Frequency Domain Analysis of Control System Based on Implicit Lyapunov Function*

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***Abstract*— A frequency domain analysis of implicit Lyapunov function-based control system [14] is developed for the case of finite-time stabilizing feedback law. The Gang of Four and loop transfer function are considered for practical implementation of the control via frequency domain control design. The effectiveness of this control scheme is demonstrated on an illustrative example of roll control for a vectored thrust aircraft.**

I. INTRODUCTION

Modern control theory based on state space approach to feedback design usually does not provide frequency domain analysis, which is applied in control engineering practice. Frequently, this is the reason why control engineers use completely different tools than scientists. Due to simplicity of tuning and frequency analysis, PID controllers are still the most popular feedbacks in the process control today. Therefore, there exists a need in the study of existing nonlinear control systems in the sense of simplicity of practical realization and applicability by engineers.

The quantitative feedback theory is one of the most popular tools for frequency domain approach to control system design (see for example, [6]–[12]), when the performance of a system is given in terms of the frequency response between an input and output. The set of transfer functions called the Gang of Six (Gang of Four if the feedback is restricted to operate on the error signal) [6], [11], [12] is commonly used to determine performances of feedback systems, such as the ability to follow reference signals, effects of measurement noise and load disturbances and effects of process variations. Thus, analysis, shaping and designing of the Gang of Six (Gang of Four) are subjects of intensive researches in the last years; e.g., classical PID-based or lead-lag control techniques ([6]–[8]), closed- and open-loop shaping ([11]–[13]), etc. Moreover, the analysis of the Gang of Four also gives a good explanation for the fact that control systems can be designed based on simplified models [6]. This issue is extremely useful as engineers usually do not know the complete object model. Multiple Input Multiple Output (MIMO) processes can also be considered by regarding frequency domain design methods [6], [10].

Modern technological processes need control systems of very high quality. The modern control theory should provide effective tools for tuning of different performance indexes of the control system. For example, a lot of controlled processes must have a finite or prescribed transient time. That is why, finite-time stability and stabilization problems have been intensively studied last years (see, for example, [1]–[5]).

The paper [14] is devoted to development of a finite-time control for multiple integrators together with implicit Lyapunov function of closed-loop system. The problem of control design for chain of integrators is quite significant due to the fact that nominal models have the form of multiple integrators in many applications (for instance, different mechanical and electromechanical systems, see [16], [17]). An extension of results [14] for MIMO systems is presented in [15].

In order to make a proper assessment of a feedback law developed in [14] and to provide tuning rules, the present paper provides the analysis of all transfer functions in the Gang of Four. The control in [14] is essentially nonlinear and even non-Lipschitz, so frequency domain analysis of the Gang of Four and loop transfer function is complicated. Fortunately, the control law is designed based on Implicit Lyapunov Function (ILF) technique [14], which allows us to interpret the nonlinear feedback as a family of linear feedbacks properly parameterized by means of the Lyapunov function, and to develop a frequency domain analysis of the ILF-based control system. A solution of the roll control design problem for a vectored thrust aircraft is presented as an example of the analysis applicability.

For processes with multiple inputs and multiple outputs [15], the obtained results may be easily extended.

The paper is organized as follows. Notation used in the paper is presented in Section II. Some preliminaries about finite-time stability, finite-time stabilization method of the multiple integrators system and homogeneity are considered in Section III. Problem statement is introduced in Section IV. Section V presents some aspects for practical implementation of the finite-time stabilization method. Frequency domain analysis of the finite-time control scheme is presented in Section VI. Section VII presents problem solution of roll control for a vectored thrust aircraft based on frequency domain analysis. Finally, conclusions with some remarks and possible directions for further works are given.

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II. NOTATION

Through the paper the following notation will be used:

- $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x > 0\}$, where $\mathbb{R}$ is the set of real number;
- the inequality $P > 0$ ($P < 0, P \geq 0, P \leq 0$) means that $P = P^T \in \mathbb{R}^{n\times n}$ is symmetric and positive (negative) definite (semi-definite);
- the brackets $\lfloor \cdot \rfloor$ mean rounding up to the nearest integer downwards;
- $diag\left\{\lambda_i\right\}_{i=1}^{n}$ is the diagonal matrix with the elements λ_i on the main diagonal;
- a continuous function $\sigma\colon \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ belongs to the class $\mathcal{K}$ if $\sigma(0) = 0$ and the function is strictly increasing.

III. PRELIMINARIES

A. Finite-time Stability

Denote a nonlinear vector field $f(t,x)\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, which can be discontinuous with respect to the state variable $x \in \mathbb{R}^n$. Then consider the system of the form

$$\dot{x} = f(t,x), \quad x(0) = x_0, \tag{1}$$

the solutions $x(t,x_0)$ of which are interpreted in the sense of Filippov [18] $f(t,0) = 0$.

According to Filippov definition [18] an absolutely continuous function $x(t,x_0)$ is the Cauchy problem solution associated to (1) if $x(0,x_0) = x_0$ and it satisfies the following differential inclusion

$$\dot{x} \in K[f](t,x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(N)=0} co\ f(t, B(x,\varepsilon)\backslash N),$$

where $co(M)$ is the convex closure of the set M, $B(x,\varepsilon)$ is the ball of the radius ε with the center at $x \in \mathbb{R}^n$ and the equality $\mu(N) = 0$ means that the set N has zero measure.

Assume that the origin is an equilibrium point of the system (1) and it has uniqueness of solutions in forward time.

Definition 1 ([19], [20], [21]) The origin of system (1) is globally finite-time stable if:

- **Finite-time attractivity**: there exists a function T : $\mathbb{R}^n\backslash\{0\} \to \mathbb{R}_+$ called settling time function such that $\lim_{t\to T(x_0)} x(t,x_0) = 0$ for any $x_0 \in \mathbb{R}^n\backslash\{0\}$.
- **Lyapunov stability**: there exists a function $\delta \in \mathcal{K}$ such that $\|x(t,x_0)\| \leq \delta(\|x_0\|)$ for any $x_0 \in \mathbb{R}^n$.

Notice that finite-time stability assumes an "infinite eigenvalue assignation" for the system at the origin.

B. Homogeneity

Homogeneity [22]–[24] is an intrinsic property of an objects such as functions or vector fields, which remains consistent with respect to some scaling operation called a dilation.

For fixed $r_i \in \mathbb{R}_+$, $i = \overline{1,n}$ and $\lambda > 0$ one can define the vector of weights $r = (r_1, ..., r_n)^T$ and the dilation matrix $D(\lambda) = diag\{\lambda^{r_i}\}_{i=1}^{n}$. Note that $D(\lambda)x = (\lambda^{r_1}x_1, ..., \lambda^{r_n}x_n)^T$ represents a mapping $x \mapsto D(\lambda)x$ called a dilation for $x \in \mathbb{R}^n$.

Definition 2 [24] A function $g\colon \mathbb{R}^n \to \mathbb{R}$ (vector field $f\colon \mathbb{R}^n \to \mathbb{R}^n$) is said to be r-homogeneous of degree m iff $g(D(\lambda)x) = \lambda^m g(x)$ ($f(D(\lambda)x) = \lambda^m D(\lambda) f(x)$) for all $\lambda > 0$ and $x \in \mathbb{R}^n$.

Theorem 1 [25] *Let f be a r-homogeneous continuous vector field on $\mathbb{R}^n$ with a negative degree. Then if the system $\dot{x} = f(x)$ is a locally asymptotically stable it is globally finite-time stable.*

In addition the homogeneity theory provides many advantages to analysis and design of nonlinear control system (for example, see [26], [27]).

C. Finite-time Stabilization for Multiple Integrators

Consider a single input control system of the following form

$$\dot{x} = Ax + bu + d(t,x), \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is the control input,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and the function $d(t,x)\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ describes the system uncertainties and disturbances. Note that for $d(t,x) \equiv 0$ the system (2) describes a chain of integrators.

Denote Lyapunov function V implicitly defined by the following function

$$Q(V,x) := x^T D(V^{-1}) P_Q D(V^{-1}) x - 1, \tag{3}$$

where $P_Q = P_Q^T \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix and $D(\lambda)$ is the dilation matrix of the form

$$D(\lambda) = diag\{\lambda^{1+(n-i)\mu}\}, \quad 0 < \mu \leq 1.$$

Denote the matrix $H_\mu = diag\left\{1 + (n-i)\mu\right\}_{i=1}^{n}$.

Theorem 2 [14], [15]. *If*

1) the system of matrix inequalities:

$$\begin{cases} AX + XA^T + by + y^T b^T + H_\mu X + X H_\mu + \beta I_n \leq 0, \\ X H_\mu + H_\mu X > 0, \quad X > 0, \end{cases} \tag{4}$$

is feasible for some $\mu \in (0,1]$, $\beta \in (0,1)$, $X \in \mathbb{R}^{n\times n}$ and $y \in \mathbb{R}^{1\times n}$;

2) the control $u(V,x)$ has the form

$$u(V,x) = V^{1-\mu} k D(V^{-1}) x, \tag{5}$$

where $k = (k_1, ..., k_n) = yX^{-1}$,

$$V \in \mathbb{R}_+ \colon Q(V,x) = 0$$

and $Q(V,x)$ is presented by (3) with $P_Q = X^{-1}$;

3) the disturbance function $d(t,x)$ satisfies the following inequality

$$\begin{aligned} &d^T(t,x) (D(V^{-1}))^2 d(t,x) \leq \\ &\beta^2 V^{-2\mu} x^T D(V^{-1}) (H_\mu P_Q + P_Q H_\mu) D(V^{-1}) x; \end{aligned} \tag{6}$$

Then the system (2) is globally finite-time stable and the settling time function estimate has the form

$$T_{st}(x_0) \le \frac{V_0^{\mu}}{\mu(1-\beta)}, \tag{7}$$

where $V_0 \in \mathbb{R}_+ \colon Q(V_0, x_0) = 0$.

In disturbance-free case β tends to zero, the inequality (6) gives $d(t,x) \equiv 0$, and the conditions of Theorem 2 coincide with Theorem 3 in [14].

IV. PROBLEM STATEMENT

A block diagram of a basic feedback scheme for systems where feedback is restricted to operate on the error signal is shown in Fig. 1, where P is the process to be controlled, C is the control block, u is the control signal, η is the process output, y is the plant output, and the external signals are denoted by the reference signal r, the measurement noise n and the load disturbance d.

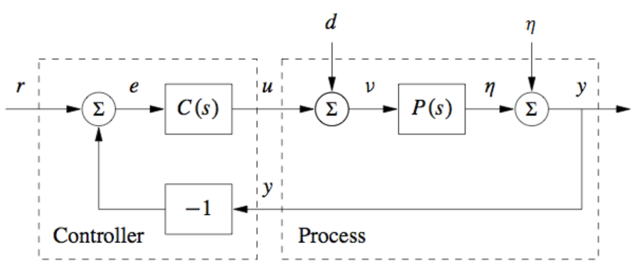

Fig. 1. Block diagram of a linear feedback loop

In this case the system in Fig. 1 is completely characterized by four transfer functions called the Gang of Four: the sensitivity function S, the complementary sensitivity function T, the load disturbance sensitivity function PS and the noise sensitivity function CS. The closed-loop specifications are typically defined in terms of inequalities on amplitude frequency responses of the Gang of Four transfer functions (moreover, the main specifications in practice are either based on the Gang of Four, or can be translated to it [9]):

- Noise attenuation at the plant output

$$|S(j\omega)| = \left|\frac{1}{1+P(j\omega)C(j\omega)}\right| \le \beta_S(\omega), \quad \forall \omega > 0. \tag{8}$$

  The sensitivity function S gives the response of the plant output to the noise measurement and describes how noises are attenuated by closing the feedback loop (noises are attenuated if $|S(j\omega)| < 1$ and amplified if $|S(j\omega)| > 1$ and the maximum sensitivity $M_s = \max_{\omega} |S(j\omega)|$ on some frequency ω_{ms} corresponds to the largest amplification of the noises; the frequency where $|S(j\omega)| = 1$ is called the sensitivity crossover frequency ω_{sc}).
- Stability

$$|T(j\omega)| = \left|\frac{P(j\omega)C(j\omega)}{1+P(j\omega)C(j\omega)}\right| < \lambda_T, \quad \forall \omega > 0. \tag{9}$$

  The complementary sensitivity function T gives the response of the control variable to the load disturbance. Inequality (9) implies that the closed-loop system is stable for substantial variations in the process dynamics, namely, variations can be large for those frequencies ω where $|T(j\omega)|$ is small and smaller variations are permitted for frequencies ω where the value $|T(j\omega)|$ is large.
- Disturbance rejection

$$|PS(j\omega)| = \left|\frac{P(j\omega)}{1+P(j\omega)C(j\omega)}\right| \le \beta_{PS}(\omega), \quad \forall \omega > 0. \tag{10}$$

  The load disturbance sensitivity function PS gives the response of the process output to the load disturbance. Since load disturbances typically have low frequencies, $|PS(j\omega)|$ typically has smaller value at low frequencies.
- Noise rejection

$$|CS(j\omega)| = \left|\frac{C(j\omega)}{1+P(j\omega)C(j\omega)}\right| \le \beta_{CS}(\omega), \quad \forall \omega > 0. \tag{11}$$

  The noise sensitivity function CS gives the response of the control variable to the noise. Since noise signals typically have high frequencies, $|CS(j\omega)|$ typically has smaller value at high frequencies and the smaller the value $|CS(j\omega)|$ the less the effect of noise on the process.

The main aim of this paper is to adapt the analysis of the Gang of Four to the nonlinear feedback system (2), (5) and to provide tuning rules in order to meet the desired closed-loop specifications.

V. PRACTICAL IMPLEMENTATION OF THE ILF CONTROL

The implementation of the control scheme presented in Theorem 2 requires to solve the equation $Q(V,x) = 0$ in order to find the ILF value V. In some cases the function $V(x)$ can be calculated analytically (for instance, the paper [14] contains an example of analytical calculation of V for $n = 2$). However, generally, these calculations are very cumbersome.

In this case the discrete-time version of the control scheme can be implemented with using a simple numerical procedure in order to find a corresponding value of V_i at the time instant t_i [14], [15].

Denote an arbitrary sequence of time instances $\{t_i\}_{i=0}^{+\infty}$, where $0 = t_0 < t_1 < t_2 < \ldots$.

Corollary 1 [15] *Let the conditions of Theorem 2 hold then the origin of the system (2) is asymptotically stable with the switching control*

$$u(x) = u(V_i, x) \quad \textit{for} \quad t \in [t_i, t_{i+1}), \tag{12}$$

where $V_i > 0 \colon Q(V_i, x(t_i)) = 0$.

The Corollary 1 shows that the sampled-time control in the form (5) keeps the robust stability property of the closed-loop system (2) independently on the sampling interval. For any fixed V between two switching instants the system (2) becomes a linear, where $V \in [V_{\min}, V_0]$, $V_0 \in \mathbb{R}_+ : Q(V_0, x_0) = 0$. The parameter $V_{\min}$ defines lower possible value of V and cannot be selected arbitrary small due to finite numerical precision of digital devices.

The parameter $V_{\min}$ can be selected by control engineers using the frequency domain analysis to be developed below with considering V as a scalar parameter of the linear control.

Thus, despite the fact that the system (2), (5) is nonlinear, for frequency domain analysis (next section) the original nonlinear plant can be replaced by a set of linear time-invariant (LTI) plants with different values of $V \in [V_{\min}, V_0]$. Note, that transformation of the nonlinear control problem in a LTI equivalent control problem is often used in nonlinear quantitative feedback theory (see, for example [28], [29]).

VI. Frequency Domain Analysis

A. The Gang of Four Derivation

From (2), (5) the transfer functions in the blocks according to the Fig. 1 at each time instance take the following form

$$P = \frac{1}{s^n}$$

and

$$C = -s^{n-1}V^{-\mu}k_n - s^{n-2}V^{-2\mu}k_{n-1} - \cdots - V^{-n\mu}k_1.$$

Amplitude frequency responses for the Gang of Four transfer functions (8)–(11) take the forms:

$$|S(j\omega)| = \frac{1}{\sqrt{(1-a)^2 + b^2}}, \tag{13}$$

$$|T(j\omega)| = \frac{\sqrt{c^2 + d^2}}{\omega^n \sqrt{(1-a)^2 + b^2}}, \tag{14}$$

$$|PS(j\omega)| = \frac{1}{\omega^n \sqrt{(1-a)^2 + b^2}}, \tag{15}$$

$$|CS(j\omega)| = \frac{\sqrt{c^2 + d^2}}{\sqrt{(1-a)^2 + b^2}}, \tag{16}$$

where

$$a = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i k_{n+1-2i}}{\omega^{2i} V^{2i\mu}},$$

$$b = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor - 1} \frac{(-1)^{i+1} k_{n-2i}}{\omega^{2i+1} V^{(2i+1)\mu}},$$

$$c = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor - 1} \frac{(-1)^i \omega^{2i} k_{2i+1}}{V^{(n-2i)\mu}},$$

$$d = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^{i-1} \omega^{2i-1} k_{2i}}{V^{(n-2i+1)\mu}}.$$

B. The Gang of Four Analysis

Since V parameterizes the control, let us consider the limits of the Gang of Four (13)–(16) for $\omega \to +\infty$ and $\omega \to 0$ when $V = const$:

$$\lim_{\omega\to+\infty} |S(j\omega)| = 1, \quad \lim_{\omega\to+\infty} |T(j\omega)| = 0,$$
$$\lim_{\omega\to+\infty} |PS(j\omega)| = 0, \quad \lim_{\omega\to+\infty} |CS(j\omega)| = \infty,$$
$$\lim_{\omega\to0} |S(j\omega)| = 0, \quad \lim_{\omega\to0} |T(j\omega)| = 1,$$
$$\lim_{\omega\to0} |PS(j\omega)| = \frac{V^{-n\mu}}{|k_1|}, \quad \lim_{\omega\to0} |CS(j\omega)| = 0,$$

and limits for $V \to 0$ when $\omega = const$:

$$\lim_{V\to0} |S(j\omega)| = 0, \quad \lim_{V\to0} |T(j\omega)| = 1,$$
$$\lim_{V\to0} |PS(j\omega)| = 0, \quad \lim_{V\to0} |CS(j\omega)| = \omega^n.$$

Looking at limits for CS, one can conclude that the system reduces the influence of noise only at significantly low frequencies, then there is a little consolation as noise usually tends to be present at high frequencies. However, it is well known that good reference signal tracking and disturbance rejection has to be traded off against suppression of process noise (thus, the control scheme allows good performance and quality parameters to be obtained in the case of the noise free system). If there is noise in the system, to reduce the value $|CS(j\omega)|$ at some frequencies the case of increasing the parameter $V_{\min}$ can be considered. Consequently, hereinafter we consider only S, T and PS transfer functions.

Let us consider the load disturbance sensitivity function PS. Since load disturbances typically have low frequencies, it is natural to focus on the behavior of the transfer function at low frequencies. As $\lim_{\omega\to0} |T(j\omega)| = 1$ we have the following approximation for small ω:

$$PS = \frac{T}{C} \approx \frac{1}{C}. \tag{17}$$

Thus the greater $|k_i|$, $i = \overline{1,n}$ or/and smaller V, the smaller the value of PS. Finaly as $\lim_{V\to0} |PS(j\omega)| = 0$ the influence of the load disturbance disappears completely. Based on continuity of the Gang of Four functions in $\mathbb{R}^n \setminus \{0\}$ the next claim can be achieved.

Claim 1 *For any β_{PS} and $\omega_c > 0$ there is V_c such that $|PS(\omega)| \leq \beta_{PS}$ for all $\omega \leq \omega_c$ and $V \leq V_c$.*

The gain curve of the load disturbance sensitivity function for $n = 4$, $k = \begin{pmatrix} -100.4298 & -95.5530 & -36.5809 & -5.8000 \end{pmatrix}$, $\mu = 0.9$ and different values of V is shown in Fig. 2.

Now consider the sensitivity function S.

Proposition 1 *Amplitude frequency response of the sensitivity function $|S(\omega, V^{-1})|$ is homogeneous of zero degree with the vector of weights $r = (1, \mu^{-1})^T$.*

Proof. Let us look at a and b as functions of two variables ω and $\tau = V^{-1}$. Then

$$a(\lambda\omega, \lambda^{\frac{1}{\mu}}\tau) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^i k_{n+1-2i} (\lambda^{\frac{1}{\mu}}\tau)^{2i\mu}}{(\lambda\omega)^{2i}} = a(\omega, \tau),$$

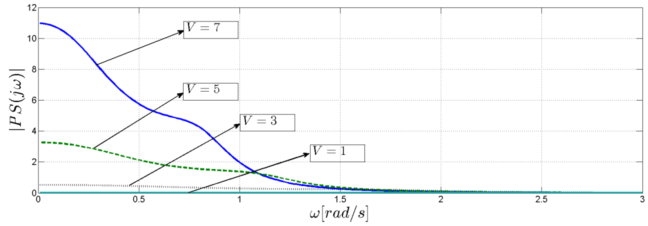

Fig. 2. Gain curve of the load disturbance sensitivity function PS

$$b(\lambda\omega, \lambda^{\frac{1}{\mu}}\tau) = \sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor-1} \frac{(-1)^{i+1}k_{n-2i}(\lambda^{\frac{1}{\mu}}\tau)^{(2i+1)\mu}}{(\lambda\omega)^{2i+1}} = b(\omega,\tau)$$

and thus,

$$|S(\omega,\tau)| = |S(\lambda\omega, \lambda^{\frac{1}{\mu}}\tau)|,$$

i.e. the function $|S(\omega, V^{-1})|$ is homogeneous of zero degree with the vector of weights $r = (1, \mu^{-1})^T$. $\square$

Based on this proposition, one can conclude that at each time instance t_i the sensitivity function $S(\omega, V_i)$ can be obtained through the previous one:

$$|S(\omega, V_i)| = \left| S\left(\left(\frac{V_i}{V_{i-1}}\right)^{\mu}\omega, V_{i-1}\right)\right|.$$

Note, that results similar to Proposition 1 can be obtained for other Gang of Four functions:

Proposition 2 *Amplitude frequency response of the complementary sensitivity function $|T(\omega, V^{-1})|$ is homogeneous of zero degree with the vector of weights $r = (1, \mu^{-1})^T$.*

Proposition 3 *Amplitude frequency response of the load disturbance sensitivity function $|PS(\omega, V^{-1})|$ is homogeneous of degree $-n$ with the vector of weights $r = (1, \mu^{-1})^T$.*

Proposition 4 *Amplitude frequency response of the noise sensitivity function $|CS(\omega, V^{-1})|$ is homogeneous of degree n with the vector of weights $r = (1, \mu^{-1})^T$.*

Proofs of Propositions 2 – 4 are based on the proof of Proposition 1 and the following expressions

$$c(\lambda\omega, \lambda^{\frac{1}{\mu}}\tau) = \lambda^n c(\omega,\tau)$$

and

$$d(\lambda\omega, \lambda^{\frac{1}{\mu}}\tau) = \lambda^n d(\omega,\tau).$$

Based on the Proposition 1, one can conclude, that the sensitivity crossover frequency ω_{sc} and the maximum value frequency ω_{ms} can be obtained through previous values and the maximum sensitivity M_s is constant independently on the ILF value V. Due to continuity of the function $|S(j\omega)|$ the next claim can be achieved.

Claim 2 *For any $\omega_c > 0$ there is $V_c > 0$ such that $|S(j\omega)| < \beta_S$ for all $\omega \le \omega_c$ and $V \le V_c$.*

The gain curve of the sensitivity function for the same parameters is shown in Fig. 3.

The similar conclusions can be made about the complementary sensitivity function T (Fig. 4).

Since there are many processes that can be described by the second order plants (for instance, mechanical planar

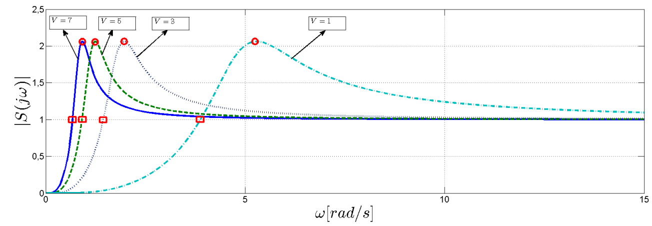

Fig. 3. Gain curve of the sensitivity function S (the sensitivity crossover frequency ω_{sc} and the maximum sensitivity M_s are designated by $\square$ and O correspondingly)

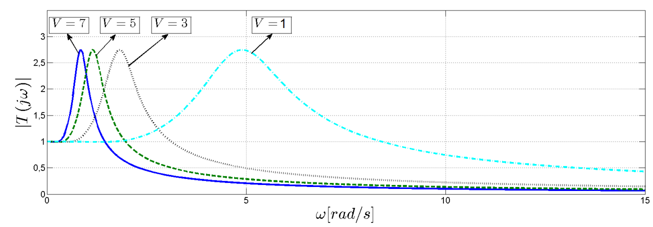

Fig. 4. Gain curve of the complementary sensitivity function T

systems), more detailed results are presented bellow for $n = 2$.

C. Second-Order Plant

According to [15] the first matrix inequality of the system (4) for $d(t,x) \equiv 0$, $\beta = 0$ can be replaced with the equality

$$\begin{cases} X_{i\ i+1} + [1+\mu(n-i)]X_{i\ i} = 0, \\ X_{i+1\ j} + X_{i\ j+1} + [2+\mu(2n-i-j)]X_{i\ j} = 0, \\ X_{i+1\ n} + [2+\mu(n-i)]X_{i\ n} + y_i = 0, \\ X_{n\ n} + y_n = 0, \end{cases} \quad (18)$$

where $j > i = \overline{1, n-1}$, and according to (18) for $n = 2$ the matrices X and y take the forms

$$X = \begin{pmatrix} X_{11} & -(1+\mu)X_{11} \\ -(1+\mu)X_{11} & X_{22} \end{pmatrix},$$

$$y = \begin{pmatrix} (2+\mu)(1+\mu)X_{11} - X_{22} & -X_{22} \end{pmatrix},$$

where X_{11} and X_{22} are chosen in order to $X > 0$ and $XH_\mu + H_\mu X > 0$, i.e. $X_{11} > 0$ and $X_{22} > X_{11}(1+\mu)^2$.

As $k = (k_1, k_2) = yX^{-1}$ one can obtain $k_1 = -\frac{X_{22}}{X_{11}}$ and $k_2 = -2-\mu$. Thus, for $n = 2$ the coefficients k_1 and k_2 can take the following values:

$$k_1 < -0.25(1+\mu)(2+\mu)^2$$

and

$$k_2 = -2-\mu.$$

Thus, according to (17), decreasing k_1 allows to get better attenuation of load disturbances at low frequencies (see Fig. 5, where $\mu = 0.5$, $V = 5$).

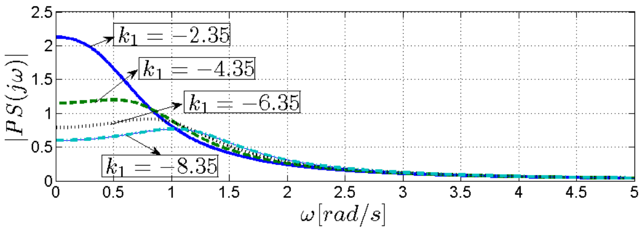

Fig. 5. Gain curve of the load disturbance sensitivity function PS for the second-order plant

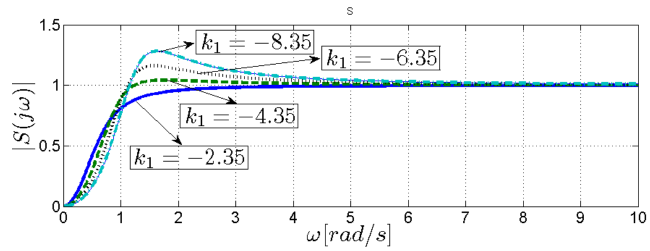

Fig. 6. Gain curve of the sensitivity function S for the second-order plant

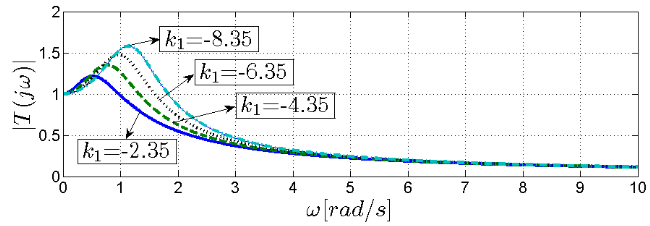

Fig. 7. Gain curve of the complementary sensitivity function T for the second-order plant

However, as for $n = 2$ the amplitude frequency response for sensitivity function according to (13) takes the form:

$$|S(j\omega)| = \frac{\omega^2}{\sqrt{\omega^4 + (2k_1 + k_2^2)V^{-2\mu}\omega^2 + k_1^2 V^{-4\mu}}},$$

one can conclude that selection of $k_1 \in (-0.5(2+\mu)^2, -0.25(1+\mu)(2+\mu)^2)$ implies noises attenuation $|S(j\omega)| < 1$ for all finite frequencies (for example, see Fig. 6). For $|k_1| > 0.5(2+\mu)^2$ the function $|S(j\omega)|$ has maximum $M_s = \frac{-2k_1}{\sqrt{-4k_1k_2^2 - k_2^4}}$ for any value V at frequency $\omega_{ms} = \sqrt{\frac{-2k_1^2V^{-2\mu}}{2k_1 + k_2^2}}$. Thus, the greater $|k_1| > 0.5(2+\mu)^2$, the greater value M_s at greater frequency ω_{ms}.

The amplitude frequency response for complementary sensitivity function T takes the form:

$$|T(j\omega)| = \frac{\sqrt{\omega^2 V^{-2\mu}k_2^2 + V^{-4\mu}k_1^2}}{\sqrt{\omega^4 + (2k_1 + k_2^2)V^{-2\mu}\omega^2 + k_1^2 V^{-4\mu}}}$$

and

$$|T(j\omega)|_{\max} = \sqrt{\frac{k_2^4\sqrt{k_1^2 - 2k_2^2k_1}}{-2k_1^3 + 4k_2^2k_1^2 + \sqrt{k_1^2 - 2k_2^2k_1}(-2k_1^2 + 2k_1k_2^2 + k_2^4)}}$$

for any value V at frequency $\omega = -k_2^{-1}V^{-\mu}\sqrt{-k_1^2 + \sqrt{k_1^4 - 2k_2^2k_1^3}}$. Therefore as $|T(j\omega)|_{\max}$ does not depend on the values of ω and V we can rewrite the specification (9)

$$\sqrt{\frac{k_2^4\sqrt{k_1^2 - 2k_2^2k_1}}{-2k_1^3 + 4k_2^2k_1^2 + \sqrt{k_1^2 - 2k_2^2k_1}(-2k_1^2 + 2k_1k_2^2 + k_2^4)}} \le \lambda_T$$

and the greater $|k_1| > 0.25(1+\mu)(2+\mu)^2$, the greater value $|T(j\omega)|_{\max}$ at greater frequency ω_{mt}. Thereby, to fulfill the specification (9) the selection of the parameter $k_1 < -0.25(1+\mu)(2+\mu)^2$ close to its maximum value is preferable.

The gain curve of the complementary sensitivity function T for the same values as in Fig. 5 and Fig. 6 is shown in Fig. 7.

VII. NUMERICAL EXAMPLE

The loop transfer function $L = PC = \frac{T}{S}$ also can be considered for frequency domain design using the same finite-time control scheme. As an example, let us consider the problem of roll control for a vectored thrust aircraft presented in [6]. The simplified model represents the double integrator system with some gain in the form

$$P(s) = \frac{r}{Js^2},$$

where $r = 0.25m$ is the force moment arm and $J = 0.0475kg \cdot m^2$ is the vehicle inertia.

Assume that the system has to meet the following performance specifications:

- the error in steady state is less than 1%;
- the tracking error is less than 10% up to 10 rad/s.

To achieve this performance specifications it is necessary to increase the crossover frequency ω_{sc} in order to have a gain at least 10 at a frequency of $10rad/s$, where $\omega_{sc} = \{\omega \in \mathbb{R}_+ : |L(j\omega)| = 1\}$ for the loop transfer function L. Absolute value of the loop transfer function L takes the form

$$|L(j\omega)| = \frac{r}{J}\sqrt{\omega^{-4}V^{-4\mu}k_1^2 + \omega^{-2}V^{-2\mu}k_2^2}.$$

It is easy to see that the value of the crossover frequency is increased for V tending to zero (see Fig. 8) and for greater values of $|k_1|$ (see Fig. 9). However, in the consideration of an argument of the loop transfer function

$$\angle L(j\omega) = \arctan\left(\frac{\omega V^{\mu}k_2}{k_1}\right)$$

one can conclude, that rather large values of $|k_1|$ give a very low phase margin (see Fig. 9) and as the functions $|L(j\omega, V)|$ and $\angle L(j\omega, V)$ are homogeneous of zero degree then phase margin is constant independently on the ILF value V (see Fig. 8).

Thus, to satisfy the performance specifications and providing of high phase margin the coefficient k_1 should be chosen close to the value $-0.25(1+\mu)(2+\mu)^2$.

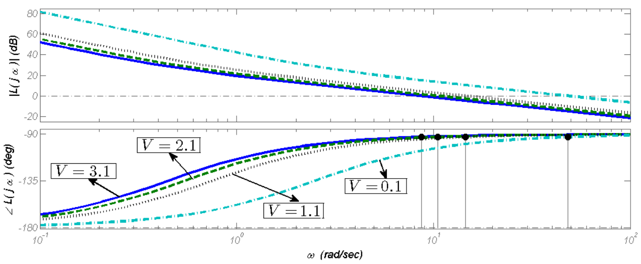

Fig. 8. Bode plot for the the loop transfer function L with different values of V

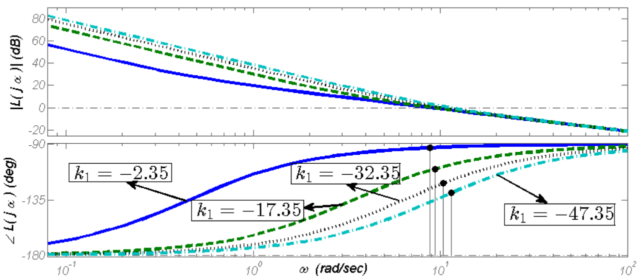

Fig. 9. Bode plot for the the loop transfer function L with different values of k_1

VIII. CONCLUSIONS

The paper presents frequency domain analysis of the finite-time control algorithm presented in [14] in order to make it more attractive for practical implementation. The analysis is performed in order to fulfill different performance specifications based on frequency representation of the Gang of Four and loop transfer function. As an example, the solution of the problem of roll control for a vectored thrust aircraft is presented.

Despite the fact that the control scheme is designed for the multiple integrators system, the frequency domain analysis can be useful in implementation of the control extended to the case of MIMO systems presented in [15]. Moreover, the analysis may be some impetus in studying the same control scheme in terms of the quantitative feedback theory, where the ILF value V is presented as $V \in [V_{\min}, V_0]$. Also it would be of interest to study other specifications, such as specifications based on the Gang of Six, for instance, and to create a special toolbox that can be useful for engineers during designing controllers. These and other aspects are selected as possible directions for further research.

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