T}(t) \in \mathcal{M}$, then $\mathcal{T}(0) \in \mathcal{T} \subset \mathcal{M}$, a contradiction. $\square$

Proposition 15. *Any maximal equilibrium manifold $\mathcal{M} \not\subset \mathcal{C}$ of System 3 where $n \in \mathbb{N}\backslash\{1\}$ under Algorithm 6 is unstable.*

Proof. Reasoning as in the proof of Proposition 14 we find there is a smooth trajectory $\mathcal{T} : (0,\infty) \to (\mathcal{S}^n)^N$ where $\mathcal{T}(0)$ can be chosen arbitrarily close to an unstable equilibrium $\{\mathbf{x}_i\}_{i=1}^N \in \mathcal{M}$ that leaves an ε-neighborhood of $\{\mathbf{x}_i\}_{i=1}^N$ for some fixed $\varepsilon \in (0,\infty)$. Suppose that this trajectory stays in an ε-neighborhood of $\mathcal{M}$. It must converge to an equilibrium that does not belong to $\mathcal{M}$ by Proposition 7 and 14. Hence there is a second equilibrium manifold $\mathcal{N}$ at a distance at most ε from $\mathcal{M}$. Since ε is arbitrary, there must be some point where $d(\mathcal{N}, \mathcal{M}) = 0$. Since $\mathcal{M}$ is closed, this contradicts the assumption of $\mathcal{M}$ being maximal. $\square$

Remark 16. *The reasoning of Proposition 14 and 15 can be generalized to establish that stability and attractiveness are equivalent properties of maximal equilibrium manifolds for gradient systems.*

V. Main Result

Theorem 17. *The consensus manifold $\mathcal{C}$ of System 3 where $n \in \mathbb{N}\backslash\{1\}$ under Algorithm 6 is uniquely asymptotically stable.*

Proof. The proof follows directly by verifying that the requirements of Definition 2 are satisfied as is evident by Theorem 8, Proposition 14, and Proposition 15. □

Remark 18. *It is shown in [15] that certain graph topologies generate stable equilibria on $\mathcal{S}^1$. The circle is a subset of any n-Sphere yet this result does not contradict Theorem 17 since any configuration that is confined to a circle can be perturbed into a hemisphere when $n \in \mathbb{N}\backslash\{1\}$ whereby Theorem 8 guarantees a consensus. It would seem the additional degrees of freedom is the source of qualitatively different stability properties on $\mathcal{S}^1$ and the higher dimensional spheres. This suggests that the result of [19] concerning the failure of basic consensus protocols to yield almost global consensus on $\mathsf{SO}(3)$ may not apply on $\mathsf{SO}(n)$ for $n \in \mathbb{N}\backslash\{2, 3\}$.*

VI. Simulation

The question of whether Theorem 17 can be extended to the case of almost global consensus remains unanswered. This section provides simulations that argue in favor of such a result. It also compares the global performance of two consensus protocols on $\mathcal{S}^2$ and $\mathsf{SO}(3)$ respectively in simulation.

A. A Case for Almost Global Consensus on $\mathcal{S}^2$

Let $\mathcal{R}$ denote the union of the regions of attraction over all equilibrium manifolds except the consensus manifold. Suppose the probability measure of $\mathcal{R}$ on $(\mathcal{S}^2)^N$ is $\varepsilon \in [0, 1]$. Consider the statement '$\mathcal{R}$ has measure zero' as a null hypothesis. Let the random variable X denote the number of draws from $\mathcal{R}$ after m draws from $\mathcal{U}(\mathcal{S}^2)$. If $X > 0$ then the null hypothesis is rejected. The probability of the null hypothesis not being rejected is $P(X = 0) = (1 - p)^m$, where p is the probability of an outcome belonging to $\mathcal{R}$, *i.e.*, $p = \varepsilon/(4\pi N)$. If m is large and the null hypothesis is not rejected, then it is very likely that p is small. Of course, the expression for $P(X = 0)$ does not account for the effect of numerical errors, but convergence to the desired manifold in the presence of errors is anyhow a positive quality.

Good candidates of graphs that results in stable non-consensus equilibria on the 2-sphere are the Platonic solids and the fullerene graphs. The Platonic solids are symmetric three-dimensional bodies whose corners lie on the surface of a sphere. There are a total of five platonic bodies, the tetrahedron, hexahedron, octahedron, icosahedron and dodecahedron, see Figure 1 and 2. A planar depiction of all graphs is given in Table I. If the number of agents equals the number of nodes of a Platonic graph and are positioned as on the corners of the corresponding Platonic solid, then the multi-agent systems is at an equilibrium since the contributions of all neighboring agents on each tangent space cancel for reasons of symmetry, see Figure 1.

Table I displays the outcome of running 10^6 trials of Algorithm 6 with $w_{ij} = 5$ for all $(i, j) \in \mathcal{E}$ on System 3 for the five platonic graphs. The initial conditions are drawn uniformly from the sphere using that $\mathbf{x} \in \mathcal{N}(\mathbf{0}, \mathbf{I})$ implies that $\mathbf{x}/\|\mathbf{x}\| \in \mathcal{U}(\mathcal{S}^n)$, see [30]. By inspection of Table I

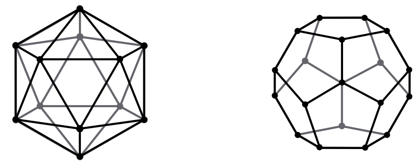
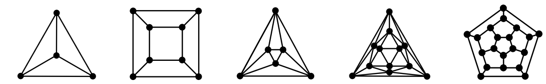


Fig. 2. The icosahedral (left) and dodecahedral (right) graph depicted in three dimensions.

there were no failures to reach consensus. There is hence no reason to reject the null hypothesis. It can be shown, as a consequence of Proposition 11, that the platonic body configurations in the case of the tetrahedron and octahedron are unstable equilibria on $\mathcal{S}^2$, see also Figure 1. It appears that almost global consensus holds for all the platonic graphs.

TABLE I. Number of failures to reach consensus on $\mathcal{S}^2$ over 10^6 trials for the five platonic graphs.



0	0	0	0	0

B. A Comparison of Consensus on $\mathcal{S}^2$ and $\mathsf{SO}(3)$

Consider the following multi-agent system on the special orthogonal group $\mathsf{SO}(3)$.

System 19. *The system is given by N agents, an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, agent states $\mathbf{R}_i \in \mathsf{SO}(3)$, and dynamics $\dot{\mathbf{R}}_i = \mathbf{\Omega}_i \mathbf{R}_i$ where $\mathbf{\Omega}_i \in \mathsf{so}(3)$ for all $i \in \mathcal{V}$. It is assumed that $\mathcal{G}$ is connected and that the system can be actuated on a kinematic level, i.e., $\mathbf{\Omega}_i$ is the input signal of agent i.*

Recall that Algorithm 6 is derived by taking the gradient of the potential function in (5). A related consensus protocol on $\mathsf{SO}(3)$ can be derived by taking the gradient of the corresponding potential function, $V : \mathsf{SO}(3) \times \mathsf{SO}(3) \to [0, \infty)$, given by

$$V(\mathbf{R}_i, \mathbf{R}_j) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} w_{ij} \|\mathbf{R}_i - \mathbf{R}_j\|_F^2,$$

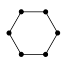
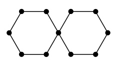
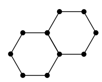
where the Euclidean metric $\|\mathbf{x}_i - \mathbf{x}_j\|$ on $\mathcal{S}^2$ has been replaced by the Frobenius metric $d(\mathbf{R}_i, \mathbf{R}_j) = \|\mathbf{R}_i - \mathbf{R}_j\|_F$ on $\mathsf{SO}(3)$. As such, Algorithm 6 is rather similar to the following algorithm on System 19.

Algorithm 20. *The feedback is given by $\mathbf{\Omega}_i = \sum_{j \in \mathcal{N}_i} w_{ij} (\mathbf{R}_i^\top \mathbf{R}_j - \mathbf{R}_j^\top \mathbf{R}_i)$, where $w_{ij} \in (0, \infty)$ and $w_{ij} = w_{ji}$ for all $(i, j) \in \mathcal{E}$.*

Table II displays the outcome of running 10^6 trials of Algorithm 6 with $w_{ij} = 5$ for all $(i, j) \in \mathcal{E}$ on System 3 and Algorithm 20 with $w_{ij} = 5$ for all $(i, j) \in \mathcal{E}$ on System 19 for the five platonic graphs. The method is also used to draw uniform samples from $\mathcal{S}^2$ is also used to draw from $\mathcal{U}(\mathsf{SO}(3))$ by first generating a uniform distribution on the unit sphere in quaternion space, *i.e.*, drawing from $\mathcal{U}(\mathcal{S}^3)$, and then mapping the sample to $\mathsf{SO}(3)$.

By inspection of Table II, note that almost global consensus does not hold for Algorithm 20 on System 19 over $\mathsf{SO}(3)$. The findings of our simulations agree with those of [19]. However, there were no failures to reach consensus on the graphs in on $\mathcal{S}^2$ despite a factor of 10^2 more trials. Neither Table I nor II give us reason to reject the null hypothesis formulated in the previous section. It is hence not unreasonable to conjecture that Algorithm 6 may give almost global consensus on the 2-sphere for all graph topologies in the case of all weights being equal.

TABLE II. Number of failures to reach consensus on $\mathcal{S}^2$, and $\mathsf{SO}(3)$ over 10^6 and 10^4 trials respectively.

			
$\mathcal{S}^2$	0	0	0
$\mathsf{SO}(3)$	695	54	82

VII. Future Work

This paper defines and establishes the uniquely asymptotical stability of the consensus manifold for of a basic consensus protocol for systems that evolve on the n-sphere in the case of $n \in \mathbb{N}\backslash\{1\}$ and for all connected, undirected graph topologies. Future work plans include the consideration of a larger class of feedback laws and the case of undirected graph topologies. It would also be interesting to consider the case of $\mathsf{SO}(n)$ when $n \in \mathbb{N}\backslash\{1,2,3\}$.

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