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Extraction of cylinders and cones from minimal point sets

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Abstract

We propose new algebraic methods for extracting cylinders and cones from minimal point sets, including oriented points. More precisely, we are interested in computing efficiently cylinders through a set of three points, one of them being oriented, or through a set of five simple points. We are also interested in computing efficiently cones through a set of two oriented points, through a set of four points, one of them being oriented, or through a set of six points. For these different interpolation problems, we give optimal bounds on the number of solutions. Moreover, we describe algebraic methods targeted to solve these problems efficiently.

Keywords: Mixed set of 3D points, cylinders, cones, interpolation.

1. Introduction

Extracting geometric primitives from 3D point clouds is an important problem in reverse engineering. These 3D point clouds are typically obtained from accurate 3D scanners and there exist several methods for extracting 3D geometric primitives [2]. An important category among these methods is based on the RANSAC approach [8, 11, 2]. For such methods, the primitives are extracted directly from the input point cloud. The basic idea is to extract a particular elementary type of shape, such as planes, spheres, cylinders, cones or tori, from the smallest possible set of points and then to judge if this extracted primitive is relevant to the full point cloud. Therefore, for this category of methods it is very important to compute a particular type of shape through the smallest possible number of points, including normals if available. If extracting planes and spheres is easy, the cases of cylinders and cones are more involved. In this paper we provide new methods for extracting these geometric primitives from the smallest possible number of points, counting multiplicities of oriented points (i.e. points given with their normal vector). These methods are intended to serve the larger goal of improving speed and numerical accuracy in data extraction from graphical information. As far as we know, and surprisingly, the above-mentioned problems have not appeared in the existing literature with the exception of [6, 9]. Instead, the classical approaches to these interpolation problems usually extract, actually we should say estimate, these geometric primitives from an overdetermined number of points, counting multiplicities (e.g. [10]).

An *oriented point* is a couple of a point and a nonzero vector. A surface is said to interpolate an oriented point if the point belongs to the surface and its associated vector

is collinear to the normal of the surface at this point, we do not assume that the orientation of the normal of the point is the same as the orientation of the surface since often in the data sets normals are unoriented. Moreover, it is important to deal with inhomogeneous data, that is to say some points are oriented but not all, in order to take into account the estimated accuracy of oriented point clouds that are generated by means of normal estimation algorithms. Data made of points and oriented points will be called a *mixed set of points*.

We emphasize that interpolating at a point imposes a single algebraic condition on a given shape whereas interpolating at an oriented 3D point imposes three algebraic conditions. Typically, a 3D plane is uniquely defined either by three distinct points or by one oriented point. A sphere is uniquely defined either by four points or by one oriented point and an additional point. In these two cases, it turns out that there is a unique shape that interpolates a mixed set of points corresponding to the number of parameters of this shape (a plane is determined by three parameters and the sphere is determined by four parameters). In this paper, we will treat interpolation of two other basic shapes, namely cylinders and cones for which the situation is more involved.

Our approach is inspired by effective methods in algebraic geometry. We consider two families of unknowns. The first one corresponds to the parameters needed to describe all features of the targeted surface (e.g. the radius and axis of a circular cylinder) and hence its equation. The second family consists of auxiliary unknowns (e.g. such as a special point on that axis) which permit us to describe a collection of geometric constructions. These constructions are designed to establish a complete link between the input and the first family of unknowns. Then, we translate

algebraically the collection of constraints attached to these geometric constructions into a system of polynomial equations that we further analyze and simplify, discarding spurious solutions if necessary. Since the input and output are (and should be) real approximate data, we designed efficient algorithms to compute very accurate real solutions of these systems of equations. Indeed, in all the considered cases, we were able to express the results as the solutions of (generalized) eigenvalue problems together with close formulas. These expressions allow us to rely on classical matrix computation software and achieve accuracy and efficiency. Prototypes of our algorithms are implemented in the computer algebra system MAPLE, and we provide some statistics and timings (which are quite satisfactory).

2. Interpolation of cylinders

A cylinder (more precisely a right circular cylinder) is defined as the set of points in the three-dimensional affine space \mathbb{R}^3 located at a fixed distance (called the *radius* of the cylinder) of a given straight line (called the *axis* of the cylinder). It is hence defined by means of five parameters : four parameters describe a line in \mathbb{R}^3 and an additional parameter measures the radius.

A popular determination of a cylinder is done by interpolating two points with normals, which imposes six conditions (instead of five). So, a priori no cylinder interpolates this data; therefore some approximations are necessary. In this section, we will give new methods to compute cylinders using just five independent conditions. There are two possible such minimal configurations, either an oriented point and two other distinct points, or five distinct points.

2.1. Cylinders through a mixed minimal point set

We seek the cylinders that interpolate a given mixed minimal set of points \mathcal{P} . Since a cylinder is given by 5 parameters, \mathcal{P} is assumed to be composed of an oriented point p_1 with its normal vector n_1 and two other distinct points p_2, p_3 in \mathbb{R}^3 .

First, by a linear change of coordinates, one can assume that $p_1 = (0, 0, 0)$ and $n_1 = (0, 0, 1)$ and we set $p_2 = (x_2, y_2, z_2)$ and $p_3 = (x_3, y_3, z_3)$. Then, the axis of a cylinder interpolating \mathcal{P} must be orthogonal to the z -axis and must intersect it. It follows that a normal plane Π contains the z -axis, hence is given by an equation of the form $lx + my = 0$ where $t := (l, m, 0)$ is the corresponding direction of the axis. Observe that these directions are in correspondence with a projective line \mathbb{P}^1 . For simplicity, we set $\rho := \sqrt{l^2 + m^2} = \|t\| > 0$.

Now, we compute the orthogonal projections q_1 and q_2 of p_2 and p_3 onto the plane Π . Π contains the point p_1 and is generated by the two orthogonal vectors n_1 and

$v = n_1 \wedge t = (-m, l, 0)$. The matrix

$$M = \begin{pmatrix} \frac{l}{\rho} & \frac{m}{\rho} & 0 \\ -\frac{m}{\rho} & \frac{l}{\rho} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

defines the change of coordinates from the current coordinate system (x, y, z) to a new coordinate system (x', y', z') , with the same origin p_1 , defined by the three vectors t/ρ , v/ρ , n_1 , where Π has equation $x' = 0$. It follows that the coordinates (x'_i, y'_i, z'_i) of p_i , $i = 2, 3$, in this new coordinate system are given by

$$(x'_i, y'_i, z'_i) = \left(x_i \frac{l}{\rho} + y_i \frac{m}{\rho}, -x_i \frac{m}{\rho} + y_i \frac{l}{\rho}, z_i \right).$$

Therefore, the coordinates of the orthogonal projections q_i , $i = 2, 3$ are given by

$$q_i = \left(-x_i \frac{m}{\rho} + y_i \frac{l}{\rho}, z_i \right) \in \Pi$$

in the basis $v/\rho, n_1$.

The existence of a cylinder interpolating \mathcal{P} is equivalent to the fact that the points p_1 , q_2 and q_3 all belong to a circle whose center c is located on the z -axis, say $c = (0, 0, r)$. Such a circle has an equation of the form $y'^2 + (z' - r)^2 = r^2$, or equivalently $y'^2 + z'^2 - 2rz' = 0$. Therefore, this cocyclicity condition can be written as

$$0 = \begin{vmatrix} y_2'^2 + z_2'^2 & z_2 \\ y_3'^2 + z_3'^2 & z_3 \end{vmatrix} = \frac{1}{\rho^2} \begin{vmatrix} (-x_2m + y_2l)^2 + (l^2 + m^2)z_2^2 & z_2 \\ (-x_3m + y_3l)^2 + (l^2 + m^2)z_3^2 & z_3 \end{vmatrix}.$$

Since $\rho > 0$, the expansion of this latter determinant allows us to rewrite this condition as a degree 2 homogeneous equation $al^2 + blm + cm^2$ where the coefficients a, b, c are given by the following closed formulas

$$a := \begin{vmatrix} y_2^2 + z_2^2 & z_2 \\ y_3^2 + z_3^2 & z_3 \end{vmatrix}, \quad b := -2 \begin{vmatrix} x_2y_2 & z_2 \\ x_3y_3 & z_3 \end{vmatrix},$$

$$c := \begin{vmatrix} x_2^2 + z_2^2 & z_2 \\ x_3^2 + z_3^2 & z_3 \end{vmatrix}.$$

Unless $a = b = c = 0$, this equation has two roots, counting multiplicities, in the field of complex numbers. If a real solution is found, that is to say the direction of a real cylinder interpolating \mathcal{P} (observe that one can impose $\rho = 1$ since the condition is homogeneous in l, m), then the remaining parameter r is uniquely determined by one of the formulas

$$2z_2r = y_2'^2 + z_2^2 = \frac{1}{\rho^2} (-x_2m + y_2l)^2 + z_2^2, \quad (1)$$

$$2z_3r = y_3'^2 + z_3^2 = \frac{1}{\rho^2} (-x_3m + y_3l)^2 + z_3^2, \quad (2)$$

depending on whether $z_2 \neq 0$ or $z_3 \neq 0$. We notice that if $z_2 = z_3 = 0$ then $a = b = c = 0$.

Theorem 1. *Given a mixed set of points \mathcal{P} composed of an oriented point p_1, n_1 and two other points p_2, p_3 such that*

- i) p_1, p_2, p_3 are all distinct,
- ii) p_1, p_2, p_3 do not belong to a common plane that is normal to n_1 ,
- iii) p_2 and p_3 are not symmetric with respect to the line through p_1 and generated by n_1 ,

then there exist at most 2 real cylinders interpolating \mathcal{P} . Otherwise, there exists a cylinder (possibly "flat", i.e. with infinite radius) interpolating \mathcal{P} in any direction that is normal to n_1 .

Proof. Following the above discussion, this theorem will be proved if we show that $a = b = c = 0$ if and only if at least one of the three conditions i), ii), iii) holds. It is not hard to check that if one of the three latter conditions holds then $a = b = c = 0$. To prove the converse, we observe that by a linear change of coordinate, we can assume that $x_2 = 0$ in addition to the fact that $p_1 = (0, 0, 0)$ and $n_1 = (0, 0, 1)$. Then, we have $b = 2x_3y_3z_2$ and hence three cases to analyze.

If $x_3 = 0$, then $c = z_2z_3(z_2 - z_3)$ so that $c = 0$ implies that $z_2 = 0$, or $z_3 = 0$ or $z_2 = z_3$. If $x_3 = z_2 = 0$, then $a = y_2^2z_3$ so that $a = 0$ implies that i) or ii) hold. Similarly, if $x_3 = z_3 = 0$ then $a = y_3^2z_2$ so that $a = 0$ implies that i) or ii) hold. Finally, if $x_3 = 0$ and $z_2 = z_3$ then $a = z_2(y_2^2 - y_3^2)$ so that $a = 0$ implies that i) or iii) hold.

The case corresponding to $y_3 = 0$ can be treated exactly as the previous case $x_3 = 0$, exchanging x_2 with x_3 and y_2 with y_3 leaves b unchanged and permute a and c .

Finally, if $z_2 = 0$, then $c = 0$ (recall $x_2 = 0$) and $a = y_2^2z_3$. So $a = 0$ if either $y_2 = 0$ or $z_3 = 0$. But $y_2 = 0$ means that $p_1 = p_2$, i.e. i) holds, and $z_3 = 0$ means that ii) holds. \square

When a mixed set of points \mathcal{P} satisfies conditions i), ii) and iii) in Theorem 1, then a, b and c are not all zero and hence whether there are zero, one or two homogeneous real solutions to the equation $al^2 + blm + cm^2 = 0$ is decided by means of the discriminant $\Delta := b^2 - 4ac$ which depends on the coordinates of p_2 and p_3 . If $\Delta < 0$ then there is no real homogeneous solution, if $\Delta = 0$ then there exists a double homogeneous solution and if $\Delta > 0$ then there exists two distinct homogeneous solutions. As we have already observed in the proof of Theorem 1, it is possible to assume, without loss of generality, that $x_2 = 0$ in addition of $p_1 = (0, 0, 0)$ and $n_1 = (1, 0, 0)$. Then, a straightforward computation shows that

$$\Delta = 4z_2z_3 \left(y_2^2 (x_3^2 + z_3^2) + z_2^2 (x_3^2 + y_3^2) - z_2z_3 (x_3^2 + y_2^2 + y_3^2 + (z_2 - z_3)^2) \right).$$

From this equation we see directly that there are no real cylinders interpolating \mathcal{P} if p_2 and p_3 are not on the same side of the plane through p_1 and normal to n_1 (i.e. z_2 and z_3 have opposite signs). Another interesting case is to assume that p_2 belongs to the plane through p_1 and normal to n_1 (i.e. $z_2 = 0$). Indeed, in this case $\Delta = 0$ so there is a unique cylinder (counted with multiplicity two) through \mathcal{P} : its direction is given by $t = (-b, 2a, 0) = (2x_2y_2z_3, 2y_2^2z_3, 0)$ and r is still defined by (1) (z_3 is assumed to be nonzero for otherwise $a = b = c = 0$).

Finally, we notice that if $a = 0$ then the directions are given by the equation $blm + cm^2 = 0$. It follows that these directions are given by $(1, 0, 0)$ and $(-c, b, 0)$. Of course, if in addition $b = 0$ then there is a single interpolating cylinder that appears with multiplicity 2 (the discriminant vanishes). The radius r_1 or r_2 is then deduced from (1). Similarly, if $c = 0$ then the directions of the interpolating cylinders are given by $(0, 1, 0)$ and $(-b, a, 0)$.

Extraction algorithm. From the above analysis, we deduce the following extraction algorithm for cylinders from a mixed point set \mathcal{P} :

1. Perform a linear change of coordinates so that the oriented point is at the origin and its normal vector is $(0, 0, 1)$. If $z_2z_3 < 0$ then there is no solution.
2. Compute the quantities a, b and c . If $a = b = c = 0$ then stop because there are infinitely many interpolating cylinders.
3. If $a = 0$ or $c = 0$ then compute the interpolating cylinders as explained above and stop.
4. Otherwise, solve the equation $al^2 + blm + cm^2 = 0$ and keep only those roots that are real numbers (up to numerical precision). For each such root, compute the corresponding radius by means of (1) and return the interpolating cylinder.

This algorithm has been implemented with the **Maple** software and all timings are measured on a Mac laptop equipped with Intel Core i7 CPU @ 2.8GHz, 16 GB memory. We observed that computing the cylinders through a random set of points takes on average 3.5ms (including all the steps in the above algorithm) and is almost constant, i.e. independent of the point set. The proportion of the number of cylinders found through random point sets is given in Table 1 and an illustrative picture is given in Figure 1.

Number of cylinders	0	2
Proportion (%)	46.87	53.13

Table 1: Proportion of the number of cylinders found through a thousand random point sets.

2.2. Cylinders through five points

The problem of extracting a cylinder passing through five points has already been treated in the literature, see

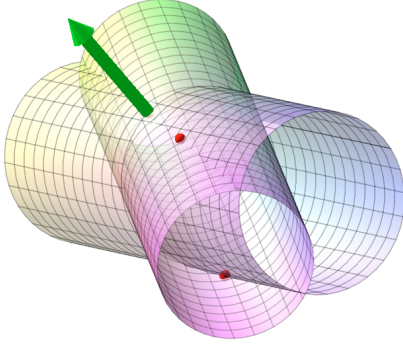


Figure 1: A general mixed set of points with the two interpolated cylinders.

[6, 3, 9] (see also [4, 1]). These works solve the problem by relying on a polynomial system solver. In this section, we briefly review the model already described in [6] and then we propose an improvement to gain efficiently. Our key ingredient removes some spurious solutions by means of additional algebraic manipulations.

2.2.1. Geometric analysis

Given a set \mathcal{P} of five points p_1, p_2, p_3, p_4, p_5 we seek cylinders through \mathcal{P} . First of all, by a linear change of coordinates, we assume without loss of generality that the coordinate system (x, y, z) is such that

$$p_1 = (0, 0, 0), \quad p_2 = (x_2, 0, 0), \quad p_3 = (x_3, y_3, 0), \quad (3)$$

$$p_4 = (x_4, y_4, z_4), \quad p_5 = (x_5, y_5, z_5).$$

Let us pick a nonzero vector $t = (l, m, n)$ (which will represent the direction of the axis) and denote by Π the plane through the origin which is orthogonal to t . Let us also denote by q_i the orthogonal projection of p_i onto Π , for all $i = 1, \dots, 5$. Then, p_1, \dots, p_5 belongs to a cylinder of direction t if and only if the coplanar points q_1, \dots, q_5 are cocyclic. In order to make explicit this cocyclic condition, we consider a new system of coordinates (x', y', z') whose third axis is parallel to t . More precisely, we choose the coordinates system (x', y', z') which is obtained from the coordinate system (x, y, z) by the change of coordinates defined by the orthogonal matrix

$$M = \begin{pmatrix} \frac{\sqrt{m^2+n^2}}{\|t\|} & -\frac{lm}{\|t\|\sqrt{m^2+n^2}} & -\frac{ln}{\|t\|\sqrt{m^2+n^2}} \\ 0 & \frac{n}{\sqrt{m^2+n^2}} & -\frac{m}{\sqrt{m^2+n^2}} \\ \frac{l}{\|t\|} & \frac{m}{\|t\|} & \frac{n}{\|t\|} \end{pmatrix}.$$

Therefore, the coordinates of q_i in the system (x', y', z') are given by

$$\begin{pmatrix} x_i \frac{\sqrt{m^2+n^2}}{\|t\|} - \frac{lm}{\|t\|\sqrt{m^2+n^2}} y_i - \frac{ln}{\|t\|\sqrt{m^2+n^2}} z_i, \\ \frac{n}{\sqrt{m^2+n^2}} y_i - \frac{m}{\sqrt{m^2+n^2}} z_i, 0 \end{pmatrix} =: (x'_i, y'_i, 0).$$

Now the points q_1, q_2, q_3 and q_4 are cocyclic (or aligned, which corresponds to a circle of infinite radius) providing

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x'_1 & x'_2 & x'_3 & x'_4 \\ y'_1 & y'_2 & y'_3 & y'_4 \\ x'^2_1 + y'^2_1 & x'^2_2 + y'^2_2 & x'^2_3 + y'^2_3 & x'^2_4 + y'^2_4 \end{vmatrix} = 0.$$

Since $x'_1 = y'_1 = 0$, $x'_2 = \frac{\sqrt{m^2+n^2}}{\|t\|} x_2$, $y'_2 = 0$ and

$$x'^2_i + y'^2_i = \|q_i\|^2 = \|p_i\|^2 - \frac{(t \cdot p_i)^2}{\|t\|^2}$$

for $i = 3, 4$, we deduce that the points p_1, p_2, p_3, p_4 all belong to a cylinder of direction t if and only if

$$C_{p_1, p_2, p_3, p_4}(l, m, n) = 0$$

where, after some calculations,

$$C_{p_1, p_2, p_3, p_4}(l, m, n) := x_2^2(m^2 + n^2) \begin{vmatrix} l & x_3 & x_4 \\ m & y_3 & y_4 \\ n & 0 & z_4 \end{vmatrix} - x_2 \begin{vmatrix} m & y_3 & y_4 \\ n & 0 & z_4 \\ 0 & \|t\|^2 \|p_3\|^2 - (t \cdot p_3)^2 & \|t\|^2 \|p_4\|^2 - (t \cdot p_4)^2 \end{vmatrix}.$$

This condition is given by a homogeneous polynomial of degree 3 in l, m, n . Observe that this polynomial is satisfied for the six particular directions corresponding to the lines $(p_i p_j)$, $1 \leq i < j \leq 4$. To determine the cylinders through the five points p_1, \dots, p_5 , we consider the polynomial system of equations

$$C_{p_1, p_2, p_3, p_4}(l, m, n) = C_{p_1, p_2, p_3, p_5}(l, m, n) = 0 \quad (4)$$

Equation 4 corresponds geometrically to the intersection of two cubic curves, which is composed of nine points whenever this intersection is finite. As already noted, the directions corresponding to the lines $(p_1 p_2)$, $(p_1 p_3)$ and $(p_2 p_3)$ are roots of these two equations, but these roots do not imply the existence of a cylinder interpolating \mathcal{P} , they have to be removed once (if there is indeed a cylinder in this direction, then it will appear as a multiple solution). Moreover, when a direction of a cylinder through \mathcal{P} is found, then it is completely determined since its radius and a point on its axis are given by the center and the radius of the circle through the points q_1, q_2, q_3, q_4, q_5 . Therefore, we have the following result.

Theorem 2 ([6]). *If the algebraic system of equations (4) has a finite number of solutions, then there may be zero, two, four or six real cylinders through \mathcal{P} .*

In [6, 3, 9], various polynomial system solvers are used in order to extract the directions of the cylinders through \mathcal{P} . We will describe a new approach where we first simplify this algebraic system and then rely on eigen-computations.

To begin, we observe that the three extraneous directions (p_1p_2) , (p_1p_3) and (p_2p_3) which we write as

$$(x_2, 0, 0), (x_3, y_3, 0), (x_3 - x_2, y_3, 0),$$

are the common roots of both algebraic equations

$$F := nx_2 = 0, \quad G := (mx_3 - mx_2 - ly_3)(mx_3 - ly_3)mx_2 = 0.$$

Then a result in algebraic geometry says that the two polynomials C_{p_1, p_2, p_3, p_j} , $j = 4, 5$, belong to the ideal generated by F and G ; and suggests explicitly computing these membership relations. Computing the Euclidean pseudo-division of C_{p_1, p_2, p_3, p_4} and C_{p_1, p_2, p_3, p_5} by the polynomial x_2n , we get two homogeneous polynomials of degree 2,

$$\begin{aligned} D_j(l, m, n) = & (-y_j^2 y_3^2 + y_j^2 y_3 + z_j^2 y_3) l^2 \\ & + ((2y_j y_3 x_3 - 2y_j x_j y_3) m + z_j (x_2 y_3 - 2x_j y_3) n) l + \\ & (x_2 y_j x_3 - x_2 x_j y_3 - y_j x_3^2 + x_j^2 y_3 + z_j^2 y_3) m^2 \\ & + z_j (-x_2 x_3 + x_3^2 + y_3^2 - 2y_j y_3) nm + \\ & (x_2 y_j x_3 - x_2 x_j y_3 - y_j x_3^2 + x_j^2 y_3 - y_j y_3^2 + y_j^2 y_3) n^2 \end{aligned}$$

where $j = 4, 5$, such that

$$\begin{pmatrix} D_4 & z_4 \\ D_5 & z_5 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} C_{p_1, p_2, p_3, p_4} \\ C_{p_1, p_2, p_3, p_5} \end{pmatrix}.$$

Proposition 3. *If $z_4 \neq 0$ (resp. $z_5 \neq 0$), then the directions of the cylinders interpolating \mathcal{P} correspond to all the common roots of both equations $C_{p_1, p_2, p_3, p_4}(l, m, n) = 0$ and $\Delta(l, m, n) := z_5 D_4(l, m, n) - z_4 D_5(l, m, n) = 0$ (resp. $C_{p_1, p_2, p_3, p_5}(l, m, n) = 0$ and $\Delta(l, m, n) = 0$).*

Proof. If $z_4 \neq 0$ then the system (4) is obviously equivalent to $C_{p_1, p_2, p_3, p_4} = 0$ and

$$z_4 C_{p_1, p_2, p_3, p_5}(l, m, n) - z_5 C_{p_1, p_2, p_3, p_4}(l, m, n) = nx_2(z_4 D_5 - z_5 D_4) = 0.$$

The solutions corresponding to $n = 0$ satisfy $n = G = 0$ so that they define exactly the three extraneous directions (p_1p_2) , (p_1p_3) and (p_2p_3) . Therefore, the remaining polynomial system

$$C_{p_1, p_2, p_3, p_4}(l, m, n) = 0, \quad \Delta(l, m, n) = 0 \quad (5)$$

gives exactly the six solutions of interest. The case $z_5 \neq 0$ is treated similarly. \square

Notice that the case where $z_4 = z_5 = 0$ corresponds geometrically to five coplanar points. If these points are also aligned, there are infinitely many cylinders. If they belong to an ellipse, there are two non-degenerate cylinders through these points (they are symmetric with respect to the plane $z = 0$); if this ellipse is a circle these two solutions coincide. Otherwise there are no cylinders through these points. In all cases, these points belong to a degenerate “flat” cylinder (infinite radius).

2.2.2. Solving via eigencomputations

Now we describe a method that allows us to compute efficiently the roots of the system of algebraic equations given in Proposition 3 as the eigenvalues of a pencil of matrices given in closed form. This approach is based on known techniques (see e.g. [12], [5, §1]) that allow us to recover the solutions in a single computation step (similar to [6, Appendix] or [3]) with good control on numerical stability and accuracy [7]. Hereafter, we provide, for the convenience of the reader, a short review, adapted to our context, of this matrix-based solution method.

First, in order to treat separately the case $n = 0$, observe that the common roots of $C_{p_1, p_2, p_3, p_5} = 0$ and $\Delta = 0$ such that $n = 0$ are easily computed since these roots are among the three directions (p_1p_2) , (p_1p_3) and (p_2p_3) (i.e. the roots of $n = G = 0$) and the decision is given by the evaluation of the equation $\Delta(l, m, n)$ at these directions. For instance, the direction $(1, 0, 0)$ is a solution if and only if

$$\begin{aligned} \Delta(1, 0, 0) = & z_5 (-y_4 y_3^2 + y_4^2 y_3 + z_4^2 y_3) \\ & - z_4 (-y_5 y_3^2 + y_5^2 y_3 + z_5^2 y_3) = 0. \end{aligned}$$

Therefore, from now on we set $n = 1$.

We consider the Sylvester Matrix S of the polynomials $C_{p_1, p_2, p_3, p_5}(l, m, 1)$ and $\Delta(l, m, 1)$ seen as univariate polynomials in the variable m . This matrix is a polynomial matrix of degree 2 in l . More precisely, this matrix is of the form $M_2 l^2 + M_1 l + M_0$ where each M_i is a 5×5 -matrix whose coefficients are given in closed forms in terms of \mathcal{P} . For instance,

$$M_2 := \begin{pmatrix} 0 & 0 & x_2 y_3^2 z_4 & b & 0 \\ 0 & 0 & 0 & x_2 y_3^2 z_4 & b \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & a \end{pmatrix}$$

where

$$\begin{aligned} a := & -y_3^2 y_4 z_5 + y_3^2 z_4 y_5 + y_3 y_4^2 z_5 - y_3 z_4 y_5^2 \\ & + y_3 z_4^2 z_5 - y_3 z_4 z_5^2 \end{aligned}$$

and

$$b := -x_2 y_3^2 y_4 + x_2 y_3 y_4^2 + x_2 y_3 z_4^2.$$

Now, following [3], we linearize this polynomial matrix by considering its companion matrices that are defined by

$$A = \begin{pmatrix} 0 & I \\ M_0^t & M_1^t \end{pmatrix}, \quad B = \begin{pmatrix} I & 0 \\ 0 & -M_2^t \end{pmatrix}$$

where I stands for the 5×5 -identity matrix and M_i^t stands for the transpose of the matrix M_i . These matrices are of

size 10×10 and their main feature and importance is the fact that for all $\lambda \in \mathbb{C}$ and all vector $v \in \mathbb{C}^m$ we have

$$S^t(\lambda)v = 0 \Leftrightarrow (A - \lambda B) \begin{pmatrix} v \\ \lambda v \end{pmatrix} = 0.$$

In other words, the solutions to the system

$$C_{p_1, p_2, p_3, p_4}(l, m, 1) = 0, \quad \Delta(l, m, 1) = 0$$

can be computed from the eigenvalues and eigenvectors of the pencil A, B which is given in closed form in terms of the input data \mathcal{P} . We refer the reader to [3] for more details on these computations.

2.2.3. Extraction algorithm

From the above analysis, we deduce the following extraction algorithm for cylinders through a point set \mathcal{P} :

1. Perform a linear change of coordinates so that the five points are of the form (3).
2. If $z_4 = z_5 = 0$ the points are coplanar and the algorithm stops here.
3. Assume $z_4 \neq 0$. Then instantiate the pre-computed matrices M_0, M_1, M_2 , and hence A, B , with the coordinates of the points $p_i, i = 1, \dots, 5$.
4. Compute the six finite eigenvalues of the pencil A, B (see [3] for details), and sort them in order to keep only those that are real numbers (up to a given precision).
5. For each real eigenvalue obtained in Step 4, compute the remaining coordinate of the direction by means of the associated eigenvectors (see [3] for details); keep those directions that are given by real numbers.
6. For each real direction of a cylinder through \mathcal{P} , compute q_1, q_2, q_3 then the radius and the center of their circumcircle.

This algorithm has been implemented with the **Maple** software. We observed that the computation of the cylinders through a random set of points takes on average 15ms (including all the steps in the above algorithm) and is almost constant, i.e. independent of the point set. The proportion of the number of cylinders found through random point sets is given in Table 2; we notice that we recover essentially those that appear in [9, §4.1]. Some illustrative configurations are presented in Figure 2.

Number of cylinders	0	2	4	6
Proportion (%)	22.7	53.9	21.4	2

Table 2: Proportion of the number of cylinders found through a thousand random point sets.

3. Interpolation of cones

A cone (more precisely a right circular cone) is a three-dimensional geometric shape that consists of the locus of

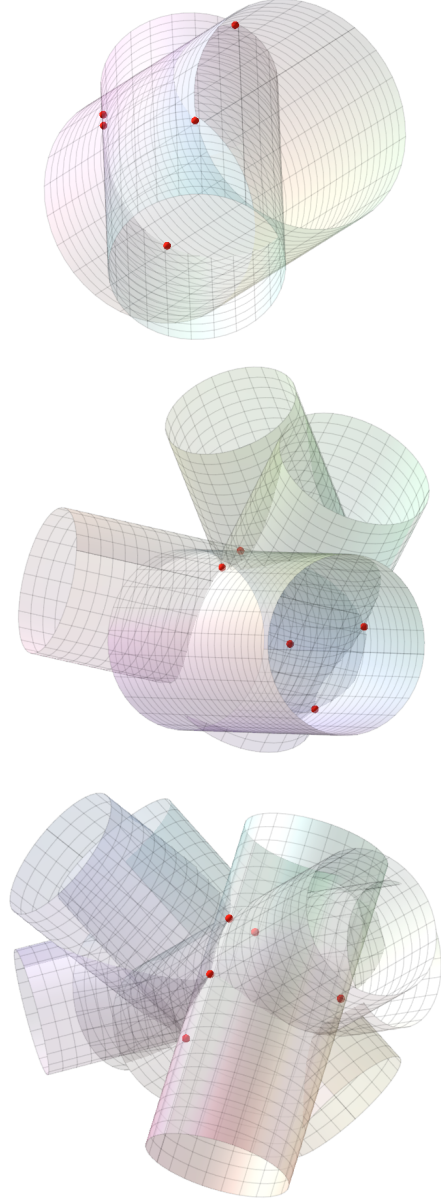


Figure 2: Three configurations of \mathcal{P} with 2, 4 and 6 cylinders passing through \mathcal{P} .

all straight lines (the generatrices) joining a fixed point, called the *apex* to the points of a circle, the apex lying on the line, called the *axis*, passing through the center of the circle and normal to the plane containing this circle. A cone is determined by six parameters : four parameters for the axis, an additional parameter for the apex, which is a point on the axis and a last parameter for the angle of the cone, that is to say the angle made at the apex between the axis and any of the straight lines generating the cone.

This approach is classical to determine a cone from three oriented points. Indeed, each normal plane to each oriented point must contains the apex, so the apex can be determined as the intersection of three such planes. Nevertheless, this approach is overdetermined in the sense that

it will not yield a cone in general because three general oriented points give nine conditions whereas a cone has only six parameters. Hereafter, we provide methods for interpolating a cone through a mixed point set that define exactly six conditions. There are three possibilities for such mixed point sets : either two oriented points, or one oriented point and three other distinct points, or six distinct points.

3.1. Cones through two oriented points

We seek for cones through a set \mathcal{P} of two oriented points that we will denote by p_1, n_1 and p_2, n_2 . By a linear change of coordinates, one can assume without loss of generality that $p_1 := (0, 0, 0)$ and $n_1 := (0, 0, 1)$. For the remaining data, we set $p_2 = (x_2, y_2, z_2)$ and $n_2 = (a_2, b_2, c_2)$.

Assuming that n_2 is not proportional to n_1 , i.e. $a_2 \neq 0$ or $b_2 \neq 0$, the normal plane to n_1 through p_1 and the normal plane Π_2 to n_2 through p_2 intersect along a line \mathcal{L} . A key remark is that the apex of a cone through \mathcal{P} must be on \mathcal{L} . This line is defined by the equations

$$\mathcal{L} : \begin{cases} z = 0 \\ a_2x + b_2y + c_2z - p_2 \cdot n_2 = 0. \end{cases}$$

If n_2 is proportional to n_1 , then we get a degenerate situation. Indeed, either $z_2 \neq 0$ and there is no cone through \mathcal{P} , or $z_2 = 0$ and there are infinitely many cones through \mathcal{P} (the line through p_1 and p_2 define a generatrix of the cone and then the apex can be freely chosen on this line, as well as the angle of the cone). So from now on we assume that n_2 is not proportional to n_1 , i.e. $a_2^2 + b_2^2 > 0$.

The axis of a cone through \mathcal{P} must intersect the line \mathcal{L}_1 through p_1 with direction n_1 and also intersect the line \mathcal{L}_2 through p_2 with direction n_2 . So to characterize this axis, we pick a point on each of these lines, say $q_1 = \lambda_1 n_1 \in \mathcal{L}_1$ and $q_2 = p_2 + \lambda_2 n_2 \in \mathcal{L}_2$ where $\lambda_1, \lambda_2 \in \mathbb{R}$, and we consider the line \mathcal{A} through q_1 and q_2 . The only special case when \mathcal{A} is not well defined is when \mathcal{L}_1 and \mathcal{L}_2 intersect and both q_1 and q_2 are located at this intersection point. For the sake of clarity, we will treat this case separately and we assume for the moment that \mathcal{L}_1 and \mathcal{L}_2 do not intersect.

The line \mathcal{A} will be the axis of a cone through \mathcal{P} providing the two following conditions hold:

- \mathcal{A} intersects \mathcal{L} , in which case the intersection point is the apex of the cone,
- the angle between n_1 and \mathcal{A} must be equal to the angle between n_2 and \mathcal{A} .

\mathcal{A} can be parameterized as the set of points $(1 - \lambda)q_1 + \lambda q_2$ for $\lambda \in \mathbb{R}$. It is necessary to have $z_2 + \lambda_2 c_2 - \lambda_1 \neq 0$ for otherwise \mathcal{A} is parallel to the plane $z = 0$ and hence it does not intersect \mathcal{L} , unless $\lambda_1 = z_2 + \lambda_2 c_2 = 0$ but in this case the apex would be p_1 which is impossible (normal vector is not well defined at the apex). So, assuming $z_2 + \lambda_2 c_2 - \lambda_1 \neq$

0, the intersection point between \mathcal{A} and the plane $z = 0$ is the point with coordinates

$$\left(-\lambda_1 \frac{x_2 + \lambda_2 a_2}{z_2 + \lambda_2 c_2 - \lambda_1}, -\lambda_1 \frac{y_2 + \lambda_2 b_2}{z_2 + \lambda_2 c_2 - \lambda_1}, 0 \right). \quad (6)$$

Substituting this point into \mathcal{L} , we get the following first constraint on λ_1 and λ_2 :

$$\lambda_1 \lambda_2 (a_2^2 + b_2^2) - \lambda_1 z_2 c_2 + (p_2 \cdot n_2)(z_2 + \lambda_2 c_2) = 0. \quad (7)$$

Now denote by q the vector $q_2 - q_1$; the second condition is obtained by imposing $q \cdot n_1 = \pm q \cdot n_2 / \|n_2\|$, the sign change arises because the normals are supposed to be oriented with respect to the interpolating cone. More explicitly, we obtain the equation

$$\rho(z_2 + \lambda_2 c_2 - \lambda_1) = \varepsilon(-c_2 \lambda_1 + \rho^2 \lambda_2 + p_2 \cdot n_2), \quad (8)$$

where $\rho = \|n_2\| = \sqrt{a_2^2 + b_2^2 + c_2^2}$ and $\varepsilon = \pm 1$. This latter condition being linear in λ_1 and λ_2 , one can solve the system of equations (7) and (8) in closed form. More precisely, once ε is fixed, we get the following two solutions :

$$\begin{cases} \lambda'_1 = \frac{a_2^2 z_2 - a_2 c_2 x_2 + b_2^2 z_2 - b_2 c_2 y_2}{a_2^2 + b_2^2} \\ \lambda'_2 = -\frac{a_2 x_2 + b_2 y_2}{a_2^2 + b_2^2} \end{cases}, \quad \begin{cases} \lambda_1 = \frac{p_2 \cdot n_2}{c_2 + \varepsilon \rho} \\ \lambda_2 = -\frac{z_2}{c_2 + \varepsilon \rho} \end{cases}.$$

Observe that $|c_2| \neq \rho$ since it is assumed that n_2 is not parallel to n_1 . Another important observation is that the solution (λ'_1, λ'_2) is independent of ε and moreover satisfies $z_2 + \lambda'_2 c_2 - \lambda'_1 = 0$. Therefore, (λ'_1, λ'_2) is not a valid solution. So we are left with two solutions to our geometric interpolation problem, namely

$$\begin{cases} \lambda_1 = \frac{p_2 \cdot n_2}{c_2 - \rho} \\ \lambda_2 = -\frac{z_2}{c_2 - \rho} \end{cases}, \quad \begin{cases} \lambda_1 = \frac{p_2 \cdot n_2}{c_2 + \rho} \\ \lambda_2 = -\frac{z_2}{c_2 + \rho} \end{cases}. \quad (9)$$

Once such a solution is chosen, then one can determine a unique cone through \mathcal{P} : its apex is given by (6), its direction is given by $q = q_2 - q_1$ and its angle is given by the angle between the vectors $p_1 - \text{apex}$ and q .

It remains to treat the case where the lines \mathcal{L}_1 and \mathcal{L}_2 intersect. Let us denote by Π the plane that contains these two lines, by \mathcal{N}_1 the normal line in Π to \mathcal{L}_1 through p_1 , by \mathcal{N}_2 the normal line in Π to \mathcal{L}_2 through p_2 and by ω the intersection point between \mathcal{N}_1 and \mathcal{N}_2 . We notice that ω is nothing but the intersection point between Π and \mathcal{L} ; see Figure 3 for a geometric illustration.

First, we observe that there are always two cones through \mathcal{P} . These two cones have the same apex ω and their axes are the perpendicular bisectors of the lines \mathcal{N}_1 and \mathcal{N}_2 through ω . Therefore, these two cones are symmetric with respect to the plane Π and their intersection is composed of the two lines $\mathcal{N}_1 \cup \mathcal{N}_2$. There are no other cones through

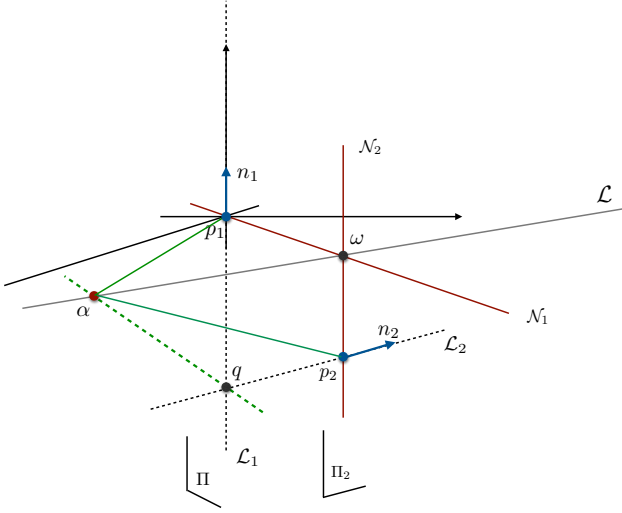


Figure 3: Geometric construction in the case where the lines \mathcal{L}_1 and \mathcal{L}_2 intersect.

\mathcal{P} whose axis is contained in Π (equivalently whose apex is ω).

Next we look for cones through \mathcal{P} whose apex α is such that $\alpha \in \mathcal{L} \setminus \{\omega\}$. Since the axis of such a cone must intersect \mathcal{L}_1 and \mathcal{L}_2 and go through α which is not in Π , then the axis must be the line through α and $q := \mathcal{L}_1 \cap \mathcal{L}_2 \in \Pi$ (recall that it is assumed that n_2 and n_1 are not proportional). The two triangles αqp_1 and αqp_2 are right angled at p_1 , respectively at p_2 , since α belongs to $\mathcal{L} = \Pi_2 \cap \{z = 0\}$. Therefore, a necessary and sufficient condition for the existence of a cone through \mathcal{P} , and whose axis is the line through α and q , is that the angles $(\alpha q, \alpha p_1)$ and $(\alpha q, \alpha p_2)$ are the same. This condition is equivalent to the equality $\|qp_1\| = \|qp_2\|$ since the two right triangles αqp_1 and αqp_2 share the edge αq .

In summary, when the lines \mathcal{L}_1 and \mathcal{L}_2 intersect there always exists two cones through \mathcal{P} , whose axis are contained in Π . If $\|qp_1\| \neq \|qp_2\|$ then there are no more cones through \mathcal{P} , otherwise there are infinitely many cones through \mathcal{P} , more precisely a 1-dimensional family of cones which is parameterized by \mathcal{L} . A last observation is that if $\|qp_1\| = \|qp_2\|$ then one of the two perpendicular bisectors of \mathcal{N}_1 and \mathcal{N}_2 through ω goes through q , so that there is a continuity in this family of cones through \mathcal{P} when α is moving on the line \mathcal{L} .

Theorem 4. *Given an oriented set of two distinct points \mathcal{P} , denote by \mathcal{L} the line through p_1 with direction n_1 and by \mathcal{L}_2 the line through p_2 with direction n_2 . If the following two conditions hold :*

- i) n_1 and n_2 are not proportional,
- ii) if the lines \mathcal{L}_1 and \mathcal{L}_2 intersect at a point q then $\|qp_1\| \neq \|qp_2\|$,

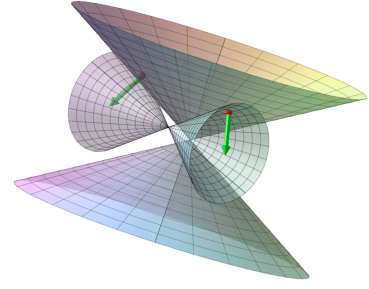


Figure 4: Two cones through a general set of two oriented points.

then there are exactly two real cones through \mathcal{P} . Moreover, the intersection of these two cones consists only of the points p_1 and p_2 , except if they share the same apex in which case their intersection consists of the lines \mathcal{N}_1 and \mathcal{N}_2 .

If n_1 is parallel to n_2 then there is no cone through \mathcal{P} unless p_2 belongs to the line \mathcal{N}_1 , in which case there are infinitely many cones through \mathcal{P} .

If the lines \mathcal{L}_1 and \mathcal{L}_2 intersect at a point q such that $\|qp_1\| = \|qp_2\|$, then there are infinitely many cones through \mathcal{P} .

Extraction algorithm. From the above analysis, we deduce the following extraction algorithm for cones through a set of two oriented points \mathcal{P} .

1. Perform a linear change of coordinates so that $p_1 = (0, 0, 0)$ and $n_1 = (0, 0, 1)$.
2. If n_1 and n_2 are proportional then check whether p_2 belongs to \mathcal{N}_1 and return that either there is no cone through \mathcal{P} , or there are infinitely many cones through \mathcal{P} .
3. If \mathcal{L}_1 intersects \mathcal{L}_2 , then if $\|qp_1\| \neq \|qp_2\|$ return the two cones through \mathcal{P} , otherwise if $\|qp_1\| = \|qp_2\|$ return that there are infinitely many cones through \mathcal{P} .
4. Now, since n_1 is not proportional to n_2 and $\mathcal{L}_1 \cup \mathcal{L}_2 = \emptyset$, there are exactly two cones through \mathcal{P} that are computed by means of the closed formulas (9).

This algorithm has been implemented with the **Maple** software. We observed that computing the cones through a random set of points takes in average 3.8ms (including all the steps in the above algorithm) and is almost constant, i.e. independent of the point set. Some illustrative configurations are shown in Figure 4 and Figure 5.

3.2. Cones through a mixed minimal point set

We seek cones through a set \mathcal{P} that is composed of an oriented point (p_1, n_1) and three other simple points p_2, p_3, p_4 . By a linear change of coordinates, one can assume without loss of generality that $p_1 := (0, 0, 0)$ and $n_1 := (0, 0, 1)$. For the remaining data, we set $p_2 = (x_2, y_2, z_2)$, $p_3 = (x_3, y_3, z_3)$ and $p_4 = (x_4, y_4, z_4)$.

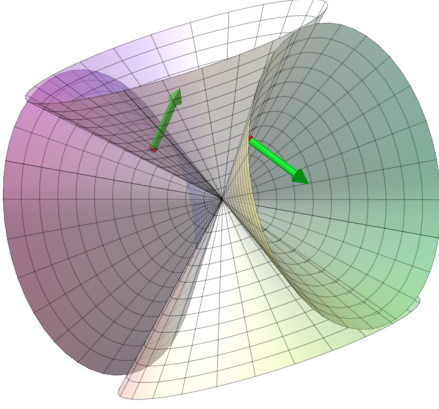


Figure 5: Two cones intersecting along two lines and sharing the same apex.

The axis of a cone through \mathcal{P} must intersect the line generated by n_1 through p_1 in a point $q = (0, 0, r)$. Observe that r must be nonzero otherwise p_1 would be the apex of such a cone, which is impossible since it is a non-singular point. In addition, the apex of a cone through \mathcal{P} must belong to the normal plane to n_1 through the point p_1 and we choose to represent the coordinates of this point by $p = (ar, br, 0)$. Again, observe for the same reason that a and b must be nonzero, i.e. $a^2 + b^2 \neq 0$.

Now a point $m = (x, y, z)$ belonging to a cone through \mathcal{P} must satisfy

$$(p\vec{m} \cdot p\vec{q})^2 = \|pm\|^2 \|pq\|^2 \kappa,$$

that is to say

$$r^2 (-a(x - ar) - b(y - br) - z)^2 = r^2 ((x - ar)^2 + (y - br)^2 + z^2) (a^2 + b^2 + 1) \kappa$$

where $\kappa > 0$ is a constant that corresponds to the angle of the cone. The factor r^2 can be cancelled out. Moreover, the cone goes through p_1 so we deduce that

$$r^2 (a^2 + b^2)^2 = r^2 (a^2 + b^2) (a^2 + b^2 + 1) \kappa.$$

Substituting into the previous equation, we finally deduce that the point $m = (x, y, z)$ belonging to a cone through \mathcal{P} must satisfy $F(x, y, z) = 0$ where

$$F(x, y, z) = 2(a^2 + b^2) zr + (a^2 + b^2 - 1) z^2 - 2(ax + by)z + (bx - ay)^2.$$

The three parameters a, b and r are to be determined and we will use the three points p_2, p_3 and p_4 for that purpose.

Assume that one of the points p_2, p_3, p_4 , say p_2 , is not in the plane $z = 0$ (the case where all the points are in this plane is a special case of the situation discussed at the end of Section 3.3). Hence $z_2 \neq 0$. Then there exists a cone through \mathcal{P} if and only if

$$F(x_2, y_2, z_2) = 0, \quad z_3 F(x_2, y_2, z_2) - z_2 F(x_3, y_3, z_3) = 0,$$

$$z_4 F(x_2, y_2, z_2) - z_2 F(x_4, y_4, z_4) = 0.$$

It turns out that the two last equations are independent of r ; for $i = 3, 4$ we have

$$\begin{aligned} z_i F(x_2, y_2, z_2) - z_2 F(x_i, y_i, z_i) = & (y_2^2 z_i - y_i^2 z_2 + z_2^2 z_i - z_2 z_i^2) a^2 - (2x_2 y_2 z_i - 2x_i y_i z_2) ab \\ & - (2x_2 z_2 z_i - 2x_i z_2 z_i) a + (x_2^2 z_i - x_i^2 z_2 + z_2^2 z_i - z_2 z_i^2) b^2 \\ & - (2y_2 z_2 z_i - 2y_i z_2 z_i) b - z_2^2 z_i + z_2 z_i^2. \end{aligned}$$

Since these two equations are quadratic in a and b , they have four common roots. Once a and b are determined then r is uniquely defined by the equation $F(x_2, y_2, z_2) = 0$ which is linear in r . So, we have proved the following result.

Theorem 5. *Given a general mixed point set \mathcal{P} , there are 0, 2 or 4 cones through \mathcal{P} .*

Extraction algorithm. From the above analysis, the extraction algorithm for cones through a mixed point set \mathcal{P} composed of an oriented point and three other distinct points relies on solving two algebraic equations in two variables. To solve such systems, we proceed as in Section 2.2.2 by means of eigen-computations from a pencil of matrices which is in closed form in terms of the input data. This pencil A, B of 8×8 matrices is obtained as the companion matrices of both equations $z_i F(x_2, y_2, z_2) - z_2 F(x_i, y_i, z_i) = 0$, $i = 3, 4$.

1. Perform a linear change of coordinates so that $p_1 = (0, 0, 0)$ and $n_1 = (0, 0, 1)$.
2. Instantiate the pencil of matrices A, B with the coordinates of the input points.
3. Compute the eigenvalues and eigenvectors and deduce the real cones through \mathcal{P} .

This algorithm has been implemented with the **Maple** software. We observed that computing the cones through a random set of points takes in average 7.5ms (including all the steps in the above algorithm) and is almost constant, i.e. independent of the point set. The proportion of the number of cones found through random point sets is given in Table 3 and some illustrative configurations are shown in Figure 6.

Number of cones	0	2	4
Proportion (%)	6.9	85.9	7.2

Table 3: Proportion of the number of cones found through a thousand random point sets.

3.3. Cones through six points

We seek cones through a set \mathcal{P} composed of six simple points $p_1, p_2, p_3, p_4, p_5, p_6$. By a linear change of coordinates, we can assume without loss of generality that

$$\begin{aligned} p_1 &= (0, 0, 0), \quad p_2 = (x_2, 0, 0), \quad p_3 = (x_3, y_3, 0), \\ p_4 &= (x_4, y_4, z_4), \quad p_5 = (x_5, y_5, z_5), \quad p_6 = (x_6, y_6, z_6). \end{aligned} \quad (10)$$

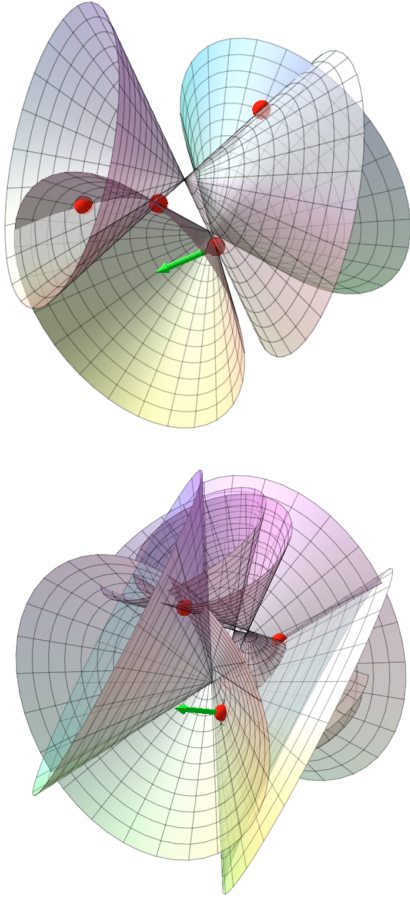


Figure 6: Two configurations of \mathcal{P} with 2 and 4 cones passing through \mathcal{P} .

Geometric analysis. The general equation of a cone depends on 6 parameters:

$$F(x, y, z) := (x-a)^2 + (y-b)^2 + (z-c)^2 - (l(x-a) + m(y-b) + n(z-c))^2 \quad (11)$$

where (a, b, c) are the coordinates of the apex and the vector (l, m, n) encodes the direction of the axis as well as the angle of the cone ($l^2 + m^2 + n^2 = 1/\cos^2(\theta)$, θ being the angle of the cone).

We introduce the quantity $k := al + bm + cn$. Since p_1 is the origin, we get the equation

$$F(p_1) = a^2 + b^2 + c^2 - k^2 = 0. \quad (12)$$

Moreover, we observe that the polynomial

$$F(x, y, z) - F(p_1) = 2(lx + my + nz)k - 2(ax + by + cz) - 2lmxy - 2lnxz - 2mnyz - l^2x^2 - m^2y^2 - n^2z^2 + x^2 + y^2 + z^2 \quad (13)$$

is linear in a, b, c and k . From the equation

$$F(p_2) - F(p_1) = 2lx_2k - 2x_2a - l^2x_2^2 + x_2^2 = 0$$

we get, assuming $x_2 \neq 0$, i.e. $p_2 \neq p_1$,

$$a = lk + \frac{x_2}{2}(1 - l^2). \quad (14)$$

Now we have that

$$F(p_3) - F(p_1) = 2my_3k - 2y_3b + x_3l^2x_2 - l^2x_3^2 - 2lmx_3y_3 - m^2y_3^2 - x_3x_2 + x_3^2 + y_3^2 = 0$$

and we get, assuming $y_3 \neq 0$, i.e. assuming that p_1, p_2 and p_3 are not aligned,

$$b = mk + \frac{1}{2y_3} (x_3l^2x_2 - l^2x_3^2 - 2lmx_3y_3 - m^2y_3^2 - x_3x_2 + x_3^2 + y_3^2). \quad (15)$$

Continuing this way, we get the equation

$$\begin{aligned} F(p_4) - F(p_1) = & 2z_4nk - 2z_4c \\ & + \frac{1}{y_3} (-l^2x_2x_3y_4 + l^2x_2x_4y_3 + l^2x_3^2y_4 - l^2x_4^2y_3 \\ & + 2lmx_3y_3y_4 - 2lmx_4y_4y_3 - 2lnx_4z_4y_3 + m^2y_3^2y_4 \\ & - m^2y_4^2y_3 - 2mny_4z_4y_3 - n^2z_4^2y_3 + x_2x_3y_4 - x_2x_4y_3 \\ & - x_3^2y_4 + x_4^2y_3 - y_3^2y_4 + y_4^2y_3 + z_4^2y_3) = 0 \end{aligned}$$

and we get, assuming that $z_4 \neq 0$, i.e. assuming that p_1, p_2, p_3 and p_4 are not coplanar,

$$\begin{aligned} c = & nk + \frac{1}{2y_3z_4} (-l^2x_2x_3y_4 + l^2x_2x_4y_3 + l^2x_3^2y_4 - l^2x_4^2y_3 \\ & + 2lmx_3y_3y_4 - 2lmx_4y_4y_3 - 2lnx_4z_4y_3 + m^2y_3^2y_4 - m^2y_4^2y_3 \\ & - 2mny_4z_4y_3 - n^2z_4^2y_3 + x_2x_3y_4 - x_2x_4y_3 - x_3^2y_4 + x_4^2y_3 \\ & - y_3^2y_4 + y_4^2y_3 + z_4^2y_3). \quad (16) \end{aligned}$$

From here, we substitute a, b and c by the above quantity that we found in both equations $F(p_i) - F(p_1) = 0$ with $i = 5, 6$. Notice that these two equations do not depend on k because of the cancellation of the two terms $2(lx + my + nz)k - 2(ax + by + cz)$ in (13) by this substitution. We get two equations $H_i(l, m, n)$, $i = 5, 6$, that satisfy

$$\begin{aligned} y_3z_4H_i(l, m, n) = & (x_2x_3y_4z_i - x_2x_3y_iz_4 - x_2x_4y_3z_i + x_2x_iz_3z_4 \\ & - x_3^2y_4z_i + x_3^2y_iz_4 + x_4^2y_3z_i - x_i^2y_3z_4) l^2 \\ & + (-2x_3y_3y_4z_i + 2x_3y_3y_iz_4 + 2x_4y_3y_4z_i - 2x_iz_3y_3z_4) lm \\ & + (2x_4y_3z_4z_i - 2x_iz_3z_4z_i) ln \\ & + (-y_3^2y_4z_i + y_3^2y_iz_4 + y_3y_4^2z_i - y_i^2y_3z_4) m^2 \\ & + (2y_3y_4z_4z_i - 2y_3y_iz_4z_i) mn + (y_3z_4^2z_i - z_i^2y_3z_4) n^2 \\ & - x_2x_3y_4z_i + x_2x_3y_iz_4 + x_2x_4y_3z_i - x_2x_iz_3z_4 \\ & + x_3^2y_4z_i - x_3^2y_iz_4 - x_4^2y_3z_i + x_i^2y_3z_4 + y_3^2y_4z_i \\ & - y_3^2y_iz_4 - y_3y_4^2z_i + y_i^2y_3z_4 - y_3z_4^2z_i + z_i^2y_3z_4 = 0. \end{aligned}$$

These equations H_5 and H_6 are polynomials of degree 2 in l, m, n and only the constant term and monomials of degree 2 have nonzero coefficients. The equation H_5 depends on the coefficients of p_2, p_3, p_4 and p_5 whereas H_6 depends on the coefficients of p_2, p_3, p_4 and p_6 . Applying the same substitutions and replacing k by its defining value in (12), we get another equation $H_0(l, m, n)$ which is of degree 6 in l, m, n and such that only monomials of even degree have nonzero coefficients. Observe that H_0 depends only on the coordinates of p_2, p_3 and p_4 . To give an idea, we have

$$\begin{aligned} 4y_3^2 z_4^2 (l^2 + m^2 + n^2 - 1) H_0 = \\ (-x_2^2 x_3^2 y_4^2 - x_2^2 x_3^2 z_4^2 + 2x_2^2 x_3 x_4 y_3 y_4 - x_2^2 x_4^2 y_3^2 \\ + 2x_2 x_3^3 y_4^2 + 2x_2 x_3^3 z_4^2 - 2x_2 x_3^2 x_4 y_3 y_4 \\ - 2x_2 x_3 x_4^2 y_3 y_4 + 2x_2 x_4^3 y_3^2 - x_3^4 y_4^2 - x_3^4 z_4^2 \\ + 2x_3^2 x_4^2 y_3 y_4 - x_4^4 y_3^2) l^6 + \dots \end{aligned}$$

We notice that the quantity $l^2 + m^2 + n^2 - 1$ can be assumed to be nonzero because cones such that $l^2 + m^2 + n^2 = 1$ degenerate to straight lines.

The common roots of the equations H_0, H_5 and H_6 yield exactly all the cones through \mathcal{P} . If finite, this number is at most $2 \times 2 \times 6 = 24$ because of the Bézout theorem. However, each cone through \mathcal{P} yields two solutions of this polynomial system $H_0 = 0, H_5 = 0, H_6 = 0$, namely (l_0, m_0, n_0) and $(-l_0, -m_0, -n_0)$. This latter property is a consequence of the fact that H_0, H_5, H_6 can be written as monomials of even total degree in l, m, n . Therefore, we deduce that there are at most 12 cones through \mathcal{P} . Nevertheless, in order to devise an efficient algorithm for computing those cones through \mathcal{P} , we apply a transformation (called a σ -process or a *blow-up* in birational geometry) in order to break the above central symmetry and reduce the degree of the equations so that each cone will correspond to a unique root of the new polynomial system.

Consider the transformation

$$\begin{aligned} \sigma : \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ (l, m, n) &\mapsto \left(u := \frac{l}{n}, v := \frac{m}{n}, n \right). \end{aligned}$$

Since the equations H_0, H_5 and H_6 contain only even degree monomials, their transform under σ will be a polynomial in n^2 (and not only a polynomial in n). More precisely, setting $w := n^2$, we have

$$\begin{aligned} H_0(un, vn, n) &= w^3 P(u, v) + w^2 Q(u, v) + w R(u, v) + S, \\ H_5(un, vn, n) &= w H(u, v) + C, \\ H_6(un, vn, n) &= w K(u, v) + D, \end{aligned}$$

where H, K, P, Q, R are polynomials in u, v of degree 2, 2, 6, 4, 2 respectively, and where C, D, S are constants that depend only on the x_i 's, y_i 's and z_i 's. Since the last two equations are linear in w we can simplify this polynomial

system in w , assuming $w \neq 0$, to get

$$\begin{aligned} E_0(u, v) &= C^3 P(u, v) - C^2 Q(u, v) H(u, v) \\ &\quad + CR(u, v) H(u, v)^2 - SH(u, v)^3, \\ E_1(u, v) &= DH(u, v) - CK(u, v), \\ E_2(u, v, w) &= wK(u, v) + D. \end{aligned} \tag{17}$$

The two equations E_0 and E_1 are bivariate polynomials in u, v of degree 6 and 2 respectively. Therefore, they define at most 12 solutions. In addition, for each solution the value of w can be computed from E_2 . As a consequence, we have just proved the following result.

Theorem 6. *Given a point set \mathcal{P} composed of six distinct points then, if finite, there is an even number of cones through \mathcal{P} , which is possibly 0 and at most 12.*

Proof. Solving the polynomial system given by the equations $E_0(u, v) = 0$ and $E_1(u, v) = 0$ leads to an even number of real solutions with a maximum of $12 = 6 \times 2$ solutions. So the only thing to show is that there is a bijection between these solutions and the cones through \mathcal{P} . For that purpose, we first observe that the value of w is uniquely determined from each of those roots by equation E_2 . And then the solutions (u, v, w) uniquely pull back to a solution (l, m, n) under σ .

So it remains to show that once a solution (l, m, n) is computed, then the parameters a, b and c are uniquely determined. This is indeed the case because a, b and c can actually be given in closed form in terms of l, m, n and the coordinates of the input points because the three equations (14), (15) and (16) yield a linear system in a, b, c after replacing k by its defining value $al + bm + cn$. \square

In Figure 7 it is shown that all the possible solutions are reached in practice. Finally, observe that the particular case $w = n^2 = 0$ can be treated independently in the same vein as above. Indeed, the remaining variables l, m can be computed from the equations $H_5(l, m, 0) = 0$ and $H_6(l, m, 0) = 0$. Moreover, to solve efficiently this system one can perform a transformation similar to σ by considering the equations $H_5(um, m, 0) = 0$ and $H_6(um, m, 0) = 0$ that are linear equations in m^2 . At the end, the computed solutions must satisfy the equation H_0 in order to validate that they correspond to cones through the given set of points \mathcal{P} .

Extraction algorithm. Summarizing the above analysis, solving for the cones through \mathcal{P} can be done as follows.

1. Perform a linear change of coordinates so that the five points are of the form (10).
2. If x_2, y_3 or z_4 is equal to zero then we are in a particular configuration (see Section 3.3) so this algorithm stops here.
3. Build the system of equations (17) which is given in closed form in terms of the coordinates of the input points.

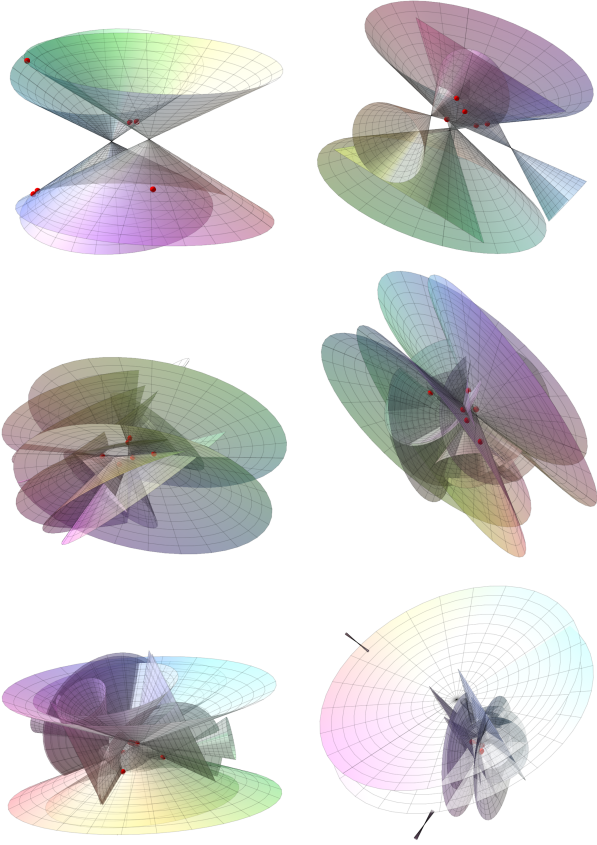


Figure 7: Configurations of points \mathcal{P} with 2, 4, 6, 8, 10 and 12 cones passing through \mathcal{P} .

4. Solve the system $E_0(u, v) = E_1(u, v) = 0$ in the two variables u, v by means of eigenvectors and eigenvalues computations, as explained in Section 2.2.2 (see [3] for details). This computation returns a list of an even number between 0 and 12 of real solutions (up to a given precision).
5. For each above solution (u_i, v_i) , compute the corresponding value of w_i using the equation $E_2(u_i, v_i, w_i) = 0$. Then pullback the solution (u_i, v_i, w_i) under σ to get the solutions $(l_i = u_i n_i, m_i = v_i n_i, n_i = \sqrt{w_i})$ of the polynomial system $H_0 = H_5 = H_6 = 0$.
6. For each solution (l_i, m_i, n_i) , compute the corresponding (a_i, b_i, c_i) (as explained in the proof of Theorem 6) and return the corresponding cone through \mathcal{P} .

This algorithm has been implemented with the **Maple** software. We observed that computing the cones through a random set of points takes in average 80ms (including all the steps in the above algorithm) and is almost constant, i.e. independent of the point set. In Table 4, we provide the distribution of the number of cones through a set of points \mathcal{P} for a random sample of a thousand point sets \mathcal{P} .

Some particular configurations. If every subset of four points in \mathcal{P} is coplanar then all the six points in \mathcal{P} are necessarily coplanar. In such a configuration, there exists a

Number of cones	0	2	4	6
Proportion (%)	1	10,5	28,7	36,3
Number of cones	8	10	12	
Proportion (%)	18,7	3,9	0,9	

Table 4: Proportion of the number of cones found through a thousand random point sets.

cone through \mathcal{P} if and only if \mathcal{P} can be interpolated by a conic section (parabola, hyperbola or ellipse). If this is the case, then there are infinitely many cones through \mathcal{P} . When a subset of five points in \mathcal{P} are coplanar then we get interesting particular configurations.

Theorem 7. *Given a point set \mathcal{P} where five points are coplanar but not six. There exists a cone through \mathcal{P} if these five points are located on an ellipse or a hyperbola. Moreover, if this is the case there are at most four cones through \mathcal{P} .*

More specifically, if the five points are co-circular, then there are two cones through \mathcal{P} and if the five points are on a parabola then there are at most 3 cones through \mathcal{P} .

Proof. It is clear that the five coplanar points must be on a conic section if there exists a cone through \mathcal{P} . By a change of coordinate system, one can assume that this conic section lies in the plane $z = 0$ and is in its canonical form. We examine below the three cases that correspond to an ellipse, a hyperbola and a parabola.

Ellipse. The equation of the conic section is of the form

$$\alpha x^2 + \beta y^2 - 1 = 0, \quad \alpha, \beta > 0$$

and it must coincide with the intersection of the implicit equation of a cone (11) after the substitution $z = 0$. Now we proceed by identifying the monomial coefficients of these two equations. the coefficient of the monomial xy yields the condition $2lm = 0$ so that $l = 0$ or $m = 0$. If $l = 0$, then the coefficients of y^2 yields the condition $m^2 = 1 - \beta/\alpha$ and if $m = 0$ then the coefficients of x^2 yields the condition $l^2 = 1 - \alpha/\beta$. Depending on the ratio α/β , only one case leads to real solutions. From now on we assume that $\alpha/\beta \geq 1$, the other case can be treated similarly. So we have that

$$l = 0, \quad m^2 = 1 - \frac{\beta}{\alpha} \geq 0.$$

The coefficient of the monomial x yields the condition $a = 0$ and the coefficient of y gives the condition

$$b = -\frac{cmn}{m^2 - 1} = \frac{\alpha}{\beta} cmn \quad (18)$$

(observe that $m^2 - 1 \neq 0$). Finally, the coefficients of the monomial 1 yields the condition

$$(\alpha n^2 - \beta) c^2 - \frac{\beta}{\alpha} = 0. \quad (19)$$

Now let (x_0, y_0, z_0) be the coordinates of the sixth point. Evaluating (11) at this point together with the conditions $l = 0, a = 0$ and (18) yields the condition

$$c^2 \left(n^2 - \frac{\beta}{\alpha} \right) - 2cz_0 \left(n^2 - \frac{\beta}{\alpha} \right) + \varphi_2(n) = 0 \quad (20)$$

(recall that $\alpha, \beta > 0$ and in particular $m^2 - 1 \neq 0$) where

$$\varphi_2(n) := (-m^2 z_0^2 + z_0^2) n^2 + (-2m^3 y_0 z_0 + 2m y_0 z_0) n - m^4 y_0^2 + m^2 x_0^2 + 2m^2 y_0^2 + m^2 z_0^2 - x_0^2 - y_0^2 - z_0^2$$

is a degree 2 polynomial in n that is independent of c . The cones through \mathcal{P} are in correspondence with the solutions of the equations (19) and (20) in the variables c and n . Using (19), Equation (20) becomes

$$2cz_0 (\alpha n^2 - \beta) = \alpha \varphi_2(n) + \frac{\beta}{\alpha}$$

and hence we get

$$\left(\alpha \varphi_2(n) + \frac{\beta}{\alpha} \right)^2 = 4c^2 z_0^2 (\alpha n^2 - \beta)^2 = 4z_0^2 \frac{\beta}{\alpha} (\alpha n^2 - \beta)$$

which is an equation of degree 4 in n . Therefore, once m is chosen from the condition $m^2 = 1 - \beta/\alpha$, we obtain at most 4 values of n and all the other parameters are uniquely determined. We conclude that we have at most four cones through \mathcal{P} , as claimed.

If the conic section is a circle, i.e. $\alpha = \beta$, we have $l = m = 0$. In this case, $\varphi_2(n) = z_0^2(n^2 - 1) - (x_0^2 + y_0^2)$ and hence the degree 4 equation in n is actually an equation in n^2 . Therefore, the 4 solutions come in pairs of opposite solutions and we deduce that we have two cones through \mathcal{P} .

Hyperbola. The equation of the conic section is of the form

$$\alpha x^2 - \beta y^2 - 1 = 0, \quad \alpha, \beta > 0$$

and we proceed similarly to the case of the ellipse. The identification of xy implies that $2lm = 0$ so we have two cases to consider, namely $l = 0$ and $m = 0$.

If $l = 0$ then the coefficient of y^2 yields $m^2 = 1 + \beta/\alpha$. The coefficient of x shows that $a = 0$ and the coefficient of y gives

$$b = -\frac{\alpha}{\beta} cmn.$$

Then the constant term yields the equality

$$b^2 + c^2 - (bm + cn)^2 = -\frac{1}{\alpha}$$

that becomes, after substituting b ,

$$c^2 \left(1 + \frac{\alpha}{\beta} n^2 \right) = -\frac{1}{\alpha}.$$

Therefore, there are no real solutions in this case.

Now if $m = 0$, then $l^2 = 1 + \alpha/\beta$. The coefficient of y shows that $b = 0$ and the coefficient of x gives

$$a = -\frac{ncl}{l^2 - 1} = -\frac{\beta}{\alpha} ncl.$$

Then the constant coefficient yields the equation

$$\frac{c^2(l^2 + n^2 - 1)}{l^2 - 1} = \frac{1}{\beta}.$$

From here, we use the sixth point as in the case of the ellipse and we get again 4 possible real cones through \mathcal{P} .

Parabola. The equation of the conic section is of the form

$$x^2 - \alpha y = 0, \quad \alpha > 0.$$

Again we inspect the coefficients of (11) after substituting $z = 0$. The coefficient of x^2 gives $l = 0$ and the coefficient of y^2 gives $m^2 = 1$. The coefficient of x gives $a = 0$ and the coefficient of y gives $2cmn = -\alpha$. Finally, the constant coefficient yields

$$b = -\frac{c^2(1 - n^2)}{\alpha}.$$

Now choosing $m = 1$ and using the sixth point of coordinates (x_0, y_0, z_0) , Equation (11) gives an equation of degree 3 in n , namely

$$-2z_0^2 n^3 + (-\alpha z_0 - 4y_0 z_0) n^2 + (-2y_0 \alpha + 2x_0^2 + 2z_0^2) n + \alpha z_0 = 0$$

and from here c can be uniquely determined. \square

4. Conclusion

We have presented several methods in order to extract efficiently cylinders and cones from minimal point sets. We have also provided a detailed analysis of these interpolation problems and we have given optimal bounds on the number of solutions. Our approach relies on closed algebraic formulas that have been computed and experimented with the help of a computer algebra system. In the near future, we plan to incorporate these methods into an efficient C++ library in order to experiment our new methods in the framework of a RANSAC-based extraction algorithm. Another future research direction will be extracting tori from a minimal point set, tori being described by means of seven parameters.

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