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COCO: The Bi-objective Black Box Optimization Benchmarking (bbob-biobj) Test Suite

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Abstract

The `bbob-biobj` test suite contains 55 bi-objective functions in continuous domain which are derived from combining functions of the well-known single-objective noiseless `bbob` test suite. Besides giving the actual function definitions and presenting their (known) properties, this documentation also aims at giving the rationale behind our approach in terms of function groups, instances, and potential objective space normalization.

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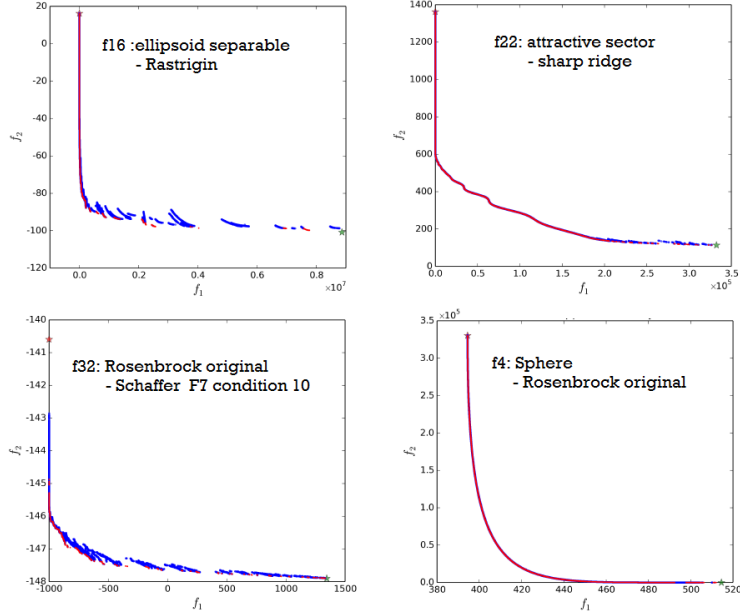


Fig. 1: Example plots of the Pareto front approximation, found by NSGA-II on selected bbob-biobj functions. In blue the non-dominated points at the end of different independent runs, in red the points that are non-dominated over all runs.

1 Introduction

In the following, we consider bi-objective, unconstrained **minimization** problems of the form

$$\min_{x \in \mathbb{R}^D} f(x) = (f_\alpha(x), f_\beta(x)),$$

where D is the number of variables of the problem (also called the problem dimension), $f_\alpha : \mathbb{R}^D \rightarrow \mathbb{R}$ and $f_\beta : \mathbb{R}^D \rightarrow \mathbb{R}$ are the two objective functions, and the min operator is related to the standard *dominance relation*. A solution $x \in \mathbb{R}^D$ is thereby said to *dominate* another solution $y \in \mathbb{R}^D$ if $f_\alpha(x) \leq f_\alpha(y)$ and $f_\beta(x) \leq f_\beta(y)$ hold and at least one of the inequalities is strict.

Solutions which are not dominated by any other solution in the search space are called *Pareto-optimal* or *efficient solutions*. All Pareto-optimal solutions constitute the *Pareto set* of which an approximation is sought. The Pareto set's image in the objective space $f(\mathbb{R}^D)$ is called *Pareto front*.

The objective of the minimization problem is to find, with as few evaluations of f as possible, a set of non-dominated solutions which is (i) as large as possible and (ii) has f -values as close to the Pareto front as possible.¹

¹ Distance in f -space is defined here such that nadir and ideal point have in each coordinate distance one. Neither of these points is however freely accessible to the optimization algorithm.

1.1 Definitions and Terminology

We remind in this section different definitions.

function instance, problem Each function within COCO $f^\theta : \mathbb{R}^D \rightarrow \mathbb{R}^m$ is parametrized with parameter values $\theta \in \Theta$. A parameter value determines a so-called *function instance*. For example, θ encodes the location of the optimum of single-objective functions, which means that different instances have shifted optima. In the `bbob-biobj` test suite, $m = 2$ and the function instances are determined by the instances of the underlying single-objective functions.

A *problem* is a function instance of a specific dimension D .

ideal point The ideal point is defined as the vector in objective space that contains the optimal f -value for each objective *independently*. More precisely let $f_\alpha^{\text{opt}} := \inf_{x \in \mathbb{R}^D} f_\alpha(x)$ and $f_\beta^{\text{opt}} := \inf_{x \in \mathbb{R}^D} f_\beta(x)$, the ideal point is given by

$$z_{\text{ideal}} = (f_\alpha^{\text{opt}}, f_\beta^{\text{opt}}).$$

nadir point The *nadir point* (in objective space) consists in each objective of the worst value obtained by a Pareto-optimal solution. More precisely, let \mathcal{PO} be the set of Pareto optimal points. Then the nadir point satisfies

$$z_{\text{nadir}} = \left(\sup_{x \in \mathcal{PO}} f_\alpha(x), \sup_{x \in \mathcal{PO}} f_\beta(x) \right).$$

In the case of two objectives with a unique global minimum each (that is, a single point in the search space maps to the global minimum)

$$z_{\text{nadir}} = (f_\alpha(x_{\text{opt},\beta}), f_\beta(x_{\text{opt},\alpha})),$$

where $x_{\text{opt},\alpha} = \arg \min f_\alpha(x)$ and $x_{\text{opt},\beta} = \arg \min f_\beta(x)$.

2 Overview of the Proposed `bbob-biobj` Test Suite

The `bbob-biobj` test suite provides 55 bi-objective functions in six dimensions (2, 3, 5, 10, 20, and 40) with a large number of possible instances. The 55 functions are derived from combining a subset of the 24 well-known single-objective functions of the `bbob` test suite [HAN2009] which has been used since 2009 in the `BBOB workshop series`. While concrete details on each of the 55 `bbob-biobj` functions are given in Section *The `bbob-biobj` Test Functions and Their Properties*, we will detail here the main rationale behind them together with their common properties.

2.1 The Single-objective `bbob` Functions

The `bbob-biobj` test suite is designed to be able to assess performance of algorithms with respect to well-identified difficulties in optimization typically occurring in real-world problems. A

multi-objective problem being a combination of single-objective problems, one can obtain multi-objective problems with representative difficulties by simply combining single objective functions with representative difficulties observed in real-world problems. For this purpose we naturally use the single-objective `bbob` suite [HAN2009].

Combining all 24 `bbob` functions in pairs thereby results in $24^2 = 576$ bi-objective functions overall. We however assume that multi-objective optimization algorithms are not sensitive to permutations of the objective functions such that combining the 24 `bbob` functions and taking out the function (g_2, g_1) if the function (g_1, g_2) is present results in $24 + \binom{24}{2} = 24 + (24 \times 23)/2 = (24 \times 25)/2 = 300$ functions.

Some first tests, e.g. in [BRO2015], showed that having 300 functions is impracticable in terms of the overall running time of the benchmarking experiment. We then decided to exploit the organization of the `bbob` functions into classes to choose a subset of functions. More precisely, the 24 original `bbob` functions are grouped into five function classes where each class gathers functions with similar properties, namely

1. separable functions
2. functions with low or moderate conditioning
3. functions with high conditioning and unimodal
4. multi-modal functions with adequate global structure,
5. multi-modal functions with weak global structure.

To create the `bbob-biobj` suite, we choose two functions within each class. This way we do not introduce any bias towards a specific class. In addition within each class, the functions are chosen to be the most representative without repeating similar functions. For example, only one Ellipsoid, one Rastrigin, and one Gallagher function are included in the `bbob-biobj` suite although they appear in separate versions in the `bbob` suite. Finally our choice of 10 `bbob` functions for creating the `bbob-biobj` test suite is the following:

- Separable functions
 - Sphere (function 1 in `bbob` suite)
 - Ellipsoid separable (function 2 in `bbob` suite)
- Functions with low or moderate conditioning
 - Attractive sector (function 6 in `bbob` suite)
 - Rosenbrock original (function 8 in `bbob` suite)
- Functions with high conditioning and unimodal
 - Sharp ridge (function 13 in `bbob` suite)
 - Sum of different powers (function 14 in `bbob` suite)
- Multi-modal functions with adequate global structure

- Rastrigin (function 15 in `bbob suite`)
- Schaffer F7, condition 10 (function 17 in `bbob suite`)
- Multi-modal functions with weak global structure
 - Schwefel $x \cdot \sin(x)$ (function 20 in `bbob suite`)
 - Gallagher 101 peaks (function 21 in `bbob suite`)

Using the above described pairwise combinations, this results in having $10 + \binom{10}{2} = 55$ bi-objective functions in the final *bbob-biobj* suite. These functions are denoted f_1 to f_{55} in the sequel.

2.2 Function Groups

From combining the original `bbob` function classes, we obtain 15 function classes to structure the 55 bi-objective functions of the `bbob-biobj` test suite. Each function class contains three or four functions. We are listing below the function classes and in parenthesis the functions that belong to the respective class:

1. separable - separable (f_1, f_2, f_{11})
2. separable - moderate (f_3, f_4, f_{12}, f_{13})
3. separable - ill-conditioned (f_5, f_6, f_{14}, f_{15})
4. separable - multi-modal (f_7, f_8, f_{16}, f_{17})
5. separable - weakly-structured ($f_9, f_{10}, f_{18}, f_{19}$)
6. moderate - moderate (f_{20}, f_{21}, f_{28})
7. moderate - ill-conditioned ($f_{22}, f_{23}, f_{29}, f_{30}$)
8. moderate - multi-modal ($f_{24}, f_{25}, f_{31}, f_{32}$)
9. moderate - weakly-structured ($f_{26}, f_{27}, f_{33}, f_{34}$)
10. ill-conditioned - ill-conditioned (f_{35}, f_{36}, f_{41})
11. ill-conditioned - multi-modal ($f_{37}, f_{38}, f_{42}, f_{43}$)
12. ill-conditioned - weakly-structured ($f_{39}, f_{40}, f_{44}, f_{45}$)
13. multi-modal - multi-modal (f_{46}, f_{47}, f_{50})
14. multi-modal - weakly structured ($f_{48}, f_{49}, f_{51}, f_{52}$)
15. weakly structured - weakly structured (f_{53}, f_{54}, f_{55})

More details about the single functions can be found in Section *The bbob-biobj Test Functions and Their Properties*. We however first describe their common properties in the coming sections.

2.3 Normalization of Objectives

None of the 55 `bbob-biobj` functions is explicitly normalized and the optimization algorithms therefore have to cope with objective values in different ranges. Typically, different orders of magnitude between the objective values can be observed.

However, to facilitate comparison of algorithm performance over different functions, we normalize the objectives based on the ideal and nadir points before calculating the hypervolume indicator [BRO2016biperf]. Both points can be computed, because the global optimum is known and is unique for the 10 `bbob` base functions. In the black-box optimization benchmarking setup, however, the values of the ideal and nadir points are not accessible to the optimization algorithm [HAN2016ex].

2.4 Instances

Our test functions are parametrized and instances are instantiations of the underlying parameters (see [HAN2016co]). The instances for the bi-objective functions are using instances of each single objective function composing the bi-objective one. In addition, we assert two conditions:

1. The Euclidean distance between the two single-objective optima (also called the extreme optimal points) in the search space is at least 10^{-4} .
2. The Euclidean distance between the ideal and the nadir point in the non-normalized objective space is at least 10^{-1} .

We associate to an instance, an instance-id which is an integer. The relation between the instance-id, K_{id}^f , of a bi-objective function $f = (f_\alpha, f_\beta)$ and the instance-ids, $K_{id}^{f_\alpha}$ and $K_{id}^{f_\beta}$, of its underlying single-objective functions f_α and f_β is the following:

- $K_{id}^{f_\alpha} = 2K_{id}^f + 1$ and
- $K_{id}^{f_\beta} = K_{id}^{f_\alpha} + 1$

If we find that above conditions are not satisfied for all dimensions and functions in the `bbob-biobj` suite, we increase the instance-id of the second objective successively until both properties are fulfilled. For example, the `bbob-biobj` instance-id 8 corresponds to the instance-id 17 for the first objective and instance-id 18 for the second objective while for the `bbob-biobj` instance-id 9, the first instance-id is 19 but for the second objective, instance-id 21 is chosen instead of instance-id 20.

Exceptions to the above rule are, for historical reasons, the `bbob-biobj` instance-ids 1 and 2 in order to match the instance-ids 1 to 5 with the ones proposed in [BRO2015]. The `bbob-biobj` instance-id 1 contains the single-objective instance-ids 2 and 4 and the `bbob-biobj` instance-id 2 contains the two instance-ids 3 and 5.

For each bi-objective function and given dimension, the `bbob-biobj` suite contains 10 in-

stances.²

3 The bbob–biobj Test Functions and Their Properties

In the following, we detail all 55 bbob–biobj functions and their properties.

The following table gives an overview and quick access to the functions, inner cell IDs refer to the bbob–biobj functions, outer column and row annotations refer to the single-objective bbob functions.

	f_1	f_2	f_6	f_8	f_{13}	f_{14}	f_{15}	f_{17}	f_{20}	f_{21}
f_1	$f1$	$f2$	$f3$	$f4$	$f5$	$f6$	$f7$	$f8$	$f9$	$f10$
f_2		$f11$	$f12$	$f13$	$f14$	$f15$	$f16$	$f17$	$f18$	$f19$
f_6			$f20$	$f21$	$f22$	$f23$	$f24$	$f25$	$f26$	$f27$
f_8				$f28$	$f29$	$f30$	$f31$	$f32$	$f33$	$f34$
f_{13}					$f35$	$f36$	$f37$	$f38$	$f39$	$f40$
f_{14}						$f41$	$f42$	$f43$	$f44$	$f45$
f_{15}							$f46$	$f47$	$f48$	$f49$
f_{17}								$f50$	$f51$	$f52$
f_{20}									$f53$	$f54$
f_{21}										$f55$

3.1 Some Function Properties

In the description of the 55 bbob–biobj functions below, several general properties of objective functions will be mentioned that are defined here in short. It depends on these properties whether the optimization problem is easy or hard to solve.

A *separable* function does not show any dependencies between the variables and can therefore be solved by applying D consecutive one-dimensional optimizations along the coordinate axes while keeping the other variables fixed. Consequently, *non-separable* problems must be considered. They are much more difficult to solve. The typical well-established technique to generate non-separable functions from separable ones is the application of a rotation matrix \mathbf{R} to x , that is $x \in \mathbb{R}^D \mapsto g(\mathbf{R}x)$, where g is a separable function.

A *unimodal* function has only one local minimum which is at the same time also its global one. A *multimodal* function has at least two local minima which is highly common in practical optimization problems.

² In principle, as for the instance generation for the bbob suite, the number of possible instances for the bbob–biobj suite is unlimited [HAN2016co]. However, running some tests with too few instances will render the potential statistics and their interpretation problematic while even the tiniest observed difference can be made statistically significant with a high enough number of instances. A good compromise to avoid either pitfall seems to lie between, say, 9 and 19 instances.

Ill-conditioning is another typical challenge in real-parameter optimization and, besides multimodality, probably the most common one. In a general case, we can consider a function as ill-conditioned if for solution points from the same level-set “the minimal displacement [...] that produces a given function value improvement differs by orders of magnitude” [HAN2011]. Conditioning can be rigorously formalized in the case of convex quadratic functions, $f(x) = \frac{1}{2}x^T Hx$ where H is a symmetric positive definite matrix, as the condition number of the Hessian matrix H . Since contour lines associated to a convex quadratic function are ellipsoids, the condition number corresponds to the square root of the ratio between the largest axis of the ellipsoid and the shortest axis.

The proposed `bbob-biobj` testbed contains ill-conditioned functions with a typical conditioning of 10^6 . We believe this is a realistic requirement, while we have seen practical problems with conditioning as large as 10^{10} .

3.2 Domain Bounds

All bi-objective functions provided in the `bbob-biobj` suite are unbounded, i.e., defined on the entire real-valued space \mathbb{R}^D . The search domain of interest is defined as $[-100, 100]^D$, outside of which non-dominated solutions are quite unlikely to be found.³ The majority of non-dominated solutions are likely to lie even within $[-5, 5]^D$.

While we believe that the domain of interest contains the Pareto set, due to the nature of the `bbob-biobj` function definitions, there is no guarantee that this is always the case. However, the extremal solutions and their neighborhood ball of radius one are guaranteed to lie within $[-5, 5]^D$.

3.3 The 55 `bbob-biobj` Functions

3.3.1 f_1 : Sphere/Sphere

Combination of two sphere functions (f_1 in the `bbob` suite).

Both objectives are unimodal, highly symmetric, rotational and scale invariant. The Pareto set is known to be a straight line and the Pareto front is convex. Considered as the simplest bi-objective problem in continuous domain.

Contained in the *separable - separable* function class.

3.3.2 f_2 : Sphere/Ellipsoid separable

Combination of the sphere function (f_1 in the `bbob` suite) and the separable ellipsoid function (f_2 in the `bbob` suite).

³ The functions `coco_problem_get_smallest_value_of_interest` and `coco_problem_get_largest_value_of_interest` of the COCO platform allow the optimizer to retrieve the *search domain of interest* from the `coco_problem_t`, for example to generate the initial search points.

Both objectives are unimodal and separable. While the first objective is truly convex-quadratic with a condition number of 1, the second objective is only globally quadratic with smooth local irregularities and highly ill-conditioned with a condition number of about 10^6 .

Contained in the *separable - separable* function class.

3.3.3 f_3 : Sphere/Attractive sector

Combination of the sphere function (f_1 in the *bbob suite*) and the attractive sector function (f_6 in the *bbob suite*).

Both objective functions are unimodal, but only the first objective is separable and truly convex quadratic. The attractive sector function is highly asymmetric, where only one *hypercone* (with angular base area) with a volume of roughly $(1/2)^D$ yields low function values. The optimum of it is located at the tip of this cone.

Contained in the *separable - moderate* function class.

3.3.4 f_4 : Sphere/Rosenbrock original

Combination of the sphere function (f_1 in the *bbob suite*) and the original, i.e., unrotated Rosenbrock function (f_8 in the *bbob suite*).

The first objective is separable and truly convex, the second objective is partially separable (tri-band structure). The first objective is unimodal while the second objective has a local optimum with an attraction volume of about 25%.

Contained in the *separable - moderate* function class.

3.3.5 f_5 : Sphere/Sharp ridge

Combination of the sphere function (f_1 in the *bbob suite*) and the sharp ridge function (f_{13} in the *bbob suite*).

Both objective functions are unimodal. In addition to the simple, separable, and differentiable first objective, a sharp, i.e., non-differentiable ridge has to be followed for optimizing the (non-separable) second objective. The gradient towards the ridge remains constant, when the ridge is approached from a given point. Approaching the ridge is initially effective, but becomes ineffective close to the ridge when the ridge needs to be followed in direction to its optimum. The necessary change in *search behavior* close to the ridge is difficult to diagnose, because the gradient towards the ridge does not flatten out.

Contained in the *separable - ill-conditioned* function class.

3.3.6 f_6 : Sphere/Sum of different powers

Combination of the sphere function (f_1 in the `bbob` suite) and the sum of different powers function (f_{14} in the `bbob` suite).

Both objective functions are unimodal. The first objective is separable, the second non-separable. When approaching the second objective's optimum, the difference in sensitivity between different directions in search space increases unboundedly.

Contained in the *separable - ill-conditioned* function class.

3.3.7 f_7 : Sphere/Rastrigin

Combination of the sphere function (f_1 in the `bbob` suite) and the Rastrigin function (f_{15} in the `bbob` suite).

In addition to the simple sphere function, the prototypical highly multimodal Rastrigin function needs to be solved which has originally a very regular and symmetric structure for the placement of the optima. Here, however, transformations are performed to alleviate the original symmetry and regularity in the second objective.

The properties of the second objective contain non-separability, multimodality (roughly 10^D local optima), a conditioning of about 10, and a large global amplitude compared to the local amplitudes.

Contained in the *separable - multi-modal* function class.

3.3.8 f_8 : Sphere/Schaffer F7, condition 10

Combination of the sphere function (f_1 in the `bbob` suite) and the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite).

In addition to the simple sphere function, an asymmetric, non-separable, and highly multimodal function needs to be solved to approach the Pareto front/Pareto set where the frequency and amplitude of the modulation in the second objective vary. The conditioning of the second objective and thus the entire bi-objective function is low.

Contained in the *separable - multi-modal* function class.

3.3.9 f_9 : Sphere/Schwefel $x \cdot \sin(x)$

Combination of the sphere function (f_1 in the `bbob` suite) and the Schwefel function (f_{20} in the `bbob` suite).

While the first objective function is separable and unimodal, the second objective function is partially separable and highly multimodal—having the most prominent 2^D minima located comparatively close to the corners of the unpenalized search area.

Contained in the *separable - weakly-structured* function class.

3.3.10 f_{10} : Sphere/Gallagher 101 peaks

Combination of the sphere function (f_1 in the *bbob suite*) and the Gallagher function with 101 peaks (f_{21} in the *bbob suite*).

While the first objective function is separable and unimodal, the second objective function is non-separable and consists of 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function). The conditioning around the global optimum of the second objective function is about 30.

Contained in the *separable - weakly-structured* function class.

3.3.11 f_{11} : Ellipsoid separable/Ellipsoid separable

Combination of two separable ellipsoid functions (f_2 in the *bbob suite*).

Both objectives are unimodal, separable, only globally quadratic with smooth local irregularities, and highly ill-conditioned with a condition number of about 10^6 .

Contained in the *separable - separable* function class.

3.3.12 f_{12} : Ellipsoid separable/Attractive sector

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the attractive sector function (f_6 in the *bbob suite*).

Both objective functions are unimodal but only the first one is separable. The first objective function, in addition, is globally quadratic with smooth local irregularities, and highly ill-conditioned with a condition number of about 10^6 . The second objective function is highly asymmetric, where only one *hypercone* (with angular base area) with a volume of roughly $(1/2)^D$ yields low function values. The optimum of it is located at the tip of this cone.

Contained in the *separable - moderate* function class.

3.3.13 f_{13} : Ellipsoid separable/Rosenbrock original

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the original, i.e., unrotated Rosenbrock function (f_8 in the *bbob suite*).

Only the first objective is separable and unimodal. The second objective is partially separable (tri-band structure) and has a local optimum with an attraction volume of about 25%. In addition, the first objective function shows smooth local irregularities from a globally convex quadratic function and is highly ill-conditioned with a condition number of about 10^6 .

Contained in the *separable - moderate* function class.

3.3.14 f_{14} : Ellipsoid separable/Sharp ridge

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the sharp ridge function (f_{13} in the *bbob suite*).

Both objective functions are unimodal but only the first one is separable.

The first objective is globally quadratic but with smooth local irregularities and highly ill-conditioned with a condition number of about 10^6 . For optimizing the second objective, a sharp, i.e., non-differentiable ridge has to be followed.

Contained in the *separable - ill-conditioned* function class.

3.3.15 f_{15} : Ellipsoid separable/Sum of different powers

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the sum of different powers function (f_{14} in the *bbob suite*).

Both objective functions are unimodal but only the first one is separable.

The first objective is globally quadratic but with smooth local irregularities and highly ill-conditioned with a condition number of about 10^6 . When approaching the second objective's optimum, the sensitivities of the variables in the rotated search space become more and more different.

Contained in the *separable - ill-conditioned* function class.

3.3.16 f_{16} : Ellipsoid separable/Rastrigin

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the Rastrigin function (f_{15} in the *bbob suite*).

The objective functions show rather opposite properties. The first one is separable, the second not. The first one is unimodal, the second highly multimodal (roughly 10^D local optima). The first one is highly ill-conditioning (condition number of 10^6), the second one has a conditioning of about 10. Local non-linear transformations are performed in both objective functions to alleviate the original symmetry and regularity of the two baseline functions.

Contained in the *separable - multi-modal* function class.

3.3.17 f_{17} : Ellipsoid separable/Schaffer F7, condition 10

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the Schaffer F7 function with condition number 10 (f_{17} in the *bbob suite*).

Also here, both single objectives possess opposing properties. The first objective is unimodal, besides small local non-linearities symmetric, separable and highly ill-conditioned while the second objective is highly multi-modal, asymmetric, and non-separable, with only a low conditioning.

Contained in the *separable - multi-modal* function class.

3.3.18 f_{18} : Ellipsoid separable/Schwefel $x*\sin(x)$

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the Schwefel function (f_{20} in the *bbob suite*).

The first objective is unimodal, separable and highly ill-conditioned. The second objective is partially separable and highly multimodal—having the most prominent 2^D minima located comparatively close to the corners of the unpenalized search area.

Contained in the *separable - weakly-structured* function class.

3.3.19 f_{19} : Ellipsoid separable/Gallagher 101 peaks

Combination of the separable ellipsoid function (f_2 in the *bbob suite*) and the Gallagher function with 101 peaks (f_{21} in the *bbob suite*).

While the first objective function is separable, unimodal, and highly ill-conditioned (condition number of about 10^6), the second objective function is non-separable and consists of 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function). The conditioning around the global optimum of the second objective function is about 30.

Contained in the *separable - weakly-structured* function class.

3.3.20 f_{20} : Attractive sector/Attractive sector

Combination of two attractive sector functions (f_6 in the *bbob suite*). Both functions are unimodal and highly asymmetric, where only one *hypercone* (with angular base area) per objective with a volume of roughly $(1/2)^D$ yields low function values. The objective functions' optima are located at the tips of those two cones.

Contained in the *moderate - moderate* function class.

3.3.21 f_{21} : Attractive sector/Rosenbrock original

Combination of the attractive sector function (f_6 in the *bbob suite*) and the Rosenbrock function (f_8 in the *bbob suite*).

The first function is unimodal but highly asymmetric, where only one *hypercone* (with angular base area) with a volume of roughly $(1/2)^D$ yields low function values (with the optimum at the tip of the cone). The second objective is partially separable (tri-band structure) and has a local optimum with an attraction volume of about 25%.

Contained in the *moderate - moderate* function class.

3.3.22 f_{22} : Attractive sector/Sharp ridge

Combination of the attractive sector function (f_6 in the *bbob* suite) and the sharp ridge function (f_{13} in the *bbob* suite).

Both objective functions are unimodal and non-separable. The first objective is highly asymmetric in the sense that only one *hypercone* (with angular base area) with a volume of roughly $(1/2)^D$ yields low function values (with the optimum at the tip of the cone). For optimizing the second objective, a sharp, i.e., non-differentiable ridge has to be followed.

Contained in the *moderate - ill-conditioned* function class.

3.3.23 f_{23} : Attractive sector/Sum of different powers

Combination of the attractive sector function (f_6 in the *bbob* suite) and the sum of different powers function (f_{14} in the *bbob* suite).

Both objective functions are unimodal and non-separable. The first objective is highly asymmetric in the sense that only one *hypercone* (with angular base area) with a volume of roughly $(1/2)^D$ yields low function values (with the optimum at the tip of the cone). When approaching the second objective's optimum, the sensitivities of the variables in the rotated search space become more and more different.

Contained in the *moderate - ill-conditioned* function class.

3.3.24 f_{24} : Attractive sector/Rastrigin

Combination of the attractive sector function (f_6 in the *bbob* suite) and the Rastrigin function (f_{15} in the *bbob* suite).

Both objectives are non-separable, and the second one is highly multi-modal (roughly 10^D local optima) while the first one is unimodal. Further properties are that the first objective is highly asymmetric and the second has a conditioning of about 10.

Contained in the *moderate - multi-modal* function class.

3.3.25 f_{25} : Attractive sector/Schaffer F7, condition 10

Combination of the attractive sector function (f_6 in the `bbob` suite) and the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite).

Both objectives are non-separable and asymmetric. While the first objective is unimodal, the second one is a highly multi-modal function with a low conditioning where frequency and amplitude of the modulation vary.

Contained in the *moderate - multi-modal* function class.

3.3.26 f_{26} : Attractive sector/Schwefel $x \cdot \sin(x)$

Combination of the attractive sector function (f_6 in the `bbob` suite) and the Schwefel function (f_{20} in the `bbob` suite).

The first objective is non-separable, unimodal, and asymmetric. The second objective is partially separable and highly multimodal—having the most prominent 2^D minima located comparatively close to the corners of the unpenalized search area.

Contained in the *moderate - weakly-structured* function class.

3.3.27 f_{27} : Attractive sector/Gallagher 101 peaks

Combination of the attractive sector function (f_6 in the `bbob` suite) and the Gallagher function with 101 peaks (f_{21} in the `bbob` suite).

Both objective functions are non-separable but only the first is unimodal. The first objective function is furthermore asymmetric. The second objective function has 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function). The conditioning around the global optimum of the second objective function is about 30.

Contained in the *moderate - weakly-structured* function class.

3.3.28 f_{28} : Rosenbrock original/Rosenbrock original

Combination of two Rosenbrock functions (f_8 in the `bbob` suite).

Both objectives are partially separable (tri-band structure) and have a local optimum with an attraction volume of about 25%.

Contained in the *moderate - moderate* function class.

3.3.29 f_{29} : Rosenbrock original/Sharp ridge

Combination of the Rosenbrock function (f_8 in the `bbob` suite) and the sharp ridge function (f_{13} in the `bbob` suite).

The first objective function is partially separable (tri-band structure) and has a local optimum with an attraction volume of about 25%. The second objective is unimodal and non-separable and, for optimizing it, a sharp, i.e., non-differentiable ridge has to be followed.

Contained in the *moderate - ill-conditioned* function class.

3.3.30 f_{30} : Rosenbrock original/Sum of different powers

Combination of the Rosenbrock function (f_8 in the `bbob` suite) and the sum of different powers function (f_{14} in the `bbob` suite).

The first objective function is partially separable (tri-band structure) and has a local optimum with an attraction volume of about 25%. The second objective function is unimodal and non-separable. When approaching the second objective's optimum, the sensitivities of the variables in the rotated search space become more and more different.

Contained in the *moderate - ill-conditioned* function class.

3.3.31 f_{31} : Rosenbrock original/Rastrigin

Combination of the Rosenbrock function (f_8 in the `bbob` suite) and the Rastrigin function (f_{15} in the `bbob` suite).

The first objective function is partially separable (tri-band structure) and has a local optimum with an attraction volume of about 25%. The second objective function is non-separable and highly multi-modal (roughly 10^D local optima).

Contained in the *moderate - multi-modal* function class.

3.3.32 f_{32} : Rosenbrock original/Schaffer F7, condition 10

Combination of the Rosenbrock function (f_8 in the `bbob` suite) and the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite).

The first objective function is partially separable (tri-band structure) and has a local optimum with an attraction volume of about 25%. The second objective function is non-separable, asymmetric, and highly multi-modal with a low conditioning where frequency and amplitude of the modulation vary.

Contained in the *moderate - multi-modal* function class.

3.3.33 f_{33} : Rosenbrock original/Schwefel $x \cdot \sin(x)$

Combination of the Rosenbrock function (f_8 in the `bbob` suite) and the Schwefel function (f_{20} in the `bbob` suite).

Both objective functions are partially separable. While the first objective function has a local optimum with an attraction volume of about 25%, the second objective function is highly multimodal—having the most prominent 2^D minima located comparatively close to the corners of its unpenalized search area.

Contained in the *moderate - weakly-structured* function class.

3.3.34 f_{34} : Rosenbrock original/Gallagher 101 peaks

Combination of the Rosenbrock function (f_8 in the `bbob` suite) and the Gallagher function with 101 peaks (f_{21} in the `bbob` suite).

The first objective function is partially separable, the second one non-separable. While the first objective function has a local optimum with an attraction volume of about 25%, the second objective function has 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function). The conditioning around the global optimum of the second objective function is about 30.

Contained in the *moderate - weakly-structured* function class.

3.3.35 f_{35} : Sharp ridge/Sharp ridge

Combination of two sharp ridge functions (f_{13} in the `bbob` suite).

Both objective functions are unimodal and non-separable and, for optimizing them, two sharp, i.e., non-differentiable ridges have to be followed.

Contained in the *ill-conditioned - ill-conditioned* function class.

3.3.36 f_{36} : Sharp ridge/Sum of different powers

Combination of the sharp ridge function (f_{13} in the `bbob` suite) and the sum of different powers function (f_{14} in the `bbob` suite).

Both functions are uni-modal and non-separable. For optimizing the first objective, a sharp, i.e., non-differentiable ridge has to be followed. When approaching the second objective's optimum, the sensitivities of the variables in the rotated search space become more and more different.

Contained in the *ill-conditioned - ill-conditioned* function class.

3.3.37 f_{37} : Sharp ridge/Rastrigin

Combination of the sharp ridge function (f_{13} in the `bbob` suite) and the Rastrigin function (f_{15} in the `bbob` suite).

Both functions are non-separable. While the first one is unimodal and non-differentiable at its ridge, the second objective function is highly multi-modal (roughly 10^D local optima).

Contained in the *ill-conditioned - multi-modal* function class.

3.3.38 f_{38} : Sharp ridge/Schaffer F7, condition 10

Combination of the sharp ridge function (f_{13} in the `bbob` suite) and the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite).

Both functions are non-separable. While the first one is unimodal and non-differentiable at its ridge, the second objective function is asymmetric and highly multi-modal with a low conditioning where frequency and amplitude of the modulation vary.

Contained in the *ill-conditioned - multi-modal* function class.

3.3.39 f_{39} : Sharp ridge/Schwefel $x \cdot \sin(x)$

Combination of the sharp ridge function (f_{13} in the `bbob` suite) and the Schwefel function (f_{20} in the `bbob` suite).

While the first objective function is unimodal, non-separable, and non-differentiable at its ridge, the second objective function is highly multimodal—having the most prominent 2^D minima located comparatively close to the corners of its unpenalized search area.

Contained in the *ill-conditioned - weakly-structured* function class.

3.3.40 f_{40} : Sharp ridge/Gallagher 101 peaks

Combination of the sharp ridge function (f_{13} in the `bbob` suite) and the Gallagher function with 101 peaks (f_{21} in the `bbob` suite).

Both objective functions are non-separable. While the first objective function is unimodal and non-differentiable at its ridge, the second objective function has 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function). The conditioning around the global optimum of the second objective function is about 30.

Contained in the *ill-conditioned - weakly-structured* function class.

3.3.41 f_{41} : Sum of different powers/Sum of different powers

Combination of two sum of different powers functions (f_{14} in the `bbob` suite).

Both functions are uni-modal and non-separable where the sensitivities of the variables in the rotated search space become more and more different when approaching the objectives' optima.

Contained in the *ill-conditioned - ill-conditioned* function class.

3.3.42 f_{42} : Sum of different powers/Rastrigin

Combination of the sum of different powers functions (f_{14} in the `bbob` suite) and the Rastrigin function (f_{15} in the `bbob` suite).

Both objective functions are non-separable. While the first one is unimodal, the second objective function is highly multi-modal (roughly 10^D local optima).

Contained in the *ill-conditioned - multi-modal* function class.

3.3.43 f_{43} : Sum of different powers/Schaffer F7, condition 10

Combination of the sum of different powers functions (f_{14} in the `bbob` suite) and the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite).

Both objective functions are non-separable. While the first one is unimodal with an increasing conditioning once the optimum is approached, the second objective function is asymmetric and highly multi-modal with a low conditioning where frequency and amplitude of the modulation vary.

Contained in the *ill-conditioned - multi-modal* function class.

3.3.44 f_{44} : Sum of different powers/Schwefel $x \cdot \sin(x)$

Combination of the sum of different powers functions (f_{14} in the `bbob` suite) and the Schwefel function (f_{20} in the `bbob` suite).

Both objectives are non-separable. While the first objective function is unimodal, the second objective function is highly multimodal—having the most prominent 2^D minima located comparatively close to the corners of its unpenalized search area.

Contained in the *ill-conditioned - weakly-structured* function class.

3.3.45 f_{45} : Sum of different powers/Gallagher 101 peaks

Combination of the sum of different powers functions (f_{14} in the `bbob` suite) and the Gallagher function with 101 peaks (f_{21} in the `bbob` suite).

Both objective functions are non-separable. While the first objective function is unimodal, the second objective function has 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function). The conditioning around the global optimum of the second objective function is about 30.

Contained in the *ill-conditioned - weakly-structured* function class.

3.3.46 f_{46} : Rastrigin/Rastrigin

Combination of two Rastrigin functions (f_{15} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal (roughly 10^D local optima).

Contained in the *multi-modal - multi-modal* function class.

3.3.47 f_{47} : Rastrigin/Schaffer F7, condition 10

Combination of the Rastrigin function (f_{15} in the `bbob` suite) and the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal.

Contained in the *multi-modal - multi-modal* function class.

3.3.48 f_{48} : Rastrigin/Schwefel $x*\sin(x)$

Combination of the Rastrigin function (f_{15} in the `bbob` suite) and the Schwefel function (f_{20} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal where the first has roughly 10^D local optima and the most prominent 2^D minima of the second objective function are located comparatively close to the corners of its unpenalized search area.

Contained in the *multi-modal - weakly-structured* function class.

3.3.49 f_{49} : Rastrigin/Gallagher 101 peaks

Combination of the Rastrigin function (f_{15} in the `bbob` suite) and the Gallagher function with 101 peaks (f_{21} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal where the first has roughly 10^D local optima and the second has 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function).

Contained in the *multi-modal - weakly-structured* function class.

3.3.50 f_{50} : Schaffer F7, condition 10/Schaffer F7, condition 10

Combination of two Schaffer F7 functions with condition number 10 (f_{17} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal.

Contained in the *multi-modal - multi-modal* function class.

3.3.51 f_{51} : Schaffer F7, condition 10/Schwefel $x \cdot \sin(x)$

Combination of the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite) and the Schwefel function (f_{20} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal. While frequency and amplitude of the modulation vary in an almost regular fashion in the first objective function, the second objective function possesses less global structure.

Contained in the *multi-modal - weakly-structured* function class.

3.3.52 f_{52} : Schaffer F7, condition 10/Gallagher 101 peaks

Combination of the Schaffer F7 function with condition number 10 (f_{17} in the `bbob` suite) and the Gallagher function with 101 peaks (f_{21} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal. While frequency and amplitude of the modulation vary in an almost regular fashion in the first objective function, the second has 101 optima with position and height being unrelated and randomly chosen (different for each instantiation of the function).

Contained in the *multi-modal - weakly-structured* function class.

3.3.53 f_{53} : Schwefel $x \cdot \sin(x)$ /Schwefel $x \cdot \sin(x)$

Combination of two Schwefel functions (f_{20} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal where the most prominent 2^D minima of each objective function are located comparatively close to the corners of its unpenalized search area. Due to the combinatorial nature of the Schwefel function, it is likely in low dimensions that the Pareto set goes through the origin of the search space.

Contained in the *weakly-structured - weakly-structured* function class.

3.3.54 f_{54} : Schwefel $x \cdot \sin(x)$ /Gallagher 101 peaks

Combination of the Schwefel function (f_{20} in the `bbob` suite) and the Gallagher function with 101 peaks (f_{21} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal. For the first objective function, the most prominent 2^D minima are located comparatively close to the corners of its unpenalized search area. For the second objective, position and height of all 101 optima are unrelated and randomly chosen (different for each instantiation of the function).

Contained in the *weakly-structured - weakly-structured* function class.

3.3.55 f_{55} : Gallagher 101 peaks/Gallagher 101 peaks

Combination of two Gallagher functions with 101 peaks (f_{21} in the `bbob` suite).

Both objective functions are non-separable and highly multi-modal. Position and height of all 101 optima in each objective function are unrelated and randomly chosen and thus, no global structure is present.

Contained in the *weakly-structured - weakly-structured* function class.

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