Interval Differentiators: on-line estimation of differentiation accuracy

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***Abstract*—In this work an interval observer is proposed for on-line estimation of differentiation errors in some class of high-order differentiators (like a high-gain differentiator from [26], or homogeneous nonlinear differentiator from [23], or super-twisting differentiator [16]). The results are verified and validated on the telescopic link of a robotic arm for forestry applications in which the mentioned approaches are used to estimate the extension velocity while the interval observer gives bounds to this estimation.**

I. Introduction

State estimation is an important problem in many areas of the control engineering science dealing with plant regulation, synchronization or fault detection [2], [20], [11]. In many cases, if the model of the system is highly uncertain, then a design of conventional Luenberger-like observers is not possible and various model-free estimation techniques can be used [10]. Many of them are based on estimation of derivatives since a large class of systems can be transformed in output canonical forms where the state is represented as the output and its derivatives. That is why many differentiation algorithms are proposed in the literature [13], some of them have a form of nonlinear (Luenberger-like) observer [16], [23], [26]. One of the main characteristics of differentiators is their sensitivity or robustness with respect to measurement noise, for almost all existing differentiation techniques there exist estimates providing qualitative [16], [23], and sometimes quantitative [26], estimates of errors caused by a non-differentiable noise presence. Even existent, these estimates are a kind of "worst-case asymptotic" bounds, and more accurate derivations are appreciated in applications.

Interval observers, proposed in [12] and developed for instance in [1], [4], [8], [7], [18], [21], follow the ideas of set-membership estimation theory [14], [15], where for each instant of time a set of admissible values for the state vector is evaluated. The diameter of this set is proportional to the system uncertainty. Thus, the interval observers generate the estimate of the state and simultaneously evaluate the current error of this estimation.

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The objective of this work is to propose an interval observer for estimation error of differentiators from [16], [23], [26], Super Twisting (ST), Homogeneous Differentiator (HOMD) and High Gain Differentiator (HGD) respectively. The coefficients of these nth order differentiators have to be tuned taking into account the maximum value of the $n+1$ derivative of the signal to be differentiated, which is a kind of uncertain signal in the differential equations of estimation error. Another source of uncertainty is the measurement noise, which is supposed to be almost bounded with a known upper and lower bound (*i.e.* bounded for all t for a set of zero Lebesgue measure). Taking all these constraints, the interval observer has to evaluate on-line the set of admissible values for the error of differentiation.

To show the effectiveness of the proposed approach an on-line estimation of the extension velocity is carried out for the telescopic link of a hydraulic actuated industrial crane in which just the position can be measured. Such industrial equipment is widely used in forestry and the automation of tasks is the subject of many researches [22].

II. Preliminaries

Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}$ ($\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$) the symbol $||u||_{[t_0,t_1]}$ denotes its L_∞ norm:

$$||u||_{[t_0,t_1]} = ess\sup\{|u(t)|, t \in [t_0, t_1]\},$$

if $t_1 = +\infty$ then we will simply write $||u||$. We will denote as $\mathcal{L}_\infty$ the set of all inputs u with the property $||u|| < \infty$. Denote the sequence of integers $1, ..., k$ as $\overline{1,k}$. The symbols I_n, $E_{n\times m}$ and E_p denote the identity matrix with dimension $n \times n$, the matrix with all elements equal 1 with dimensions $n \times m$ and $p \times 1$ respectively. For a matrix $A \in \mathbb{R}^{n\times n}$ the vector of its eigenvalues is denoted as $\lambda(A)$.

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elemen-twise. The relation $P \prec 0$ ($P \succ 0$) means that the matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive) definite. Given a matrix $A \in \mathbb{R}^{m\times n}$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ (similarly for vectors) and denote the matrix of absolute values of all elements by $|A| = A^+ + A^-$.

Lemma 1. [6] *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$.*

(1) *If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then*

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \quad (1)$$

(2) *If $A \in \mathbb{R}^{m\times n}$ is a matrix variable and $\underline{A} \le A \le \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then*

$$\begin{aligned} &\underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \le Ax \\ &\le \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-. \end{aligned} \tag{2}$$

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are nonnegative. Any solution of linear system

$$\begin{aligned} \dot{x} &= Ax + B\omega(t),\ \omega : \mathbb{R}_+ \to \mathbb{R}^q_+, \\ y &= Cx + D\omega(t), \end{aligned} \tag{3}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$ and a Metzler matrix $A \in \mathbb{R}^{n\times n}$, is elementwise nonnegative for all $t \ge 0$ provided that $x(0) \ge 0$ and $B \in \mathbb{R}^{n\times q}_+$ [9], [25]. The output solution $y(t)$ is nonnegative if $C \in \mathbb{R}^{p\times n}_+$ and $D \in \mathbb{R}^{p\times q}_+$. Such dynamical systems are called cooperative (monotone) or nonnegative if only initial conditions in $\mathbb{R}^n_+$ are considered [9], [25].

The L_1 and L_∞ gains for nonnegative systems (3) have been studied in [3], [5], for this kind of systems these gains are interrelated.

Lemma 2. [3] *Let the system (3) be nonnegative (i.e. A is Metzler, $B \ge 0$, $C \ge 0$ and $D \ge 0$), then it is asymptotically stable if and only if there exist $\lambda \in \mathbb{R}^n_+ \backslash \{0\}$ and a scalar $\gamma > 0$ such that the following Linear Programming (LP) problem is feasible:*

$$\begin{bmatrix} A\lambda + BE_q \\ C\lambda - \gamma E_p + DE_q \end{bmatrix} < 0.$$

Moreover, in this case the L_∞ gain of the operator $\omega \to y$ is lower than γ.

III. Interval differentiator

The differentiators from [16], [23], [26] can be presented in the following generalized form:

$$\begin{aligned} \dot{x}_i(t) &= -\chi_i[t, x_1(t) - y(t)] + x_{i+1}(t),\ i = \overline{1, n}; \\ \dot{x}_{n+1}(t) &= -\chi_{n+1}[t, x_1(t) - y(t)], \\ y(t) &= f(t) + v(t),\ t \ge 0, \\ x_1(0) &= y(0),\ x_k(0) = 0,\ k = \overline{2, n+1}, \end{aligned} \tag{4}$$

where $x(t) = [x_1(t), \dots, x_{n+1}(t)]^{\mathrm{T}} \in \mathbb{R}^{n+1}$ is the differentiator state; $y(t) \in \mathbb{R}$ is the signal available for measurements, $f(t) \in \mathbb{R}$ is the signal to be differentiated n times and it is supposed that it has $n+1$ derivatives; $v(t) \in \mathbb{R}$ is the measurement noise, $v \in \mathcal{L}_\infty$; the locally bounded functions $\chi_i : \mathbb{R}^2 \to \mathbb{R}$ are varying depending on the differentiator. It is supposed that $x_i(t)$ corresponds to an estimate of $f^{(i-1)}(t)$ for $i = \overline{1, n+1}$, and also the following assumptions are adopted in this work.

Assumption 1. *There is a known $V > 0$ such that $|v(t)| \le V$ for almost all $t \ge 0$.*

Assumption 2. *There are known functions $\underline{f}^{(j)}, \overline{f}^{(j)} \in \mathcal{L}_\infty$, $j = \overline{0, n}$ such that for almost all $t \ge 0$*

$$\begin{aligned} \underline{f}^{(j)}(0) \le f^{(j)}(0) \le \overline{f}^{(j)}(0). \\ \underline{f}^{(n+1)}(t) \le f^{(n+1)}(t) \le \overline{f}^{(n+1)}(t). \end{aligned}$$

Under these assumptions we are going to design an interval observer for differentiation errors

$$e_i(t) = x_i(t) - f^{(i-1)}(t),\ i = \overline{1, n+1},$$

first for a generic order n, and next this solution will be detailed for $n = 1$. Define $e = [e_1, \dots, e_{n+1}]^{\mathrm{T}}$.

A. High order case

The dynamics of differentiation errors can be presented as follows:

$$\begin{aligned} \dot{e}_i(t) &= -\chi_i[t, e_1(t) + v(t)] + e_{i+1}(t),\ i = \overline{1, n}; \\ \dot{e}_{n+1}(t) &= -\chi_{n+1}[t, e_1(t) + v(t)] - f^{(n+1)}(t), \\ \underline{e}_0 &\le e(0) \le \overline{e}_0, \\ \underline{e}_0 &= [-V, -\overline{f}^{(1)}(0) \dots, -\overline{f}^{(n)}(0)]^{\mathrm{T}}, \\ \overline{e}_0 &= [V, -\underline{f}^{(1)}(0) \dots, -\underline{f}^{(n)}(0)]^{\mathrm{T}}, \end{aligned}$$

where the signal $\psi(t) = e_1(t) + v(t)$ is available for measurements, or equivalently

$$\begin{aligned} \dot{e}_i(t) &= \rho_i[t, \psi(t)] - a_i e_1(t) + a_i v(t) + e_{i+1}(t),\ i = \overline{1, n}; \\ \dot{e}_{n+1}(t) &= \rho_{n+1}[t, \psi(t)] - a_{n+1} e_1(t) + a_{n+1} v(t) - f^{(n+1)}(t), \end{aligned}$$

where $\rho_i[t, \psi(t)] = a_i \psi(t) - \chi_i[t, \psi(t)]$ for $i = \overline{1, n+1}$ and the coefficients $a = [a_1, \dots, a_{n+1}]^{\mathrm{T}}$ satisfy the following requirement.

Assumption 3. *The matrix*

$$A = \begin{bmatrix} -a_1 & 1 & 0\dots0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & 0 & 0\dots0 & 1 \\ -a_{n+1} & 0 & 0\dots0 & 0 \end{bmatrix}$$

is Hurwitz and there exists a nonsingular matrix $S \in \mathbb{R}^{(n+1)\times(n+1)}$ such that the matrix $R = S^{-1}AS$ is Metzler.

The conditions of existence of such a S for a Hurwitz matrix A are studied in [24], a time-varying similarity transformation $S(t)$ is proposed in [18]. In the vector representation $e = [e_1, \dots, e_{n+1}]^{\mathrm{T}}$ we obtain

$$\dot{e}(t) = Ae(t) + \rho[t, \psi(t)] + av(t) + bf^{(n+1)}(t), \tag{5}$$

where $\rho(t, \psi) = [\rho_1(t, \psi), \dots, \rho_{n+1}(t, \psi)]^{\mathrm{T}}$ and $b = [0, 0, \dots, 0, -1]^{\mathrm{T}}$. To design an interval observer for (5) we need to transform this system to its positive counterpart [24], for this purpose introduce new coordinates $\epsilon = S^{-1}e$, then

$$\dot{\epsilon}(t) = R\epsilon(t) + \eta[t, \psi(t)] + \alpha v(t) + \beta f^{(n+1)}(t), \tag{6}$$

where $\eta[t, \psi(t)] = S^{-1}\rho[t, \psi(t)]$, $\alpha = S^{-1}a$ and $\beta = S^{-1}b$. Using (1) we obtain:

$$\begin{aligned} \underline{\beta f}^{(n+1)}(t) &\le \beta f^{(n+1)}(t) \le \overline{\beta f}^{(n+1)}(t), \\ \underline{\beta f}^{(n+1)}(t) &= \beta^+ \underline{f}^{(n+1)}(t) - \beta^- \overline{f}^{(n+1)}(t), \\ \overline{\beta f}^{(n+1)}(t) &= \beta^+ \overline{f}^{(n+1)}(t) - \beta^- \underline{f}^{(n+1)}(t), \\ -|\alpha|V &\le \alpha v(t) \le |\alpha|V. \end{aligned}$$

Then an interval observer for (6) takes the form:

$$\dot{\underline{\epsilon}}(t) = R\underline{\epsilon}(t) + \eta[t, \psi(t)] - |\alpha|V + \underline{\beta f}^{(n+1)}(t), \quad (7)$$
$$\dot{\overline{\epsilon}}(t) = R\overline{\epsilon}(t) + \eta[t, \psi(t)] + |\alpha|V + \overline{\beta f}^{(n+1)}(t),$$
$$\underline{\epsilon}(0) = [S^{-1}]^{+}\underline{e}_0 - [S^{-1}]^{-}\overline{e}_0,$$
$$\overline{\epsilon}(0) = [S^{-1}]^{+}\overline{e}_0 - [S^{-1}]^{-}\underline{e}_0,$$

where $\underline{\epsilon}(t)$ and $\overline{\epsilon}(t)$ are lower and upper estimates for the vector $\epsilon(t)$, and

$$\underline{e}(t) = S^{+}\underline{\epsilon}(t) - S^{-}\overline{\epsilon}(t), \quad (8)$$
$$\overline{e}(t) = S^{+}\overline{\epsilon}(t) - S^{-}\underline{\epsilon}(t).$$

Theorem 1. *Let assumptions 1–3 be satisfied. Then in the differentiator (4) the differentiation errors $e_i(t)$, $i = \overline{1, n+1}$ satisfy the inequalities:*

$$\underline{e}(t) \leq e(t) \leq \overline{e}(t) \quad \forall t \geq 0 \quad (9)$$

and $\underline{\epsilon}, \overline{\epsilon}, \underline{e}, \overline{e} \in \mathcal{L}_\infty^n$ in (7), (8) provided that $e_1 \in \mathcal{L}_\infty$.

In general just boundedness of estimates is not enough, and some optimality in (9) should be obtained using, for example, result of Lemma 2. However, it is easy to see that the solving problem is highly nonlinear, and to propose some LMIs for its solutions some constraint have to be imposed or some variables have to be fixed, as in the following result.

Proposition 1. *Giving a Metzler matrix $R \in \mathbb{R}^{(n+1)\times(n+1)}$, let there exist $\lambda \in \mathbb{R}_+^{n+1} \setminus \{0\}$ and a scalar $\gamma > 0$ such that the following LP problem is feasible:*

$$\begin{bmatrix} R\lambda + Xb \\ \lambda - \gamma E_{n+1} \end{bmatrix} < 0, \; Xb \geq 0,$$
$$RX = XA_0 - wc^T, \; c = [1, 0, \ldots, 0]^T,$$
$$A_0 = \begin{bmatrix} 0 & 1 & 0\ldots0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0\ldots0 & 1 \\ 0 & 0 & 0\ldots0 & 0 \end{bmatrix},$$

where $X \in \mathbb{R}^{(n+1)\times(n+1)}$ is a nonsingular matrix and $w \in \mathbb{R}^{n+1}$, then $S = X^{-1}$ and $a = X^{-1}w$ satisfy Assumption 3. In addition, the L_∞ gain of the operator $f^{(n+1)} \to \epsilon$ is lower than γ in (6).

B. First order derivative estimation

Let us consider with more details the case of the first derivative estimation, then

$$\dot{x}_1(t) = -\chi_1[t, x_1(t) - y(t)] + x_2(t),$$
$$\dot{x}_2(t) = -\chi_2[t, x_1(t) - y(t)], \quad (10)$$
$$y(t) = f(t) + v(t), \; t \geq 0,$$
$$x_1(0) = y(0), \; x_2(0) = 0,$$

and let assumptions 1 and 2 be satisfied. To check Assumption 3 note that

$$A = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix}$$

and for any $a_1 > 0$ and $a_2 > 0$ it is Hurwitz and has eigenvalues

$$\lambda(A) = \frac{1}{2}\begin{bmatrix} -a_1 + \sqrt{a_1^2 - 4a_2} \\ -a_1 - \sqrt{a_1^2 - 4a_2} \end{bmatrix},$$

which are real and distinct for $a_1 \geq 2\sqrt{a_2}$. The corresponding eigenvectors form the matrix for given $r_1 > 0$, $r_2 > 0$,

$$S = \begin{bmatrix} \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2r_1 a_2} & \frac{\sqrt{a_1^2 - 4a_2} - a_1}{2r_2 a_2} \\ r_1^{-1} & -r_2^{-1} \end{bmatrix},$$

which admits the conditions of Assumptions 3:

$$S^{-1}AS = R = \frac{1}{2}\begin{bmatrix} -a_1 - \sqrt{a_1^2 - 4a_2} & 0 \\ 0 & -a_1 + \sqrt{a_1^2 - 4a_2} \end{bmatrix}.$$

Since

$$S^{-1} = \begin{bmatrix} \frac{a_2 r_1}{\sqrt{a_1^2 - 4a_2}} & \frac{r_1}{2}\left(1 - \frac{a_1}{\sqrt{a_1^2 - 4a_2}}\right) \\ \frac{a_2 r_2}{\sqrt{a_1^2 - 4a_2}} & -\frac{r_2}{2}\left(1 + \frac{a_1}{\sqrt{a_1^2 - 4a_2}}\right) \end{bmatrix},$$

then

$$\beta = S^{-1}b = \begin{bmatrix} \frac{r_1}{2}\left(\frac{a_1}{\sqrt{a_1^2 - 4a_2}} - 1\right) \\ \frac{r_2}{2}\left(1 + \frac{a_1}{\sqrt{a_1^2 - 4a_2}}\right) \end{bmatrix} \geq 0$$

and the pair (R, β) forms a nonnegative system. According to Lemma 2, this system has L_∞ gain of the transfer function $f^{(n+1)} \to \epsilon$ less than $\gamma > 0$ if for some $\lambda \in \mathbb{R}^2$, $\lambda > 0$ we have

$$\begin{bmatrix} R\lambda + \beta \\ \lambda - \gamma E_p \end{bmatrix} < 0,$$

but this LP problem has always a solution if the following restrictions on r_1 and r_2 are satisfied:

$$r_1 < \sqrt{a_1^2 - 4a_2}\frac{a_1 + \sqrt{a_1^2 - 4a_2}}{a_1 - \sqrt{a_1^2 - 4a_2}}\gamma,$$
$$r_2 < \sqrt{a_1^2 - 4a_2}\frac{a_1 - \sqrt{a_1^2 - 4a_2}}{a_1 + \sqrt{a_1^2 - 4a_2}}\gamma$$

for a given $\gamma > 0$.

From (8), it is necessary to minimize L_∞ norms of S^+ and S^- to ensure a good L_∞ gain for the transfer $[\underline{f}^{(n+1)} \; \overline{f}^{(n+1)}] \to [\underline{e}^{\mathrm{T}} \; \overline{e}^{\mathrm{T}}]$. For this purpose, define

$$r_1 = \varsigma_1\sqrt{a_1^2 - 4a_2}\frac{a_1 + \sqrt{a_1^2 - 4a_2}}{a_1 - \sqrt{a_1^2 - 4a_2}}\gamma,$$
$$r_2 = \varsigma_2\sqrt{a_1^2 - 4a_2}\frac{a_1 - \sqrt{a_1^2 - 4a_2}}{a_1 + \sqrt{a_1^2 - 4a_2}}\gamma$$

for some $\varsigma_1, \varsigma_2 \in (0, 1)$, then

$$S^+ = \frac{a_1 - \sqrt{a_1^2 - 4a_2}}{\gamma\varsigma_1\sqrt{a_1^2 - 4a_2}}\begin{bmatrix} \frac{1}{2a_2} & 0 \\ \frac{1}{a_1 + \sqrt{a_1^2 - 4a_2}} & 0 \end{bmatrix},$$
$$S^- = \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{\gamma\varsigma_2\sqrt{a_1^2 - 4a_2}}\begin{bmatrix} 0 & \frac{1}{2a_2} \\ 0 & \frac{1}{a_1 - \sqrt{a_1^2 - 4a_2}} \end{bmatrix}$$

and

$$||S^{+}||_{\infty} = \frac{a_1 - \sqrt{a_1^2 - 4a_2}}{\gamma \varsigma_1 \sqrt{a_1^2 - 4a_2}} \max\left\{\frac{1}{2a_2}, \frac{1}{a_1 + \sqrt{a_1^2 - 4a_2}}\right\},$$

$$||S^{-}||_{\infty} = \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{\gamma \varsigma_2 \sqrt{a_1^2 - 4a_2}} \max\left\{\frac{1}{2a_2}, \frac{1}{a_1 - \sqrt{a_1^2 - 4a_2}}\right\}.$$

In order to minimize these norms it is necessary to take $\varsigma_1 = \varsigma_2 \simeq 1$ and since for $\varsigma_1 = \varsigma_2$

$$||S^{-}||_{\infty} > ||S^{+}||_{\infty},$$

then the problem of minimization of the function

$$\varpi(a_1, a_2) = \max\{\varpi_1(a_1, a_2), \varpi_2(a_1, a_2)\}$$

$$\varpi_1(a_1, a_2) = \frac{1}{a_2} \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2\sqrt{a_1^2 - 4a_2}},$$

$$\varpi_2(a_1, a_2) = \frac{2}{a_1 - \sqrt{a_1^2 - 4a_2}} \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{2\sqrt{a_1^2 - 4a_2}}$$

can be posed. Computing the partial derivatives of ϖ_i, $i = 1, 2$ we obtain that $\frac{\partial \varpi_1}{\partial a_1}$ and $\frac{\partial \varpi_2}{\partial a_1}$ are always negative and $\frac{\partial \varpi_1}{\partial a_2} = 0$ for

$$a_2 = \frac{3}{16} a_1^2,$$

$\frac{\partial \varpi_2}{\partial a_2} = 0$ for

$$a_2 = \frac{1 + \sqrt{2}}{6 + 4\sqrt{2}} a_1^2.$$

Both these solutions correspond to minimums of ϖ_i, $i = 1, 2$. After substitution of this optimal selection of a_2 in ϖ_i we obtain

$$\varpi(a_1) = a^{-1} \max\left\{\frac{5\sqrt{2} + 7}{\sqrt{2\sqrt{2} + 3}}, 8a_1^{-1}\right\},$$

then

$$a_2 = a_1^2 \begin{cases} \frac{3}{16} & \text{if } a_1 \leq 8\frac{\sqrt{2}+1}{5\sqrt{2}+7}, \\ \frac{1+\sqrt{2}}{6+4\sqrt{2}} & \text{otherwise.} \end{cases} \quad (11)$$

Therefore, increasing the value of a_1 and taking a_2 from (11) reduces the L_∞ gain for the transfer $[\underline{f}^{(n+1)} \ \overline{f}^{(n+1)}] \to [\underline{e}^{\mathrm{T}} \ \overline{e}^{\mathrm{T}}]$, but increasing a_1 and a_2 augments the same gain with respect to the noise $v(t)$. To evaluate the gain with respect to noise, note that

$$\alpha = S^{-1} a = \gamma \frac{a_2}{2} \begin{bmatrix} \varsigma_1 \frac{\left(a_1 + \sqrt{a_1^2 - 4a_2}\right)^2}{a_1 - \sqrt{a_1^2 - 4a_2}} \\ \varsigma_2 \frac{\left(a_1 - \sqrt{a_1^2 - 4a_2}\right)^2}{a_1 + \sqrt{a_1^2 - 4a_2}} \end{bmatrix}$$

is a nonnegative vector, thus L_∞ gain $\gamma_v > 0$ of the transfer $v \to \epsilon$ can be evaluated using Lemma 2 as follows:

$$\begin{bmatrix} R\lambda + \alpha \\ \lambda - \gamma_v E_p \end{bmatrix} < 0, \ \lambda > 0,$$

where $\lambda \in \mathbb{R}^2$. This LP problem has a solution if

$$\gamma a_2 \varsigma_1 \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{a_1 - \sqrt{a_1^2 - 4a_2}} < \lambda_1 < \gamma_v,$$

$$\gamma a_2 \varsigma_2 \frac{a_1 - \sqrt{a_1^2 - 4a_2}}{a_1 + \sqrt{a_1^2 - 4a_2}} < \lambda_2 < \gamma_v$$

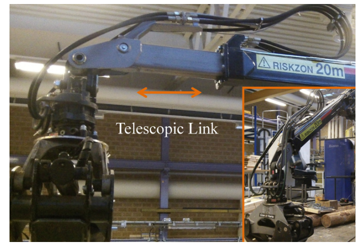

Figure 1. Hydraulic Forestry Crane.

that implies

$$\gamma_v = \gamma a_2 \frac{a_1 + \sqrt{a_1^2 - 4a_2}}{a_1 - \sqrt{a_1^2 - 4a_2}}.$$

Thus, L_∞ gain of the error $\epsilon(t)$ with respect to the noise $v(t)$ is higher (worse) than that with respect to $f^{(n+1)}(t)$. Substituting (11) we obtain

$$\gamma_v = \gamma a_1^2 \begin{cases} \frac{9}{16} & \text{if } a_1 \leq 8\frac{\sqrt{2}+1}{5\sqrt{2}+7}, \\ 0.5 & \text{otherwise.} \end{cases}$$

Normally in applications $||v|| \ll ||f^{(n+1)}||$, thus it is reasonable to limit the value of a_1 using the last expression assuming that the influence of the noise $v(t)$ on the errors should not exceed the influence of $f^{(n+1)}(t)$, *i.e.* $0.5\gamma a_1^2 V \leq \gamma ||f^{(n+1)}||$, then

$$a_1 \leq \sqrt{2\frac{||f^{(n+1)}||}{V}}. \quad (12)$$

IV. FORESTRY-STANDARD MOBILE-HYDRAULIC CRANE

The on-line velocity estimation problem is an important issue in mobile hydraulics where instrumentation is limited. The system under study is the telescopic link of an industrial hydraulic forestry crane, see Fig. 1, which consists of a double-acting single-side hydraulic cylinder and a solid load which is attached to a piston of the cylinder. Such industrial equipment is widely used in forestry and the automation is a subject of many researches, see [22].

The position of the link x varies from 0 to 1.55m; positive velocity $\dot{x}$ corresponds to extraction of the cylinder. This link can be described as a 1-DOF mechanical system actuated by a hydraulic force, and the equation of the motion is

$$m\ddot{x} = f_h - f_g - f_{fric}, \quad (13)$$

where m is the mass, f_h is the generated hydraulic force, f_g is the gravity force and f_{fric} is the friction force. The friction is modeled as a Coulomb friction plus a viscous friction: $f_{fric} = f_c \operatorname{sign}(\dot{x}) + f_v \dot{x}$. The force generated by the hydraulics is presented below:

$$f_h = P_a A_a - P_b A_b, \quad (14)$$

where the piston areas A_a and A_b are known geometric parameters, P_a and P_b are the measured pressures in chambers A and B of the cylinder. The dynamics of the pressures is given, see [19, Sec. 3.8], by

$$\dot{P}_a = \frac{\beta}{V_a(x)}\left(-\dot{x}A_a + q_a\right), \quad \dot{P}_b = \frac{\beta}{V_b(x)}\left(\dot{x}A_b - q_b\right), \quad (15)$$

where $V_a(x) = V_{a0}+x\, A_a$ and $V_b(x) = V_{b0}-x\, A_b$ are volumes of the chambers A and B at the given piston position x, V_{a0} and V_{b0} are known geometric constants, β is a known bulk modulus, q_a and q_b are flows to the chamber A and from the chamber B.

A. Bounds of $\ddot{x}$

Differentiating (14) and substituting (15) leads to

$$\dot{x} = \eta_0(x, q_a, q_b) - \eta_1(x)\, \dot{f}_h,$$

where

$$\eta_0(x, q_a, q_b) = \frac{A_a\, V_b(x)\, q_a + A_b\, V_a(x)\, q_b}{A_a^2\, V_b(x) + A_b^2\, V_a(x)},$$
$$\eta_1(x) = \frac{V_a(x)\, V_b(x)\, \beta^{-1}}{A_a^2\, V_b(x) + A_b^2\, V_a(x)}.$$

Note that since x is bounded, η_0 and η_1 are bounded. Substituting the equation above, $\dot{x}$, in (13) it follows

$$\ddot{x} = -c_0(x, \dot{x}, q_a, q_b) + c_1 f_h + c_2(x)\, \dot{f}_h - f_g, \quad (16)$$

with $c_0(\cdot) = \frac{f_c}{m}\text{sign}\dot{x} + \frac{f_v}{m}\eta_0$, $c_1 = \frac{1}{m}$, $c_2(\cdot) = -\frac{f_v}{m}\eta_1$. The pressures are measured with installed pressure transducers that allow the hydraulic force to be estimated, equation (14), and precisely this measurement in conjunction with equation (16) provides the lower and upper bounds for the second derivative as follows:

$$-L(t) \leq \ddot{x} \leq L(t)$$

where $L(t)$ is a continuous positive function

$$L(t) = \gamma_0 + \gamma_1|f_h| + \gamma_2\zeta(f_h),$$

where parameters γ_0, γ_1 and γ_2 are positive constants; the rate of variation of f_h, is given by $\zeta(f_h)$, which is a positive function that depends on the available pressure measurements. One option is:

$$\zeta(f_h) = \frac{|f_h(t-\tau_1) - f_h(t-\tau_2)|}{\tau_2 - \tau_1},$$

with $\tau_2 > \tau_1 > 0$.

Both pressures P_a and P_b are bounded by the tank pressure P_t and the supply pressure P_s. However it is not a realistic practical situation when both pressures have extreme contrary values simultaneously. Due to internal restrictions the practical bound is $|f_h| \leq \bar{f}_h$. Both flows q_a and q_b are bounded by a factory-set level of a maximum flow through a valve, $|q_{a,b}| \leq \bar{q}$. Moreover, the flows cannot go in the same direction simultaneously, *i.e.* they always are of the same sign. A practical bound of the velocity is $|\dot{x}| \leq 1.1\text{m/s}$, obtained by experiments. From measurements an off-line estimation of $\ddot{x}$ is obtained and $L(t)$ is computed as shown in red in Fig. 2 that indeed overcomes $\ddot{x}$.

B. Simulations and Experimental results

The proposed interval observer (7) has been tested for three differentiators, The first one proposed in [17] by Levant (ST),

$$\begin{aligned} \dot{\zeta}_1(t) &= -1.5L(t)|e(t)|^{0.5}\text{sign}(e(t)) + \zeta_2(t) \\ \dot{\zeta}_2(t) &= -1.1L(t)\text{sign}(e(t)) \end{aligned}, \quad (17)$$

the second one proposed by Vázquez *et al.* [27] (HGD),

$$\begin{aligned} \dot{\zeta}_1(t) &= -\bar{\alpha}_1 e(t) + \zeta_2(t) \\ \dot{\zeta}_2(t) &= -\bar{\alpha}_2 e(t) - 1.1L(t)\text{sign}(e(t)) \end{aligned}, \quad (18)$$

the last one presented in [23] by Perruquetti *et al.* (HOMD) has the form

$$\begin{aligned} \dot{\zeta}_1(t) &= -\underline{\alpha}_1|e(t)|^{0.75}\text{sign}(e(t)) + \zeta_2(t) \\ \dot{\zeta}_2(t) &= -\underline{\alpha}_2|e(t)|^{0.5}\text{sign}(e(t)) \end{aligned}. \quad (19)$$

The differentiators have clearly the structure presented in (4), in each of them $e(t) = \zeta_1(t) - y(t)$; they should process position data from the robotic platform Forestry-Standard Mobile-Hydraulic crane. The position of the telescopic link is measured with a wire-actuated encoder. The encoder provides 2381 counts for the range from 0 to 1.55m; the quantization interval is $Q = 0.651\text{mm}$. The measured signal x represents the position signal with an additive uniform noise with a variance $\frac{Q^2}{12}$. It is worth to remark that the proposed method works *on-line* and the differentiation by spline is just a priori step which allows to characterize the variables $V = 0.0005$ and $|\ddot{x}| \leq 104$ adopted in the experiments.

Then, a_1 and a_2 are chosen following (11) and (12), for this particular set of experiments $a_1 = 150$. It is worth to remark that the coefficients for the three differentiators must be chosen to achieve the best performances from the differentiators themselves. In the experiments the coefficients are $\bar{\alpha}_1 = 165$, $\bar{\alpha}_2 = 5638$, $\underline{\alpha}_1 = 45$ and $\underline{\alpha}_2 = 12\underline{\alpha}_1$. In Fig. 2 the behavior of the three differentiators are presented in black and the performances of the interval differentiator are shown for each of them for the entire length of the dataset that is $120s$ ($\bar{\epsilon}(t)$ in red and $\underline{\epsilon}(t)$ in blue (8)) . Two different zooming options are shown in Fig. 3 for particular parts of the dataset in which the change of velocity is abrupt, it can be clearly seen that the interval observer gives the upper and lower bound to the estimation following the velocity profile keeping the actual estimation in between as desired.

To quantify the performances of the interval observer let us introduce two variables $\Gamma^-(t) = |\bar{\epsilon}(t) - \underline{\epsilon}(t)|/2$ and $\Gamma^+(t) = |\bar{\epsilon}(t) + \underline{\epsilon}(t)|/2$, with the respective means and standard deviations whose values are shown in Table I. The variable Γ^- reveals that the proposed Interval Observer is independent from the differentiator used as soon it has the form specified in (4), indeed the values for μ_{Γ^-} and σ_{Γ^-} are equals for the three different techniques. Moreover, Γ^+ gives information about the quality of the interval observer: the lower μ_{Γ^+} the better the overall differentiator quality, the lower σ_{Γ^+} the better is the behavior with respect to the oscillation.

V. Conclusion

This paper presents the construction of an interval observer for the estimation error for differentiation techniques. The

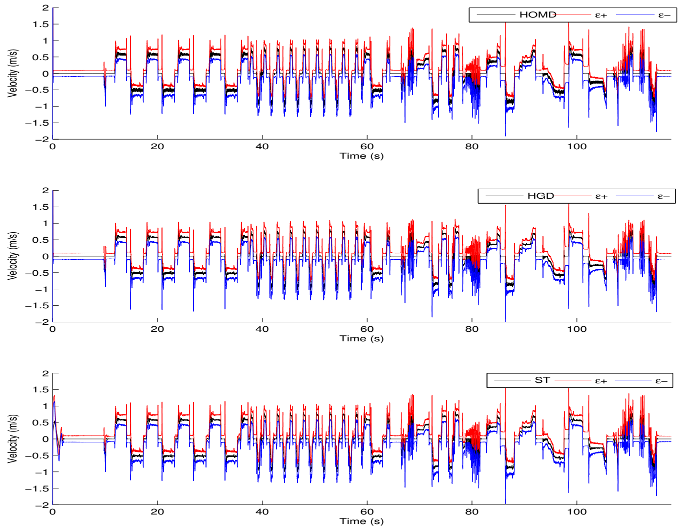

Figure 2. Interval differentiators performances for Homogeneus Differentiator (HOMD), High Gain Differentaitor (HGD) and Super Twisting (ST)

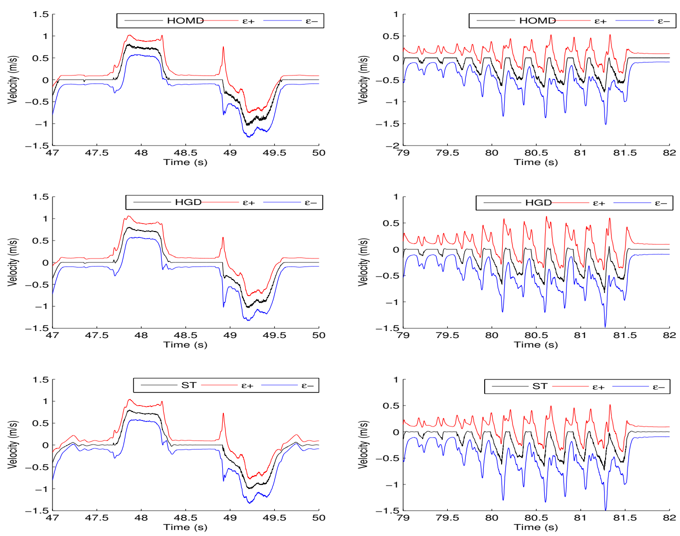

Figure 3. Interval differentiators performances (zooming)

main results are presented for the high order case whereas, for the application and experiments, a first order derivative estimation is carried out considering three different techniques.

The method has been applied on the velocity estimation on the telescopic link of a hydraulic actuated robotic crane used in forestry. The results obtained show the efficiency of the proposed method which bounds the error of estimation and it is shown to be independent with respect to the differentiation method chosen.

	$\mu_{\Gamma-}$	$\sigma_{\Gamma-}$	$\mu_{\Gamma+}$	$\sigma_{\Gamma+}$
Levant [17]	0.1776	0.1193	0.0229	0.0937
Vázquez [27]	0.1776	0.1193	0.0099	0.0761
Perruquetti [23]	0.1776	0.1193	0.0235	0.0893

Table I
COMPARISON BETWEEN THE THREE INTERVAL OBSERVER PERFORMANCES

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