Closeness Centralization Measure for Two-mode Data of Prescribed Sizes*

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Abstract

We confirm a conjecture by Everett, Sinclair, and Dankelmann [Some Centrality results new and old, J. Math. Sociology 28 (2004), 215–227] regarding the problem of maximizing closeness centralization in two-mode data, where the number of data of each type is fixed. Intuitively, our result states that among all networks obtainable via two-mode data, the largest closeness is achieved by simply locally maximizing the closeness of a node. Mathematically, our study concerns bipartite graphs with fixed size bipartitions, and we show that the extremal configuration is a rooted tree of depth 2, where neighbors of the root have an equal or almost equal number of children.

1 Introduction

A social network is often conveniently modeled by a graph: nodes represent individual persons and edges represent the relationships between pairs of individuals. Our work focuses on simple unweighted graphs: our graph only tells us, for a given (binary) relation R, which pairs of individual are in relation according to R.

Centrality is a crucial concept in studying social networks [8, 12]. It can be seen as a measure of how central is the position of an individual in a social network. Various node-based measures of the centrality have been proposed to determine the relative importance of a node within a graph (the reader is referred to the work of Koschützki et al. [9] for an overview). Some widely used centrality measures are the degree centrality, the betweenness centrality, the closeness centrality and the eigenvector centrality (definitions and extended discussions are found in the book edited by Brandes and Erlebach [5]).

We focus on closeness centrality, which measures how close a node is to all other nodes in the graph: the smaller the total distance from a node v to all other nodes, the more important the node v is. Various closeness-based measures have been developed [1, 2, 4, 13, 11, 14, 16, 13].

Let us see an example: suppose we want to place a service facility, e.g., a school, such that the total distance to all inhabitants in the region is minimal. This would make the chosen location as convenient as possible for most inhabitants. In social network analysis the centrality index based on this concept is called *closeness centrality*.

Formally, for a node v of a graph G, the *closeness* of v is defined to be

$$C_G(v) := \frac{1}{\sum_{u \in V(G)} \mathrm{dist}_G(v,u)}, \tag{1}$$

where $\mathrm{dist}_G(u, v)$ is the *distance* between u and v in G, that is, the length of a shortest path in G between nodes u and v. We shall use the shorthand $W_G(v) \coloneqq \sum_{u \in V(G)} d(v, u)$. In both notations, we may drop the subscript when there is no risk of confusion.

While centrality measures compare the importance of a node within a graph, the associated notion of *centralization*, as introduced by Freeman [8], allows us to compare the relative importance of nodes within their respective graphs. The closeness centralization of a node v in a graph G is given by

$$C_1(v; G) \coloneqq \sum_{u \in V(G)} [C(v) - C(u)]. \tag{2}$$

Further, we set $C_1(G) \coloneqq \max \{C_1(v; G) : v \in V(G)\}$.

It is important to note that the parameter C_1 is really tailored to compare the centralization of nodes in different graphs. If only one graph is involved, then one readily sees that maximizing $C_1(v; G)$ over the nodes of a graph G amounts to minimizing W_G. Indeed, suppose that G is a graph and v a node of G such that $W_G(v) \leqslant W_G(u)$ for every $u \in V(G)$. Then for every node x of G,

$$\begin{aligned} C_1(v; G) - C_1(x; G) &= (n-1)\left(\frac{1}{W_G(v)} - \frac{1}{W_G(x)}\right) - \left(\frac{1}{W_G(x)} - \frac{1}{W_G(v)}\right) \\ &= n\left(\frac{1}{W_G(v)} - \frac{1}{W_G(x)}\right) \\ &\geqslant 0. \end{aligned}$$

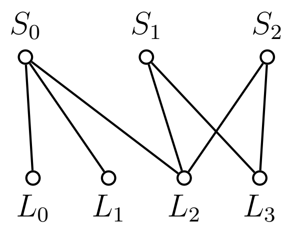
In what follows, we use the the following notation. The *star graph* of order n, sometimes simply known as an n-*star*, is the tree on $n+1$ nodes with one node having degree n. The star graph is thus a complete bipartite graph with one part of size 1. Everett, Sinclair, and Dankelmann [7] established that over all graphs with a fixed number of nodes, the closeness is maximized by the star graph.

Theorem 1. *If G is a graph with n nodes, then*

$$C_1(u; S_{n-1}) \geqslant C_1(G),$$

where u is the node of S_{n-1} of maximum degree.

They also considered the problem of maximizing centralization measures for two-mode data [7]. In this context, the relation studied links two different types of data (e.g., persons and events) and we are interested in the centralization of one type of data only (e.g., the most central person). Thus the graph obtained is *bipartite*: its nodes can be partitioned into two parts so that all the edges join nodes belonging to different parts. A toy example is depicted in Figure 1, where one type of data consists of students and the other of classes: edges link the students to the classes they attended. (The sole purpose of this example is to make sure the reader is at ease with the definitions of C and C_1.) Closeness centrality is maximized at the student "S_0" for one part and at the class "L_2" for the other. An example of a real-world two-mode network N on 89 edges with partition sizes $|P_1| = 18$ and $|P_2| = 14$, borrowed from [6]



$v \in V(N)$	$C_N(v)$	$C_1(v, N)$
S_0	1/10	0.1222
S_1	1/12	0.0055
S_2	1/12	0.0055
L_0	1/15	−0.1111
L_1	1/15	−0.1111
L_2	1/9	0.2000
L_3	1/15	−0.1111

Figure 1: A two-mode network N with 7 nodes (3 in one part, 4 in the other) and 7 edges, with the corresponding values for C_N and C_1.

is depicted on Figure 2. On the figure, one can observe a frequency of interparticipation of a group of women in social events in Old City, 1936. On Tables 1 and 2, one can observe closeness centralization for partitions P_1 and P_2 and notice that closeness centrality (and hence centralization) is maximized at "Mrs. Evelyn Jefferson" and the event from "September 16th", respectively.

Everett et al. formulated an interesting conjecture, which was later proved by Sinclair [15]. To state it, we first need a definition.

Definition 2. *Let $H(u; n_0, n_1)$ be the tree with node bipartition (A_0, A_1) such that*

- $|A_i| = n_i$ *for* $i \in \{0, 1\}$*;*
- *there exists a node* $u \in A_0$ *such that* $N_G(u) = A_1$*; and*
- $\deg(w) \in \left\{1 + \left\lceil \frac{n_0-1}{n_1} \right\rceil, 1 + \left\lfloor \frac{n_0-1}{n_1} \right\rfloor \right\}$ *for all nodes* $w \in A_1$.

The node u is called the root *of $H(u; n_0, n_1)$.*

The aforementioned conjecture was that the pair $(H(u; n_0, n_1), u)$ is an *extremal pair* for the problem of maximizing *betweenness centralization* in bipartite graphs with a fixed sized bipartition into parts of sizes n_0 and n_1. Recall that for two-mode data, we are only interested in one type of data: in graph-theoretic terms, we look only at nodes that belong to the part of size n_0, and we want to know which of these nodes has the largest closeness in the graph. In other words, letting A_0 be the part of size n_0 of $V(G)$, we want to determine $\max\{C_1(v; G) : v \in A_0\}$.

Everett et al. also suggested that the same pair is extremal for closeness and eigenvector centralization measures. In this paper, we confirm the conjecture for the closeness centralization measure. That is, we prove that the pair $H(v; n_0, n_1)$ is extremal for the problem of maximizing closeness centralization in bipartite graphs with parts of size n_0 and n_1,where v is the root.

We point out that a similar study for the centrality measure of eccentricity was led recently [10]. In addition, Bell [3] worked on closely related notions, namely subgroup centrality measures. Similarly as for two-mode data, a susbet S of the nodes is fixed (called a *group*) and the aim is to find a node in S with largest centrality.

$v \in P_1$	$C_N(v)$	$C_1(v,N)$
Mrs. Evelyn Jefferson	0.01667	0.07779
Miss Theresa Anderson	0.01667	0.07779
Mrs. Nora Fayette	0.01667	0.07779
Mrs. Sylvia Avondale	0.01613	0.06058
Miss Laura Mandeville	0.01515	0.02930
Miss Brenda Rogers	0.01515	0.02930
Miss Katherine Rogers	0.01515	0.02930
Mrs. Helen Lloyd	0.01515	0.02930
Miss Ruth DeSand	0.01471	0.01504
Miss Verne Sanderson	0.01471	0.01504
Miss Myra Liddell	0.01429	0.00160
Miss Frances Anderson	0.01389	−0.01110
Miss Eleanor Nye	0.01389	−0.01110
Miss Pearl Oglethorpe	0.01389	−0.01110
Mrs. Dorothy Murchison	0.01351	−0.02311
Miss Charlotte McDowd	0.01250	−0.05555
Mrs. Olivia Carleton	0.01220	−0.06530
Mrs. Flora Price	0.01220	−0.06530

Table 1: Nodes from the group of women and their closeness values.

$v \in P_2$	label on Fig. 2	$C_N(v)$	$C_1(v,N)$
September 16th	P8	0.01923	0.15984
April 8th	P9	0.01786	0.11588
March 15th	P7	0.01667	0.07779
May 19th	P6	0.01562	0.04445
February 25th	P5	0.01351	−0.02311
April 12th	P3	0.01282	−0.04529
April 7th	P12	0.01282	−0.04529
June 10th	P10	0.01250	−0.05555
September 26th	P4	0.01220	−0.06530
February 23rd	P11	0.01220	−0.06530
June 27th	P1	0.01190	−0.07459
March 2nd	P2	0.01190	−0.07459
November 21st	P13	0.01190	−0.07459
August 3rd	P14	0.01190	−0.07459

Table 2: Nodes from the partition of social events from 1936, reported in *Old City Herald*, and their closeness values.

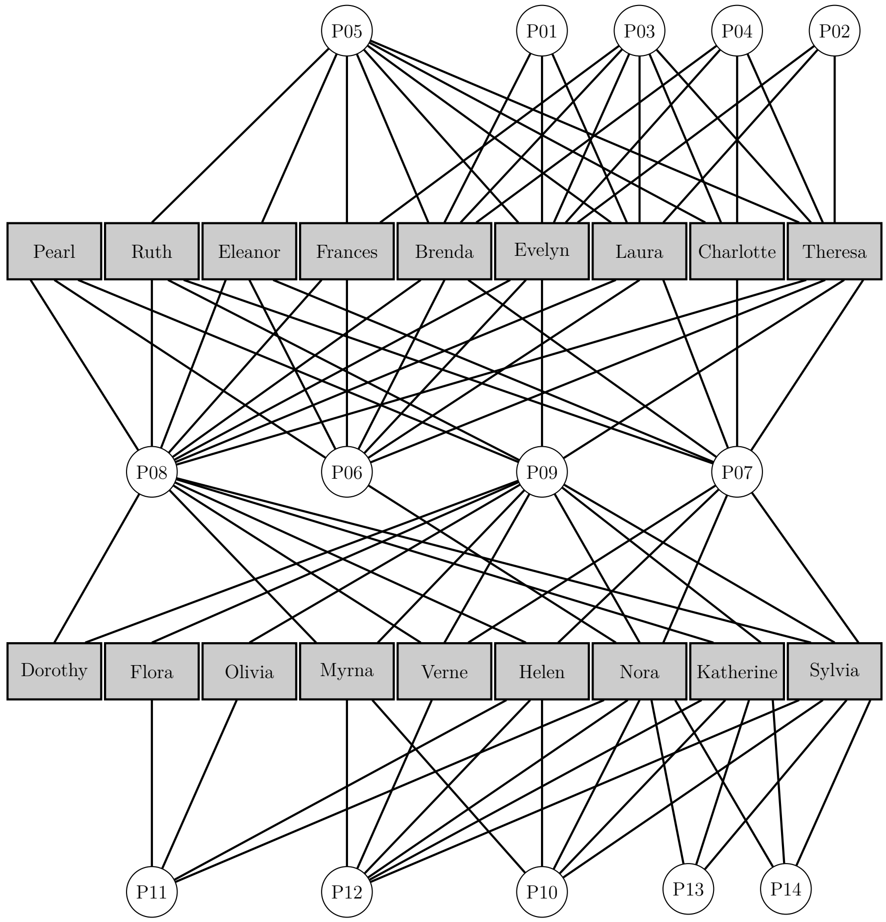


Figure 2: A two-mode network N on 89 edges with partition sizes $n_0 = 18$ and $n_1 = 14$. The network represents the participation of a given set of people in the social events from 1936 reported in Old City Herald, where circles represent social events while rectangles represent women (see Tables 1 and 2).

However, unlike in the standard centrality notion, the centrality itself is computed using distances only to the nodes in S (*local centrality*) or to the nodes outside S (*global centrality*). Note that the standard notion, which is used in this work, takes into account the distances to all other nodes in the graph.

2 Bipartite Networks With Fixed Number of Nodes

Theorem 3. *Let G be a bipartite graph with node parts A_0 and A_1 sizes n_0 and n_1, respectively. Then for each $v \in A_0$,*

$$C_1(u; H(u; n_0, n_1)) \geqslant C_1(v; G).$$

To prove Theorem 3, suppose that G is a bipartite graph with bipartition (A_0, A_1)

where $|A_i| = n_i$ for $i \in \{0, 1\}$, and u is a node in A_0 such that $C_1(u; G) \geqslant C_1(v; H(v; n_0, n_1))$. We prove that this inequality must actually be an equality by showing that any such extremal pair $C_1(u; G)$ must satisfy the following three properties:

(P1) G is a tree;

(P2) $\deg_G(u) = n_1$; and

(P3) $|\deg_G(w_1) - \deg_G(w_2)| \leqslant 1$ whenever $w_1, w_2 \in A_1$.

Property (P1) is relatively straightforward to check and so is (P3) if we assume that (P2) holds. Thus the majority of the discussion below will be devoted to proving that (P2) holds, which we do last. For convenience, we define V to be $V(G)$.

We start by establishing (P1); namely, that the graph G is a tree. Assume, for the sake of contradiction, that G is not a tree and let T be a breadth-first-search tree of G rooted at u. Note that $W_G(u) = W_T(u)$ and $W_T(x) \geqslant W_G(x)$ for any node $x \in V(G)$. In addition, there exist at least two nodes for which the above inequality is strict. It follows that $C_1(u; T) > C_1(u; G)$, a contradiction.

We now establish that (P3) holds if (P2) does. Thus we know that G is a tree and we assume that $N_G(u) = A_1$, therefore also all nodes from $A_0 \setminus \{u\}$ are leaves. Suppose, for the sake of contradiction, that there exist nodes $w_1, w_2 \in A_1$ such that $\deg(w_1) \geqslant \deg(w_2) + 2$. Let z be a neighbor of w_1 different from u and consider the graph G' obtained by deleting the edge w_1z and replacing it with w_2z. Note that $W_{G'}(u) = W_G(u)$ and that $W_{G'}(x) = W_G(x)$ unless $x \in N_G[w_1] \cup N_G[w_2]$, that is unless x belongs to the closed neighborhood of either w_1 or w_2. So

$$C_1(u; G') - C_1(u; G) = \sum_{x \in N_G[w_1] \cup N_G[w_2]} \frac{1}{W_G(x)} - \sum_{x \in N_G[w_1] \cup N_G[w_2]} \frac{1}{W_{G'}(x)}. \quad (3)$$

Now, let $N_G(w_1) = \{u, z, x_1, \ldots, x_t\}$ and $N_G(w_2) = \{u, y_1, \ldots, y_s\}$ where, by assumption, $t > s$.

Recalling that G is a tree, observe that the following hold for every $i \in \{1, \ldots, t\}$ and every $j \in \{1, \ldots, s\}$ (for better illustration, see Figure 3).

(i). $W_{G'}(x_i) = W_G(x_i) + 2$;

(ii). $W_{G'}(y_j) = W_G(y_j) - 2$;

(iii). $W_G(y_j) = W_G(x_i) + 2(t - s + 1) > W_G(x_i) + 2$;

(iv). $W_{G'}(z) = W_G(z) + 2(t - s) > W_G(z)$;

(v). $W_{G'}(w_1) = W_G(w_1) + 2$; and

(vi). $W_{G'}(w_2) = W_G(w_2) - 2$.

From (i)–(iii), we infer that for any $j \in \{1, \ldots, s\}$,

$$\frac{1}{W_{G'}(x_j)} + \frac{1}{W_{G'}(y_j)} < \frac{1}{W_G(x_j)} + \frac{1}{W_G(y_j)},$$

and similarly by (v) and (vi),

$$\frac{1}{W_{G'}(w_1)} + \frac{1}{W_{G'}(w_2)} < \frac{1}{W_G(w_1)} + \frac{1}{W_G(w_2)}.$$

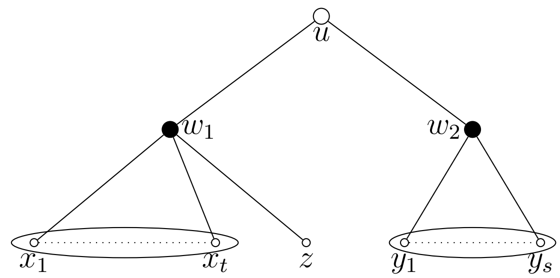


Figure 3: The subtree of G induced by $N_G[w_1] \cup N_G[w_2]$.

Thus the right side of (3) is greater than

$$\frac{1}{W_G(z)} - \frac{1}{W_{G'}(z)} + \sum_{j=s+1}^{t} \frac{1}{W_G(x_j)} - \frac{1}{W_{G'}(x_j)},$$

which is positive by (i) and (iv). This contradiction shows that (P3) holds provided (P2) does.

It remains to prove that (P2) holds to complete the proof. First, if $n_1 = 1$, then the tree G must be an n_0-star, hence the second property is satisfied. Now consider the case where $n_1 = 2$. Then there is precisely one node x that is adjacent to both nodes in A_1. Moreover, $W_G(x) \leqslant W_G(w)$ if $w \in A_0$ since, if $w \in A_0 \setminus \{x\}$ then $W_G(w) \geqslant 2(n_0 - 1) + 4 = 2n_0 + 2$ while $W_G(x) = 2 + 2(n_0 - 1) = 2n_0 + 1$. Thus $u = x$ and hence $\deg_G(u) = n_1 = 2$, as wanted.

From now on, we assume that $n_1 \geqslant 3$. As in the proof of (P3), we argue that if (P2) does not hold then $C_1(u; G)$ can be increased by altering the graph G. In this case, however, we find it necessary to use our assumption that $C_1(u; G)$ itself is at least as large as $C_1(v; H(v; n_0, n_1))$. This shall allow us to have a lower bound on $C_1(u; G)$, by the next lemma.

Lemma 4. $C_1(u; H(u; n_0, n_1)) \geqslant \frac{n_1 - 1}{2(2n_1 - 1)}$.

Proof. We establish the inequality via a direct computation. Unfortunately, the expressions involved force a lengthy computation.

We set $m := n_0 - 1$ and we write $m = pn_1 + r$ where $0 \leqslant r < n_1$. Let us now calculate $W(x)$ for each node x of $H(u; n_0, n_1)$.

1. $W(u) = n_1 + 2m$.
2. Consider the neighbors of u: there are
   (a) r neighbors x for which $W(x) = \lceil m/n_1 \rceil + 1 + 2(n_1 - 1) + 3(m - \lceil m/n_1 \rceil)$; and
   (b) $n_1 - r$ neighbors x for which $W(x) = \lfloor m/n_1 \rfloor + 1 + 2(n_1 - 1) + 3(m - \lfloor m/n_1 \rfloor)$.
3. Consider the nodes at distance two from u: there are
   (a) $r \lceil m/n_1 \rceil$ nodes x for which $W(x) = 1 + 2\lceil m/n_1 \rceil + 3(n_1 - 1) + 4(m - \lceil m/n_1 \rceil)$; and

(b) $(n_1 - r)\lfloor m/n_1 \rfloor$ nodes x for which $W(x) = 1 + 2\lfloor m/n_1 \rfloor + 3(n_1 - 1) + 4(m - \lfloor m/n_1 \rfloor)$.

Since $\lfloor m/n_1 \rfloor = (m-r)/n_1$ and, for $r > 0$, we have $\lceil m/n_1 \rceil = (m + n_1 - r)/n_1$, it follows that if $r > 0$ then

$$\begin{aligned} C_1(u) &= \frac{n_1+m}{n_1+2m} - \frac{rn_1}{3mn_1 - 2m + 2n_1^2 - 3n_1 + 2r} - \frac{n_1(n_1-r)}{3mn_1 - 2m + 2n_1^2 - n_1 + 2r} \\ &\quad - \frac{r(m+n_1-r)}{4mn_1 - 2m + 3n_1^2 - 4n_1 + 2r} - \frac{(n_1-r)(m-r)}{4mn_1 - 2m + 3n_1^2 - 2n_1 + 2r} \qquad (4) \\ &\geqslant \frac{n_1+m}{n_1+2m} - \frac{n_1^2}{3mn_1 - 2m + 2n_1^2 - 3n_1 + 2r} - \frac{n_1 m}{4mn_1 - 2m + 3n_1^2 - 4n_1 + 2r}, \qquad (5) \end{aligned}$$

where we used that $n_1 > 0$ to derive (5).
One notes that (5) is still true if $r = 0$. Indeed, in this case $\lceil \frac{m}{n_1} \rceil = \lfloor \frac{m}{n_1} \rfloor = \frac{m}{n_1}$, so

$$C_1(u) = \frac{n_1+m}{n_1+2m} - \frac{n_1^2}{3mn_1 - 2m + 2n_1^2 - n1} - \frac{n_1 m}{4mn_1 - 2m + 3n_1^2 - 2n_1},$$

so that (5) stays true.

As is seen from (4), if n_1 is fixed and n_0 tends to infinity (hence, so does m), then $C_1(u)$ approaches $1/2 - n_1/(4n_1 - 2) = \frac{n_1-1}{4n_1-2}$.

Let us now subtract $\frac{n_1-1}{4n_1-2}$ from the right side of (5) and show that the difference is non-negative. After cross-multiplying and simplifying, we obtain a fraction with positive denominator (since each denominator in the right side of (5) is positive), and with numerator equal to

$$\begin{aligned} &m^2(10n_1^4 - 44n_1^3 + 12n_1^2 r + 30n_1^2 - 8n_1 r - 4n_1) \\ &\quad + m(15n_1^5 - 77n_1^4 + 38n_1^3 r + 74n_1^3 - 54n_1^2 r - 14n_1^2 + 8n_1 r^2 + 8n_1 r) \\ &\qquad + (6n_1^6 - 35n_1^5 + 22n_1^4 r + 45n_1^4 - 48n_1^3 r - 12n_1^3 + 12n_1^2 r^2 + 14n_1^2 r - 4n_1 r^2). \qquad (6) \end{aligned}$$

This expression increases with n_1 and is clearly positive when $n_1 = 6$ (to see it quickly just compare, in each parenthesis, every (maximal) sequence of consecutive negative terms with the (maximal) sequence of positive terms preceding it). Further, a direct calculation ensures that (6) is actually positive even when $n_1 = 5$.

However, if $n_1 \in \{3, 4\}$, then (6) could take on negative values for certain values of m. To deal with these two cases we revert back to the initial equation (4).

Assume that $n_1 = 3$. Then subtracting $\frac{n_1-1}{4n_1-2}$ from both sides of (4) yields that $C_1(u) - \frac{n_1-1}{4n_1-2}$ is at least

$$\frac{m+3}{2m+3} - \frac{3r}{7m+9+2r} - \frac{9-3r}{7m+15+2r} - \frac{r(m+3-r)}{10m+15+2r} - \frac{(3-r)(m-r)}{10m+21+2r} - \frac{1}{5}. \qquad (7)$$

Placing (7) under one (positive) denominator, the numerator becomes

$$\begin{aligned}1540m^4 + 2m^3(9075 - 1016r + 588r^2) + 6m^2(10605 - 1047r + 937r^2 + 112r^3)\\ + m(88155 - 3816r + 9828r^2 + 2408r^3 + 96r^4)\\ + (42525 + 1350r + 6174r^2 + 2280r^3 + 184r^4),\end{aligned} \quad (8)$$

which is clearly positive as $r \leqslant n_1 - 1 = 2$.

A similar calculation yields the conclusion when $n_1 = 4$. In this case, the difference of (4) and $\frac{n_1-1}{4n_1-2}$ yields that $C_1(u) - \frac{n_1-1}{4n_1-2}$ is at least

$$\frac{m+4}{2m+4} - \frac{2r}{5m+10+r} - \frac{8-2r}{5m+14+r} - \frac{r(m+4-r)}{14m+32+2r} - \frac{(4-r)(m-r)}{14m+40+2r} - \frac{3}{14},$$

whose numerator, when placed under a common (positive) denominator, is

$$\begin{aligned}1855m^4 + 4m^3(5855 - 82r + 100r^2) + 2m^2(52090 + 206r + 1405r^2 + 80r^3)\\ + 4m(49180 + 2022r + 1793r^2 + 194r^3 + 4r^4)\\ + 3(44800 + 4080r + 2204r^2 + 332r^3 + 13r^4).\end{aligned}$$

This is non-negative as $r \leqslant n_1 - 1 = 3$. This concludes the proof. $\square$

It remains to demonstrate that (P2) holds. To this end, we consider the tree G to be rooted at u and, for a node x, we let T_x be the subtree of G rooted at x. To avoid unnecessary notation later, let us observe immediately that if $\deg_G(u) = 1$ then (P2) holds. For otherwise, $n_1 \geqslant 2$ and there exists a node u' at distance two from u such that $\deg_G(u') \geqslant 2$. As a result, $W_G(u) \geqslant W_G(u') + |V(T_{u'})| - 1 > W_G(u')$, which implies that $C_1(u'; G) > C_1(u; G)$, a contradiction.

We also note that if $\mathrm{dist}_G(u, x) \leqslant 2$ for all $x \in V(G)$, then (P2) is satisfied. So assume that there exists some child of u whose subtree has depth at least 2. Among all such children of u, let z be such that $|V(T_z)|$ is maximum, that is,

$$|V(T_z)| = \max\left\{|V(T_v)| \,:\, v \text{ child of } u \text{ and } T_v \text{ has depth at least } 2\right\}.$$

We now give some notations, which are illustrated in Figure 4. Let $y_1, \ldots, y_t$ be the nodes of T_z with depth 2 and set $Y := \cup_{i=1}^{t} V(T_{y_i})$. Note that, by definition, $t \geqslant 1$ and $\mathrm{dist}_G(u, y_i) = 3$ whenever $1 \leqslant i \leqslant t$. Let $p_1, \ldots, p_\ell$ be the children of z (in T_z) with degree more than 1 and set $P := \{p_1, \ldots, p_\ell\}$. Let P' be the set of children of z with degree 1 and set $k := |P'|$.

Note that for any $w \in N(u)$, the definition of z ensures that T_w is a star whenever $|V(T_w)| > |V(T_z)|$. The graph G' is obtained from G as follows. (An illustration is given in Figure 5.) For convenience, we set $n := n_0 + n_1 = |V(G)|$.

(a). For each $i \in \{1, \ldots, t\}$, the edge uy_i is added.

(b). For each $i \in \{1, \ldots, \ell\}$, the edge zp_i is removed and all other edges incident to p_i but one are removed. Thus the vertices $p_1, \ldots, p_\ell$ become leaves of G', each being attached to one of the vertices $y_1, \ldots, y_t$.

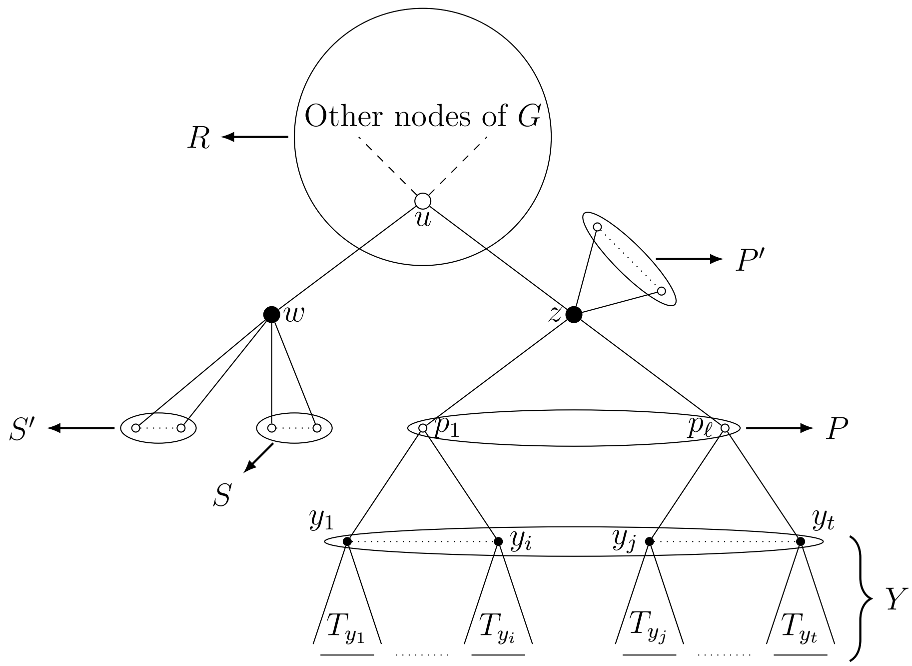


Figure 4: Figurative view of the subsets of nodes of G. Recall that $S' \coloneqq V(T_w) \setminus \{w\}$ if $S = \varnothing$.

(c). If there exists a child w of u different from z with $|V(T_w)| \geqslant n/2$, then we select an arbitrary set $S \subset V(T_w) \setminus \{w\}$ of size $|V(T_w)| - \lfloor n/2 \rfloor$ and we set $S' \coloneqq V(T_w) \setminus (S \cup \{w\})$. Then for each $s \in S$, we replace the edge sw by the edge sz.

(d). If there is no node w as in (c), then we let w be a child of u different from z such that $|V(T_w)|$ is as large as possible, and we define S' to be $V(T_w) \setminus \{w\}$. (Recall that $\deg_G(u) \geqslant 2$, hence such a child always exists.) Moreover, we set $S \coloneqq \varnothing$ for convenience.

As noted earlier, if (c) applies then T_w is a star. Moreover, if $S \neq \varnothing$, then one can see that $W_G(w) < W_G(u)$ and hence $C_1(w; G) > C_1(u; G)$. However, this is not a contradiction since $C_1(u; G) = \max\{C_1(v; G) : v \in A_0\}$ and $w \in A_1$.

Regardless of whether (c) or (d) applies, $|S'| \leqslant \lfloor \frac{n}{2} \rfloor - 1$. Actually, it is important to notice that, in G', no child of u different from z has more than $\lfloor n/2 \rfloor - 1$ children itself. Even more, for any such child x we know that $|V(T_x)| \leqslant \lfloor n/2 \rfloor$. This follows from our previous remark if T_x has depth at most 2, and from the fact that $|V(T_x)| \leqslant |V(T_z)|$ otherwise. Also, setting $R \coloneqq V \setminus V(T_z) \cup V(T_w)$, we observe that for every node $p_i \in P$

$$\mathrm{dist}_G(p_i, x) = \begin{cases} \mathrm{dist}_G(u, x) - 2 & \text{if } x \in V(T_{p_i}) \\ \mathrm{dist}_G(u, x) + 2 & \text{if } x \in R \cup V(T_w) \\ \mathrm{dist}_G(u, x) & \text{otherwise.} \end{cases}$$

Therefore, $W(p_i) \leqslant W(u) - 2(|V(T_{p_i})| - (|R| + |V(T_w)|))$. Since the definition of u implies that $W(p_i) \geqslant W(u)$, it follows that the size of $V(T_{p_i})$ is at most $\lfloor n/2 \rfloor$.

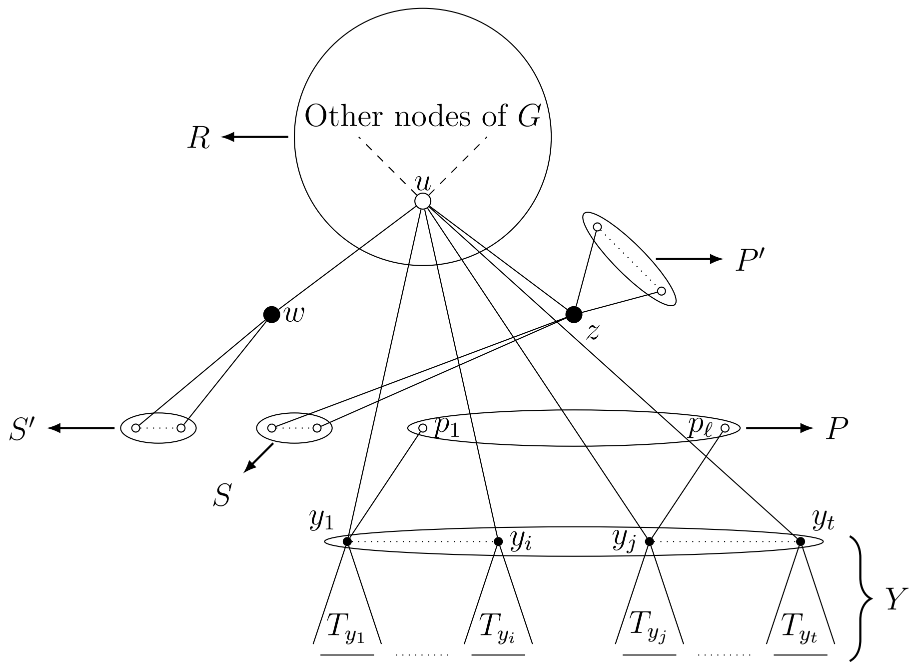


Figure 5: Obtaining G' from G. Recall that $S' \coloneqq V(T_w) \setminus \{w\}$ if $S = \varnothing$.

Note that G' is a tree, which we see rooted at u, and G and G' have the same node set, which we call V. In addition, G and G' have the same bipartition (A_0, A_1). Our next task is to compare the total distance of nodes in G and in G', that is, we compare $W_G(x)$ and $W_{G'}(x)$. For readability purposes, let us set $W(x) \coloneqq W_G(x)$, $W'(x) \coloneqq W_{G'}(x)$, and let T'_x be the subtree of G' rooted at x. We now make a few statements about $W(x)$ and $W'(x)$ for various nodes. We shall often use that

$$n = |V| = |R| + |Y| + |P| + |P'| + |S| + |S'| + 2.$$

Lemma 5. *The following hold.*

(i). *If $x \in R$, then $W(x) - W'(x) = 2|Y|$.*

(ii). *If $x \in \{z\} \cup P'$, then $W'(x) \geqslant W(x) - 2|S|$.*

(iii). *If $x \in \{w\} \cup S'$, then $W'(x) = W(x) + 2|S| - 2|Y|$.*

(iv). *If $x \in P \cup S$, then $W'(x) \geqslant W(x)$.*

(v). *If $S \neq \varnothing$, then $W(x_1) > W(x_2)$ and $W'(x_1) > W'(x_2)$ whenever $x_1 \in P'$ and $x_2 \in S'$.*

(vi). *If $x \in Y$, then $W'(x) \leqslant W(x)$.*

(vii). *$W'(x) \geqslant W'(u)$ for every node $x \in Y \cup R \cup S' \cup \{w\}$.*

Proof. We prove all the statements in order.

(i). If $x \in R$, then the distance from x to any node not in Y is unchanged. In addition, $\operatorname{dist}_{G'}(x, y) = \operatorname{dist}_G(x, y) - 2$ whenever $y \in Y$, hence the conclusion.

(ii). If $x \in \{z\} \cup P'$, then $\operatorname{dist}_{G'}(x, v) \geqslant \operatorname{dist}_G(x, v)$ for each $v \in V \setminus S$. In addition, if $s \in S$, then $\operatorname{dist}_{G'}(x, s) = \operatorname{dist}_G(x, s) - 2$, which yields the conclusion.

(iii). It suffices to observe that if $x \in \{w\} \cup S'$, then

$$\operatorname{dist}_{G'}(x, v) = \begin{cases} \operatorname{dist}_G(x, v) & \text{if } v \in V \setminus (S \cup Y) \\ \operatorname{dist}_G(x, v) - 2 & \text{if } v \in Y \\ \operatorname{dist}_G(x, v) + 2 & \text{if } v \in S. \end{cases}$$

(iv). First note that if $x \in P$, then the definition of G' ensures that $\operatorname{dist}_{G'}(x, v) \geqslant \operatorname{dist}_G(x, v)$ for each $v \in V$, which implies that $W'(x) \geqslant W(x)$.

Now let $x \in S$. Observe that if $v \in V$, then $\operatorname{dist}_{G'}(x, v) \geqslant \operatorname{dist}_G(x, v) - 2$. In addition, if $v \in S' \cup \{w\}$, then $\operatorname{dist}_{G'}(x, v) = \operatorname{dist}_G(x, v) + 2$. Consequently,

$$W'(x) - W(x) \geqslant 2\left|S' \cup \{w\}\right| - 2\left|V \setminus (\{x, w\} \cup S')\right|,$$

which is non-negative since $|S' \cup \{w\}| = \lfloor |V|/2 \rfloor$ when $S \neq \varnothing$, and $x \notin S' \cup \{w\}$.

(v). Let $x_1 \in P'$ and $x_2 \in S'$. First note that every node in $V(T_w) \setminus \{x_1\}$ is two units closer to x_1 than to x_2. Similarly, every node in $V(T_z) \setminus \{x_2\}$ is two units closer to x_2 than to x_1. Since, in addition, every remaining node (different from x_1 and x_2) is at the same distance from x_1 and x_2, we deduce that

$$W(x_1) - W(x_2) = 2(|S| + |S'| - |P| - |P'| - |Y|).$$

This quantity is positive since, as $S \neq \varnothing$, we know that $|S| + |S'| \geqslant \lfloor n/2 \rfloor - 1$ while $|P| + |P'| + |Y| \leqslant n - |S| - |S'| - 3 < \lfloor n/2 \rfloor - 2$.

A similar analysis in G' yields that

$$W'(x_1) - W'(x_2) = 2(|S'| - |S| - |P'|),$$

because every node not in $S' \cup S \cup P' \cup \{x_1, x_2\}$ is at the same distance (in G') from x_1 and x_2. Again, $|S'| - |S| - |P'|$ is positive since $|S'| = \lfloor n/2 \rfloor - 1$ while $|P'| + |S'| \leqslant n - |S'| - 3 \leqslant \lfloor n/2 \rfloor - 2$.

(vi). Let $x \in Y$. Observe that if $\operatorname{dist}_{G'}(x, v) > \operatorname{dist}_G(x, v)$, then v must be the child of z that is an ancestor of x (that is, $v \in P$ and $x \in V(T_v)$). Furthermore, in this instance, the distance increases by exactly 2. As the distance from x to any node in R decreases by 2 (and $|R| \geqslant 1$), it follows that $W'(x) \leqslant W(x)$.

(vii). For readability, the proof is split intro four cases depending on whether $x \in \{w\}$, $x \in R$, $x \in S'$ or $x \in Y$. The interested reader will notice that a similar argument is used in all these cases, however, proceeding with cases simplifies the verification and gives a better vision of the situation.

We start by showing that $W'(w) \geqslant W'(u)$. Since $\operatorname{dist}_{G'}(w, u) = 1$, we know that

$$\operatorname{dist}_{G'}(w, v) = \begin{cases} \operatorname{dist}_{G'}(u, v) - 1 & \text{if } v \in V(T_w) \setminus S = S' \cup \{w\} \\ \operatorname{dist}_{G'}(u, v) + 1 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} W'(w) - W'(u) &= \left|V \setminus (S' \cup \{w\})\right| - \left|S' \cup \{w\}\right| \\ &= |V| - 2(\left|S'\right| + 1), \end{aligned}$$

which is non-negative since $|S'| \leqslant \lfloor n/2 \rfloor - 1$.

A similar reasoning applies to the nodes in R. Let $x \in R \setminus \{u\}$. Set $d \coloneqq \mathrm{dist}_{G'}(x, u)$ and let x' be the child of u on the unique path between u and x in G. Note that $T'_{x'} = T_{x'}$. Since

$$\mathrm{dist}_{G'}(x, v) = \mathrm{dist}_{G'}(u, v) + d \qquad \text{if } v \in V \setminus V(T_{x'})$$

and

$$\mathrm{dist}_{G'}(x, v) \geqslant \mathrm{dist}_{G'}(u, v) - d \qquad \text{if } v \in V(T_{x'}),$$

we observe that

$$W'(x) - W'(u) \geqslant d \cdot \left(|V \setminus V(T_{x'})| - |V(T_{x'})|\right).$$

This yields the desired inequality since, as reported earlier, $|V(T_{x'})| \leqslant n/2$.

We now deal with the nodes in S'. Let $x \in S'$. First, if $S \neq \varnothing$, then S' is composed of precisely $\lfloor n/2 \rfloor - 1$ nodes, which are all children of w. The definition of G' thus implies that $\mathrm{dist}_{G'}(x, v) \geqslant \mathrm{dist}_{G'}(u, v)$ whenever $v \neq x$, hence $W'(x) \geqslant W'(u)$, as asserted. Assume now that $S = \varnothing$. The situation can then be dealt with in the very same way as for the nodes in R. Indeed, in this case,

$$W'(x) - W'(u) \geqslant \mathrm{dist}_{G'}(x, u) \cdot \left(|V \setminus V(T_w)| - |V(T_w)|\right),$$

and T_w contains at most $n/2$ nodes since $S = \varnothing$.

Finally, let $x \in Y$. Similarly as before, set $d \coloneqq \mathrm{dist}_{G'}(x, u)$. For every $v \in V$,

$$\mathrm{dist}_{G'}(x, v) \geqslant \mathrm{dist}_{G'}(u, v) - d.$$

Let y_i be the ancestor of x among $\{y_1, \ldots, y_t\}$. If $v \notin V(T'_{y_i})$, then

$$\mathrm{dist}_{G'}(x, v) = \mathrm{dist}_{G'}(u, v) + d.$$

Consequently,

$$W'(x) - W'(u) \geqslant d \cdot \left(\left|V \setminus V(T'_{y_i})\right| - \left|V(T'_{y_i})\right|\right).$$

Now let p_k be the father of y_i in G. Then $V(T'_{y_i}) \subseteq V(T_{p_k})$. As reported earlier, $|V(T_{p_k})| \leqslant \lfloor n/2 \rfloor$, which yields that $W'(x) - W'(u) \geqslant 0$. $\square$

The next lemma in particular bounds $C_1(u; G)$ from below.

Lemma 6. *If $x \in Y$, then $0 \leqslant \frac{W(x) - W'(x)}{W(x)} < 2C_1(u; G)$.*

Proof. Assume that $x \in V(T_{y_i})$. Lemma 5(vi) ensures that $W'(x) \leqslant W(x)$, thereby proving that $\frac{W(x)-W'(x)}{W(x)}$ is non-negative.

Let D be the set of those nodes whose distance to x is greater in G than in G', that is, $D \coloneqq \{v \in V : \mathrm{dist}_G(v,x) > \mathrm{dist}_{G'}(v,x)\}$. Observe that $W(x) - W'(x) \leqslant 2\,|D|$, since $\mathrm{dist}_{G'}(x,v) \geqslant \mathrm{dist}_G(x,v) - 2$ for every $v \in V$.

We partition D into parts $D_1, \ldots, D_m$ where $v \in D_j$ if and only if $v \in D$ and $\mathrm{dist}_G(x,v) = j$. Note that $D_1 = \varnothing = D_2$. In addition, $D_3 = \{u\}$ if $x = y_i$ while $D_3 = \varnothing$ if $x \neq y_i$. Finally, if $x \neq y_i$, then $D_4 \subseteq \{u\}$, while otherwise D_4 is contained in $A_1 \setminus \{x, z\}$. In both cases, we deduce that $|D_4| \leqslant n_1 - 2$, since $n_1 \geqslant 3$. Thus

$$W(x) - W'(x) \leqslant 2\sum_{i=3}^{m} |D_i| \tag{9}$$

and, since G contains at least one node at distance 2 from x,

$$W(x) \geqslant 1 + 2 + \sum_{i=3}^{m} i\,|D_i| \tag{10}$$

Since we assume that $C_1(u;G) \geqslant C_1(v;H(v;n_0,n_1))$, it follows from Lemma 4 that $C_1(u;G) \geqslant \frac{n_1-1}{2(2n_1-1)}$. Therefore,

$$\begin{aligned}
\frac{W(x)-W'(x)}{W(x)} - 2C_1(u;G) &\leqslant \frac{W(x)-W'(x)}{W(x)} - \frac{n_1-1}{2n_1-1} \\
&\leqslant \frac{2\sum_{i=3}^{m}|D_i|}{W(x)} - \frac{n_1-1}{2n_1-1} \\
&\leqslant \frac{2(2n_1-1)\sum_{i=3}^{m}|D_i| - (n_1-1)(3+\sum_{i=3}^{m} i\,|D_i|)}{(2n_1-1)W(x)} \\
&= \frac{-3n_1+3+\sum_{i=3}^{m}|D_i|\,(n_1(4-i)-2+i)}{(2n_1-1)W(x)} \\
&\leqslant \frac{-3n_1+3+|D_3|\,(n_1+1)+2\cdot|D_4|}{(2n_1-1)W(x)} \\
&\leqslant \frac{-3n_1+3+(n_1+1)+2(n_1-2)}{(2n_1-1)W(x)} \\
&= 0,
\end{aligned}$$

where the second line follows from (9), the third line from (10), and the fifth and seventh lines from our assumption that $n_1 \geqslant 3$. □

To complete the proof of Theorem 3, what remains is to show that $C_1(u;G') > C_1(u;G)$ which contradicts the choice of (G,u). We define

$$\gamma \coloneqq \sum_{u \in \{w\} \cup S'} \frac{2\,|S|}{W(u)W'(u)} - \sum_{u \in \{z\} \cup P'} \frac{2\,|S|}{W(u)W'(u)}.$$

By Lemma 5(v) and the fact that $|S' \cup \{w\}| \geqslant |P' \cup \{z\}|$ whenever $S \neq \varnothing$, we infer that γ is always non-negative (noticing that $\gamma = 0$ if $S = \varnothing$).

Note that

$$C_1(u;G') - C_1(u;G) = \sum_{v\in V}\left[\frac{1}{W'(u)} - \frac{1}{W(u)} - \left(\frac{1}{W'(v)} - \frac{1}{W(v)}\right)\right]$$
$$= \sum_{v\in V}\left[\frac{W(u)-W'(u)}{W(u)W'(u)} - \frac{W(v)-W'(v)}{W(v)W'(v)}\right].$$

For readability, set $f(v) \coloneqq \frac{W(u)-W'(u)}{W(u)W'(u)} - \frac{W(v)-W'(v)}{W(v)W'(v)}$ and $g(v) \coloneqq \frac{1}{W(v)W'(v)}$ for each node $v \in V$.

By Lemma 5(i) and (iii),

$$f(v) = \begin{cases} 2\,|Y|\,(g(u)-g(v)) & \text{if } v \in R \\ 2\,|Y|\,(g(u)-g(v)) + 2\,|S|\,g(v) & \text{if } v \in S' \cup \{w\}. \end{cases}$$

In addition, if $v \in P \cup S$ then $W'(v) \geqslant W(v)$, by Lemma 5(iv), so $f(v) \geqslant 2\,|Y|\,g(u)$. In total, we infer that $C_1(u;G') - C_1(u;G)$ is at least

$$\sum_{v\in Y\cup(\{z\}\cup P')} f(v) + \sum_{v\in R\cup S'\cup\{w\}} 2\,|Y|\cdot(g(u)-g(v)) + 2\,|Y| \sum_{v\in P\cup S} g(u) + \sum_{v\in S'\cup\{w\}} 2\,|S|\cdot g(v).$$

Notice that $g(u) > \frac{1}{W'(u)}\left(\frac{1}{W(u)} - \frac{1}{W(v)}\right)$ for every node $v \in V$. Moreover by Lemma 5(i), (vi), (vii) and Lemma 6 we know that

$$\sum_{v\in Y} f(v) = 2\,|Y| \sum_{v\in Y} g(u) - \sum_{v\in Y}(W(v)-W'(v))g(v)$$
$$\geqslant 2\,|Y| \sum_{v\in Y} g(u) - \frac{1}{W'(u)} \sum_{v\in Y} \frac{W(v)-W'(v)}{W(v)}$$
$$> 2\,|Y| \sum_{v\in Y} g(u) - \frac{|Y|}{W'(u)} \cdot 2C_1(u;G)$$
$$> \frac{2\,|Y|}{W'(u)} \sum_{v\in Y}\left(\frac{1}{W(u)} - \frac{1}{W(v)}\right) - \frac{2\,|Y|\,C_1(u;G)}{W'(u)}.$$

So we infer that $C_1(u;G') - C_1(u;G)$ is greater than

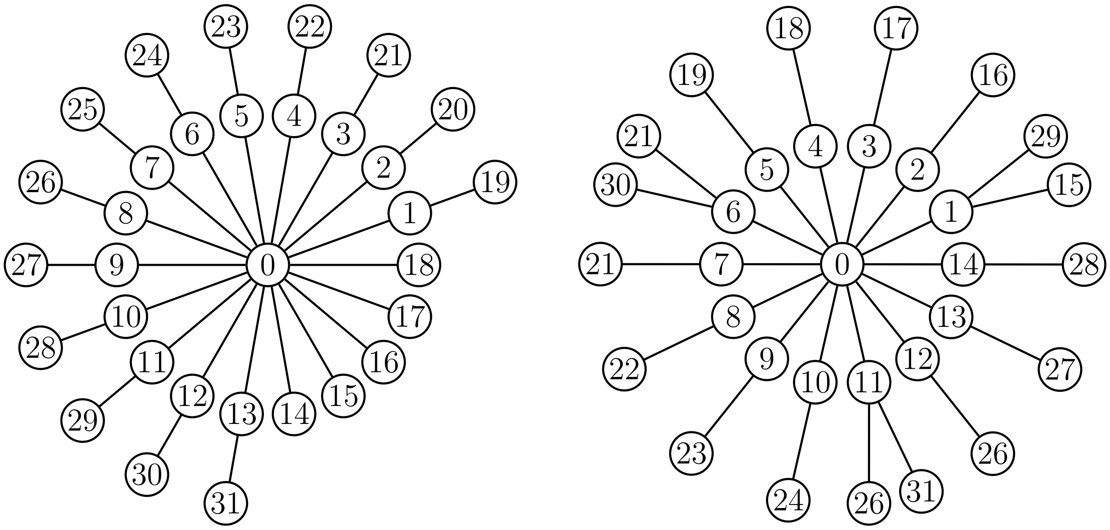
$$\sum_{v\in P'\cup\{z\}} f(v) + 2\,|Y| \sum_{v\in R\cup S'\cup\{w\}} (g(u)-g(v)) + \frac{2\,|Y|}{W'(u)} \sum_{v\in Y\cup P\cup S}\left(\frac{1}{W(u)} - \frac{1}{W(v)}\right)$$
$$+ 2\,|S| \sum_{v\in\{w\}\cup S'} g(v) - 2\,|Y|\,\frac{C_1(u;G)}{W'(u)}.$$

Thanks to Lemma 5(vii), if $v \in R \cup S' \cup \{w\}$ then

$$g(u) - g(v) \geqslant \frac{1}{W'(u)}\left(\frac{1}{W(u)} - \frac{1}{W(v)}\right).$$

In addition, by Lemma 5(ii) if $v \in P' \cup \{z\}$, then

$$f(v) \geqslant 2\,|Y|\,g(u) - 2\,|S|\,g(v) > \frac{2\,|Y|}{W'(u)}\left(\frac{1}{W(u)} - \frac{1}{W(v)}\right) - 2\,|S|\,g(v).$$



(a) $H(0, 14, 18)$ (b) $H(0, 18, 14)$

Figure 6: The two graphs that maximize closeness centralization among all bipartite graphs with partition sizes 14 and 18. Note that in both cases the root is node 0.

Consequently, we deduce that

$$\begin{aligned} C_1(u; G) - C_1(u; G') &> \frac{2\,|Y|}{W'(u)} \sum_{v \in V} \left(\frac{1}{W(u)} - \frac{1}{W(v)} \right) - \frac{2\,|Y|}{W'(u)} C_1(u; G) + \gamma \\ &\geqslant \frac{2\,|Y|}{W'(u)} (C_1(u; G) - C_1(u; G)) \\ &= 0. \end{aligned}$$

This completes the proof of Theorem 3.

3 Concluding remarks and future work

In Figure 2 we have a bipartite network N on 89 edges with partition sizes $|P_1| = 18$ and $|P_2| = 14$ that maximizes closeness centralization at nodes corresponding to "Mrs. Evelyn Jefferson" and to the event from "September 16th", respectively. Their closeness values are approximately equal to 0.0167 and 0.0192, while their closeness centralization values are approximately equal to 0.078 and 0.160, respectively. As shown in the paper, the graphs $H(0, 18, 14)$ and $H(0, 14, 18)$ maximize closeness centralization among all bipartite graphs with partition sizes 11 and 28 (regarding from which partition we are measuring). These graphs are depicted on Figure 6. In both graphs the maximum closeness centralization is attained at the node labeled 0 with values $C_1(H(0, 14, 18), 0) \approx 0.329$ and $C_1(H(0, 11, 28), 0) \approx 0.299$, respectively.

We showed that among all two-mode networks with fixed size bipartitions n_0 and n_1, the largest closeness centralization is achieved by a rooted tree of depth 2, where neighbors of the root have an equal or almost equal number of children, namely at node v of a graph $H(v, n_0, n_1)$. This confirms a conjecture by Everett, Sinclair, and Dankelmann [7] regarding the problem of maximizing closeness centralization in two-mode data, where the number of data of each type is fixed. A similar statement for the centrality measure of eccentricity was recently established [10]. However, the same conjecture remains open for the eigenvalue centrality C_e.

Conjecture 7. *Let $\mathcal{B}(n_0, n_1)$ be the class of all bipartite graphs with bipartition P_0 and P_1, such that $|P_i| = n_i$ for $i \in \{0, 1\}$. Then*

$$\max_{G \in \mathcal{B}(n_0,n_1)} \max_{v \in P_0} C_e(v, G) = C_e\left(v, H\left(v, n_0, n_1\right)\right).$$

A centrality measure $\mathcal{C}$ is said to satisfy the *max-degree property* in the family $\mathcal{F}$ if for every graph $G \in \mathcal{F}$ and every node $v \in V(G)$,

$$\mathcal{C}_G(v) = \max_{u \in V(G)} \mathcal{C}_G(u) \implies \deg_G(v) = \max_{u \in V(G)} \deg_G(u).$$

While degree centrality trivially satisfies the max-degree property in $\mathcal{G}_n$, one can easily observe that this is not true for closeness centrality. Still, it is interesting to observe that the maximizing family for bipartite graphs $H\left(v, |P_0|, |P_1|\right)$ (or stars, for connected graphs $\mathcal{G}_n$ in general) satisfies the max-degree property.

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